Towards Patterns for Heaps and Imperative Lambdas

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Abstract

In functional programming, point-free relation calculi have been fruitful for general theories of program construction, but for specific applications pointwise expressions can be more convenient and comprehensible. In imperative programming, refinement calculi have been tied to pointwise expression in terms of state variables, with the curious exception of the ubiquitous but invisible heap. To integrate pointwise with point-free, de Moor and Gibbons [dMG00] extended lambda calculus with non-injective pattern matching interpreted using relations. This article gives a semantics of that language using “ideal relations” between partial orders, and a second semantics using predicate transformers. The second semantics is motivated by its potential use with separation algebra, for pattern matching in programs acting on the heap. Laws including lax beta and eta are proved in these models and a number of open problems are posed.

Dedicated to José Nuno Oliveira on the occasion of his 60th birthday.

1. Introduction

An important idea in the mathematics of program construction is to embed the programming language of interest into a richer language with additional features that are useful for writing specifications and for reasoning. Functional programs can be embedded in the calculus of relations, which provides two key benefits: converse functions as specifications and intersection of specifications. An example of the first benefit is parsing. Let $show : Tree \rightarrow String$ be the function that maps an ordered tree with strings at its leaves to the “inorder” catenation of the leaves. Its converse, $show^o$, is a relation but not a function. One seeks to derive, by algebraic reasoning in the calculus of relations, a total function $parse$ such that $show^o \supseteq parse$. See Bird and de Moor [BdM96] for many more examples. Imperative programs can be embedded in a refinement calculus [BvW98, Mor94], by augmenting the language with assumptions and angelic choice, or “specification statements” in some other form. These can be modeled using weakest precondition predicate transformers. An imperative program $prog$ satisfies specification $spec$ just if $spec \sqsubseteq prog$ where $\sqsubseteq$ is the pointwise order on predicate transformers, and again one seeks to derive $prog$ from $spec$. 

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Many authors have pointed out useful and elegant aspects of the calculus of relations for programming. Relations cater for the development of general theory by facilitating a “point free” style in which algebraic calculation is not encumbered by manipulation of bound variables and substitutions (e.g., see [Oli09]).

Although pointfree style is elegant and effective for development of general theory, it can be awkward and cryptic for developing and expressing specific algorithms. Functional programmers tend to prefer a mix of pointfree and pointwise expressions, “pointwise” meaning the use of variables and other expressions that denote data elements—application rather than composition. Pointwise reasoning involves logical quantifiers and is the norm in imperative program construction. For example, refinement laws for assignment statements involve conditions on free variables, and specifications are expressed in terms of state variables and formulas with quantifiers.

Conventional pattern matching can help raise the abstraction level in pointwise programs, by directly expressing data structure of interest. Non-injective patterns have been proposed by de Moor and Gibbons [dMG00] as a way to achieve pointwise programming with relations. Imperative programmers draw graphs to express patterns of pointer structure, but their programs are written in impoverished notation that amounts to little more than load and store instructions.

This article contributes to the long term goal of a unified theory of programming in which one may move freely between pointwise or pointfree reasoning as suits the occasion. For example, requirements might be formalized in a transparent pointwise specification that is then transformed to a pointfree equivalent from which an efficient solution is derived by algebraic calculation. A unified theory will also enable effective mixes of functional, imperative, and other styles both in program structure and in reasoning.

This article describes one approach to a programming calculus integrating functional and imperative styles, addressing some aspects of pointwise and pointfree reasoning. Some of the technical results were published in a conference paper by the author [Nau01], from which much of the material is adapted. The introductory sections have been rewritten using different examples. This article provides full details of the main semantic definitions and some results only mentioned sketchily in the conference paper, namely beta and eta laws. We cannot expect beta and eta equalities to hold unrestrictedly, as they fail already in by-value functional languages. Inequational laws are mentioned but not proved in [dMG00] and [Nau01]. Here we prove weak beta and eta laws for both relational and predicate transformer semantics. We also pose several open problems.

Outline. The rest of this article is organized as follows. Section 2 begins with motivation, focusing on higher types and the idea of non-injective patterns. We show by example how non-injective patterns could be used in imperative programming including pointer programs. This idea helps motivate the predicate transformer semantics but is not otherwise developed in this article. Section 2
also surveys related work on alternate approaches to programming calculi integrating pointwise with pointfree and functional with imperative styles.

Section 3 reviews the standard semantics of simply typed lambda calculus in a cartesian closed category, in particular Poset. Section 4 describes the category of ideal relations, motivated by difficulties with semantics in [dMG00]. Section 5 gives our relational semantics. Section 6 gives a simulation connecting relational and functional semantics, and proves the lax beta and eta laws that are our main results for relational semantics. Section 7 gives the predicate transformer model and semantics. Section 8 proves the main results for transformer semantics. Section 9 assesses the work and discusses open problems.

For Section 3 onwards, the reader should be familiar with predicate transformer semantics [BvW98] and with basic category theory including adjunctions and cartesian closure [Gun92]. Span constructions and lax adjunctions are only mentioned in passing, and “laxity” appears only as an informal term that indicates the weakening of equations to inequations.

2. Motivation and background

Motivation. One attraction of pointfree style is that it facilitates derivation of programs that are “polypthic”, i.e., generic in some sense with respect to type constructors [BJJM98]. For example, a polynomial functor on a category of data types may have a fixpoint; its values are trees of some form determined by the particular functor. If the element type has a well ordering, one can define the function repmin that sends tree $t$ to the tree $t'$ of the same shape but where each leaf of $t'$ is the minimum of the leaves of $t$. De Moor gives a pointfree derivation of repmin, at this level of generality, using type constructions and equational laws that can be interpreted in functions or in relations [dM96].

Relations can model demonic nondeterminacy [dMG00] or angelic nondeterminacy (as in automata theory and in logic programming), but not both — unless states or data values are replaced by richer structures such as predicates. The present author showed that the algebraic structure needed for the polypthic repmin derivation exists in the setting of monotonic predicate transformers [Nau98a].

Although the repmin problem only involves first order data (trees with primitive, ordered data), the derived solution involves higher order: It traverses the input tree to build a closure that, when applied to a value, builds a tree of the same shape with that value at its leaves. This brings us to a question about how to embed a programming language in a richer calculus for specification and derivation. In the language of categories, taking data types as objects and programs as arrows, the question is what objects to use for arrow types. For each pair $B, C$ of objects, a function space $B \Rightarrow C$ exists as an object in the category Rel of binary relations, and indeed as an object in the category of monotonic predicate transformers. But $B \Rightarrow C$ is not the “internal hom” or exponent in Rel. There should be some account of what it means to reason with exponents in Rel if the derived program is interpreted as a functional one.
Pointfree reasoning is not without its shortcomings. De Moor and Gibbons observe that for many specific programming problems a pointwise formulation is easier to understand. They extend pointwise functional notation to relations by means of non-injective patterns. As a simple example, the following is intended to define a relation that performs an arbitrary rotation of a list:

\[ \text{rotate}(x + + y) = y + + x \quad (1) \]

An input list \( w \) relates to all \( y + + x \) such that \( w = x + + y \). Let us consider in detail how a pattern term gives rise to a relation from a set \textit{in} of inputs to a set \textit{out} of outputs. The pattern is an expression with free variables \textit{vars} that are also free in the result expression. The situation looks like this:

As indicated by the dotted arrow, the semantics of the pattern term is the composition \( \text{pattern}^{\circ} ; \text{expr} \). Throughout this article, \( (\cdot) \) denotes forwards relational composition.

This way of obtaining relations is connected to one systematic approach to embedding programs in richer calculi. A \textit{span} in a category is a pair of arrows with common source, like \textit{pattern} and \textit{expr} above. The category \textbf{Rel} is equivalent to the category whose arrows are spans over the category \textbf{Fun} of sets and functions. We refrain from elaborating on the construction but note that there are several variations, one of which is a lax span construction that not only gives \textbf{Rel} from \textbf{Fun} but also monotonic predicate transformers from \textbf{Rel} \[GMdM94, Nau98b\].

Let us connect spans with familiar elements of imperative programming. Consider the humble assignment \( x := x + 1 \). To specify it, beginners often write \( x = x + 1 \) as postcondition, but only miraculous or divergent programs establish postcondition \textit{false}. What is needed is an auxiliary variable ("logical constant") \( u \), used in the specification

\[ \text{pre} : \quad x = u \quad \text{post} : \quad x = u + 1 \]

This is interpreted by \( u \) being universally quantified over \textit{pre} and \textit{post}, one instance being the useful one that makes \( u \) serve to name the initial value of \( x \). Imperative specification notations often feature notation for the special case where the auxiliary is merely equated with an initial value, but the general form is needed, for example to obtain complete laws for sequential composition of specifications \[Mor94\]. The general form comprises \textit{pre} and \textit{post} that are relations between the program state and auxiliary state.

\[ \text{pre} \leftarrow \text{aux} \rightarrow \text{post} \]

\[ (2) \]
The leftward dotted arrow indicates that imperative programs in refinement calculus can be modeled by predicate transformers, i.e., from postconditions to preconditions. Still more, the category of monotonic predicate transformers is equivalent to the lax span category over \( \text{Rel} \) \cite{GMdM94}.

One way to view \( \xi \) is that one agent (the angel) chooses the auxiliary value for which the other agent (the demonic program) chooses an output. Indeed, non-injective patterns can express nondeterministic choice, e.g., the choice between functions \( f \) and \( g \) can be written as \( h \) where \( h(fst(x,y)) = f \) if \( y \) else \( g \) \( x \). Here \( fst \) is the left projection from \( A \times \text{bool} \) where \( A \) is the input type of \( f \) and \( g \). In predicate transformer semantics this choice turns out to be angelic, which is consistent with the standard semantics, using \( \exists \), of logical constants \cite{Mor94}.

Prior work on the approach explored here. Although the lax span perspective provides an elegant connection between \( \text{Fun}, \text{Rel} \), and predicate transformers, it does not directly account for exponents in these categories. Because we are interested in refinement—an order relation on arrows—the exponents, as internal homs, should reflect the ordering. This led to the investigation of how, by starting from \( \text{Poset} \) instead of \( \text{Fun} \), one obtains a more robust category of “ideal” relations and then a category—called \( \text{Tran} \) in the sequel—of monotonic predicate transformers where predicates themselves are monotonic \cite{Nau98a}. One step towards a programming calculus based on \( \text{Tran} \) is to show \cite{Nau98c} that (lax) polynomial functors have unique fixpoints; the lax exponent serves to derive a variation on Lawvere’s parameterized recursion theorem. Further steps are taken in \cite{Nau98a} which adds the pointfree theory of containers/membership \cite{HdM00} and uses that, together with the recursion theorem, to recreate a derivation of de Moor’s \cite{dM96} pointfree \textit{repmin} solution at the level of \( \text{Tran} \). What is still not achieved after all these years is to derive a solution that uses shared pointer structure: updating all leaves of the tree to point to a single cell that is updated with the minimum of the tree, once that has been determined.

Towards refinement calculus for pointer programs. Extant refinement calculi for imperative programming do not address pointer programs except by explicit encodings of the heap using arrays. For \textit{post hoc} program verification there has been a great deal of progress using separation logic. The assertion language features the separating conjunction, \( * \), and assertions are often in the form of a top level separation \( P * Q * \ldots * R \) that expresses some way of partitioning heap cells into disjoint sets that satisfy the predicates \( P, Q, \ldots \). Informally, the sets are referred to as “footprints” of predicates. Some static analysis tools use separated conjunctions as “symbolic heaps” in symbolic execution. Underlying the semantics of the assertion language is a notion of separation algebra, whose expressions denote (partial) heaps and whose operations manipulate heaps. However, for reasoning about programs, separation logic distinguishes programs from specifications rather than embedding one in the other as in refinement calculus.

\[1\] A recent reference that emphasizes separation algebra is the book by Appel et al \cite{App14}.
And reasoning must be done in terms of assertions; put differently, in terms of expressions only of type ‘proposition’.

