Explicitly self-similar solutions for the
Euler/Navier-Stokes-Korteweg equations in $\mathbb{R}^N$

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Abstract

In this paper, we present a unified formulae for explicit self-similar solutions of the Euler/Navier-Stokes-Korteweg equations arising in the modeling of capillary fluids. The technique used here is to reduce Euler/Navier-Stokes-Korteweg equations into a series of solvable ordinary differential equations by making use of multi-dimensional self-similar ansatz and variable separation method.

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1 Introduction

We consider the Euler/Navier-Stokes-Korteweg equations arising in the modeling of capillary fluids, which read

\[ \rho_t + \nabla \cdot (\rho u) = 0, \]  
\[ (\rho u)_t + (\rho u \cdot \nabla)u + \nabla p + \mu \Delta u = \rho \nabla \left( K(\rho) \Delta \rho + \frac{1}{2} K'(\rho)|\nabla \rho|^2 \right), \]

where \( x = (x_1, x_2, \cdots, x_N)^T \in \mathbb{R}^N \) and \( u = (u_1, u_2, \cdots, u_N)^T \) are the components of the \( N \)-dimensional velocity field, \( \rho(x, t) \) is the density and \( p(x, t) \) is the pressure of the fluid. The right-hand side of the second equation modelizes capillarity forces. It is a dispersive perturbation of the classical Euler equations. If \( \mu = 0 \), the system (1.1)-(1.2) is called the Euler-Korteweg equations [1]-[4]. If \( \mu > 0 \), the system (1.1)-(1.2) is the Navier-Stokes-Korteweg equations [5]-[8]. So we call the system (1.1)-(1.2) as Euler/Navier-Stokes-Korteweg equations in this paper.

For special case when \( K(\rho) = k_2 > 0 \), the system (1.1)-(1.2) reduce to [9, 10]

\[ \rho_t + \nabla \cdot (\rho u) = 0, \]
\[ (\rho u)_t + (\rho u \cdot \nabla)u + \nabla p + \mu \Delta u = k_2 \rho \nabla \Delta \rho, \]

which further reduce to compressible Euler/Navier-Stokes equations when \( k_2 = 0 \) [11]-[22]

\[ \rho_t + \nabla \cdot (\rho u) = 0, \]
\[ (\rho u)_t + (\rho u \cdot \nabla)u + \nabla p + \mu \Delta u = 0. \]

The compressible Euler/Navier-Stokes equations (1.5)-(1.6) have been extensively investigated in the term of both weak solutions and exact analytical solutions. For instance, the global existence in critical spaces for compressible Navier-Stokes equations was shown by Danchin [11]. The existence and explicit structure of global solutions for Euler equations with antisymmetry continuous initial data was constructed by Zhang [12]. Chen proved the existence of global weak solution for Euler equation with symmetry outside of a circular core with the center at the origin [13]. For more results and details, see references [14]-[22]. There are also much work on the exact solutions of the Euler/Navier-Stokes equations. For example, In 1995, Zhang and Zheng obtained spiral solutions for the 2D compressible Euler equations with \( p = K \rho^\gamma \) and \( \gamma = 2 \) [23]. Recently, Li found Lax pairs for 2D and 3D incompressible Euler equations [24]. Lou et al proposed Backlund transformation, Darboux transformation and exact solutions for the 2D Euler equations in vorticity form [25]. Ludlow, Clarkson and Bassom found similarity reductions and exact solutions for the two-dimensional incompressible Navier-Stokes equations [26]. Yuen further obtained a class of self-similar solutions for the compressible Euler/Navier-Stokes equations in \( \mathbb{R}^N \) [27, 28]. Based on matrix theory and decomposition technique,
we theoretically show the existence of the Cartesian rotational solutions for the general \(N\)-dimensional compressible Euler/Navier-Stokes equations [29].

It is well-known that the self-similar solution is a kind of important analytical solutions. With a straightforward generalization of the self-similar ansatz a partial differential equation system of the three-dimensional Navier-Stokes equations was successfully investigated [30]. Bana et al obtained analytic self-similar solutions for the one-dimensional compressible Euler equations with heat conduction and the Navier-Stokes equations [31]. Recently, we constructed analytic self-similar solutions for the \(N\)-dimensional compressible Euler equations and the Navier-Stokes equations, such solutions are still not explicit [27, 28]. However, related to the Euler/Navier-Stokes-Korteweg equations (1.1)-(1.2) or (1.3)-(1.4), except some mathematical results on the existence and well-posedness of weak solutions in Sobolev space [1]-[10], the exact solutions, especially self-similar solution of the Euler/Navier-Stokes-Korteweg equations have been still unknown to our knowledge.

