Disjunctive Cuts for Cross-Sections of the Second-Order Cone

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Abstract

In this paper we study general two-term disjunctions on affine cross-sections of the second-order cone. Under some mild assumptions, we derive a closed-form expression for a convex inequality that is valid for such a disjunctive set, and we show that this inequality is sufficient to characterize the closed convex hull of all two-term disjunctions on ellipsoids and paraboloids and a wide class of two-term disjunctions—including split disjunctions—on hyperboloids. Our approach relies on the work of Kılınç-Karzan and Yıldız which considers general two-term disjunctions on the second-order cone.

Keywords: Mixed-integer conic programming, second-order cone programming, cutting planes, disjunctive cuts

1 Introduction

In this paper we consider the mixed-integer second-order conic set

\[ S := \{ x \in \mathbb{L}^n : Ax = b, \ x_j \in \mathbb{Z} \ \forall j \in J \} \]

where \( \mathbb{L}^n \) is the \( n \)-dimensional second-order cone \( \mathbb{L}^n := \{ x \in \mathbb{R}^n : \|(x_1; \ldots; x_{n-1})\| \leq x_n \} \), \( A \) is an \( m \times n \) real matrix of full row rank, \( d \) and \( b \) are real vectors of appropriate dimensions, \( J \subseteq \{1, \ldots, n\} \), and \( \| \cdot \| \) denotes the Euclidean norm. The set \( S \) appears as the feasible solution set or a relaxation thereof in mixed-integer second order cone programming problems. Because the structure of \( S \) can be very complicated, a first approach to solving

\[ \sup \left\{ d^\top x : x \in S \right\} \tag{1} \]

events solving the relaxed problem obtained after dropping the integrality requirements on the variables:

\[ \sup \left\{ d^\top x : x \in C \right\} \text{ where } C := \{ x \in \mathbb{L}^n : Ax = b \} \]

The set \( C \) is called the natural continuous relaxation of \( S \). Unfortunately, the continuous relaxation \( C \) is often a poor approximation to the mixed-integer conic set \( S \), and tighter formulations are needed for the development of practical strategies for solving [1]. An effective way to improve the approximation quality of the continuous relaxation \( C \) is to strengthen it with additional convex inequalities that are valid for \( S \) but not for the whole of \( C \). Such valid inequalities can be

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derived by exploiting the integrality of the variables $x_j, j \in J$, and enhancing $C$ with linear 
two-term disjunctions $l_1^T x \geq l_{1,0} \lor l_2^T x \geq l_{2,0}$ that are satisfied by all solutions in $S$. Valid 
inequalities that are obtained from disjunctions using this approach are known as disjunctive cuts. In this paper we study two-term disjunctions on the set $C$ and give closed-form expressions 
for the strongest disjunctive cuts that can be obtained from such disjunctions.

Disjunctive cuts were introduced by Balas in the context of mixed-integer linear programming 
[3] and have since been the cornerstone of theoretical and practical achievements in integer 
programming. There has been a lot of recent interest in extending disjunctive cutting-plane 
theory from the domain of mixed-integer linear programming to that of mixed-integer conic 
programming [2 7 9 11 12 15]. Kılınç-Karzan [13] studied minimal valid linear inequalities 
for general disjunctive conic sets and showed that these are sufficient to describe the associated 
closed convex hull under a mild technical assumption. Bienstock and Michalka [6] studied the 
characterization and separation of linear inequalities that are valid for the epigraph of a convex, 
differentiable function whose domain is restricted to the complement of a convex set. On the 
other hand, several papers in the last few years have focused on deriving closed-form expressions 
for nonlinear convex inequalities that fully describe the convex hull of a disjunctive second-order 
cone set in the space of the original variables. Dadush et al. [10] and Andersen and Jensen [1] 
derived split cuts for ellipsoids and the second-order cone, respectively. Modaresi et al. extended 
these results to split disjunctions on cross-sections of the second-order cone [10] and compared 
the effectiveness of split cuts against conic MIR inequalities and extended formulations [15]. 
For disjoint two-term disjunctions on cross-sections of the second-order cone and under the 
assumption that $\{ x \in C : l_1^T x = l_{1,0} \}$ and $\{ x \in C : l_2^T x = l_{2,0} \}$ are bounded, Belotti et al. 
[4 5] proved that there exists a unique cone which describes the convex hull of the disjunction. 
They also identified a procedure for identifying this cone when $C$ is an ellipsoid. Using the 
structure of minimal valid linear inequalities, Kılınç-Karzan and Yıldız [14] derived a family of 
convex inequalities which describes the convex hull of a general two-term disjunction on the 
whole second-order cone. In this paper, we pursue a similar goal: We study general two-term 
disjunctions on a cross-section $C$ of the second-order cone, namely $C = \{ x \in \mathbb{L}^n : Ax = b \}$. 
Given a disjunction $l_1^T x \geq l_{1,0} \lor l_2^T x \geq l_{2,0}$ on $C$, we let 

$$C_1 := \{ x \in C : l_1^T x \geq l_{1,0} \} \quad \text{and} \quad C_2 := \{ x \in C : l_2^T x \geq l_{2,0} \}.$$ 

In order to derive the tightest disjunctive cuts that can be obtained for $S$ from the disjunction 
$C_1 \cup C_2$, we study the closed convex hull $\text{cl} \text{cvx}(C_1 \cup C_2)$. In particular, we are interested in 
convex inequalities that may be added to the description of $C$ to obtain a characterization 
of $\text{cl} \text{cvx}(C_1 \cup C_2)$. Our starting point is the paper [14] about two-term disjunctions on the 
second-order cone $\mathbb{L}^n$. We extend the main result of [14] to cross-sections of the second-order 
cone. Such cross-sections include ellipsoids, paraboloids, and hyperboloids as special cases. Our 
results generalize the work of [10, 16] on split disjunctions on cross-sections of the second-order 
cone and [4] on disjoint two-term disjunctions on ellipsoids. We note here that general results 
on convexifying the intersection of a cross-section of the second-order cone with a non-convex 
cone defined by a single homogeneous quadratic were recently obtained independently in [5].