One of the challenges in reasoning about pointer programs is to describe structural invariants such as reachability, uniqueness of references, separation, and confinement. Separation facilitates reasoning about independence of writes, i.e., frame conditions. This is embodied in the frame rule of separation logic:

\[
\text{from \ } \{P\} \text{ \ cmd \ } \{Q\} \ \text{ infer \ } \{P \ast R\} \text{ \ cmd \ } \{Q \ast R\} .
\]

This is sound because the footprint of precondition \(P\) serves as frame condition for the command: the antecedent implies that \(\text{cmd}\) does not write outside that part of the heap. The precondition \(P \ast R\) says that \(R\) is true of a disjoint set of heap cells, so it cannot be falsified by writes to the cells that support \(P\).

By contrast with separation, confinement facilitates reasoning about independence of reads. One application is in reasoning about simulations between data representations: Confinement helps ensure that the behavior of a client of an abstract data type is independent from the internal representation because the client has no pointer into the representation. Wang, Barbosa, and Oliveira [WBO08] extend separation logic with connectives that express confinement. For example, \(P \nRightarrow Q\) says that not only are the footprints of \(P\) and \(Q\) disjoint but no cell in the footprint of \(P\) has a pointer to a cell in the footprint of \(Q\). This is an elegant advance on works which use ad hoc means to express confinement (e.g., [BN05, BN12]). Dang and Möller [DM15] take further steps in this direction.

Another challenge in reasoning about pointer programs is that programming languages are impoverished in their means to express heap operations — it amounts to little more than load and store instructions. Separation logic was a huge advance in reasoning about such programs, which involves assertions about many intermediate states in which interesting invariants are temporarily broken. Informally, what works effectively is diagrams.

Consider, for example, an iterative algorithm that reverses, \emph{in situ}, an acyclic doubly linked list. The algorithm uses pointer variable \(p\) pointing to the reversed segment and \(n\) that points to the first node of the segment that remains to be reversed. The loop body makes three heap updates and moves \(p\) and \(n\) forward, transforming the left-hand situation into the right-hand one.

\[
\begin{array}{c}
\text{left} \quad \text{middle} \quad \text{right} \\
\end{array}
\]

Each rectangle represents a cell with two pointers, which in a high level language could be fields named \textit{next} (upper box) and \textit{prev} (lower) for the forward and
backward pointers. Here is another diagram that depicts the transformation.

Dashed arrows indicate the final state. Slashed circles indicate values in the initial state that get updated. For example, the left cell’s `prev` is initially `nil` and the right cell’s `prev` gets set to `nil`.

The following notional code is a tail-recursive procedure `rev1` that uses pattern matching to perform the transformation in a single step.

\[
\text{rev1 } p \ n \ \text{heap} = \text{match heap with}
| p \mapsto q, \text{nil} * n \mapsto \text{nil}, \text{nil} \quad \mapsto
| p \mapsto q, n \ * n \mapsto p, \text{nil}
| p \mapsto q, \text{nil} * n \mapsto r, \text{nil} * r \mapsto s, n \quad \mapsto
\text{rev1 } n \ r \ (p \mapsto q, n \ * n \mapsto p, \text{nil} * r \mapsto s, \text{nil})
\]

The second pattern clause matches the diagram. (The first clause handles the situation where \(n\) points to the last node.) As usual in pattern matching, free variables on the left of the pattern arrow \(\mapsto\) are bound by the match. That is indicated in the diagram by labels \(q, r, s\).

The notation is adapted from separation logic and from OCaml. The difference from OCaml’s match construct is the big arrow \(\mapsto\) chosen to fit with other notations in this article. The expression \(x \mapsto y, z\) is inspired by the points-to predicate in separation logic. It denotes a heap with a single cell, referenced by \(x\), with \(y, z\) as the values of the `next` and `prev` fields. Think of the heap as a mapping from references to pairs, so \(x \mapsto y, z\) simply denotes a singleton mapping. Infix `*` denotes the partial function that forms the union of two heaps, provided their domains are disjoint, and is undefined otherwise. Comma, for pairing, binds tighter than \(\mapsto\), which in turn binds tighter than `*`.

The initial call to `rev1` should be `rev1 root n heap` where `root` points to the first node and \(n\) is that node’s `next` (its `prev` being `nil`).

One wishful feature of the code for `rev1` is that the patterns are local, in the sense that they describe only the relevant cells. Given that the heap is an explicit argument and return value, perhaps the pattern should match the entire heap. To see how that might look, consider the following putative definition of

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2Diagrams can mislead, of course. In the general case, the rightmost and leftmost pointers, in and out of the clouds that indicates the rest of the heap, may not exist.
reverse, which makes the top level call to rev1 for lists of plural length.

\[
\begin{align*}
\text{reverse } \text{nil } hp &= hp \\
\text{reverse } p \ (p \mapsto \text{nil},\text{nil} \ast hp) &= (p \mapsto \text{nil},\text{nil} \ast hp) \\
\text{reverse } p \ (p \mapsto n,\text{nil} \ast hp) &= \text{rev1} \ p \ n \ (p \mapsto n,\text{nil} \ast hp)
\end{align*}
\]

It would be nicer for hp to be implicit, by analogy with the frame rule (3). An obvious approach would be to use a state monad. Another possibility is to find algebraic notation and laws whereby a whole-heap operation can be derived from one written using a local-heap pattern term. Laws are also needed to transform heap patterns into some restricted form that can be automatically compiled to code.

In separation algebra the heap union operator, $\ast$, is partial. For example, $x \mapsto y, z \ast x \mapsto u, w$ is not defined. Our wishful pattern notation needs to be interpreted in a setting that admits patterns that are non-total as well as non-injective. The calculus of de Moor and Gibbons allows general relations in patterns—but there are unsettled issues concerning refinement monotonicity at higher types, as discussed in Section 3. In our relational model, the issues are addressed by restricting pattern terms to total functions. The predicate transformer model supports relational pattern terms.

Related work on other approaches. It does not appear that the specific calculus of de Moor and Gibbons has been developed beyond the original paper [dMG00].

Hinze makes nice use of explicit powersets as a way to do pointwise relational programming [Hin02].

Bunkenburg [Bun97] develops a calculus of “expression refinement”, for derivation of lazy functional programs including those that use the state monad. Ordering is taken into account by treating data types as ordered sets. Nondeterministic expressions are interpreted as monotonic functions $f : A \rightarrow UB$ where $UB$ is the set of updeals (upward closed subsets) on $B$. The set $UB$ is ordered by $\supseteq$, which coincides with the Smyth order (total correctness) restricted to upward closed sets. Upward closure is natural for outcomes of expressions. Monotonic functions $f : A \rightarrow UB$ are naturally isomorphic to ideals, as is discussed in Section 4. So there is some overlap with the present work, although our language (and [dMG00]) is by-value. Bunkenburg develops a wide spectrum language and a logic of refinement, but the language does not include pattern matching.

Uses of angelic and demonic nondeterminacy are explored by Back and von Wright [BvW98] and by Morris and Tyrrell [MT08a]. Martin, Curtis, and Rewitzky [MCR07] use multi-valued relations as an alternative to predicate transformers for modelling the combination of angelic and demonic nondeterminacy, and adapt map/fold algebra to that model [MC08].

Morris, Bunkenberg, and Tyrrell [MBT09] introduce a novel “phrase” construct, and “term transformer” semantics, as a technique for specifying and reasoning about imperative programs; it generalizes predicate transformers to terms of any type. Their approach encompasses unbounded angelic and demonic nondeterminacy together with higher order functions, for which Morris
and Tyrrell present numerous laws [MT08a]. They sketch how non-injective patterns may be reduced to nondeterminacy. They have beta-reduction as an equality, unlike our results in the sequel, owing to partitioning terms as proper or not, and use of a semantic monotonicity restriction. Morris and Tyrrell have also shown equivalence between monotonic predicate transformers, multirelations, and free completely distributive lattices [MT08b]. The phrase construct is very appealing but quite different from the functional notations explored in this article, and from conventional notations of refinement calculi.

For first order concurrent programs, concurrent Kleene algebra provides a pointfree algebraic setting for reasoning about separation, mutable state and concurrency (e.g., [HvSM+14]). Modal Kleene algebra is the basis for the extended separation logic of Dang and Möller [DM15], which they use to verify algorithms including list reversal and tree rotation, expressed in terms of load and store commands. Banerjee et al [BNH13] use explicit expressions to designate footprints of pointer structures, but only for post hoc verification.

3. Functional semantics

This section reviews the categorical semantics of simply-typed lambda calculus in the order-enriched category Poset of monotonic functions on (small) posets.

An order-enriched category is one with homsets partially ordered and composition monotonic. Each homset Poset(B, C) carries the pointwise ordering \( f \preceq g \equiv (\forall b. fb \preceq_C gb) \). Composition of \( f \in \text{Poset}(B, C) \) with \( h \in \text{Poset}(C, D) \), written \( (f \circ h) \) as with relations in general, is monotonic in both \( f \) and \( h \). We write \( \times \) for binary product and \( (f, g) : B \to C \times D \) for pairing of \( f : B \to C \) with \( g : B \to D \). For projection we write \( \pi \) with subscripts or some other indication of which projection is meant. We write \( B \Rightarrow C \) for function space as an exponent object of Poset, ordered pointwise. Let \( \Rightarrow \) bind less tightly than \( \times \). The application function is denoted by \( \text{ap}_f : (B \Rightarrow C) \times B \to C \) and currying sends \( f : B \times C \to D \) to \( \text{cur}_f : B \Rightarrow C \Rightarrow D \). The decorative subscript “\( f \)” distinguishes the functional constructs from those for ideals (i) and predicate transformers (t) to come.

**Proposition 1.** Poset is cartesian closed and the adjunctions \( (B \times \cdot) \dashv (\Rightarrow \cdot) \) are order-enriched.

Order enrichment, in the context of Proposition 1, simply means that pairing, currying, and the exponent and product functors are monotonic.

The functional terms \( M \) are those of simply-typed lambda calculus. Types are given by

\[
\sigma ::= b \mid 1 \mid \sigma \times \sigma \mid \sigma \to \sigma
\]  

where \( b \) ranges over some given set of base types. Terms are generated from given constants \( c \) and variables \( x \):

\[
M ::= x \mid c \mid \text{fst} \mid \text{snd} \mid (M, M) \mid MM \mid \lambda x. M
\]  

9
Typing judgements take the form $\Gamma \vdash M : \sigma$ where $\Gamma$ is a list $x_0 : \sigma_0, \ldots, x_n : \sigma_n$ of variable typings. The typing rules are standard and omitted [Gun92]. We allow constants at all types. The type of a constant is assumed given.

