On Regression Analysis of Spatial Proportional Data with Zero/One Values

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Abstract

Proportions including exact zero and/or one values observed at spatial locations in a study area are often encountered in environmental and ecological studies. In this paper, we propose a new spatial beta-Bernoulli mixture model that combines a beta distribution and a Bernoulli distribution. The beta component links the original response on the open unit interval $(0, 1)$ to covariates via regression and is flexible to capture a variety of shapes of the data distributions. Further, the Bernoulli component models the probability of zero/one values with regression. In addition, we propose a novel spatial generalized Tobit model which extends an existing spatial Tobit model by applying an inverse beta cumulative distribution function transformation. A composite likelihood approach is developed for parameter estimation by maximizing a pairwise likelihood function for each model. The standard errors of the parameter estimates are obtained via the inverse of the Godambe information matrix. A simulation study is conducted to evaluate the performance of the proposed models and methods, followed by an ecological data example. Connections among the spatial beta-Bernoulli mixture model, the spatial generalized Tobit model, and the spatial Tobit model are explored using both simulated and real data.

Keywords: Beta-Bernoulli mixture model; composite likelihood estimation; Gaussian latent variable; generalized Tobit model; Godambe information; spatial statistics.

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1. Introduction

Of interest is regression analysis of spatial proportional data with zero and/or one values (henceforth, zero/one values). The motivating example is an integrative landscape ecology and environmental history study that aims to assess the influence of past land-ownership characteristics on landscape structure in northern Wisconsin, USA. The raw data consist of proportions of aspen (*Populus* spp.)-paper birch (*Betula papyrifera*) (APB) aerial extent in each quarter section with about 12% of zero/one values, corresponding to no or all APB in those quarter sections (Figure 1). An objective of the study is to evaluate the influence of various ownership covariates on APB. Standard linear regression analysis may not be appropriate to relate the response to covariates, as the probability model does not restrict the range of the response to the unit interval and the spatial dependence is not accounted for. The purpose of this paper is to develop new statistical models and their inference for the regression analysis of spatial proportional data with zero/one values.

The family of beta distributions characterized by two shape parameters is flexible for modeling proportions on the open unit interval $(0, 1)$, since the density can take on a variety of shapes for different values of the shape parameters. Several recent studies have considered beta regression models. For independent data, Ferrari and Cribari-Neto (2004) proposed a regression model where the response variable follows a beta distribution parameterized by a regression mean and a precision parameter, enabling a more straightforward interpretation of the regression parameters. Simas *et al.* (2010) extended this model to allow regression for the precision parameter as well. For dependent data, Eskelson *et al.* (2011) and Guolo and Varin (2014) proposed beta regression models to accommodate spatial dependence and temporal dependence, respectively. However, the aforementioned studies did not consider the possibility of zero/one values in the data.

In practice, the proportional data may contain exact zero/one values and thus are on the closed unit interval $[0, 1]$ (or half-closed intervals $[0, 1)$ and $(0, 1]$). For independent data, Ospina and Ferrari (2010, 2012) described a class of models based on a mixture of continuous and discrete distributions with probability mass at zero/one. For spatial proportional data, Feng *et al.* (2014)

proposed a spatial Tobit model, and the distribution of zero/one values are handled by the thresholding parameter and the regression mean. Although the spatial Tobit model is a natural extension of a spatial probit model for spatial binary data (Heagerty and Lele, 1998; De Oliveira , 2000), the Gaussian latent variable restricts the distribution to be unimodal in the open interval $(0, 1)$, which may not be appropriate for some data distributions in practice (Kieschnick and McCullough, 2003).

Although beta regression models are well established for the open unit interval, it remains a challenge to model spatial proportional data observed on the closed unit interval. We will propose a new spatial beta-Bernoulli mixture model that accommodates exact zero/one values and takes into account spatial dependence in the spatial proportional data using a Gaussian latent process. Compared with the existing spatial Tobit model, the spatial beta-Bernoulli mixture model captures a wider variety of shapes of data distributions on the open interval $(0, 1)$. In addition, we extend the spatial Tobit model by applying an inverse beta cumulative distribution function (CDF) transformation on the Gaussian latent variable, which will be referred as a spatial generalized Tobit model. Similar to the spatial Tobit model, the resulting new model is based on the idea of thresholding a spatial latent process, but overcomes the limitation of the Gaussian latent variable. These two new models assume two distinct data generation mechanisms and can be applied to different situations in practice.