We first show in Section 2 that the continuous relaxation $C$ can be assumed to be the 
intersection of a lower-dimensional second-order cone with a single hyperplane. In Section 3 
we give a complete description of the convex hull of a homogeneous two-term disjunction on 
the whole second-order cone. In Section 4 we prove our main result, Theorem 3, characterizing 
$\text{cl} \text{cvx}(C_1 \cup C_2)$ under certain conditions. We end the paper with two examples which illustrate 
the applicability of Theorem 3.
Throughout the paper, we use \( \text{conv} K \), \( \overline{\text{conv}} K \), cone \( K \), and \( \text{span} K \) to refer to the convex hull, closed convex hull, conical hull, and linear span of a set \( K \), respectively. We also use \( \text{bd} K \), \( \text{int} K \), and \( \text{dim} K \) to refer to the boundary, interior, and dimension of \( K \). The dual cone of \( K \subseteq \mathbb{R}^n \) is \( K^* := \{ \alpha \in \mathbb{R}^n : x^\top \alpha \geq 0 \ \forall x \in K \} \). The second-order cone \( \mathbb{L}^n \) is self-dual, that is, \((\mathbb{L}^n)^* = \mathbb{L}^n\). Given a vector \( u \in \mathbb{R}^n \), we let \( \tilde{u} := (u_1; \ldots; u_{n-1}) \) denote the subvector obtained by dropping its last entry.

## 2 Intersection of the Second-Order Cone with an Affine Subspace

In this section, we show that the continuous relaxation \( C \) can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. Let \( E := \{ x \in \mathbb{R}^n : Ax = b \} \) so that \( C = \mathbb{L}^n \cap E \). We are going to use the following lemma to simplify our analysis.

**Lemma 1.** Let \( V \) be a \( p \)-dimensional linear subspace of \( \mathbb{R}^n \). The intersection \( \mathbb{L}^n \cap V \) is either the origin, a half-line, or a bijective linear transformation of \( \mathbb{L}^p \).

See Section 2.1 of [1] for a similar result. We do not give a formal proof of Lemma 1 but just note that it can be obtained by observing that the second-order cone is the conic hull of a (one dimension smaller) sphere, and that the intersection of a sphere with an affine space is either empty, a single point (when the affine space intersects the sphere but not its interior), or a lower dimensional sphere of the same dimension as the affine space (when the affine space intersects the interior of the sphere).

Lemma 1 implies that, when \( b = 0 \), \( C \) is either the origin, a half-line, or a bijective linear transformation of \( \mathbb{L}^{n-m} \). The closed convex hull \( \overline{\text{conv}}(C_1 \cup C_2) \) can be described easily when \( C \) is a single point or a half-line. Furthermore, the problem of characterizing \( \overline{\text{conv}}(C_1 \cup C_2) \) when \( C \) is a bijective linear transformation of \( \mathbb{L}^{n-m} \) can be reduced to that of convexifying an associated two-term disjunction on \( \mathbb{L}^{n-m} \). We refer the reader to [14] for a detailed study of the closed convex hulls of two-term disjunctions on the second-order cone.

In the remainder, we focus on the case \( b \neq 0 \). Note that, whenever this is the case, we can permute and normalize the rows of \( (A, b) \) so that its last row reads \((a_m^\top, 1)^\top\), and subtracting a multiple of \((a_m^\top, 1)^\top\) from the other rows if necessary, we can write the remaining rows of \( (A, b) \) as \((\tilde{A}, 0)^\top\). Therefore, we can assume without any loss of generality that all components of \( b \) are zero except the last one. Isolating the last row of \( (A, b) \) from the others, we can then write

\[
E = \left\{ x \in \mathbb{R}^n : \tilde{A}x = 0, \ a_m^\top x = 1 \right\}.
\]

Let \( V := \{ x \in \mathbb{R}^n : \tilde{A}x = 0 \} \). By Lemma 1, \( \mathbb{L}^n \cap V \) is the origin, a half-line, or a bijective linear transformation of \( \mathbb{L}^{n-m+1} \). Again, the first two cases are easy and not of interest in our analysis. In the last case, we can find a matrix \( D \) whose columns form an orthonormal basis for \( V \) and define a nonsingular matrix \( H \) such that \( \{ y \in \mathbb{R}^{n-m+1} : Dy \in \mathbb{L}^n \} = H\mathbb{L}^{n-m+1} \). Then
we can represent \( C \) equivalently as

\[
C = \left\{ x \in \mathbb{L}^n : x = Dy, a_m^\top x = 1 \right\}
\]

\[
= D \left\{ y \in \mathbb{R}^{n-m+1} : Dy \in \mathbb{L}^n, a_m^\top Dy = 1 \right\}
\]

\[
= D \left\{ y \in \mathbb{R}^{n-m+1} : y \in H\mathbb{L}^{n-m+1}, a_m^\top Dy = 1 \right\}
\]

\[
= DH \left\{ z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1 \right\}.
\]