We write $\mathcal{F}[-]$ for the functional semantics, but omit $\mathcal{F}$ in this section. Functional semantics is based on a given poset $\mathcal{B}$ for each base type $b$. Given a singleton set $[1]$ as well, the semantics of types is given inductively by $[\sigma \times \sigma'] = [\sigma] \times [\sigma']$ and $[\sigma \rightarrow \sigma'] = [\sigma] \hookrightarrow [\sigma']$. For $\Gamma = x_0 : \sigma_0, \ldots, x_n : \sigma_n$, we write $[\Gamma]$ for the product $\prod x \colon \sigma \vdash M : \sigma$. For terms, we assume that for each constant $c : \sigma$ a function $f : [\Gamma] \rightarrow [\sigma]$ is given, and let $[\text{fst}] : [1] \rightarrow [\sigma \times \sigma' \rightarrow \sigma]$ pick out the left projection function (as an element of the set $[\sigma \times \sigma' \rightarrow \sigma]$). The semantics of $\Gamma \vdash M : \sigma$ is a morphism $[\Gamma] \rightarrow [\sigma]$, defined in a standard way in Table 1.

4. Ideals

This section motivates and defines the category $\text{Idl}$ of ideal relations, starting from a sketch of difficulties that arise in the work of De Moor and Gibbons [dMG00] using the category $\mathcal{Rel}$ of relations on sets. We describe the embedding of $\text{Poset}$ in $\text{Idl}$, the power adjunction which gives an embedding the other way around, and the lax cartesian closed structure used in Section 5 for semantics. Results that are not proved here can be found in [Nau98b] or [Nau01].

The relational semantics of $\mathcal{Rel}$ interprets arrow types using the set $A \leadsto B$ of relations from $A$ to $B$. For relation $R : B \times C \rightarrow D$ the pre-curried relation $\text{cur } R : B \rightarrow C \leadsto D$ is defined by

$$b(\text{cur } R)S \equiv (\forall c, d : cSd \equiv (b, c)Rd) \quad (6)$$

But $\text{cur }$ is not monotonic with respect to $\supseteq$, and monotonicity with respect to refinement is crucial for a useful calculus and for recursion. De Moor and Gibbons address monotonicity by replacing $\supseteq$ with a refinement order defined by $R \preceq R' \equiv R ; \preceq \supseteq R'$ for $R, R'$ of type $B \rightarrow C \leadsto D$. (Small font is used when order relations are composed.) In terms of points: $R \preceq R'$ iff for all $b, S$,

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See, for example, [Gun92]. Our formulations use only binary products. The “appropriate projection” for $x_k$ is the evident morphism $[\Gamma] \rightarrow [x_k]$ defined using binary projections and pairing.
bR′S implies there is S′ with bRS′ and S′ ⊇ S. At higher types, refinement is defined inductively:

\[ R \preceq R' \equiv R; z \supseteq R' \]  

(7)

where the small \( \preceq \) is the refinement order on the target type. Observe that \( \preceq \) reduces to \( \supseteq \) for relations such that \( R; z = R \). The ordering has a problem: \( (Q; R) \) fails to be monotonic in Q, e.g., if \( R : C \rightarrow D \rightarrow C \rightarrow D \) sends a relation to its complement. Our solution is to rule out complementation on grounds of non-monotonicity. A function \( f : B \rightarrow C \) is monotonic iff \( f; \preceq \supseteq \preceq \) for relations such that \( R; \preceq = R \). The ordering has a problem:

\[ (Q; R) \text{ fails to be monotonic in } Q, \text{ e.g., if } R : C \rightarrow D \rightarrow C \rightarrow D \text{ sends a relation to its complement}. \]

If \( R \) is monotonic then so is \( R; \preceq \). A relation is monotonic and satisfies \( R; \preceq = R \) if it is an ideal. An ideal is a relation \( R : A \rightarrow B \) such that

\[ z_A; R; z_B \subseteq R \]

In terms of points: \( a' \preceq a, aRb, \text{ and } b \preceq b' \implies a'Rb' \).

Ideals are the morphisms of the order-enriched category \( \text{Idl} \) whose objects are all (small) posets. Composition \( (\cdot) \) in \( \text{Idl} \) is the same as in \( \text{Rel} \), but the identity on \( A \), written id, is \( z_A \). Homsets are ordered by \( \supseteq \), not \( \subseteq \), for reasons mentioned later.

**Ideals, power adjunction, and products.** A comap in an order-enriched category is an arrow \( g \) with a corresponding map or left adjoint \( g^* \). That is, \( g \) and \( g^* \) satisfy

\[ \text{id} \preceq g^*; g \quad \text{and} \quad g; g^* \preceq \text{id} \]  

(8)

It is common to order \( \text{Rel} \) by \( \subseteq \) and describe \( \text{Fun} \) as the subcategory of maps of \( \text{Rel} \); then the comap of a function \( f \) is its converse \( f^\circ \). Maps in order-enriched category \( \text{C} \) are the comaps of the arrow-dual \( \text{C}^{op} \) and they are also comaps in the order-dual \( \text{C}^{co} \) obtained by reversing the order on homsets. As \( \text{Rel} \) is isomorphic to both \( \text{Rel}^{op} \) and \( \text{Rel}^{co} \), one has several opportunities for making infelicitous choices of nomenclature. Our choices smooth some parts of the exposition, but have the unfortunate consequence that functions embed as comaps in \( \text{Idl} \). The reader should keep in mind that, in this article, the operation \( * \) gives the map for a comap, rather than the reverse. For example, in \( \text{Idl} \) we have the shunting property

\[ c; R \supseteq S \iff R \supseteq c^*; S \quad \text{for comap } c \]  

(9)

which is a standard consequence of \( [8] \), instantiating \( \preceq \) with the order \( \supseteq \) on homsets of \( \text{Idl} \).

Because functions are a special case of relations, \( \text{Fun} \) is included in \( \text{Rel} \). But monotonic functions are not a special case of ideal relations. If \( f \) is in

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4The symbols \( \subseteq \) and \( \supseteq \) always have their usual set-theoretic meaning, whereas \( \preceq \) is used generically for orderings. We write \( \text{id} \) only for identity functions, writing either \( \text{id} \) sans serif or \( \preceq \) for the identity in \( \text{Idl} \), depending on which seems most perspicuous.
\textbf{Poset}(A, B) and R is in \textbf{Idl}(B, C) then \((f; R)\) is in \textbf{Idl}(A, C). As a consequence, there is a \textit{graph functor} \(\text{Gr} : \text{Poset} \to \text{Idl}\) defined by \(\text{Gr} f = (f; \varepsilon)\) and \(\text{Gr} A = A\). We can sometimes elide \(\text{Gr}\) because

\[ \text{Gr} f ; R = f ; R \quad \text{for ideal } R. \] (10)

The graph functor is an order injection on homsets, because the pointwise ordering of functions \(f \preceq g\) is equivalent to the inclusion \((f ; \varepsilon) \supseteq (g ; \varepsilon)\).

An unfortunate feature of \(\text{Idl}\) as compared with \textbf{Rel} is that the converse of an ideal \(R : B \to C\) need not be an ideal of type \(C \to B\) unless \(B\) and \(C\) are discretely ordered. Note that \(R^* : C \to B\) is an ideal, where \(B\) is the order dual of \(B\), i.e., \(\preceq_B^\circ = (\preceq_B)^\circ\).

More importantly, for \(f : B \to C\) in \textbf{Poset}, \((\varepsilon ; f^\circ) : C \to B\) is an ideal; in fact it is the left adjoint of \(\text{Gr} f\). The \textit{opgraph functor} \(\text{Rg}\) is defined as

\[ \text{Rg} : \text{Poset}^{co \text{op}} \to \text{Idl} \quad \text{Rg} f = (\text{Gr} f)^* \]

So we have \(\text{id} \supseteq \text{Rg} f ; \text{Gr} f\) and \(\text{Gr} f ; \text{Rg} f \supseteq \text{id}\).

A functor \(G\) on order-enriched categories is an \textit{embedding} if it is an order injection on homsets, i.e., \(f \preceq h \equiv Gf \preceq Gh\) for all \(f, h\), and it is bijective on objects.

\textbf{Proposition 2.} \(\text{Gr}\) embeds \textbf{Poset} onto the comaps of \(\text{Idl}\).

Recall from Section 2 that \(UB\) is the set of updeals, i.e., upward closed subsets, of \(B\), ordered by \(\supseteq\). We use the power adjunction \([PS90]\) to characterize updeal lattices \(UB\) in \(\text{Idl}\). The converse \(\exists\) of the membership relation is an ideal \(UB \to B\); note that \((\exists ; \preceq_B) \subseteq \exists\) says that \(UB\) contains only updeals. For \(R\) in \(\text{Idl}(B, C)\), the function \(\Lambda R : B \to UC\) sends \(b\) to its direct image through \(R\) (i.e., the set of \(c\) such that \(bRc\)). The monotonic functor \(U : \text{Idl} \to \text{Poset}\) is defined on morphisms by \(UR = \Lambda R\).

\textbf{Proposition 3.} \(f = \Lambda R \equiv f ; \exists = R\), for all \(R, S\) in \(\text{Idl}\) and \(f\) in \textbf{Poset}. This is order-enriched: \(R \supseteq S \equiv \Lambda R \preceq \Lambda S\), where \(\preceq\) is the pointwise order on functions, lifted from the order \(\supseteq\) on \(UC\). Furthermore, \(U\) is an embedding of \(\text{Idl}\) in \textbf{Poset}, and \(\text{Gr}\) is left adjoint to \(U\) with counit \(\exists\) and unit the upward closure. This is order enriched: \(f \preceq g \equiv \text{Gr} f \supseteq \text{Gr} g\).

Owing to our choice of ordering on \(\text{Idl}\), the power adjunction is order-enriched (by contrast with \([BdM96]\))

For the product \(A \times B\) of posets, the projection function \(\pi\) in \textbf{Poset}(\(A \times B, A\)) gives a comap \(\text{Gr} \pi\) in \(\text{Idl}(A \times B, A)\). We overload notation and write \(\langle R, S \rangle : D \to A \times B\) for the pairing of \(R : D \to A\) with \(S : D \to B\), which is defined by \(d(R, S)(a, b) = dRa \& dSb\) just as in \textbf{Rel}. Clearly \(\langle \text{Gr} f, \text{Gr} g \rangle = \text{Gr}(f, g)\). Defining \(Q \times R\) as usual makes \(\times\) an extension of the product functor on \textbf{Poset} in the sense that \(\text{Gr} f \times \text{Gr} g = \text{Gr}(f \times g)\).
Proposition 4. $\times$ is a monotonic bifunctor on $\text{Idl}$ and

$$R \supseteq \langle R, S \rangle ; \text{Gr} \pi \quad (=, \text{if } S \text{ is a comap}) \quad (11)$$

$$S \supseteq \langle R, S \rangle ; \text{Gr} \pi' \quad (=, \text{if } R \text{ is a comap}) \quad (12)$$

$$\langle R ; \text{Gr} \pi, R ; \text{Gr} \pi' \rangle \supseteq R \quad (=, \text{if } R \text{ comap}) \quad (13)$$

Here $\pi$ and $\pi'$ are the left and right projections. We let composition ($i$) bind more tightly than the pairing comma, as in these laws used later:

$$\langle R ; S, R ; U \rangle \supseteq R ; \langle S, U \rangle \quad (=, \text{if } R \text{ comap}) \quad (14)$$

$$\langle R, S \rangle ; (U \times V) = \langle R ; U, S ; V \rangle \quad (15)$$

Relation space. In the same way that $\times$ extends to a functor on $\text{Idl}$, the sum and function-space constructs can be extended. But the function space is not an internal hom of $\text{Idl}$, which is what we need to interpret a lambda calculus. We now construct currying and application for the exponent defined by $B \leadsto C = \text{Idl}(B, C)$. The impatient reader may care to skip to Prop. 5 which summarizes what is needed in the sequel.