Maximum likelihood estimation for our spatial beta-Bernoulli mixture model and spatial generalized Tobit model requires matrix factorization and multiple integration and thus, is not feasible when the sample size is relatively large. Here, we adopt an alternative composite likelihood approach to parameter estimation and inference, where the higher-dimensional full likelihood function is projected to lower-dimensional likelihood components that are more feasible to compute (Rao, 1970; Lindsay, 1988). Unlike the existing spatial Tobit model, both models involve functional transformations related to the inverse beta CDF and closed-form score functions are not straightforward to derive. Thus we use numerical score functions and evaluate the performance of the new models via simulated and real data examples in comparison with the existing spatial Tobit model.

The remainder of the paper is organized as follows. In Section 2, we propose a spatial beta-Bernoulli mixture model. In Section 3, we take a composite likelihood approach to parameter estimation and statistical inference. In Section 4, we propose an alternative spatial generalized Tobit model, by extending the spatial Tobit model with an inverse beta CDF transformation. Section 5 presents a simulation study that evaluates the finite-sample properties of composite likelihood estimation and compares different models. The application to the data example is given in Section 6, followed by conclusions and a discussion in Section 7.

2. Spatial Beta-Bernoulli Mixture Model

For proportional data with zero/one values, we propose a spatial beta-Bernoulli mixture model. Let N denote the number of sampling sites in a study area. For $i = 1, \dots, N$, let Y_i denote the response variable at site i that follows a Bernoulli distribution with probability η and a beta distribution with probability $1 - \eta$, where $\eta \in [0, 1]$ determines the probability that Y_i follows a Bernoulli distribution.

More specifically, there are two mixture components in the spatial beta-Bernoulli mixture model, one for the Bernoulli distribution and the other for the beta distribution. For the Bernoulli component, we let

$$Y_i \sim \text{Bernoulli}(\nu_i), \quad (1)$$

with mean $\nu_i = E(Y_i)$ and variance $\text{var}(Y_i) = \nu_i(1 - \nu_i)$. Let $\mathbf{x}_i = (1, x_{i1}, \dots, x_{ip})'$ denote the vector of covariates containing the value one and p covariates at site i , and let $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_p)'$ denote a vector of regression coefficients including the intercept. We apply a probit link function to model the relation between the binary response $Y_i \in \{0, 1\}$ and the covariates at site i as

$$\nu_i = E(Y_i) = \Phi_1(\mathbf{x}_i' \boldsymbol{\beta}). \quad (2)$$

Thus, parameterized by β , the Bernoulli mass (or, density) function of Y_i is

$$f_{\text{Bern}}(y_i; \beta) = y_i \nu_i + (1 - y_i)(1 - \nu_i) = y_i \Phi_1(\mathbf{x}'_i \beta) + (1 - y_i)\{1 - \Phi_1(\mathbf{x}'_i \beta)\}, \quad (3)$$

where the probability of one-value, $P(Y_i = 1) = \Phi_1(\mathbf{x}'_i \beta)$, increases as $\mathbf{x}'_i \beta$ increases.

For the beta component, we let

$$Y_i \sim \text{Beta}(\mu_i, \phi), \quad (4)$$

with mean $\mu_i = E(Y_i) \in (0, 1)$, variance $\text{var}(Y_i) = \mu_i(1 - \mu_i)/\phi \geq 0$, and a precision parameter $\phi > 0$. We apply a logit link function to relate the proportional response $Y_i \in (0, 1)$ to the covariates,

$$\mu_i = E(Y_i) = \{1 + \exp(-\mathbf{x}'_i \gamma)\}^{-1}, \quad (5)$$

where $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_p)'$ is a vector of regression coefficients and μ_i increases as $\mathbf{x}'_i \gamma$ increases.

Then, parameterized by γ and ϕ , the beta density function of Y_i is

$$f_{\text{Beta}}(y_i; \gamma, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu_i \phi) \Gamma((1 - \mu_i) \phi)} y_i^{\mu_i \phi - 1} (1 - y_i)^{(1 - \mu_i) \phi - 1}, \quad (6)$$

where $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$ is the gamma function. The beta distribution is flexible for modeling continuous data on the open interval $(0, 1)$, since its probability density can take on a variety of shapes for different values of μ_i and ϕ including unimodal, uniform, “U”, and “J” shaped distributions (Ferrari and Cribari-Neto, 2004; Ospina and Ferrari, 2010). As $\phi \rightarrow 0$, Y_i tends to a Bernoulli random variable with success probability $P(Y_i = 1) = \mu_i$, whereas as $\phi \rightarrow \infty$, Y_i tends to a degenerate random variable with $P(Y_i = \mu_i) = 1$ (Mielke, 1975).