The set \( C = \mathbb{L}^n \cap E \) is a bijective linear transformation of \( \left\{ z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1 \right\} \). Furthermore, the same linear transformation maps any two-term disjunction in \( \left\{ z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1 \right\} \) to a two-term disjunction in \( C \) and vice versa. Thus, without any loss of generality, we can take \( m = 1 \) in (1) and study the problem of describing \( \overline{\text{conv}}(C_1 \cup C_2) \) where

\[
C = \left\{ x \in \mathbb{L}^n : a_1^\top x = 1 \right\},
\]

\[
C_1 = \left\{ x \in C : l_1^\top x \geq l_{1,0} \right\}, \quad \text{and} \quad C_2 = \left\{ x \in C : l_2^\top x \geq l_{2,0} \right\}.
\]

In Section 4 we will give a full description of \( \overline{\text{conv}}(C_1 \cup C_2) \) under certain conditions.

### 3 Homogeneous Two-Term Disjunctions on the Second-Order Cone

In this section, we study the convex hull of a homogeneous two-term disjunction \( c_1^\top x \geq 0 \lor c_2^\top x \geq 0 \) on the second-order cone. Let

\[
Q_1 := \left\{ x \in \mathbb{L}^n : c_1^\top x \geq 0 \right\} \quad \text{and} \quad Q_2 := \left\{ x \in \mathbb{L}^n : c_2^\top x \geq 0 \right\}.
\]

The main result of this section characterizes \( \text{conv}(Q_1 \cup Q_2) \). Note that \( Q_1 \) and \( Q_2 \) are closed, convex, pointed cones; therefore, \( \text{conv}(Q_1 \cup Q_2) \) is always closed (see, e.g., Rockafellar [17, Corollary 9.1.3]).

When \( Q_1 \subseteq Q_2 \), we have \( \text{conv}(Q_1 \cup Q_2) = Q_2 \). Similarly, when \( Q_1 \supseteq Q_2 \), we have \( \text{conv}(Q_1 \cup Q_2) = Q_1 \). In the remainder of this section, we focus on the case where \( Q_1 \nsubseteq Q_2 \) and \( Q_1 \nsubseteq Q_2 \).

**Assumption 1.** \( Q_1 \nsubseteq Q_2 \) and \( Q_1 \nsubseteq Q_2 \).

We also make the following technical assumption.

**Assumption 2.** \( Q_1 \cap \text{int} \mathbb{L}^n \neq \emptyset \) and \( Q_2 \cap \text{int} \mathbb{L}^n \neq \emptyset \).

This assumption will be useful later when we use Theorem 1 whose proof relies on conic duality.

By Assumption 1, we have \( Q_1, Q_2 \subseteq \mathbb{L}^n \), and by Assumption 2, we have that \( Q_1 \) and \( Q_2 \) are full-dimensional. This implies \( c_1, c_2 \notin \pm \mathbb{L}^n \), or equivalently \( \| \tilde{c}_i \|^2 > c_{i,n}^2 \), for \( i \in \{1, 2\} \). By scaling \( c_1 \) and \( c_2 \) with appropriate positive scalars if necessary, we may assume without any loss of generality that

\[
\| \tilde{c}_1 \|^2 - c_{1,n}^2 = \| \tilde{c}_2 \|^2 - c_{2,n}^2 = 1.
\]

These have the following consequences.

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Remark 1. Let $c_1$ and $c_2$ satisfy (4). Then
\[
\mathcal{M} := \|\tilde{c}_1\|^2 - c_1, n - (\|\tilde{c}_2\|^2 - c_2, n) = 0,
\]
\[
\mathcal{N} := \|\tilde{c}_1 - \tilde{c}_2\|^2 - (c_1, n - c_2, n)^2 = 2 - 2 \left( \tilde{c}_1^T \tilde{c}_2 - c_1, n c_2, n \right).
\]

Remark 2. Let $Q_1$ and $Q_2$, defined as in (3), satisfy Assumption 4. Then we have $c_1 - c_2 \notin \pm \mathbb{L}^n$. Indeed, $c_1 - c_2 \in \mathbb{L}^n$ implies that $(c_1 - c_2)^T x \geq 0$ for all $x \in \mathbb{L}^n$, and this implies $C_1 \subseteq C_2$; similarly, $c_2 - c_1 \in \mathbb{L}^n$ implies $C_2 \subseteq C_1$. Hence,
\[
\mathcal{N} = \|\tilde{c}_1 - \tilde{c}_2\|^2 - (c_1, n - c_2, n)^2 > 0.
\]

The following result from [14] gives a valid convex inequality for $\text{conv}(Q_1 \cup Q_2)$.

Theorem 1 ([14], Theorem 3 and Remark 2). Let $Q_1$ and $Q_2$ be defined as in (3). Suppose Assumptions 4 and 2 hold. Then the inequality
\[
-(c_1 + c_2)^T x \leq \sqrt{((c_1 - c_2)^T x)^2 + \mathcal{N} (x_i^2 - \|\bar{x}\|^2)}
\]
is valid for $\text{conv}(Q_1 \cup Q_2)$. Furthermore, this inequality is convex in $\mathbb{L}^n$.