In $\text{Rel}$, an exponent is given by the adjunctions $(\times B) \vdash (B \times)$. Although $\times$ does not have this property in $\text{Idl}$, there is a related functor denoted by $\kappa$ that gives $(\times B) \vdash (B \kappa)$ in $\text{Idl}$. Formally, this gives an interpretation of lambda terms, but a strange one because it does not extend $\Rightarrow$. We use $\kappa$ only as a stepping stone. The currying operation $\text{cur}_\times$ associated with $(\times B) \vdash (B \times) \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
An upward closed subset of $B \times C$ is, by definition of the order, an ideal, so $B \rightharpoonup C$ is another name for the poset $\text{Idl}(B, C)$. For ideals $R : B \times C \rightarrow D$ and $S : B \rightarrow C \rightharpoonup D$ we define $\text{cur}_R : B \rightarrow C \rightharpoonup D$ and $\text{uncur}_S : B \times C \rightarrow D$ by

$$\text{cur}_R = \text{Gr}(\Lambda(\text{cur}_x R)) \quad \text{uncur}_S = \text{uncur}_x (S ; \exists)$$

Observe that $\Lambda(\text{cur}_R)$ is the monotonic function sending $b \in B$ to the upideal $\{(c, d) \mid (c, b) \in R\}$, which is essentially “$R$ curried on $b$” (cf. (6)). Monotonicity here means that $b \preceq b'$ implies $\Lambda(\text{cur}_R)b \supseteq \Lambda(\text{cur}_R)b'$.

By definition of $\text{Gr}$ and the ordering $\supseteq$ on $C \rightharpoonup D$, an element $b$ is related by $\text{cur}_R$ to all ideals contained in $\{(c, d) \mid (b, c) \in D\}$. Clearly $\text{cur}_R$ is monotonic in $R$, as $\Lambda$ and $\text{cur}_x$ are monotonic. A crucial fact about $\text{cur}_R$ is that it is a comap, for any $R$. This follows by definition of $\text{cur}_i$ and Prop. 2. We also have $\text{uncur}_i(\text{cur}_i R) = R$ by a straightforward calculation using power and $\times$ adjunctions. For $\text{cur}_i(\text{uncur}_i S)$, however, unfolding the definitions and using $\times$ adjunction yields $\text{cur}_i(\text{uncur}_i S) = \text{Gr}(\Lambda(S ; \exists))$ and then power adjunction gives

$$\text{cur}_i(\text{uncur}_i S) = S \quad \text{if } S \text{ is a comap} \quad (18)$$

The condition is necessary; the range of $\text{cur}_i$ is comaps so it is not surjective. Unfolding the definitions at the level of points, we have $(b, c)(\text{uncur}_i S)d \iff (\exists R. bSR \wedge cRd)$, and thus $b(\text{cur}_i(\text{uncur}_i S))U \iff \{(c, d) \mid (\exists R. bSR \wedge cRd)\} \supseteq U$.

The largest such $U$ is thus a “convexification” of $S$: For given $b$, one can say that $S$ nondeterministically chooses a $V$ which for given $c$ nondeterministically chooses result $d$; whereas the largest $U$ above combines all the nondeterministic outcomes of all $V$. So $\text{cur}_i(\text{uncur}_i S)$ is $\text{Gr} f$ for the function $f$ such that $f b$ is the convex closure of $S$ for $b$. From this it is clear that

$$\text{cur}_i(\text{uncur}_i S) \supseteq S \quad \text{for all } S \quad (19)$$

Defining $\text{ap}_i = \text{uncur}_i \text{id}$ makes $\text{ap}_i$ an ideal with $(R, b)\text{ap}_i c \equiv bRc$. (It is not a comap.) Note that $\text{uncur}_i \text{id} = \text{uncur}_i \exists$ with $\exists$ taken to be an ideal $(B \leftrightharpoons C) \rightarrow B \times C$ (recall (17)). Application extends that for functions, in the sense that

$$(\text{Gr} f, a)\text{ap}_i b \iff (f, a) (\text{Gr} \text{ap}_i) b \iff fa \preceq b$$

We have $\text{uncur}_i S = (S \times \text{id}) ; \text{ap}_i$ (recall that $\text{id}$ is $\leq$). For $R : A \rightarrow B$ and $S : C \rightarrow D$, we define $R \leftrightharpoons S : B \leftrightharpoons C \rightarrow A \leftrightharpoons D$ by

$$R \leftrightharpoons S = \text{cur}_i((\text{id}_{B \rightarrow C} \times R) ; \text{ap}_i ; S)$$

as usual. This extends $\Rightarrow$ because $U(R \leftrightharpoons S)V \equiv R ; U ; S \supseteq V$. Note that $R \leftrightharpoons S$ is a comap for all $R, S$, because the range of $\text{cur}_i$ is comaps.

**Proposition 5.** $\leftrightharpoons : \text{Idl}^{op} \times \text{Idl} \rightarrow \text{Idl}$ is a monotonic functor, and for all $R, S, U$

$$\text{cur}_i R \text{ is a comap} \quad (20)$$

$$\text{cur}_i (R \times \text{id}) \text{ ; } \text{ap}_i = R \quad (21)$$

$$\text{cur}_i ((S \times \text{id}) \text{ ; } \text{ap}_i) \supseteq S \quad (=, \text{ if } S \text{ comap}) \quad (22)$$

$$\text{cur}_i ((R \times \text{id}) \text{ ; } S) \supseteq R ; \text{cur}_i S \quad (=, \text{ if } R \text{ comap}) \quad (23)$$

$$\text{cur}_i ((\text{id} \times S) \text{ ; } R ; U) = \text{cur}_i R \text{ ; } (S \leftrightharpoons U) \quad (24)$$

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**Internalizations.** To connect $B \Rightarrow C$ with $B \leadsto C$, we define the ideal $\text{gr}_{B,C}$ by

$$\text{gr}_{B,C} = \text{cur}_r(\text{Gr}_f) : (B \Rightarrow C) \rightarrow (B \leadsto C)$$

This makes $\text{gr}_{B,C}$ the internalization of the action of $\text{Gr}$ on a homset: it relates each $f$ in $B \Rightarrow C$ to the set of ideals contained in $\text{Gr}_f$. The following expresses how $\text{cur}_r$ and $\text{ap}_i$ extend their functional counterparts when applied to comaps.

$$\text{cur}_r(\text{Gr}_f) = \text{Gr}(\text{cur}_f) ; \text{gr} \ (\text{gr} \times \text{id}) ; \text{ap}_i = \text{Gr ap}_f \quad (25)$$

For any $B, C, D$, there is an ideal $\text{comp} : (B \leadsto C) \times (C \leadsto D) \rightarrow B \leadsto D$ which internalizes composition. It is defined in a standard way using $\text{cur}_r$, $\text{ap}_i$, and structural isomorphisms like associativity for products (Prop. 1 and Prop. 2).

5. Relational semantics

The language of relational terms $N$ has the same types as before, see (4). But in addition to functional constructs of (5) we add (demonic) choice $\sqcap$ and patterns:

$$N ::= M | \langle N, N \rangle | NN | \lambda x.M \sqcap N$$

In a pattern matching abstraction $\lambda x.M \sqcap N$, pattern $M$ is a functional term, not just constructors and variables — recall the rotate example (1) in Section 1. The typing rules are

$$\Gamma \vdash N : \sigma \quad \Gamma \vdash N' : \sigma \quad \Gamma, x : \sigma'' \vdash M : \sigma \quad \Gamma, x : \sigma'' \vdash N : \sigma'$$

$$\Gamma \vdash \lambda x.M \sqcap N : \sigma \rightarrow \sigma'$$

For a term $M$ of functional type, say $\sigma \rightarrow \sigma'$, one can think of the term $\lambda x.Mx \sqcap x : \sigma' \rightarrow \sigma$ as the converse of $M$.

In this section we write $[\ [-\ ]]$ for the relational interpretation $\mathcal{R}[\ [-\ ]]$. Types are interpreted as before, except for using the relational constructs: $[\sigma \times \sigma']' = [\sigma] \times [\sigma']'$ and $[\sigma \rightarrow \sigma'] = [\sigma] \leadsto [\sigma']$. We assume $[b] = \mathcal{F}[b]$ for the base types, hence if $\sigma$ is arrow-free then $[\sigma] = \mathcal{F}[\sigma]$.

For $\Gamma = x_0 : \sigma_0, \ldots, x_n : \sigma_n$, the semantics of a judgement $\Gamma \vdash N : \sigma$ is an ideal $[\Gamma] \rightarrow [\sigma]$ defined in Table 2. In addition to demonic choice, $\sqcap$, interpreted by union, one can add an operation for conjunction of specifications, interpreted by intersection which preserves the ideal property. This is omitted, following [dMG00].

As explained later, the semantics of pattern terms uses the functional semantics $\mathcal{F}[\ [-\ ]]$ of the pattern. The table omits semantics of $\text{fst}$ and $\text{snd}$ as the image under $\text{Gr}$ of their functional semantics. Prop. 3 gives a condition that is desirable to connect the relational interpretation of a constant with its functional interpretation. For arrow-free types the condition is simply that $[c : \sigma] = \text{Gr}(\mathcal{F}[c : \sigma])$.

To interpret patterns, the idea is to compose the converse of the pattern with the result term [dMG00]. This must be done internally, i.e., using morphisms that model the external operation of converse, and unlike $\text{Rel}$, $\text{Idl}$ lacks
converse. So, unlike in \[dMG00\], our patterns are restricted to functional terms. For these, the functional interpretation gives a comap that can be reversed using the internal ograph \(rg\) which is analogous to \(gr\). This is problematic because \(rg\) is anti-monotonic. Before delving into the details of the solution, let us turn aside to consider the application of relational semantics.

**Falling short of heap patterns.** Here is a simple realization of heaps. Let the base types include \(Ref\) and \(Heap\), with \([Ref]\) some set, ordered discretely. Let \([Heap]\) be the set of finite partial maps \([Ref] \rightarrow [Ref] \times [Ref]\), again ordered discretely. One constant is \(cell : Ref \times Ref \times Ref \rightarrow Heap\), interpreted as a total function that forms one-cell heaps, written \(x \mapsto y, z\) in Section \[2\]. Another constant is \(new : Heap \rightarrow Ref\), interpreted as some relation that returns references not in the domain of the heap. It could be a total function (provided that \([Ref]\) is infinite), or partial, or not even functional. The constant \(star : Heap \times Heap \rightarrow Heap\) is interpreted as a partial function that forms the union, for heaps with disjoint domains, and is otherwise undefined.

These definitions do not yet achieve the hoped-for ability to write patterns as in Section \[2\] because to appear on the left of a pattern term the constants need to have an interpretation in Poset. This might be handled by adding an “undefined element” to \([Heap]\), and using a deterministic allocator. A nicer solution is to move to predicate transformers (Section \[7\]), where the pattern term merely needs an interpretation in Idl, for which the definitions above are fine.

**Completing the semantics of pattern terms.** The rest of this section gives the technical details of the semantics of pattern matching.