From (2) and (5), the response Y_i can relate to the covariates $\mathbf{x}_i$ via β and γ in the Bernoulli and the beta component, respectively. Now, to account for possible spatial dependence among the

vector of responses at the N sampling sites, $\mathbf{Y} = (Y_1, \dots, Y_N)'$, we introduce a vector of spatial latent variables $\mathbf{Y}^* = (Y_1^*, \dots, Y_N^*)'$, where at site i ,

$$Y_i^* = Z_i + \epsilon_i, \quad (7)$$

Z_i is another spatial latent process, and ϵ_i is a measurement error, for $i = 1, \dots, N$. For $\mathbf{Z} = (Z_1, \dots, Z_N)'$, we let $\mathbf{Z} \sim N(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\mathbf{Z}})$ follow a multivariate Gaussian distribution, where $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N)'$ denotes an $N \times (p+1)$ design matrix and $\boldsymbol{\Sigma}_{\mathbf{Z}} = \text{var}(\mathbf{Z})$ is the covariance matrix of $\mathbf{Z}$. In particular, we assume an exponential covariance function $\text{cov}(Z_i, Z_{i'}) = \sigma^2 \exp\{d_{ii'} \log(\rho)\}$, or equivalently, $\text{cov}(Z_i, Z_{i'}) = \sigma^2 \rho^{d_{ii'}}$, where $d_{ii'}$ is the Euclidean distance between sites i and i' , $\rho \in (0, 1)$ is a spatial correlation parameter, and $\sigma^2 = \text{var}(Z_i)$ is a variance component. The measurement errors $\{\epsilon_i\}$ are assumed to be independently and identically distributed (iid) as $N(0, \tau^2)$ and are independent of $\mathbf{Z}$. Hence, $Y_i^* \sim N(\mathbf{x}_i' \boldsymbol{\beta}, \sigma^2 + \tau^2)$ and $\text{cov}(Y_i^*, Y_{i'}^*) = \sigma^2 \rho^{d_{ii'}}$ for $i \neq i'$. We denote the covariance matrix of $\mathbf{Y}^*$ as $\boldsymbol{\Sigma}_{\mathbf{Y}^*}$ and thus, $\mathbf{Y}^* \sim N(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\mathbf{Y}^*})$.

The Bernoulli component (1)–(2) can be re-expressed in the form of a threshold model,

$$Y_i = \mathcal{I}(Y_i^* > c), \quad (8)$$

where Y_i^* is the spatial latent variable (7), $c \in \mathbb{R}$ is a threshold value, and $\mathcal{I}(\cdot)$ is the indicator function (see, e.g., Heagerty and Lele, 1998; De Oliveira, 2000). Thus, $\nu_i = E(Y_i) = \Phi_1((\mathbf{x}_i' \boldsymbol{\beta} - c)/(\sigma^2 + \tau^2))$. For parameter identification, we set $c = 0$ and $\text{var}(Y_i^*) = \sigma^2 + \tau^2 = 1$.

The beta component (4)–(5) can be re-expressed as a transformation of the same latent variable Y_i^* ,

$$Y_i = F_{\text{Beta}}^{-1}(F_{\text{Gauss}}(Y_i^*; \boldsymbol{\beta}); \boldsymbol{\gamma}, \phi), \quad (9)$$

where $F_{\text{Beta}}^{-1}(\cdot; \boldsymbol{\gamma}, \phi)$ denotes the inverse CDF of a beta random variable parameterized by $\boldsymbol{\gamma}$ and ϕ , and $F_{\text{Gauss}}(\cdot; \boldsymbol{\beta})$ denotes the Gaussian CDF of Y_i^* parameterized by $\boldsymbol{\beta}$. It can be shown that

$F_{\text{Gauss}}(Y_i^*; \beta)$ follows a standard uniform distribution and thus $F_{\text{Beta}}^{-1}(F_{\text{Gauss}}(Y_i^*; \beta); \gamma, \phi)$ follows a beta distribution on $(0, 1)$.

In summary, for site i , the spatial beta-Bernoulli mixture model with the Bernoulli component given in (8) and the beta component given in (9) can be rewritten as

$$Y_i = \begin{cases} \mathcal{I}(Y_i^* > 0) & \text{w.p. } \eta \\ F_{\text{Beta}}^{-1}(F_{\text{Gauss}}(Y_i^*; \beta); \gamma, \phi) & \text{w.p. } 1 - \eta \end{cases} \quad (10)$$

where $\eta \in [0, 1]$. When $\eta = 1$, the spatial beta-Bernoulli mixture model (10) reduces to a spatial probit model, whereas when $\eta = 0$, it is a spatial beta model for spatial data on the open interval $(0, 1)$ (Eskelson *et al.*, 2011). Thus, the corresponding probability density function for $y_i \in [0, 1]$ is