The next proposition shows that (5) can be written in conic quadratic form in $\mathbb{L}^n$ except in the region where both clauses of the disjunction are satisfied. Its proof is a simple extension of the proofs of Propositions 3 and 4 in [14] and therefore omitted. Let
\[
r := \left( \begin{array}{c} \tilde{c}_1 - \tilde{c}_2 \\ -c_1, n + c_2, n \end{array} \right).
\]

Proposition 1 ([14], Propositions 3 and 4). Let $Q_1$ and $Q_2$ be defined as in (3). Suppose Assumptions 4 and 2 hold. Let $x' \in \mathbb{L}^n$ be such that $c_1^T x' \leq 0$ or $c_2^T x' \leq 0$. Then the following statements are equivalent:

i) $x'$ satisfies (5).

ii) $x'$ satisfies the conic quadratic inequality
\[
\mathcal{N} x - 2(c_1^T x) r \in \mathbb{L}^n.
\]

iii) $x'$ satisfies the conic quadratic inequality
\[
\mathcal{N} x + 2(c_2^T x) r \in \mathbb{L}^n.
\]

Remark 3. When $c_1$ and $c_2$ satisfy (4), the inequalities (6) and (7) describe a cylindrical second-order cone whose lineality space contains $\text{span}\{r\}$. This follows from Remark 7 by observing that
\[
\mathcal{N} = 2 - 2 \left( \tilde{c}_1^T \tilde{c}_2 - c_1, n c_2, n \right) = 2c_1^T r = -2c_2^T r.
\]

The next theorem is the main result of this section. It shows that the inequality (5) is in fact sufficient to describe $\text{conv}(Q_1 \cup Q_2)$ when $c_1$ and $c_2$ are scaled so that they satisfy (4). Because this assumption is without any loss of generality, our result settles one of the cases left open by Kılınç-Karzan and Yıldız [14], where the right-hand-sides of both terms of the disjunction are zero in (3).
Theorem 2. Let $Q_1$ and $Q_2$ be defined as in (3). Suppose Assumptions 1 and 2 hold. Assume that $c_1$ and $c_2$ have been scaled so that they satisfy (4). Then

$$\text{conv}(Q_1 \cup Q_2) = \{ x \in \mathbb{L}^n : x \text{ satisfies } (5) \}. \tag{8}$$

Proof. Let $D$ denote the set on the right-hand side of (6). We already know that (5) is valid for $\text{conv}(Q_1 \cup Q_2)$. Hence, $\text{conv}(Q_1 \cup Q_2) \subseteq D$. Let $x' \in D$. If $x' \in Q_1 \cup Q_2$, then clearly $x' \in \text{conv}(Q_1 \cup Q_2)$. Therefore, suppose $x' \in \mathbb{L}^n \setminus (Q_1 \cup Q_2)$ is a point that satisfies (5). By Proposition 1, $x'$ satisfies

$$\mathcal{N} x' - 2(c_1^T x') r \in \mathbb{L}^n \quad \text{and} \quad \mathcal{N} x' + 2(c_2^T x') r \in \mathbb{L}^n.$$ 

We are going to show that $x'$ belongs to $\text{conv}(Q_1 \cup Q_2)$.

By Remarks 2 and 3, $0 < N = 2c_1^T r = -2c_2^T r$. Let

$$\alpha_1 := \frac{-c_1^T x'}{c_1^T r}, \quad \alpha_2 := \frac{-c_2^T x'}{c_2^T r},$$

$$x_1 := x' + \alpha_1 r, \quad x_2 := x' + \alpha_2 r. \tag{9}$$

It is not difficult to see that $c_1^T x_1 = c_2^T x_2 = 0$. Furthermore, $x' \in \text{conv}\{x_1, x_2\}$ because $\alpha_2 < 0 < \alpha_1$. Therefore, the only thing we need to show is $x_1, x_2 \in \mathbb{L}^n$. By Remark 3

$$\mathcal{N} r - 2(c_1^T r) r = \mathcal{N} r + 2(c_2^T r) r = 0.$$ 

Hence,

$$\mathcal{N} x_1 - 2(c_1^T x_1) r = \mathcal{N} x' - 2(c_1^T x') r \in \mathbb{L}^n \quad \text{and} \quad \mathcal{N} x_2 + 2(c_2^T x_2) r = \mathcal{N} x' + 2(c_2^T x') r \in \mathbb{L}^n.$$ 

Now observing that $c_1^T x_1 = c_2^T x_2 = 0$ and $\mathcal{N} > 0$ shows $x_1, x_2 \in \mathbb{L}^n$. This proves $x_1 \in Q_1$ and $x_2 \in Q_2$.

In the next section, we will show that the inequality (5) can also be used to characterize $\text{conv}(C_1 \cup C_2)$ where $C_1$ and $C_2$ are defined as in (2).

4 Two-Term Disjunctions on Cross-Sections of the Second-Order Cone

4.1 The Main Result

Consider $C$, $C_1$, and $C_2$ defined as in (2). The set $C$ is an ellipsoid when $a \in \text{int} \mathbb{L}^n$, a paraboloid when $a \in \text{bd} \mathbb{L}^n$, a hyperboloid when $a \notin \pm \mathbb{L}^n$, and empty when $a \in -\mathbb{L}^n$. In this section, we prove our main result, Theorem 3, which characterizes $\text{conv}(C_1 \cup C_2)$ under some mild conditions.

When $C_1 \subseteq C_2$, we have $\text{conv}(C_1 \cup C_2) = C_2$. Similarly, when $C_1 \supseteq C_2$, we have $\text{conv}(C_1 \cup C_2) = C_1$. In the remainder we concentrate on the case where $C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.

Assumption 3. $C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.