Recall that the typing rule infers \(\Gamma \vdash \chi x. M \rightarrow N : \sigma \rightarrow \sigma'\) from the judgements \(\Gamma, x : \sigma'' \vdash M : \sigma\) and \(\Gamma, x : \sigma'' \vdash N : \sigma'\). The one fine point about typing is that it should enforce, somehow, that there is an interpretation in Poset for all constants in \(M\). We refrain from formalizing that. The semantics is defined using the semantics of the corresponding abstractions, namely

\[
\mathcal{F}[\Gamma \vdash \chi x. M : \sigma'' \rightarrow \sigma] : \mathcal{F}[\Gamma] \rightarrow \mathcal{F}[\sigma''] \Rightarrow \mathcal{F}[\sigma] \quad \text{in Poset}
\]

\[
[\Gamma \vdash \chi x. N : \sigma'' \rightarrow \sigma'] : [\Gamma] \rightarrow [\sigma''] \Rightarrow [\sigma'] \quad \text{in Idl}
\]
Consider the special case where \( \Gamma, \sigma, \sigma', \sigma'' \) are arrow-free, so the two semantics \( \mathcal{F} \) and \( \mathcal{R} \) agree on types. In that case, we want to compose the pattern term with the internal opgraph \( \text{rg} : [\sigma''] \Rightarrow [\sigma] \Rightarrow [\sigma'] \) to get

\[
\mathcal{F}[\Gamma \vdash \lambda x. M : \sigma'' \rightarrow \sigma] ; \text{rg} : [\Gamma] \rightarrow [\sigma] \sim [\sigma']
\]

which, paired with the semantics of \( N \), gives \([\Gamma] \rightarrow ([\sigma] \sim [\sigma'']) \times ([\sigma''] \sim [\sigma'])\). Following this with the internal composition \( \text{comp} \) yields what we need:

\[
(\mathcal{F}[\Gamma \vdash \lambda x. M] ; \text{rg} , [\Gamma \vdash \lambda x. N]) ; \text{comp} : [\Gamma] \rightarrow [\sigma] \sim [\sigma']
\]

This is the gist of the idea. But \( \text{rg} \) is anti-monotonic so \( \mathcal{F}[\Gamma \vdash \lambda x. M] ; \text{rg} \) need not be an ideal and the other constructs do not apply.

To solve this problem, we first extend \( \text{gr} \) to a type-indexed family of ideals \( \text{sim}_\sigma : \mathcal{F}[\sigma] \rightarrow \mathcal{R}[\sigma] \).

\[
\begin{align*}
\text{sim}_b &= \text{id} \quad \text{i.e., } \preceq_b, \text{ for base types } b \text{ (and for 1)} \\
\text{sim}_{\times \sigma'} &= \text{sim}_\sigma \times \text{sim}_{\sigma'} \\
\text{sim}_{\rightarrow \sigma'} &= \text{gr} ; (\text{sim}_\sigma \sim \text{sim}_{\sigma'})
\end{align*}
\]

For all \( \sigma \), we have that \( \text{sim}_\sigma \) is a comap, because \( \text{gr} \) is a comap, the range of \( \sim \) is comaps, and composition preserves comaps. Thus the map \( \text{sim}_\sigma^* \) exists.

Next, we define \( \text{rg} \). Other elements of the semantics are based solely on structure given by Propositions [1][2] but for \( \text{rg} \) we leave such an axiomatic treatment to future work. For any \( B, C \), let \( \text{rg}_{B,C} \) be the restriction of \( \text{Rg} \) to the homset \( \text{Poset}(B, C) \), so that \( \text{rg}_{B,C} \) is a monotonic function \( \text{rg}_{B,C} : \text{Poset}(B, C) \rightarrow (\text{Idl}(C, B))^\sim \). But the target poset, which can also be written \( (C \sim B)^\sim \), is ordered upside-down from what is needed to compose with the result term in a pattern expression. For any \( B \) let \( \text{rev}_B \) be the identity function, taken as an anti-monotonic function of type \( B \rightarrow B \). Define \( \text{rg} : (B \Rightarrow C) \rightarrow (C \sim B) \) as the anti-monotonic function \( \text{rg}_{B,C} \circ \text{rev}_{B\sim C} \). Thus we have the composite \( (26) \) as a relation of the type shown there, though it is not an ideal.

Now we can use the sandwich lemma: For relations \( A \xrightarrow{Q} B \xrightarrow{R} C \xrightarrow{S} D \) between posets,

\[
\text{if } Q, S \text{ are ideals then } Q \circ R \circ S \text{ is an ideal } A \rightarrow D \quad (28)
\]

We sandwich \( \mathcal{F}[\Gamma \vdash \lambda x. M] ; \text{rg} \) with ideals based on \( \text{sim} \) in a way that is needed anyway to reconcile the \( \mathcal{F} \) and \( \mathcal{R} \) interpretations of types. To give the details, we note first the types

\[
\begin{align*}
\mathcal{R}[\Gamma \vdash \lambda x. M' : \sigma'' \rightarrow \sigma'] ; \mathcal{R}[\Gamma] \rightarrow \mathcal{R}[\sigma''] \sim \mathcal{R}[\sigma'] \text{ in Idl} \\
\mathcal{F}[\Gamma \vdash \lambda x. M : \sigma'' \rightarrow \sigma] ; \mathcal{F}[\Gamma] \rightarrow \mathcal{F}[\sigma''] \Rightarrow \mathcal{F}[\sigma] \text{ in Poset}
\end{align*}
\]

and hence \( \mathcal{F}[\Gamma \vdash \lambda x. M : \sigma'' \rightarrow \sigma] ; \text{rg} \) is a relation \( \mathcal{F}[\Gamma] \rightarrow \mathcal{F}[\sigma] \sim \mathcal{F}[\sigma''] \).

Using appropriate instances of \( \text{sim} \) and \( \text{sim}_*^* \) we obtain

\[
\text{sim}_*^* ; \text{Gr}(\mathcal{F}[\Gamma \vdash \lambda x. M : \sigma'' \rightarrow \sigma]) ; \text{rg} ; (\text{sim}_\sigma \sim \text{sim}_{\sigma''})
\]
of type $\mathcal{R}[\Gamma] \rightarrow \mathcal{R}[\sigma] \sim \mathcal{R}[\sigma''].$ This is an ideal, by the sandwich lemma (\ref{p:sandwich}). As described earlier, for the semantics of $\chi x. M \mapsto N$ this ideal is paired with the semantics of $N$ and followed by $\text{comp}$, in accord with (\ref{p:comp}). See Table 2.

Although $\text{Gr}$ could be omitted, owing to (\ref{p:shorten}), we use it because it has a parallel in the predicate transformer semantics (Table 3).

6. Lax laws for relational semantics

This section proves a connection between relational and functional semantics, describes a stronger connection conjectured to hold, and proves new results that serve as laws of programming.

Connecting the semantics by simulation. A conservative extension result would show that, for a purely functional term, the relational (or transformer) interpretation is “the same” as the functional one, in some suitable sense. Given that the two semantics have different interpretations for arrow types, this needs to involve the embedding between those interpretations, as noted in Section 1. De Moor and Gibbons \cite{deMoor2000} state a strong conservative extension result for their relational semantics, which we explain later. For the models in this article, weaker simulation results suffice to justify program construction by stepwise refinement.

Proposition 6. \cite{Nau2002} Suppose that $\text{sim}_1 ; \mathcal{R}[c : \sigma] \supseteq \mathcal{F}[c : \sigma] ; \text{sim}_\sigma$ for all constants $c : \sigma$. Then for all functional terms in context $\Gamma \vdash M : \sigma$

$$\text{sim}_\Gamma ; \mathcal{R}[\Gamma \vdash M : \sigma] \supseteq \mathcal{F}[\Gamma \vdash M : \sigma] ; \text{sim}_\sigma$$

The right hand side is the same as $\text{Gr}(\mathcal{F}[\Gamma \vdash M : \sigma]) ; \text{sim}_\sigma$, owing to (\ref{p:shorten}). Owing to the shunting property (\cite{deMoor1990}), there is an equivalent formulation which also suggests how to obtain relational interpretations of constants from their functional interpretation:

$$\mathcal{R}[\Gamma \vdash M : \sigma] \supseteq \text{sim}_\Gamma^* ; \text{Gr}(\mathcal{F}[\Gamma \vdash M : \sigma]) ; \text{sim}_\sigma$$

Corollary: If all types in $\Gamma, \sigma$ are arrow-free, $\mathcal{R}[\Gamma \vdash M : \sigma] \supseteq \mathcal{F}[\Gamma \vdash M : \sigma]$, because $\text{sim}$ is the identity for arrow-free types and $\text{Gr}$ is increasing. This licenses development by stepwise refinement chains $\mathcal{R}[N] \supseteq \ldots \supseteq \mathcal{R}[M] \supseteq \mathcal{F}[M]$.

Connecting the semantics as an equality. The conservative extension property \cite{deMoor2000} involves two additional interpretations of types. The positive interpretation $\mathcal{P}$ replaces every arrow in positive position by a relation space and every negative arrow by a function space. The negative interpretation $\mathcal{N}$ does the

\footnote{If $\Gamma = x_0 : \sigma_0, \ldots, x_n : \sigma_n$, we write $\text{sim}_\Gamma$ for $\text{sim}(\ldots(1 \times \sigma_0) \times \ldots \times \sigma_n) : \mathcal{F}[\Gamma] \rightarrow \mathcal{R}[\Gamma]$.}
To be precise, for example $P[\langle b_0 \rightarrow b_1 \rangle \rightarrow \langle b_2 \rightarrow b_3 \rangle]$ is $\langle [b_0] \Rightarrow [b_1] \rangle \leadsto \langle [b_2] \leadsto [b_3] \rangle$ for base types $b_0, \ldots, b_3$. The definitions are mutually recursive:

\[
P[\sigma \rightarrow \sigma'] = N[\sigma \leadsto P[\sigma']]
\]

\[
P[\sigma \times \sigma'] = P[\sigma] \times P[\sigma']
\]

\[
P[b] = [b]
\]

For clarity the connection is stated for the special case that the context has a single variable. The connection is the equality of the the top and bottom paths in this diagram.

\[
\begin{array}{c}
\begin{array}{c}
N(\sigma) \\
\end{array} \\
\end{array}
\xrightarrow{\mathcal{F}[\sigma]} \begin{array}{c}
\begin{array}{c}
\mathcal{F}[x : \sigma \vdash M : \sigma'] \\
\end{array} \\
\end{array}
\xrightarrow{f_2^{\sigma'}} \begin{array}{c}
\begin{array}{c}
P(\sigma') \\
\end{array} \\
\end{array}
\end{array}
\]

To be precise, $\mathcal{G}$ should be applied to the upper path. The upper diagonals are two families of monotonic functions defined by mutual recursion on types. The lower diagonals are two families of ideals defined similarly.

\[
\begin{array}{c}
n_2f_{\sigma \rightarrow \sigma'} = f_2^\sigma \Rightarrow n_2f_{\sigma'} & f_2^{\sigma \rightarrow \sigma'} = (n_2f_{\sigma} \Rightarrow f_2^\sigma) ; \mathcal{G} \\
n_2f_{\sigma \times \sigma'} = n_2f_{\sigma} \times n_2f_{\sigma'} & f_2^{\sigma \times \sigma'} = f_2^{\sigma} \times f_2^{\sigma'} \\
n_2f_b = \text{id} & f_2^b = \text{id} \\
n_2r_{\sigma \rightarrow \sigma'} = \mathcal{G} ; (r_2^\sigma \Rightarrow n_2r_{\sigma'}) & r_2^{\sigma \rightarrow \sigma'} = n_2r_{\sigma} \Rightarrow r_2^\sigma \\
n_2r_{\sigma \times \sigma'} = n_2r_{\sigma} \times n_2r_{\sigma'} & r_2^{\sigma \times \sigma'} = r_2^{\sigma} \times r_2^\sigma \\
n_2r_b = \text{id} & r_2^b = \text{id}
\end{array}
\]

Here $\mathcal{G}$ is the function $\mathcal{G} : N[\sigma] \Rightarrow P[\sigma'] \Rightarrow N[\sigma'] \Rightarrow P[\sigma']$ and $\mathcal{G}$ is the ideal $\mathcal{G} : P[\sigma] \Rightarrow N[\sigma'] \Rightarrow P[\sigma] \Rightarrow N[\sigma']$.