$$f(y_i; \eta, \beta, \gamma, \phi) = \begin{cases} \eta f_{\text{Bern}}(y_i; \beta) & \text{if } y_i = \{0, 1\} \\ (1 - \eta) f_{\text{Beta}}(y_i; \gamma, \phi) & \text{if } y_i \in (0, 1), \end{cases} \quad (11)$$

where $f(y_i; \beta, \gamma, \phi)$ is parameterized by β when $y_i = 0$ or 1 , and is parameterized by γ and ϕ , but not β , when $y_i \in (0, 1)$. The beta density is due to $\partial P(Y_i \leq y_i)/\partial y_i = \partial P(0 < Y_i \leq y_i)/\partial y_i = (1 - \eta) \partial P(F_{\text{Gauss}}^{-1}(F_{\text{Beta}}(Y_i; \gamma, \phi); \beta) \leq F_{\text{Gauss}}^{-1}(F_{\text{Beta}}(y_i; \gamma, \phi); \beta))/\partial y_i = (1 - \eta) \partial F_{\text{Beta}}(y_i; \gamma, \phi)/\partial y_i$. For a positive β_j , the j th covariate is positively associated with the spatial latent variable, with other covariates held constant. Through thresholding (8), the larger x_{ij} corresponds to more probability mass at 1. Further, for a positive γ_j , the j th covariate is positively associated with the proportional responses, with other covariates held constant.

3. Parameter Estimation and Statistical Inference

3.1. Composite Likelihood Estimation

Let $\theta = (\eta, \beta', \sigma^2, \rho, \gamma', \phi)'$ denote the vector of model parameters, where $\beta, \gamma \in \mathbb{R}^{p+1}$, $\phi > 0$, and $\eta, \sigma^2, \rho \in (0, 1)$. Without loss of generality, we assume that the first m_1 observations,

$y_1, \dots, y_{m_1}$, are proportions on $(0, 1)$, the next m_2 observations, $y_{m_1+1}, \dots, y_{m_1+m_2}$, are zeros, and the remainder observations, $y_{m_1+m_2+1}, \dots, y_N$, are ones. The likelihood function of the parameter vector $\boldsymbol{\theta} = (\eta, \boldsymbol{\beta}, \sigma^2, \rho, \boldsymbol{\gamma}, \phi)'$ is $\mathcal{L}(\boldsymbol{\theta}) = (1 - \eta)^{m_1} \eta^{N-m_1} \partial^{m_1} P(Y_1 \leq y_1, \dots, Y_{m_1} \leq y_{m_1}, Y_{m_1+1} = 0, \dots, Y_N = 1) / \partial y_1 \cdots \partial y_{m_1}$, which can be re-expressed as

$$\mathcal{L}(\boldsymbol{\theta}) = (1 - \eta)^{m_1} \eta^{N-m_1} \frac{\partial^{m_1} P(Y_1^* \leq F_{\text{Gauss}}^{-1}(F_{\text{Beta}}(y_1; \boldsymbol{\gamma}, \phi); \boldsymbol{\beta}), \dots, Y_{m_1+1}^* \leq 0, \dots, Y_N^* > 0)}{\partial y_1 \cdots \partial y_{m_1}}. \quad (12)$$

The likelihood function (12) involves the operations of $N \times N$ matrices and multiple integrals and thus, its maximization becomes infeasible especially when N is large and when the amount of zero/one values is large.

Taking the approach of composite likelihood, we define a pairwise likelihood function for $\boldsymbol{\theta}$ as

$$\mathcal{L}_{\text{pair}}(\boldsymbol{\theta}) = \prod_{i=1}^{N-1} \prod_{i'=i+1}^N f(y_i, y_{i'}; \boldsymbol{\theta}). \quad (13)$$

In particular, for a pair of sites (i, i') , the pair of responses $(Y_i, Y_{i'})$ have three possible scenarios: both of them are integers (0 or 1), one is an integer and the other is a proportion, or both are proportions. In the first scenario, the bivariate densities involve $\eta, \boldsymbol{\beta}, \sigma^2$, and ρ , but not $\boldsymbol{\gamma}$ and ϕ as

follows:

$$\begin{aligned}
f_{\text{Bern2}}(0, 0; \eta, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 0, Y_{i'} = 0) = \eta^2 P(Y_i^* \leq 0, Y_{i'}^* \leq 0) \\
&= \eta^2 \Phi_2(-\mathbf{x}'_i \boldsymbol{\beta}, -\mathbf{x}'_{i'} \boldsymbol{\beta}; \tilde{\rho}_{ii'}), \\
f_{\text{Bern2}}(0, 1; \eta, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 0, Y_{i'} = 1) = \eta^2 P(Y_i^* \leq 0, Y_{i'}^* > 0) \\
&= \eta^2 \{ \Phi_1(-\mathbf{x}'_i \boldsymbol{\beta}) - \Phi_2(-\mathbf{x}'_i \boldsymbol{\beta}, -\mathbf{x}'_{i'} \boldsymbol{\beta}; \tilde{\rho}_{ii'}) \}, \\
f_{\text{Bern2}}(1, 0; \eta, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 1, Y_{i'} = 0) = \eta^2 P(Y_i^* > 0, Y_{i'}^* \leq 0) \\
&= \eta^2 \{ \Phi_1(-\mathbf{x}'_{i'} \boldsymbol{\beta}) - \Phi_2(-\mathbf{x}'_i \boldsymbol{\beta}, -\mathbf{x}'_{i'} \boldsymbol{\beta}; \tilde{\rho}_{ii'}) \}, \\
f_{\text{Bern2}}(1, 1; \eta, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 1, Y_{i'} = 1) = \eta^2 P(Y_i^* > 0, Y_{i'}^* > 0) \\
&= \eta^2 \Phi_2(\mathbf{x}'_i \boldsymbol{\beta}, \mathbf{x}'_{i'} \boldsymbol{\beta}; \tilde{\rho}_{ii'}),
\end{aligned}$$

where $\Phi_2(\nu_1, \nu_2; \tilde{\rho}_{ii'}) = (2\pi)^{-1} (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \int_{-\infty}^{\nu_1} \int_{-\infty}^{\nu_2} \exp\{(2 - 2\tilde{\rho}_{ii'}^2)^{-1} (t_1^2 + t_2^2 - 2\tilde{\rho}_{ii'} t_1 t_2)\} dt_1 dt_2$ is the CDF of the standard bivariate Gaussian distribution with means 0, variances 1, and correlation $\tilde{\rho}_{ii'} = \sigma^2 \rho^{d_{ii'}}$.

In the second scenario, one response is an integer and the other is a proportion. The bivariate density can be derived in a straightforward manner and involve all the parameters $\boldsymbol{\theta}$. For example, suppose $y_i = 0$ and $y_{i'} \in (0, 1)$, since

$$\begin{aligned}
P(Y_i = 0, 0 < Y_{i'} \leq y_{i'}) &= \eta(1 - \eta) P(Y_i^* \leq 0, h(Y_{i'}; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi) \leq h(y_{i'}; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi)) \\
&= \eta(1 - \eta) P(Y_i^* - \mathbf{x}'_i \boldsymbol{\beta} \leq -\mathbf{x}'_i \boldsymbol{\beta}, Y_{i'}^* - \mathbf{x}'_{i'} \boldsymbol{\beta} \leq h(y_{i'}; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi) - \mathbf{x}'_{i'} \boldsymbol{\beta}) \\
&= \eta(1 - \eta) P(Y_i^* - \mathbf{x}'_i \boldsymbol{\beta} \leq -\mathbf{x}'_i \boldsymbol{\beta}, Y_{i'}^* - \mathbf{x}'_{i'} \boldsymbol{\beta} \leq u(y_{i'}; \boldsymbol{\gamma}, \phi)) \\
&= \eta(1 - \eta) \Phi_2(-\mathbf{x}'_i \boldsymbol{\beta}, u(y_{i'}; \boldsymbol{\gamma}, \phi); \tilde{\rho}_{ii'}), \tag{14}
\end{aligned}$$

where $h(\cdot; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi) = F_{\text{Gauss}}^{-1}(F_{\text{Beta}}(\cdot, \boldsymbol{\gamma}, \phi), \boldsymbol{\beta})$ and $u(y_{i'}; \boldsymbol{\gamma}, \phi) = h(y_{i'}; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi) - \mathbf{x}'_{i'} \boldsymbol{\beta} = \Phi_1^{-1}(F_{\text{Beta}}(y_{i'}; \boldsymbol{\gamma}, \phi))$.