Therefore, in present paper we would like to present explicit self-similar solutions for the \(N\)-dimensional Euler/Navier-Stokes-Korteweg equations (1.3)-(1.4). Here we not only generalize self-similar solutions to \(N\)-dimensional case, but also provide an explicitly unified formulae for the self-similar solutions. The technique used here is to reduce Euler/Navier-Stokes-Korteweg equations into a series of solvable ordinary differential equations by making use of generalization of self-similar ansatz and variable separation method.

2 Unified formulae of self-similar solutions

To solve the Euler equations (1.5)-(1.6), Yuen showed that the conservation equation (1.5) there exist a solution in the form [27]

\[
\begin{align*}
\dot{u}_i &= \frac{\dot{a}_i(t)}{a_i(t)} x_i, \\
\rho &= \frac{h(\xi)}{\prod_{i=1}^N a_i(t)}, \\
\xi &= \sum_{i=1}^N \frac{x_i^2}{a_i^2(t)}. 
\end{align*}
\] (2.1)

Through their the conservation equations (1.3) and (1.5) are the same between the Euler/Navier-Stokes-Korteweg equation (1.3)-(1.4) and Euler equations (1.5)-(1.6), by checking computation, we find that similar transformation (2.1) is not compatible solution of the second equation (1.4) of the system (1.3)-(1.4). So it is the key in our present paper to search for a suitable similar transformation from the conservation equation (1.3), so that it can be used to construct explicit self-similar solutions for the Euler-Korteweg equations (1.3)-(1.4).

In many situations in gas dynamics, we may consider the case where the density \(\rho\) and pressure \(p\) satisfy a relation

\[ p(\rho) = k_1 \rho^\gamma, \] (2.2)
with \( k_1 > 0 \) and the constant \( \gamma = c_p / c_v \), where \( c_p, c_v \) are the specific heat capacities under constant pressure and constant volume respectively.

We can rewrite (1.3) and (1.4) in the form

\[
\begin{align*}
\rho_t + \sum_{i=1}^{N} \left( \rho x_i u_i + \rho u_{i,x_i} \right) &= 0, \\
\rho u_{i,t} + \sum_{j=1}^{N} u_j u_{i,x_j} + \frac{k_1 \gamma}{\gamma - 1} (\rho^\gamma - 1) x_i + \mu \rho^{-1} \Delta u_i &= k_2 (\Delta \rho) x_i, \quad i = 1, \ldots, N.
\end{align*}
\]

In the following we search for solutions of the system (2.3)-(2.4) in the form of self-similar ansatz

\[
\begin{align*}
\rho &= g(t) h(\xi), \quad \xi = \sum_{i=1}^{N} a_i(t) x_i,
\end{align*}
\]

where \( a_i(t), f_i(t), \ i = 1, \ldots, N, \ g(t) \) and \( h(\xi) \) are functions to be determined.

Substituting (2.5) into the first equation (2.3) yields

\[
\begin{align*}
\rho_t + \sum_{i=1}^{N} \left( \rho x_i u_i + \rho u_{i,x_i} \right) &= h(\xi) \left( \dot{g}(t) + g(t) \sum_{i=1}^{N} f_i(t) \right) + g(t) h'(\xi) \sum_{i=1}^{N} \left[ \dot{a}_i(t) + f_i(t) a_i(t) \right] x_i = 0.
\end{align*}
\]

where the notations dot \( \dot{\cdot} = \frac{\partial}{\partial t} \) denotes derivative with respect to \( t \) and the prime \( ' = \frac{\partial}{\partial \xi} \) denotes derivative with respect to \( \xi \).

It is obvious that the equation (2.6) is satisfied if we set

\[
\begin{align*}
\dot{g}(t) + g(t) \sum_{i=1}^{N} f_i(t) &= 0, \\
\dot{a}_i(t) + f_i(t) a_i(t) &= 0, \quad i = 1, \ldots, N.
\end{align*}
\]

The above system (2.7)-(2.8) is a system with \( N + 1 \) ordinary differential equations and \( 2N + 1 \) unknown functions, which always have solutions.