The result conjectured in [21] is that, for their semantics, the hexagon equality holds for all functional terms (assuming that it holds for the interpretations of constants). We conjecture that this also holds in our semantics.

*Law 7 for programming.* Stepwise refinement depends on monotonicity of program constructs. Monotonicity of pattern terms is delicate, due to the use of $\mathcal{R}$, and we leave that issue to future work on laws for patterns. Monotonicity is not addressed in [21], perhaps due to difficulties mentioned in Section [4] but it is straightforward to prove the following for relational terms in $\mathcal{I}$.

---

De Moor and Gibbons indicate that they proved the result for beta normal forms. A possibly relevant observation is that in our $\mathcal{R}$ semantics, application-free functional terms are comaps; applications are interpreted using $\mathcal{A}_P$ which is not a comap.
Theorem 7 (monotonicity). Let $C[-]$ be any context such that the hole does not occur on the left of $\mapsto$ in a pattern. Let $N$ and $N'$ be of suitable type to fill the hole. Then $R[\Gamma \vdash N : \sigma] \supseteq R[\Gamma \vdash N' : \sigma]$ implies $R[\Gamma' \vdash C[N] : \sigma'] \supseteq R[\Gamma' \vdash C[N'] : \sigma']$.

Proposition 6 licenses us to reason about functional terms using functional semantics. But the extended language would be of little interest if laws did not carry over to it. Laws for products in the relational semantics can be read directly from corresponding semantic properties, and are left to the reader.

For lambda expressions the eta law is $\lambda x.Nx \supseteq N$ (for $x$ not free in $N$).

Theorem 8 (eta law). $R[\Gamma \vdash \lambda x.Nx : \rho \to \sigma] \supseteq R[\Gamma \vdash N : \rho \to \sigma]$ provided that $x$ is not free in $N$.

Proof:

\[ R[\Gamma \vdash \lambda x.Nx : \rho \to \sigma] \]
= semantics of $\lambda$ and application
\[ \text{cur}_i((R[\Gamma, x : \rho \vdash N : \rho \to \sigma], R[\Gamma, x : \rho \vdash x : \rho]); \text{ap}_i) \]
= $x$ not free in $N$, semantics of $x$ ($\pi, \pi'$ are left, right proj.)
\[ \text{cur}_i(\langle \pi; R[\Gamma \vdash N : \rho \to \sigma], \pi' \rangle; \text{ap}_i) \]
= def $\times$
\[ \text{cur}_i(\langle R[\Gamma \vdash N : \rho \to \sigma] \times \text{id}; \text{ap}_i) \]
\[ \supseteq \text{exponent law} \]
\[ R[\Gamma \vdash N : \rho \to \sigma'] \]

The second step uses the evident semantics of a typing rule that adds variable $x$ to the context for a term in which $x$ is not free.

The semantics validates $(\lambda x.x - x)(0 \sqcap 1) = 0$, which shows that application is by-value rather than by-name. As expected in a by-value calculus, beta conversion is not an equality in the relational semantics, e.g., $(\lambda x.x - x)(0 \sqcap 1) \subseteq (0 \sqcap 1) - (0 \sqcap 1)$ is a proper inclusion because the right side includes $0 - 1$ and $1 - 0$. We do get a refinement for this and similar cases. The beta law $N[N'/x] \supseteq (\lambda x.N)N'$ holds provided that $x$ does not occur in a pattern. We say $x$ occurs in a pattern of $N$ if it occurs in $M$ for some subterm $\chi x.M$ $\mapsto N$ of $N$.

Theorem 9 (beta law). For all relational terms $N$ and $N'$, such that $x$ does not occur in a pattern in $N$, $R[\Gamma \vdash N[N'/x] : \sigma] \supseteq R[\Gamma \vdash (\lambda x.N)N' : \sigma]$. This is an equality if $N'$ denotes a comap and $N$ is pattern-free.

Note that the equality still allows nondeterminacy in $N$.

In the rest of this section we omit $R$. To prove the theorem, we observe

---

8A context $C[-]$ is a term with a missing subterm, called the hole, and $C[N]$ is the term with the hole filled by $N$. 

\[ \]
\[ [\Gamma \vdash (\lambda x. N)N'] : \sigma ] \]

= semantics of \( \lambda \) and application

\[ \langle \text{cur} \Gamma, x : \sigma' \vdash N : \sigma \rangle, [\Gamma \vdash N' : \sigma'] \rangle ; \ \text{ap}_t \]

= product law \[ \text{(15)} \]

\[ \langle \text{id}, [\Gamma \vdash N' : \sigma'] \rangle ; (\text{cur}_t [\Gamma, x : \sigma' \vdash N : \sigma] \times \text{id}) ; \ \text{ap}_t \]

= exponent law \[ \text{(21)} \]

\[ \langle \text{id}, [\Gamma \vdash N' : \sigma'] \rangle ; [\Gamma, x : \sigma' \vdash N : \sigma] \]

The last line exhibits “substitution as composition”. The proof is completed using a substitution lemma. While for functional semantics it is an equality, for ideals it weakens to an inequality.

**Lemma 10 (substitution).** For all relational terms \( N, N_0 \) such that \( x \) does not occur in a pattern of \( N \),

\[ [\Gamma \vdash N[N_0/x] : \sigma] \supseteq \langle \text{id}, [\Gamma \vdash N_0 : \sigma_0] \rangle ; [\Gamma, x : \sigma_0 \vdash N : \sigma] \]

Equality holds if \( N_0 \) denotes a comap and \( N \) is pattern-free.

An operational interpretation of the inequality goes as follows. On the right side, \( N_0 \) is evaluated once, whereas on the left there may be multiple occurrences of \( N_0 \) in \( N[N_0/x] \) which give rise to different nondeterministic choices.

The rest of this section is devoted to the proof of Lemma 10. The inequality is proved by structural induction on \( N \). We give the proof in detail, in order to pinpoint exactly what properties are needed for the inequality and for the equality. We elide \( R \) throughout, and also elide some types, including the type of \( x \).

**Case** \( N \) is a variable \( x \). Here \( N[N_0/x] \) is \( N_0 \) and we have

\[ \langle \text{id}, [\Gamma \vdash N_0] \rangle ; [\Gamma, x \vdash x] \]

= semantics of \( x \)

\[ \langle \text{id}, [\Gamma \vdash N_0] \rangle ; \ \text{Gr}_\pi \]

\[ [\Gamma \vdash N_0] \]

product law \[ \text{(11)} \], id comap

Here \( (A) \) marks a step to which we return when proving Lemma 17 for transformer semantics.

**Case** \( N \) is a choice \( N' \cap N'' \). Straightforward, using that \( ; \) distributes over \( \cup \). The straightforward cases for constants and pairing are also omitted.

**Case** \( N \) is an application \( N'N'' \).
\[
\langle \text{id}, [\Gamma \vdash N_0] \rangle ; [\Gamma, x \vdash N'N'']
\]

\[
\text{semantics of } N'N''
\]

\[
\langle \text{id}, [\Gamma \vdash N_0] \rangle ; (\langle [\Gamma, x \vdash N'], [\Gamma, x \vdash N''] \rangle ; \text{ap} _ i)
\]

\[
(B) \subseteq \text{ product law } (\text{14}), \text{; monotonic}
\]

\[
\langle \langle \text{id}, [\Gamma \vdash N_0] \rangle ; [\Gamma, x \vdash N'], \langle \text{id}, [\Gamma \vdash N_0] \rangle ; [\Gamma, x \vdash N''] \rangle ; \text{ap} _ i
\]

\[
\subseteq \text{ induction; } ; \langle \cdot, \cdot \rangle \text{ monotonic}
\]

\[
\langle [\Gamma \vdash N'[N_0/x]], [\Gamma \vdash N''[N_0/x]] \rangle ; \text{ap} _ i
\]

\[
= \text{ semantics of application, substitution}
\]

\[
[\Gamma \vdash (N'N'')[N_0/x]]
\]

Note that step (B) is an equality if \( N_0 \) denotes a comap, by (\text{14=}).

**Case** \( N \) is an abstraction \( \lambda y. N' \). Clearly the semantics models alpha conversion, so we can assume without loss of generality that \( y \) is distinct from \( x \). And \( y \) is not free in \( N_0 \). We need to use an isomorphism \( p \) that rearranges components \( x, y \) of the context. Such structural isomorphisms are available thanks to Propositions (\text{11}) and (\text{12}). Precise definitions can be found in semantics texts such as [Gun92]. We content ourselves with displaying the type of \( p \) in a diagram for an equation, that holds for arbitrary \( R : [\Gamma] \to [\sigma_0] \) and any \( \sigma'' \).

\[
\begin{align*}
(\text{id}, R) \times \text{id} & \quad \xrightarrow{\text{id} \times R} \\
([\Gamma] \times [\sigma'']) \times [\sigma''] & \quad \xrightarrow{p} ([\Gamma] \times [\sigma'']) \times [\sigma_0]
\end{align*}
\]

Using \( (\text{30}) \) we calculate

\[
\begin{align*}
\langle [\Gamma \vdash (\lambda y. N')[N_0/x]] \\
= [\Gamma \vdash \lambda y. N'[N_0/x]] & \quad \text{substitution, } x, y \text{ distinct } \\
= \text{ semantics of } \lambda \\
\subseteq \text{ cur} _ i [\Gamma, y \vdash N'[N_0/x]] & \quad \text{induction, cur} _ i \text{ monotonic} \\
\subseteq \text{ cur} _ i (\langle \text{id}, [\Gamma, y \vdash N_0] \rangle ; [\Gamma, y, x \vdash N']) & \quad y \text{ not free in } N_0, \text{ semantics} \\
\subseteq \text{ cur} _ i (\langle \text{id}, \pi ; [\Gamma \vdash N_0] \rangle ; [\Gamma, y, x \vdash N']) & \quad \text{\( p \) isomorphism} \\
\subseteq \text{ cur} _ i ((\langle \text{id}, [\Gamma \vdash N_0] \rangle \times \text{id}) ; [\Gamma, x, y \vdash N']) & \quad \text{(\text{30}) with } R := [\Gamma \vdash N_0] \\
\subseteq \langle \text{id}, [\Gamma \vdash N_0] \rangle ; \text{cur} _ i [\Gamma, x, y \vdash N'] & \quad \text{exponent law } (\text{23}) \\
\subseteq \langle \text{id}, [\Gamma \vdash N_0] \rangle ; \text{cur} _ i [\Gamma, x, y \vdash N'] & \quad \text{semantics of } \lambda
\end{align*}
\]

Step (C) is an equality if \( N_0 \) denotes a comap, by (\text{23=}).