Then, we have

$$\begin{aligned}
f_{\text{Bern,Beta}}(0, y_{i'}; \boldsymbol{\theta}) &= \partial P(Y_i = 0, Y_{i'} \leq y_{i'}) / \partial y_{i'} = \partial P(Y_i = 0, 0 < Y_{i'} \leq y_{i'}) / \partial y_{i'} \\
&= \eta(1 - \eta) \partial \Phi_2(-\mathbf{x}'_i \boldsymbol{\beta}, u(y_{i'}; \boldsymbol{\gamma}, \phi); \tilde{\rho}_{ii'}) / \partial y_{i'} \\
&= \eta(1 - \eta) \Phi_1 \left((-\mathbf{x}'_i \boldsymbol{\beta} - \tilde{\rho}_{ii'} u(y_{i'}; \boldsymbol{\gamma}, \phi)) (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \right) \\
&= \eta(1 - \eta) \Phi_1 \left((-\mathbf{x}'_i \boldsymbol{\beta} - \tilde{\rho}_{ii'} u(y_{i'}; \boldsymbol{\gamma}, \phi)) (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \right) \\
&\quad \times \phi_1(u(y_{i'}; \boldsymbol{\gamma}, \phi)) \partial u(y_{i'}; \boldsymbol{\gamma}, \phi) / \partial y_{i'} \\
&= \eta(1 - \eta) \Phi_1 \left((-\mathbf{x}'_i \boldsymbol{\beta} - \tilde{\rho}_{ii'} u(y_{i'}; \boldsymbol{\gamma}, \phi)) (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \right) \\
&\quad \times \phi_1(u(y_{i'}; \boldsymbol{\gamma}, \phi)) f_{\text{Beta}}(y_{i'}; \boldsymbol{\gamma}, \phi) / \phi_1(u(y_{i'}; \boldsymbol{\gamma}, \phi)) \\
&= \eta(1 - \eta) \Phi_1 \left((-\mathbf{x}'_i \boldsymbol{\beta} - \tilde{\rho}_{ii'} u(y_{i'}; \boldsymbol{\gamma}, \phi)) (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \right) f_{\text{Beta}}(y_{i'}; \boldsymbol{\gamma}, \phi).
\end{aligned} \tag{15}$$

Under the independence assumption (i.e., $\rho = 0$), the bivariate density (15) is simplified to the product of Bernoulli and beta densities $f_{\text{Bern,Beta}}(0, y_{i'}; \boldsymbol{\theta}) = \eta(1 - \eta) f_{\text{Beta}}(y_{i'}; \boldsymbol{\gamma}, \phi) \Phi_1(-\mathbf{x}'_i \boldsymbol{\beta})$. By analogous arguments,

$$\begin{aligned}
f_{\text{Bern,Beta}}(1, y_{i'}; \boldsymbol{\theta}) &= \eta(1 - \eta) f_{\text{Beta}}(y_{i'}; \boldsymbol{\gamma}, \phi) \Phi_1 \left((\mathbf{x}'_i \boldsymbol{\beta} + \tilde{\rho}_{ii'} u(y_{i'}; \boldsymbol{\gamma}, \phi)) (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \right), \\
f_{\text{Beta,Bern}}(y_i, 0; \boldsymbol{\theta}) &= \eta(1 - \eta) f_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) \Phi_1 \left((-\mathbf{x}'_{i'} \boldsymbol{\beta} - \tilde{\rho}_{ii'} u(y_i; \boldsymbol{\gamma}, \phi)) (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \right), \\
f_{\text{Beta,Bern}}(y_i, 1; \boldsymbol{\theta}) &= \eta(1 - \eta) f_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) \Phi_1 \left((\mathbf{x}'_{i'} \boldsymbol{\beta} + \tilde{\rho}_{ii'} u(y_i; \boldsymbol{\gamma}, \phi)) (1 - \tilde{\rho}_{ii'}^2)^{-1/2} \right).
\end{aligned}$$

In the third scenario, both responses are proportions. The bivariate density involves $\eta, \boldsymbol{\gamma}, \phi$, and

the spatial dependence parameters σ^2 and ρ , but not β ,

$$\begin{aligned}
f_{\text{Beta2}}(y_i, y_{i'}; \eta, \gamma, \phi, \sigma^2, \rho) &= \frac{\partial^2 P(Y_i \leq y_i, Y_{i'} \leq y_{i'})}{\partial y_i \partial y_{i'}} = (1 - \eta)^2 \frac{\partial^2 \Phi_2(u_i, u_{i'}; \tilde{\rho}_{ii'})}{\partial y_i \partial y_{i'}} \\
&= (1 - \eta)^2 f_{\text{Beta}}(y_i; \gamma, \phi) f_{\text{Beta}}(y_{i'}; \gamma, \phi) \\
&\quad \times \phi_2(u(y_i; \gamma, \phi), u(y_{i'}; \gamma, \phi); \tilde{\rho}_{ii'}) / \{\phi_1(u(y_i; \gamma, \phi)) \phi_1(u(y_{i'}; \gamma, \phi))\}.
\end{aligned} \tag{16}$$

Again, under the independence assumption, the bivariate density (16) is reduced to $f_{\text{Beta2}}(y_i, y_{i'}; \eta, \gamma, \phi) = (1 - \eta)^2 f_{\text{Beta}}(y_i; \gamma, \phi) f_{\text{Beta}}(y_{i'}; \gamma, \phi)$.