From (2.8), we directly obtain that

\[
\begin{align*}
f_i(t) &= -\frac{\dot{a}_i(t)}{a_i(t)}, \quad i = 1, \ldots, N.
\end{align*}
\]

Substituting (2.9) back to (2.7) yields

\[
\begin{align*}
\dot{g}(t) + g(t) \sum_{i=1}^{N} f_i(t) &= \dot{g}(t) - g(t) \sum_{i=1}^{N} \frac{\dot{a}_i(t)}{a_i(t)} \\
&= \dot{g}(t) - g(t) \frac{d}{dt} \sum_{i=1}^{N} \ln a_i(t) = \dot{g}(t) - g(t) \frac{d}{dt} \ln \prod_{i=1}^{N} a_i(t) = 0,
\end{align*}
\]
which has a solution

\[ g(t) = \exp \left( \int \frac{d}{dt} \ln \prod_{i=1}^{N} a_i(t) dt \right) = \prod_{i=1}^{N} a_i(t). \]  (2.10)

By virtue of (2.9) and (2.10), the self-similar solution (2.5) becomes

\[ u_i = -\frac{\dot{a}_i(t)}{a_i(t)} x_i, \quad \rho = h(\xi) \prod_{i=1}^{N} a_i(t), \quad \xi = \sum_{i=1}^{N} a_i(t)x_i, \]  (2.11)

which implies that it a solution of the conservation equation of mass (2.3). Here we notice that in the expression (2.11), the functions \( a_i(t), i = 1, \ldots, N \) and \( h(\xi) \) are \( N+1 \) arbitrary functions to be used in solving the second equation (2.4).

Now inserting (2.11) back into (2.4), one can write

\[
\begin{align*}
2\ddot{a}_i^2(t) - a_i(t)\dot{a}_i(t) x_i + k_1\gamma a_i(t) \left( \prod_{j=1}^{N} a_j(t) \right)^{\gamma^{-1}} h^{\gamma-2}(\xi)h'(\xi) \\
- k_2a_i(t) \prod_{j=1}^{N} a_j(t) \sum_{j=1}^{N} a_j^2(t) h'''(\xi) = 0, \quad i = 1, \ldots, N.
\end{align*}
\]  (2.12)

Observe that, the system (2.12) contains \( N \) equations and \( N+1 \) unknown functions \( a_i(t) \) and \( h(\xi) \) with respect to variables \( t \) and \( \xi \) respectively. This motivate us to split the system (2.12) into two systems.

It is obvious that the system (2.12) hold if \( a_i(t), \quad i = 1, \ldots, N \) and \( h(\xi) \) satisfy the following equations

\[
\begin{align*}
2\ddot{a}_i^2(t) - a_i(t)\dot{a}_i(t) &= 0, \quad i = 1, \ldots, N, \quad \text{(2.13)} \\
k_1\gamma \left( \prod_{j=1}^{N} a_j(t) \right)^{\gamma^{-2}} h^{\gamma-2}(\xi)h'(\xi) - k_2\sum_{j=1}^{N} a_j^2(t) h'''(\xi) &= 0. \quad \text{(2.14)}
\end{align*}
\]

The system (2.13) is a system of ordinary differential equations, which have solutions

\[ a_i(t) = \frac{1}{c_{i,1} t + c_{i,2}}, \quad i = 1, \ldots, N, \]  (2.15)

where \( c_{i,1}, c_{i,2} \) are arbitrary constants.

Substituting (2.15) into the equation (2.14) yields

\[
\begin{align*}
k_1\gamma \prod_{j=1}^{N} (b_j t + c_j)^{2-\gamma} h^{\gamma-2}(\xi)h'(\xi) - k_2\sum_{j=1}^{N} (b_j t + c_j)^{-2} h'''(\xi) &= 0. \quad \text{(2.16)}
\end{align*}
\]

We make a constraint

\[ \gamma = \frac{2}{N} + 2, \quad c_{i,1} = c_1, \quad c_{i,2} = c_2, \quad i = 1, \ldots, N, \]
where $c_1$ and $c_2$ are arbitrary constants, then (2.16) becomes

$$\frac{2k_1(N+1)}{N}h^{\frac{3}{2}}(\xi)h'(\xi) - k_2Nh'''(\xi) = 0. \quad (2.17)$$

which has a solution

$$h(\xi) = \nu\xi^{-N}, \quad \nu = \left(\frac{k_2N^2(N+2)}{2k_1}\right)^{N/2}.$$  

In this way, we find a unified formulae for analytical self-similar solutions of the $N$-dimensional Euler/Navier-Stokes-Korteweg equations

$$u_i = -\frac{c_1}{c_1t + c_2}x_i, \quad i = 1, \ldots, N;$$

$$\rho = \nu \left(\frac{c_1t + c_2}{x_1 + \cdots + x_N}\right)^N. \quad (2.18)$$