**Case** \( N \) is a pattern term \( \chi y. M \mapsto N' \). Assume w.l.o.g. that \( x \) is distinct from \( y \). Here \( N_0 \) must be a functional term in order for the substitution to yield a typeable pattern term. We write \( s \) to abbreviate \( \sim \) and a huge comma \( \cdot \) to aid parsing complicated pairings. We also elide \( \Gamma \), and the type of \( x \), on the left of \( \vdash \) throughout.
This follows by monotonicity of \((\lambda y. M)\), which is an equality, (31) is equivalent to
\[
\langle \lambda y. M \mapsto N' \rangle[0/x]
\]
Step (D) is an equality if \(N_0\) denotes a comap, by (14=). The claim is
\[
s^*; F[0/x] \supseteq (\lambda y. M[0/x]) \supseteq (\lambda y. M)[0/x]
\]
If \(x\) is not free in pattern \(M\) then \(\lambda y. M[0/x]\) is \(\lambda y. M\) and the claim can be proved using properties of the left projection \(\pi\) as follows.
\[
= \langle \lambda y. M \mapsto \lambda y. M \rangle \quad \text{x not free in } \lambda y. M, \text{ semantics}
\]
\[
= \langle \lambda y. M \supseteq \lambda y. M \rangle \quad \text{def } s, \times \text{ preserves comaps}
\]
\[
\subseteq \langle \lambda y. M \supseteq \lambda y. M \rangle \quad \pi \text{ lax natural, from (11)}
\]
\[
\subseteq (\lambda y. M)[0/x] \quad (11) \text{ and unit law}
\]
Because \(\times\) preserves comaps, \(s^*_{x \Rightarrow x} = s^*_{x^*} \times s^*_{\times^*} \). Step (F) is an equality if \(N_0\) denotes a comap, but step (E) is not.

The lemma holds as an equality if \(N_0\) is a comap and \(N\) has no patterns.
(Under those conditions, the inductive steps become equalities.)

If \(x\) occurs in \(M\), a natural attempt goes as follows. By the substitution lemma for \(F\), which is an equality, (31) is equivalent to
\[
s^*; (\lambda y. M)[0/x] \supseteq (\lambda y. M)[0/x]
\]
This follows by monotonicity of \(\langle \lambda y. M \mapsto N' \rangle\) from \(s^*; (\lambda y. M)[0/x] \supseteq (\lambda y. M)[0/x]\). Here we are in trouble. Roughly speaking, this asks for the functional semantics of \(N_0\) to contain the relational semantics, which is the reverse of Proposition 6.

7. Transformers and transformer semantics

Predicate transformers are often taken to be monotonic functions on powersets, but here we use updeal lattices. The reason is that if powersets are
used instead of updeal lattices, the internal hom does not carry a well behaved exponent structure. For example, we need the associated functor to preserve identities, which fails with powersets. This section begins with a brief explanation of why updeal lattices are sensible in programming terms. Then it proceeds to describe the model and give the semantics. Results not proved here can be found in [Nau98b] or [Nau01].

Ordered data types are needed even in first order languages, if extensible record or object types are admitted. To see the significance of ordering for imperative programs, consider an expression refinement $e \preceq e'$. This does not imply the command refinement $x := e \subseteq x := e'$, because that requires $wp(x := e)\phi \subseteq wp(x := e')\phi$ for all postconditions $\phi$, which fails for postcondition $x = e$. A more reasonable postcondition is $e \preceq x$, which is upward closed in $x$. To give a specification that exactly characterizes the assignment $x := e$, we can use auxiliary $y$ in precondition $y \preceq e$ and postcondition $y \preceq x$. These predicates are closed upward in the state variable $x$ and downward in $y$. In general, $pre$, $post$ in the span ə can be taken to be ideals from auxiliary state to program state.

Weakest-precondition functions map predicates on final states to predicates on initial states, so notation and terminology is most perspicuous if we use an opposite category. We also follow convention in predicate transformer semantics and order predicates by $\subseteq$; we write $UA$ for the lattice of updeals on $A$, ordered by $\subseteq$, as opposed to $UA$ ordered by $\supseteq$. We define $\text{Tran}$ to have all posets as objects; and the homset $\text{Tran}(A, B)$ is just $\text{Poset}((UB, UA))$, i.e., $\text{Poset}^{op}(UA, UB)$. Following the convention in refinement calculi, the symbol $\subseteq$ is used for the ordering, so $t \subseteq t'$ iff $t\alpha \subseteq t'\beta$ for all $\alpha \in UB$.

Composition in $\text{Tran}$ is just functional composition, for which we write $\circ$, so that for $t$ in $\text{Tran}(A, B)$ and $u$ in $\text{Tran}(B, C)$ we have $t \circ u = u \circ t$. The identity on $A$ is the identity function $id_{UA}$. For any ideal $R : A \rightarrow B$, the universal image $A R \in G r(A, B)$ is defined by $A R = UB(\exists / R)$, using relational quotient. At the level of points, $a \in A R X$ iff $\exists b . a R b \Rightarrow b \in X$. The universal image of an ideal is universally conjunctive. Moreover, $A$ is refinement injective: $R \supseteq S \equiv A R \subseteq A S$. Thus it preserves demonic choice: $A(R \cap S) = A R \cap A S$ where $\cap$ is pointwise intersection. Later we use $\sqcup$ for pointwise union, which models angelic choice. Define $E R$ to be the direct image of $R$, but as a function $UA \rightarrow UB$ and so distinguished from $UR : UA \rightarrow UB$. Note that $E$ is a monotonic functor $\text{Idl}^{op} \rightarrow G r$. A crucial fact is that for any $R$, $AR$ is a map with comap $ER$. Thus $A$ is onto maps. This is an unfortunate clash of terminology as $G r$ plays the same role as $A$ but $G r$ embeds onto comaps.

A bimap is a map that is also a comap. For any monotonic function $f$, $A(G r f)$ is a bimap in $\text{Tran}$. Sometimes we omit $G r$ and write simply $A f$. The situation looks as follows, using fishtail arrows for embeddings and fishhooks for inclusions.

```
Poset    ↓    Comaps(Idl)    ↓    Idl
     ↓    Bimaps(Tran)    ↓    Maps(Tran)    ↓    Tran
```

24
In fact the vertical arrows are categorical equivalences.

A transformer \( t : A \rightarrow B \) is strict if \( t\emptyset = \emptyset \) (i.e., \( t(false) = false \)) and costrict if \( t\emptyset = A \) (i.e., \( t(true) = true \), which expresses program termination). Comaps are strict and maps are costrict.

Cartesian product of underlying posets gives a weak product of predicate transformers; we overload \( \times \) and \( \langle \cdot, \cdot \rangle \) but write \( \&\pi \) explicitly for the projection lifted from Poset.

Proposition 11. \( \times \) is monotonic and for all \( t,u,v,w \),

\[
\langle t,u \rangle ; \&\pi \sqsubseteq t \quad \text{if } t \text{ strict (32)}
\]

\[
\langle t,u \rangle ; \&\pi \sqsubseteq t \quad \text{if } t \text{ costrict (33)}
\]

\[
\langle t \&\pi , t \&\pi \rangle \sqsubseteq t \quad \text{if } t \text{ map (34)}
\]

\[
\langle t \& u, (v \& w) \rangle \sqsubseteq \langle t \times v, (u \times w) \rangle \quad (=, \text{ if } u,w \text{ maps} (35))
\]

\[
\langle t \& u, (v \& w) \rangle \sqsubseteq \langle t, v \& (u \times w) \rangle \quad (=, \text{ if } u,w \text{ maps} (36))
\]

\[
\langle t \& u, t \& v \rangle \sqsubseteq t \& (u,v) \quad \text{if } t \text{ map (37)}
\]

Adequacy of predicate transformers as a model of nondeterminacy and divergence together is reflected in the behavior of products. As in Idl, the projection law (32) is not an equality in general (intuitively, \( u \) could diverge). Even the inequality depends on absence of miracles. The reverse inequality holds for costrict transformers and a fortiori for maps. The other defining law for products of functions also weakens to (34), reflecting nondeterminacy as with relations. The side condition that \( t \) is a map expresses the absence of angelic nondeterminacy; in fact the condition can be weakened to positive conjunctivity. The inequality (37) holds so long as \( t \) is finitely conjunctive, and the reverse \( \sqsubseteq \) holds if \( t \) a comap.

As with products, the exponent structure is very lax. The exponent object \( B \vdash C \) is \( \text{Tran}(B,C) \), ordered by \( \sqsubseteq \). Definitions for currying and application can be found in \cite{Nau98b} and \cite{Nau01}; we only need the following.

Proposition 12. \( \text{Nau98b} \vdash : \text{Tran}^{op} \times \text{Tran} \rightarrow \text{Tran} \) is a monotonic functor and for all \( t,u,v \)

\[
\text{cur}_{t} t \text{ is a bimap (38)}
\]

\[
(\text{cur}_{t} (t \times id)) ; \text{ap}_{t} = t \quad \text{(39)}
\]

\[
\text{cur}_{t}((t \times id) ; \text{ap}_{t}) \sqsubseteq t \quad \text{if } t \text{ map } (=, \text{if } t \text{ bimap} (40))
\]

\[
\text{cur}_{t}((t \times id) ; u) \sqsubseteq t \& \text{cur}_{t} u \quad \text{if } t \text{ map } (=, \text{if } t \text{ bimap} (41))
\]

\[
\text{cur}_{t}(id \times t) ; u \& v = \text{cur}_{t} u ; (t \rightarrow v) \quad \text{(42)}
\]

The analog of the internal graph functor \( g \) is the internal universal image \( \text{unim} : B \rightsquigarrow C \rightarrow B \leftrightarrow C \) defined by \( \text{unim} = \text{cur}_{t}(\&\pi \text{ap}_{t}) \). It is a bimap, being in the image of \( \text{cur}_{t} \). Just as \( g \) is used in (25), \( \text{unim} \) expresses how \( \text{cur}_{t} \) and \( \text{ap}_{t} \) extend their relational counterparts:

\[
\text{cur}_{t}(\&R) = \&\text{cur}_{t}(R) ; \text{unim} \quad (\text{unim} \times id) ; \text{ap}_{t} = \&\text{ap}_{t}
\]
Table 3: The predicate transformer semantics $T[-]$, given transformers $[c : \sigma] : 1 \to T[\sigma]$.

$$
\begin{align*}
[\Gamma \vdash x] &= \Delta \pi \text{ where } \pi \text{ is the appropriate projection} \\
[\Gamma \vdash \epsilon] &= \Delta \pi \cdot [c : \sigma] \text{ where } \pi \text{ is the projection } [\Gamma] \to [1] \\
[\Gamma \vdash (P, P') : \sigma \times \sigma'] &= \llbracket [\Gamma \vdash P : \sigma], [\Gamma \vdash P' : \sigma'] \rrbracket \\
[\Gamma \vdash PP' : \sigma] &= \llbracket [\Gamma \vdash P : \sigma', [\Gamma \vdash P' : \sigma')] \rrbracket \cdot \text{ap}_t \\
[\Gamma \vdash \lambda x. P : \sigma \to \sigma'] &= \text{cur}_t[\Gamma, x : \sigma \vdash P : \sigma'] \\
[\Gamma \vdash \chi x. N : \sigma \to \sigma'] &= \langle \text{psim}_t^\ast; \lambda(\mathcal{R}[\Gamma \vdash \lambda x. N]) \rangle \cdot \text{exim} \cdot (\text{psim}_t^\ast \vdash \text{psim}_t^\ast), [\Gamma \vdash \lambda x. P] \cdot \text{comp} \\
[\Gamma \vdash P \cap P' : \sigma] &= [\Gamma \vdash P : \sigma] \cap [\Gamma \vdash P' : \sigma] \\
[\Gamma \vdash P \cup P' : \sigma] &= [\Gamma \vdash P : \sigma] \cup [\Gamma \vdash P' : \sigma]
\end{align*}
$$

**Transformer semantics.** Types are interpreted as before, except for using the constructs of $\text{Tran}$: $[[\sigma \times \sigma']] = [[\sigma] \times [[\sigma']$ and $[[\sigma \to \sigma']] = [[\sigma] \to [[\sigma']]$. We assume $[[B]] = \mathcal{R}[[B]] = \mathcal{F}[[B]]$ for the base types. We extend $\text{unim}$ to a simulation $\text{psim}_\sigma : \mathcal{R}[[\sigma]] \to T[[\sigma]]$ by defining these morphisms in $\text{Tran}$:

$$
\begin{align*}
\text{psim}_B &= \text{id for base types } B \text{ (and for } 1) \\
\text{psim}_{\sigma \times \sigma'} &= \text{psim}_\sigma \times \text{psim}_{\sigma'} \\
\text{psim}_{\sigma \to \sigma'} &= \text{unim} \cdot (\text{psim}_\sigma \vdash \text{psim}_{\sigma'})
\end{align*}
$$

Because $\text{unim}$ is a bimap and the range of $\text{cur}_t$ is a bimap, each $\text{psim}$ is a bimap, and $\text{psim}^\ast$ denotes the corresponding map.