In summary, the bivariate density of the paired response $(Y_i, Y_{i'})$ is

$$f(y_i, y_{i'}; \boldsymbol{\theta}) = \begin{cases} f_{\text{Bern2}}(y_i, y_{i'}; \eta, \beta, \sigma^2, \rho) & \text{if } y_i, y_{i'} \in \{0, 1\} \\ f_{\text{Bern,Beta}}(y_i, y_{i'}; \boldsymbol{\theta}) & \text{if } y_i \in \{0, 1\}, y_{i'} \in (0, 1) \\ f_{\text{Beta,Bern}}(y_i, y_{i'}; \boldsymbol{\theta}) & \text{if } y_i \in (0, 1), y_{i'} \in \{0, 1\} \\ f_{\text{Beta2}}(y_i, y_{i'}; \eta, \gamma, \phi) & \text{if } y_i, y_{i'} \in (0, 1). \end{cases} \tag{17}$$

Since the composite likelihood function (13) contains $N(N - 1)/2$ pairs of bivariate densities, the computation can still be quite time consuming for a large sample size N . Thus, we adopt a 0-1 weighting mechanism and define a weighted composite log-likelihood function as

$$c\ell(\boldsymbol{\theta}; \mathbf{y}) = \sum_{i < i': d_{ii'} \leq r} \log \{f(y_i, y_{i'}; \boldsymbol{\theta})\}, \tag{18}$$

where r is a pre-specified radius. That is, any pair farther than r units apart is excluded via a zero weight (see, e.g., Heagerty and Lele 1998). With an exponential covariance function, the spatial correlation declines relatively fast as the distance increases between two sites and thus, this weighting scheme effectively excludes the distant pairs that have negligible correlation.

The maximum composite likelihood estimator (MCLE), denoted as $\hat{\boldsymbol{\theta}} = (\hat{\eta}, \hat{\beta}', \hat{\sigma}^2, \hat{\rho}, \hat{\gamma}', \hat{\phi})'$,

maximizes $c\ell(\boldsymbol{\theta}; \mathbf{y})$ and can be obtained by solving the score equations

$$\mathbf{S}_N(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} c\ell(\boldsymbol{\theta}; \mathbf{y}) = \mathbf{0}, \quad (19)$$

using a quasi-Newton algorithm for bounded parameter space (Byrd *et al.*, 1995). Because each individual component in (18) is a bivariate density, the resulting score equations (19) obtained from the derivative of the composite log-likelihood are unbiased (Varin *et al.*, 2011). The consistency and asymptotic normality of $\hat{\boldsymbol{\theta}}$ can be derived under suitable regularity conditions using arguments similar to Guyon (1995).

3.2. Statistical Inference

The variance of $\hat{\boldsymbol{\theta}}$ can be obtained by computing the inverse of the Godambe information matrix

$$\mathbf{G}_N^{-1}(\boldsymbol{\theta}) = \mathbf{H}_N^{-1}(\boldsymbol{\theta}) \mathbf{J}_N(\boldsymbol{\theta}) \mathbf{H}_N^{-1}(\boldsymbol{\theta}), \quad (20)$$

where $\mathbf{H}_N(\boldsymbol{\theta}) = -E\left(\frac{\partial}{\partial \boldsymbol{\theta}'} \mathbf{S}_N(\boldsymbol{\theta})\right)$ is the sensitivity matrix and $\mathbf{J}_N(\boldsymbol{\theta}) = \text{var}(\mathbf{S}_N(\boldsymbol{\theta}))$ is the variability matrix (Godambe, 1960). In practice, we approximate $\mathbf{H}_N(\boldsymbol{\theta})$ by a numerical Hessian matrix, and $\mathbf{J}_N(\boldsymbol{\theta})$ by a sample covariance matrix of the score functions, $\mathbf{S}_N(\hat{\boldsymbol{\theta}}; \mathbf{y}_1), \dots, \mathbf{S}_N(\hat{\boldsymbol{\theta}}; \mathbf{y}_M)$, denoted as $\hat{\mathbf{J}}_N(\hat{\boldsymbol{\theta}})$, where $\mathbf{y}_m = (y_{m1}, \dots, y_{mN})'$ for $m = 1, \dots, M$ are independent draws from the fitted model (Apanasovich *et al.*, 2008; Varin *et al.*, 2011). Since closed-form score functions are not straightforward to derive, we obtain the sample covariance matrix based on the numerical score functions $\hat{S}_N(\theta_j; \mathbf{y}) = \{c\ell(\theta_j + \delta; \mathbf{y}) - c\ell(\theta_j - \delta; \mathbf{y})\} / (2\delta)$, for $j = 1, \dots, 2p + 6$, where $\delta > 0$ is a small value and $p + 1$ is the number of regression coefficients for $\boldsymbol{\beta}$ or $\boldsymbol{\gamma}$. For $m = 1, \dots, M$, we draw the latent variable $\mathbf{Y}_m^* \sim N(\mathbf{X}\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\Sigma}}_{\mathbf{Y}^*})$ and then draw the response $\mathbf{y}_m$ from (10) evaluated at the parameter estimates. This procedure can be further expedited via parallel computing.