The expression (2.18) is a unified formula for the self-similar solutions of Euler/Navier-Stokes-Korteweg equations in $\mathbb{R}^N$. For example, when $N = 1$, the expression (2.17) gives a kind of self-similar solutions for one-dimensional Euler/Navier-Stokes-Korteweg equations

$$u_1 = -\frac{c_1}{c_1t + c_2}x_1, \quad \rho = \sqrt{\frac{3k_2}{2k_1}} \left(\frac{c_1t + c_2}{x_1}\right). \quad (2.19)$$

When $N = 2$, the expression (2.17) gives a kind of self-similar solutions for two-dimensional Euler/Navier-Stokes-Korteweg equations

$$u_1 = -\frac{c_1}{c_1t + c_2}x_1, \quad u_2 = -\frac{c_1}{c_1t + c_2}x_2, \quad \rho = \frac{8k_2}{k_1} \left(\frac{c_1t + c_2}{x_1 + x_2}\right)^2.$$  

3 Further discussion on one-dimensional case

Let us go back to observe the the system (2.12), where we split it into $N + 1$ solvable ordinary differentials equations (2.13) and (2.14). It is still difficult to directly split to $N$ ordinary differentials equations. However, it is can done for one-dimensional case. In this way, we may obtain more general self-similar solutions than (2.19).

In fact, for $N = 1$, then the system (2.12) become a simple partial differential equation

$$\frac{2a_1^2(t) - a_1(t)a_1'(t)}{a_1^3(t)}\xi + k_1\gamma a_1^2(t)h^{\gamma-2}(\xi)h'(\xi) - k_2a_1^4(t)h'''(\xi) = 0, \quad (3.20)$$

here we have used the key relation $\xi = a_1(t)x_1$. But there is no such relation to be sued for $N$-dimensional case.

We let $\gamma = 4$ and

$$\frac{2a_1^2(t) - a_1(t)a_1'(t)}{a_1^3(t)} = \alpha, \quad (3.21)$$
Euler equations with rotational forces

where $\alpha$ is an arbitrary constant. In this way, the equation (3.20) becomes an ordinary differential equation with respect to $\xi$

$$k_2h''(\xi) - 4k_1h^2(\xi)h'(\xi) + \alpha \xi = 0. \quad (3.22)$$

So we can find more general self-similar solutions than (2.19)

$$u_i = -\frac{\dot{a}_1(t)}{a_1(t)}x_1, \quad \rho = a_1(t)h(\xi), \quad \xi = a_1(t)x_1, \quad (3.23)$$

where $a_1(t)$ and $h(\xi)$ satisfy equations (3.21) and (3.22) respectively.

For $\alpha = 0$, solving (3.21) and (3.22), and then (3.23) is exactly the solution (2.19).

For $\alpha \neq 0$, we can get another kind of solutions. For example, we take a solution of (3.21) as

$$a_1(t) = (c_1t + c_2)^{-2/5}, \quad \alpha = -\frac{6c_1^2}{25}.$$  

We then get a solution

$$u(x,t) = \frac{2c_1}{5}(c_1t + c_2)^{-1}x, \quad \rho = (c_1t + c_2)^{-2/5}h(\xi), \quad \xi = (c_1t + c_2)^{-2/5}x,$$

where $c_1$ and $c_2$ are arbitrary constants and $h(\xi)$ is given by

$$k_2h''(\xi) - 4k_1h^2(\xi)h'(\xi) + \frac{6c_1^2}{25} \xi = 0.$$

which can be integrated once and yields

$$k_2h''(\xi) - \frac{4}{3}k_1h^3(\xi) + \frac{3}{25}c_1^2\xi^2 = 0. \quad (3.24)$$

This is a variable-coefficient nonlinear ordinary differential equation, whose explicit solutions are still difficult to be constructed here.

4 Conclusions and Remarks

The current analysis presents a procedure that establishes explicit analytical self-similar solutions of the N-dimensional Euler/Navier-Stokes-Korteweg equations (1.3)-(1.4). Besides the intrinsic mathematical interests, these solutions will be applicable to many physical disciplines, e.g. fluid mechanics and astrophysics. High speed flows from the release of a localized amount of energy typically lead to a velocity field in similarity variables for large time and large distance from the origin. These similarity variable solutions usually match the form investigated in this paper. Despite the progress made here and other works in the literature, many challenges still remain ahead:

(1) How to construct analytical self-solutions for the more general Euler/Navier-Stokes-Korteweg equations (1.1)-(1.2) in $\mathbb{R}^N$?
(2) For $N$-dimensional Euler/Navier-Stokes-Korteweg equations (1.1)-(1.2) or (1.3)-(1.4), how to utilize matrix technique for constructing more general Cartesian solutions of the form

$$u = b(t) + Ax,$$

which have been obtained for the compressible Euler/Navier-Stokes equations (1.5)-(1.6) [29].

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