We augment the language of terms with angelic choice $\sqcup$, using a new syntactic category $P$. Patterns are now relational terms:

$$
P ::= N \mid (P, P) \mid PP \mid \lambda x. P \mid \chi x. N \Rightarrow P \mid P \cap P \mid P \sqcup P
$$

The semantics is in Table 3. For semantics of patterns we overload the name $\text{comp}$ for the internal composition in $\text{Tran}$. We also use the internal existential image $\text{exim}$, which is analogous to $\text{rg}$ for the relational semantics. The analogy includes the annoying fact that it is an order-reversing function. For any $B, C$ let $\mathcal{E}_{B,C} : B \Rightarrow C \to (C \Rightarrow B)^\ast$, which yields anti-monotonic function $(\mathcal{E}_{B,C} \cdot \text{rev}) : B \Rightarrow C \to C \Rightarrow B$. Here $\text{rev}$ is the order-reversing identity function $\text{rev}_{C \Rightarrow B} : (C \Rightarrow B)^\ast \to C \Rightarrow B$ mentioned a few lines before (28).

We defined $\mathcal{A}$ only for ideals, but in fact for any $R$ with $\mathcal{A}(z) \subseteq R$, the inverse image sends updeals to updeals. Taking $R$ to be $\mathcal{E}_{B,C} \cdot \text{rev}$ and using the order $\supseteq$ as well as $\Rightarrow$ we define $\text{exim}$ to be $\mathcal{A}(z) ; \mathcal{E}_{B,C} ; \text{rev}$ which is a morphism in $\text{Tran}(B \Rightarrow C, C \Rightarrow B)$. The rest of the semantics for patterns parallels the relational semantics. Any constant that occurs on the left of a pattern needs to have an interpretation in $\mathcal{Idl}$, just as the relational semantics requires a functional interpretation of such constants.
8. Lax laws for transformer semantics

The basic connection between transformer semantics and ideals is similar to Prop. 6.

Theorem 13 ([Nau01]). Suppose that $\text{psim}_1 : T[c : \sigma] \sqsubseteq \Lambda(R[c : \sigma]) : \text{psim}_\sigma$ for all constants $c : \sigma$. Then for all relational terms $N$

$$\text{psim} : T[\Gamma \vdash N : \sigma] \sqsubseteq \Lambda(R[\Gamma \vdash N : \sigma]) : \text{psim}$$

If all types in $\Gamma, \sigma$ are arrow-free then $T[\Gamma \vdash N : \sigma] \sqsubseteq \Lambda(R[\Gamma \vdash N : \sigma])$.

The result licenses development by stepwise refinement, that is, in a chain

$$T[\Gamma \vdash P] \sqsubseteq \ldots \sqsubseteq T[\Gamma \vdash N] \sqsubseteq \Lambda(R[\Gamma \vdash N])$$

ending with a “program” that has only demonic nondeterminacy.

Refinement laws. As in the relational semantics, it is straightforward to prove the following monotonicity result for transformer terms.

Theorem 14 (monotonicity). For any context $C[\cdot]$ in which terms $P$ and $P'$ may occur, except to the left of $\mapsto$ in pattern terms, $T[\Gamma \vdash P : \sigma] \sqsupseteq T[\Gamma \vdash P' : \sigma]$ implies $T[\Gamma \vdash C[P] : \sigma'] \sqsupseteq T[\Gamma \vdash C[P'] : \sigma']$.

Laws for products in predicate transformer semantics can be read directly from corresponding semantic properties, e.g., (32) and (34) are beta and eta laws for products in $\text{Tran}$. For exponents, the eta law is a conditional refinement:

$$\lambda x. P_x \sqsubseteq P \text{ if } P \text{ is a map (and } x \text{ not free in } P).$$

Theorem 15 (eta law). $T[\Gamma \vdash \lambda x. P_x : \sigma \rightarrow \sigma'] \sqsubseteq T[\Gamma \vdash P : \sigma \rightarrow \sigma']$ if $P$ is a map and $x$ is not free in $P$.

Proof: We have $T[\Gamma \vdash \lambda x. P_x] = \text{cur}_{\sigma}(T[\Gamma \vdash P : \sigma] \times \text{id}) : \text{ap}_{\sigma}$ by semantics. If $P$ denotes a map, we can apply law (40) to obtain $\lambda x. P_x \sqsubseteq P$.

Theorem 16 (beta law). For all $\Gamma \vdash P$ and all $\Gamma \vdash P'$ such that $T[\Gamma \vdash P']$ is a map,

$$T[\Gamma \vdash P[P'/x] : \sigma] \sqsubseteq T[\Gamma \vdash (\lambda x. P)P' : \sigma]$$

provided $x$ does not occur in a pattern of $P$. This is an equality if $P'$ denotes a bimap and $P$ is pattern-free.

In particular, $P$ may have angelic and demonic nondeterminacy. Note that $(\cdot ; t)$ distributes over both $\cap$ and $\sqcup$, for any $t$. Also $E(R \cup S) = ER \sqcup ES$. As a consequence, we have $\lambda x. N \cap N' \mapsto P = (\lambda x. N \mapsto P) \sqcup (\lambda x. N' \mapsto P)$ for all $N, N', P$. This shows the angelic nature of patterns.

The proof of Theorem 16 is like the proof of Theorem 9. We show

$$T[\Gamma \vdash (\lambda x. P)P' : \sigma] = (\text{id}, T[\Gamma \vdash P' : \sigma']) : T[\Gamma, x : \sigma' \vdash P : \sigma]$$

using (36=) and (39) instead of the corresponding laws for $\text{Idl}$. The proof is completed by appeal to the following.
Lemma 17 (substitution). For all $P, P_0$ such that $T[\Gamma \vdash P_0]$ is a map and $x$ does not occur in a pattern in $P$,

$$T[\Gamma \vdash P[P_0/x] : \sigma] \sqsubseteq (id, T[\Gamma \vdash P_0 : \sigma_0]) \circ T[\Gamma; x : \sigma_0 \vdash P : \sigma]$$

This is an equality if $P_0$ denotes a bimap and $P$ is pattern-free.

The proof is similar to that for Lemma 10 using corresponding properties of product and exponent in Tran. We do not repeat the calculations; the only differences are steps marked with letters and we discuss these in turn below. We refer to the identifiers $N, N'$ in those calculations, which correspond to $P, P_0$ in the current proof. In each case, the step goes through thanks to $P_0$ denoting a map.

For the case that $N$ is $x$, the step (A) is an inequality $\sqsubseteq$ thanks to (33), provided $T[\Gamma \vdash N']$ is a map. For application, the step marked (B) is an inequality $\sqsubseteq$ if $N'$ denotes a map, thanks to (37). In fact costrictness suffices for (A) and positive conjunctivity for (B); together these are the properties of a map, i.e. universal conjunctivity. For application, the step marked (C) is an inequality $\sqsubseteq$ if $N'$ denotes a map, by (41) (and it is $\sqsubseteq$ that is needed here, because the calculation is in the reverse direction from the prior ones).

For $N$ a pattern term, the step marked (D) goes through by (37) if $N'$ denotes a map, and steps (E) and (F) can be taken using (33).

The proof for demonic choice is straightforward, using that $(t \cdot -)$ distributes over arbitrary $\sqcap$ if $t$ is a map, and the $t$ in this case is $T[\Gamma \vdash N']$. The proof for angelic choice uses the fact that $(t \cdot u) \sqcup (t \cdot v) \sqsubseteq t \cdot (u \sqcup v)$ for all $t, u, v$ (by monotonicity of $t$).

If $N'$ denotes a bimap, the bimap conditions are met for steps (A), (B), and (C), and bimaps distribute over both $\sqcup$ and $\sqcap$ on the left. The lemma holds as an equality for pattern-free $N$ and bimap $N'$.

9. Discussion

Following de Moor and Gibbons [dMG00], we have shown how to internalize the span factorizations of ideal relations and of predicate transformers, and to use these models for two semantics of lambda terms extended with non-injective and non-total pattern matching. We have shown that not only the language of lambda calculus but also its basic laws are available, although in weakened form. One notable result is that beta refinement $P[P'/x] \sqsubseteq (\lambda x. P)P'$ holds in the transformer model even if $P$ combines demonic and angelic nondeterminacy and $P'$ has demonic nondeterminacy.

The language of de Moor and Gibbons includes fixpoints. Our models support recursive definitions, because homsets are complete lattices, but we leave thorough investigation as future work. The authors say “the semantics we have sketched leaves many questions unanswered”. It would be particularly interesting to check which operations are monotonic with respect to their refinement order, and whether terms in patterns need to be restricted as in our models.

Here are some other open problems.
• What are the interesting general laws for patterns? One way to investigate would be to focus on heap patterns: design a subset of those that is easy to compile, and find requisite simplification laws.

• To investigate pattern matching for heap operations, one could try to derive a version of \textit{repmin} with shared objects. Another example close at hand would be to derive conventional code for \textit{in situ} list reversal, starting from the abstract version sketched in Section 2. A further exercise would be to specify and derive a program that maps an operation over a list of root pointers to disjoint heap structures. For example, map \textit{repmin} over a list of trees. The pattern could use a confined separation operator along the lines of Wang et al. [WBO08].

• For a usable calculus of imperative programs it seems desirable to avoid explicit threading of state through expressions. This suggests combining pattern terms with a monad to encapsulate state, perhaps drawing on ideas from Hoare Type Theory [NMB08].

• Our presentation emphasizes algebraic structure: the proofs and constructions are pointfree and based on Propositions 1, 5, 11, 12—with a few exceptions. The semantic definitions for pattern terms rely on manipulation of non-monotonic functions \textit{rg} and \textit{exim}. The proof of (19) is not point-free. The problem is to fix these blemishes so that the Propositions can be taken as axioms.

• The last problem is to prove the hexagon equality (29), an attractive chiasmus connecting \( F \) with \( R \). And do the same for \( R \) and \( T \). For the semantics of [dMG00], Benton sketched a proof for all functional terms in an unpublished note (2001) [Ben00], using a form of logical relation. In 2015, according to personal communication with the Benton, de Moor, and Gibbons, there has been no further development, but also little doubt that the result should hold.

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Special thanks to José for eloquently advocating and teaching pointfree relation algebra and for celebrating my student’s remark that those who like maths also enjoy music and maps. Following the symbols as they danced along the backward arrows of predicate transformers, I arrived at a canon cancrizans of \textit{co/maps which I offer in celebration of the work of José Nuno Oliveira.}

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[Ben00] Nick Benton. A proof sketch of something which may possibly be a conjecture of Oege de Moor. Private communication, December 2000.