We also make the following assumption.
**Assumption 4.** \( C_1 \cap \text{int} \mathbb{L}^n \neq \emptyset \) and \( C_2 \cap \text{int} \mathbb{L}^n \neq \emptyset \).

This assumption will be useful later when we again use Theorem 1 whose proof relies on conic duality. The following simple observation underlies our approach.

**Observation 1.** Let \( C, C_1, \) and \( C_2 \) be defined as in (2). Then \( C_1 = \{ x \in C : (\beta_1 l_1 + \gamma_1 a)^\top x \geq \beta_1 l_1, 0 + \gamma_1 \} \) for any \( \beta_1 > 0 \) and \( \gamma_1 \in \mathbb{R} \). Similarly, \( C_2 = \{ x \in C : (\beta_2 l_2 + \gamma_2 a)^\top x \geq \beta_2 l_2, 0 + \gamma_2 \} \) for any \( \beta_2 > 0 \) and \( \gamma_2 \in \mathbb{R} \).

Observation 1 allows us to conclude

\[
C_1 = \left\{ x \in C : (l_1 - l_1, 0)^\top x \geq 0 \right\} \quad \text{and} \quad C_2 = \left\{ x \in C : (l_2 - l_2, 0)^\top x \geq 0 \right\}.
\]

By Assumption 3, we have \( C_1, C_2 \subseteq C \), and by Assumption 4 we have \( \dim C_1 = \dim C_2 = n - 1 \). This implies \( l_i - l_i, 0 a \not\in \pm \mathbb{L}^n \), or equivalently \( \|l_i - l_i, 0 a\|^2 > (l_{i,n} - l_{i,0} a_n)^2 \), for \( i \in \{1, 2\} \). Let

\[
c_i := \lambda_i (l_i - l_i, 0 a) \quad \text{where} \quad \lambda_i := \frac{1}{\sqrt{\|l_i - l_i, 0 a\|^2 - (l_{i,n} - l_{i,0} a_n)^2}} \quad \text{for} \quad i \in \{1, 2\}.
\]

Because \( \lambda_1, \lambda_2 > 0 \), we can write

\[
C_1 = \left\{ x \in C : c_1^\top x \geq 0 \right\} \quad \text{and} \quad C_2 = \left\{ x \in C : c_2^\top x \geq 0 \right\}.
\]

This scaling ensures that \( c_1 \) and \( c_2 \) satisfy (4).

Let \( Q_1 \) and \( Q_2 \) be the relaxations of \( C_1 \) and \( C_2 \) to the whole cone \( \mathbb{L}^n \):

\[
Q_1 := \left\{ x \in \mathbb{L}^n : c_1^\top x \geq 0 \right\} \quad \text{and} \quad Q_2 := \left\{ x \in \mathbb{L}^n : c_2^\top x \geq 0 \right\}.
\]

It is clear that \( Q_1 \) and \( Q_2 \) satisfy Assumptions 1, 2 because \( C_1 \) and \( C_2 \) satisfy Assumptions 3, 4. Define \( \mathcal{N}, \mathcal{M}, \) and \( r \) as in Section 3 using \( c_1 \) and \( c_2 \). Noting that \( Q_1 \) and \( Q_2 \) satisfy Assumptions 1, 2 and \( c_1 \) and \( c_2 \) satisfy (4), all results of Section 3 hold for \( Q_1 \) and \( Q_2 \). In particular, Theorem 1 implies that the inequality (5) is valid for \( \text{cone} \{C_1 \cup C_2\} \). In Theorem 3, we are going to show that (5) is also sufficient to describe \( \text{cone} \{C_1 \cup C_2\} \) when the sets \( C_1 \) and \( C_2 \) satisfy certain conditions. The proof of Theorem 3 requires the following technical lemma.

**Lemma 2.** Let \( C_1 \) and \( C_2 \) be defined as in (2). Suppose Assumptions 3 and 4 hold. Let \( c_1 \) and \( c_2 \) be defined as in (10). Suppose \( a^\top r \neq 0 \), and let \( x^* := \frac{r}{a^\top r} \). Let \( x' \in C \setminus (C_1 \cup C_2) \) satisfy (5).

a) If \( a^\top r > 0 \), then \( c_1^\top (x' - x^*) < 0 \). If in addition

\[
(a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \text{or} \quad (-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \text{or}
\]

\[
(-a + \text{cone}\{c_2\}) \cap -\mathbb{L}^n \neq \emptyset,
\]

then \( c_2^\top (x' - x^*) \geq 0 \).

b) If \( a^\top r < 0 \), then \( c_2^\top (x' - x^*) < 0 \). If in addition

\[
(a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \text{or} \quad (-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \text{or}
\]

\[
(-a + \text{cone}\{c_1\}) \cap -\mathbb{L}^n \neq \emptyset,
\]

then \( c_1^\top (x' - x^*) \geq 0 \).
Proof. By Remarks 2 and 3, \(N = 2c_1^\top r = -2c_2^\top r > 0\). From this, we get

\[
N x^* - 2(c_1^\top x^*)r = \frac{1}{a^\top r} \left(N - 2c_1^\top r\right) r = \frac{1}{a^\top r} \left(N + 2c_2^\top r\right) r = 0,
\]

Furthermore, \(a^\top x' = a^\top x^* = 1\).

a) Having \(x' \notin C_1\) implies \(c_1^\top x' < 0\). Furthermore, it follows from \(c_1^\top r = \frac{N}{2} > 0\) that

\[
c_1^\top x^* = \frac{c_1^\top r}{a^\top r} > 0.
\]

Thus, we get \(c_1^\top (x' - x^*) < 0\).

Now suppose \((a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset\). Then there exist \(\lambda \geq 0\) and \(0 \leq \theta \leq 1\) such that \(a + \lambda(\theta c_1 + (1 - \theta)c_2) \in \mathbb{L}^n\). The point \(x'\) does not belong to either \(C_1\) or \(C_2\) and satisfies (5). By Proposition 1, it satisfies (7) as well. Using (14), we can write

\[
N(x' - x^*) + 2c_2^\top (x' - x^*) r \in \mathbb{L}^n.
\]

Because \(\mathbb{L}^n\) is self-dual, we get

\[
0 \leq (a + \lambda(\theta c_1 + (1 - \theta)c_2))^\top (N(x' - x^*) + 2c_2^\top (x' - x^*) r)
\]

\[
= 2c_2^\top (x' - x^*) a^\top r + \lambda(\theta c_1 + (1 - \theta)c_2)^\top (N(x' - x^*) + 2c_2^\top (x' - x^*) r)
\]

\[
= 2c_2^\top (x' - x^*) a^\top r + \lambda c_2^\top (N(x' - x^*) + 2c_2^\top (x' - x^*) r)
\]

\[
\geq (2a^\top r + \lambda N)c_2^\top (x' - x^*) + \lambda N c_2^\top (x' - x^*)
\]

where we have used \(a^\top (x' - x^*) = 0\) to obtain the first equality, \(N + 2c_2^\top r = 0\) to obtain the third equality, and \((c_1 - c_2)^\top r = N\) to obtain the fifth equality. Now it follows from \(2a^\top r + \lambda N > 0, c_1^\top (x' - x^*) < 0\), and \(\lambda N \geq 0\) that \(c_2^\top (x' - x^*) \geq 0\).

Now suppose \((a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset\), and let \(\lambda \geq 0\) and \(0 \leq \theta \leq 1\) be such that \(-a + \lambda(\theta c_1 + (1 - \theta)c_2) \in \mathbb{L}^n\). By Proposition 1 \(x'\) satisfies (6), and using (13), we can write

\[
N(x' - x^*) - 2c_1^\top (x' - x^*) r \in \mathbb{L}^n.
\]

As before, because \(\mathbb{L}^n\) is self-dual, we get

\[
0 \leq (-a + \lambda(\theta c_1 + (1 - \theta)c_2))^\top (N(x' - x^*) - 2c_1^\top (x' - x^*) r).
\]

The right-hand side of this inequality is identical to

\[
(2a^\top r + \lambda(1 - \theta)N)c_1^\top (x' - x^*) + \lambda(1 - \theta)N c_2^\top (x' - x^*)
\]

It follows from \(2a^\top r + \lambda(1 - \theta)N > 0, c_1^\top (x' - x^*) < 0\), and \(\lambda(1 - \theta)N \geq 0\) that \(c_2^\top (x' - x^*) \geq 0\).
Finally suppose \((-a + \text{cone}(c_2)) \cap \mathbb{L}^n \neq \emptyset\), and let \(\theta \geq 0\) be such that \(-a + \theta c_2 \in -\mathbb{L}^n\). Then using \((15)\),

\[
0 \geq (-a + \theta c_2)\top (N(x' - x^*) + 2c_2\top (x' - x^*)r) \\
= -2c_2\top (x' - x^*)a\top r + \theta c_2\top (x' - x^*)(N + 2c_2\top r) \\
= -2c_2\top (x' - x^*)a\top r.
\]

It follows from \(a\top r > 0\) that \(c_2\top (x' - x^*) \geq 0\).

b) If \(a\top r < 0\), then \(a\top (-r) > 0\). Since \(-r = \begin{pmatrix} \tilde{c}_2 - \tilde{c}_1 \\ -c_{2,n} + c_{1,n} \end{pmatrix}\), part (b) follows from part (a) by interchanging the roles of \(C_1\) and \(C_2\).

\[\square\]

In the next result we show that the inequality \((5)\) is sufficient to describe \(\text{conv}(C_1 \cup C_2)\) when conditions \((11)\) and \((12)\) hold.

**Theorem 3.** Let \(C_1\) and \(C_2\) be defined as in \((2)\). Suppose Assumptions \((3)\) and \((4)\) hold. Let \(c_1\) and \(c_2\) be defined as in \((10)\). Suppose also that one of the following conditions is satisfied:

a) \(a\top r = 0\),

b) \(a\top r > 0\) and \((11)\) holds,

c) \(a\top r < 0\) and \((12)\) holds.

Then

\[
\text{conv}(C_1 \cup C_2) = \{ x \in C : x \text{ satisfies (5)} \}.
\]  

\((16)\)

**Proof.** Let \(D\) denote the set on the right-hand side of \((16)\). The inequality \((5)\) is valid for \(\text{conv}(C_1 \cup C_2)\) by Theorem \((1)\). Hence, \(\text{conv}(C_1 \cup C_2) \subseteq D\). Let \(x' \in D\). If \(x' \in C_1 \cup C_2\), then clearly \(x' \in \text{conv}(C_1 \cup C_2)\). Therefore, suppose \(x' \in C \setminus (C_1 \cup C_2)\) is a point that satisfies \((5)\). By Proposition \((1)\) it satisfies \((6)\) and \((7)\) as well. We are going to show that in each case \(x'\) belongs to \(\text{conv}(C_1 \cup C_2)\).

a) Suppose \(a\top r = 0\). By Remarks \((2)\) and \((3)\), \(N = 2c_1\top r = -2c_2\top r > 0\). Define \(\alpha_1\), \(\alpha_2\), \(x_1\), and \(x_2\) as in \((9)\). It is not difficult to see that \(a\top x_1 = a\top x_2 = 1\) and \(c_1\top x_1 = c_2\top x_2 = 0\). Furthermore, \(x' \in \text{conv}\{x_1, x_2\}\) because \(\alpha_2 < 0 < \alpha_1\). One can show that \(x_1, x_2 \in \mathbb{L}^n\) using the same arguments as in the proof of Theorem \((2)\). This proves \(x_1 \in C_1\) and \(x_2 \in C_2\).

b) Suppose \(a\top r > 0\) and \((11)\) holds. Let \(x^* := \frac{r}{a\top r}\). Then by Lemma \((2)\) \(c_1\top (x' - x^*) < 0\) and \(c_2\top (x' - x^*) \geq 0\).

First, suppose \(c_2\top (x' - x^*) > 0\), and let

\[
\alpha_1 := \frac{-c_1\top x'}{c_1\top (x' - x^*)}, \quad \alpha_2 := \frac{-c_2\top x'}{c_2\top (x' - x^*)}, \\
x_1 := x' + \alpha_1(x' - x^*), \quad x_2 := x' + \alpha_2(x' - x^*).
\]  

\((17)\)
As in part a), $a^\top x_1 = a^\top x_2 = 1$, $c_1^\top x_1 = c_2^\top x_2 = 0$, and $x' \in \text{conv}\{x_1, x_2\}$ because $\alpha_1 < 0 < \alpha_2$. To show $x_1, x_2 \in \mathbb{L}_n$, first note $N x^* - 2(c_1^\top x^*)r = N x^* + 2(c_2^\top x^*)r = 0$ as in (13) and (14). Using this and $c_1^\top x_1 = c_2^\top x_2 = 0$, we get

$$N x_1 = N x_1 - 2(c_1^\top x_1)r = (1 + \alpha_1)(N x^* - 2(c_1^\top x^*)r),$$

$$N x_2 = N x_2 + 2(c_2^\top x_2)r = (1 + \alpha_2)(N x^* + 2(c_2^\top x^*)r).$$

Clearly, $1 + \alpha_2 > 0$, so $N x_2 \in \mathbb{L}_n$. Furthermore,

$$1 + \alpha_1 = \frac{-c_1^\top x^*}{c_1^\top (x' - x^*)} = \frac{-c_1^\top r}{(a^\top r) c_1^\top (x' - x^*)} = \frac{-N}{2(a^\top r) c_1^\top (x' - x^*)} > 0$$

where we have used the relationships $N > 0$, $a^\top r > 0$, and $c_1^\top (x' - x^*) < 0$ to reach the inequality. It follows that $N x_1 \in \mathbb{L}_n$ as well. Because $N > 0$, we get $x_1, x_2 \in \mathbb{L}_n$. This proves $x_1 \in C_1$ and $x_2 \in C_2$.

Now suppose $c_2^\top (x' - x^*) = 0$, and define $\alpha_1$ and $x_1$ as in (17). All of the arguments that we have just used to show $\alpha_1 < 0$ and $x_1 \in C_1$ continue to hold. Using $N x^* + 2c_2^\top x^* r = 0$, we can write

$$N(x' - x^*) = N(x' - x^*) + 2c_2^\top (x' - x^*) r \in \mathbb{L}_n.$$

Because $N > 0$, we get $x' - x^* \in \mathbb{L}_n$. Together with $c_2^\top (x' - x^*) = 0$ and $a^\top (x' - x^*) = 0$, this implies $x' - x^* \in \text{rec} C_2$. Then $x' = x_1 - \alpha_1(x' - x^*) \in C_1 + \text{rec} C_2$ because $\alpha_1 < 0$. The claim now follows from the fact that the last set is contained in $\text{conv}(C_1 \cup C_2)$ (see, e.g., [17 Theorem 9.8]).

c) Suppose $a^\top r < 0$ and (12) holds. Since $-r := \begin{pmatrix} \hat{c}_2 - \hat{c}_1 \\ -c_{2,n} + c_{1,n} \end{pmatrix}$, part (c) follows from part (b) by interchanging the roles of $C_1$ and $C_2$.

The following result shows that when $C$ is an ellipsoid or a paraboloid, the closed convex hull of any two-term disjunction can be obtained by adding the cut (5) to the description of $C$.

**Corollary 1.** Let $C_1$ and $C_2$ be defined as in (2). Suppose Assumptions 3 and 4 hold. Let $c_1$ and $c_2$ be defined as in (10). If $a \in \mathbb{L}_n$, then (16) holds.

**Proof.** The result follows from Theorem 3 after observing that conditions (11) and (12) are trivially satisfied for any $c_1$ and $c_2$ when $a \in \mathbb{L}_n$.

The case of a split disjunction is particularly relevant in the solution of mixed-integer second-order cone programs, and it has been studied by several groups recently, in particular Dadush et al. [10], Andersen and Jensen [1], Belotti et al. [4], and Modaresi et al. [16]. Theorem 3 has the following consequence for a split disjunction.

**Corollary 2.** Consider $C_1$ and $C_2$ defined by a split disjunction on $C$ as in (2). Suppose Assumptions 3 and 4 hold. Let $c_1$ and $c_2$ be defined as in (10). Then (16) holds.
Proof. Let \( t_1^T x \geq l_{1,0} \lor t_2^T x \geq l_{2,0} \) define a split disjunction on \( C \) with \( l_2 = -tl_1 \) for some \( t > 0 \). Then we have \( tl_{1,0} > -l_{2,0} \) so that \( C_1 \cup C_2 \neq C \). Let \( \lambda_1, \lambda_2, c_1, \) and \( c_2 \) be defined as in (10). Let \( \theta_2 := \frac{1}{\lambda_2(tl_{1,0}+l_{2,0})} \) and \( \theta_1 := \frac{tl_2\theta_2}{\lambda_1} \). Then

\[
a + \theta_1c_1 + \theta_2c_2 = a + \lambda_2\theta_2(t(l_1 - l_{1,0}a) + (l_2 - l_{2,0}a)) = 0 \in \mathbb{L}^n.
\]

The result now follows from Theorem 3 after observing that \( \theta_1, \theta_2 \geq 0 \) implies that conditions (11) and (12) are satisfied.

When the sets \( C_1 \) and \( C_2 \) do not intersect, except possibly on their boundary, Proposition 1 says that (5) can be expressed in conic quadratic form and directly implies the following result.

Corollary 3. Let \( C_1 \) and \( C_2 \) be defined as in (2). Suppose Assumptions 3 and 4 hold. Let \( c_1 \) and \( c_2 \) be defined as in (10). Suppose that one of the conditions a), b), or c) of Theorem 3 holds. Suppose, in addition, that

\[
\{x \in C : c_1^T x > 0, c_2^T x > 0\} = \emptyset.
\]

Then

\[
\text{conv}(C_1 \cup C_2) = \{x \in C : x \text{satisfies (6)}\} = \{x \in C : x \text{satisfies (7)}\}.
\]

Remark 4. Conditions (11) and (12) are directly related to the sufficient conditions that guarantee the closedness of the convex hull of a two-term disjunction on \( \mathbb{L}^n \) explored in [14]. In particular, one can show that the convex hull of a disjunction \( h_1^T x \geq h_{1,0} \lor h_2^T x \geq h_{2,0} \) on the whole second-order cone \( \mathbb{L}^n \) is closed if

i) \( h_{1,0} = h_{2,0} \in \{\pm 1\} \) and there exists \( 0 < \mu < 1 \) such that \( \mu h_1 + (1-\mu)h_2 \in \mathbb{L}^n \), or

ii) \( h_{1,0} = h_{2,0} = -1 \) and \( h_1, h_2 \in \text{int} \mathbb{L}^n \).

In our present context, exploiting (i) and (ii) after letting \( h_i := a + \theta_i c_i \) and \( h_{i,0} := 1 \) (or, \( h_i := -a + \theta_i c_i \) and \( h_{i,0} := -1 \)) for some \( \theta_i > 0 \) leads to (11) and (12).

4.2 Two Examples

In this section we illustrate Theorem 3 with two examples.

4.2.1 A Two-Term Disjunction on a Paraboloid

Consider the disjunction \(-2x_1 - x_2 - 2x_4 \geq 0 \lor x_1 \geq 0\) on the paraboloid \( C := \{x \in \mathbb{L}^4 : x_1 + x_4 = 1\} \). Let \( C_1 := \{x \in C : -2x_1 - x_2 - 2x_4 \geq 0\} \) and \( C_2 := \{x \in C : x_1 \geq 0\} \). Noting that \( C \) is a paraboloid and \( C_1 \) and \( C_2 \) are disjoint, we can use Corollary 3 to characterize \( \text{conv}(C_1 \cup C_2) \) with a conic quadratic inequality:

\[
\text{conv}(C_1 \cup C_2) = \{x \in C : 3x + x_1(-3; -1; 0; 2) \in \mathbb{L}^4\}.
\]

Figure 1 depicts the paraboloid \( C \) in mesh and the disjunction \( C_1 \cup C_2 \) in blue. The conic quadratic disjunctive cut added to convexify this set is shown in red.
4.2.2 A Two-Term Disjunction on a Hyperboloid

Consider the disjunction $-2x_1 - x_2 \geq 0 \lor \sqrt{2}x_1 - x_3 \geq 0$ on the hyperboloid $C := \{x \in \mathbb{L}^3 : x_1 = 2\}$. Let $C_1 := \{x \in C : -2x_1 - x_2 \geq 0\}$ and $C_2 := \{x \in C : \sqrt{2}x_1 - x_3 \geq 0\}$. Note that, in this setting,

$$a^\top r = \frac{1}{10} (1; 0; 0)^\top \left( -2\sqrt{5} + 5\sqrt{2}; -\sqrt{5}; -5 \right) < 0,$$

but none of the conditions (12) are satisfied. The conic quadratic inequality

$$(5 + 2\sqrt{10})x + (\sqrt{2}x_1 - x_3) \left( -2\sqrt{5} + 5\sqrt{2}; -\sqrt{5}; -5 \right) \in \mathbb{L}^3$$ (18)

of Theorem 3 is valid for $C_1 \cup C_2$ but not sufficient to describe its closed convex hull. Indeed, the inequality $x_2 \leq 2$ is valid for $\text{conv}(C_1 \cup C_2)$ but is not implied by (18). Figure 2 depicts the hyperboloid $C$ in mesh and the disjunction $C_1 \cup C_2$ in blue. The conic quadratic disjunctive cut (18) is shown in red.

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References


Figure 2: The disjunctive cut obtained from a two-term disjunction on a hyperboloid.