For model comparison, we consider a composite likelihood information criterion (CLIC) (Varin and Vidoni, 2005), which is a generalization of Akaike's information criterion (AIC) for model

assessment (Akaike, 1973). Specifically, for (18), the CLIC is given by

$$\text{CLIC}(\hat{\boldsymbol{\theta}}) = -2cl(\hat{\boldsymbol{\theta}}) + 2 \text{tr}(\hat{\mathbf{J}}_N(\hat{\boldsymbol{\theta}})\hat{\mathbf{H}}_N^{-1}(\hat{\boldsymbol{\theta}})). \quad (21)$$

When the composite likelihood is the true likelihood, (21) is the same as AIC, because $\mathbf{J}_N(\boldsymbol{\theta}) = \mathbf{H}_N(\boldsymbol{\theta})$ and the trace corresponds to the number of parameters in the model.

4. Spatial Generalized Tobit Model

In Sections 2 and 3, spatial proportional data with zero/one values are modeled by mixing a beta distribution and a Bernoulli distribution. Alternatively, we consider imposing a functional transformation on the latent variable. In particular, we propose a spatial generalized Tobit model, which extends a spatial Tobit model by applying an inverse beta CDF transformation to the latent variable. That is, let

$$Y_i = \begin{cases} 1 & \text{if } Y_i^* > \alpha \\ F_{\text{Beta}}^{-1}(Y_i^*/\alpha; \boldsymbol{\gamma}, \phi) & \text{if } 0 < Y_i^* \leq \alpha \\ 0 & \text{if } Y_i^* \leq 0, \end{cases} \quad (22)$$

where Y_i^* is a spatial latent variable given in (7). Recall that $\mathbf{Y}^* = (Y_1^*, \dots, Y_N^*)'$, $\mathbf{Y}^* \sim N(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\mathbf{Y}^*})$, and the (i, j) th term of the covariance matrix $\boldsymbol{\Sigma}_{\mathbf{Y}^*}$ is $\sigma^2 \rho^{d_{ij}}$ for $i, j = 1, \dots, N$. Further, we consider a logit link function for the mean of the beta distribution $\mu_i = E(Y_i) = \{1 + \exp(-\mathbf{x}_i' \boldsymbol{\gamma})\}^{-1}$ for $Y_i \in (0, 1)$, and $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_p)'$ is a vector of regression coefficients.

In (22), the parameter $\alpha \geq 0$ is a threshold value and controls the probability of $Y_i = 1$. The point mass at one is larger as α decreases. For $\alpha = 0$, model (22) is reduced to a spatial probit model for binary responses, while for $\alpha > 0$, the latent variable Y_i^* between 0 and α is transformed to be between 0 and 1. For Beta(0.5, 2) with $\mu = 0.5$ and $\phi = 2$, the beta distribution is the

standard uniform distribution and thus, (22) is reduced to a spatial Tobit model. The inverse beta CDF transformation $F_{\text{Beta}}^{-1}(Y_i^*/\alpha; \gamma, \phi)$ is a new idea aimed to overcome the unimodality implied by the Gaussian latent variable in the existing spatial Tobit model.

The vector of model parameters is $\boldsymbol{\theta} = (\alpha, \boldsymbol{\beta}', \sigma^2, \rho, \boldsymbol{\gamma}', \phi)'$, where $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^{p+1}$, $\phi > 0$, $\alpha \geq 0$, and $\sigma^2, \rho \in (0, 1)$. The corresponding probability density function at site i is given by

$$f(y_i; \boldsymbol{\theta}) = \begin{cases} \Phi_1(\mathbf{x}_i' \boldsymbol{\beta} - \alpha) & \text{if } y_i = 1 \\ \alpha f_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) \phi_1(\alpha F_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) - \mathbf{x}_i' \boldsymbol{\beta}) & \text{if } y_i \in (0, 1) \\ \Phi_1(-\mathbf{x}_i' \boldsymbol{\beta}) & \text{if } y_i = 0, \end{cases} \quad (23)$$

where $\phi_1(x) = (2\pi)^{-1/2} \exp(x^2/2)$ is the density of the standard Gaussian distribution. The density for $y_i \in (0, 1)$ is due to $\partial P(Y_i \leq y_i)/\partial y_i = \partial P(\alpha F_{\text{Beta}}(Y_i; \boldsymbol{\gamma}, \phi) \leq \alpha F_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi))/\partial y_i = \partial P(\alpha F_{\text{Beta}}(Y_i; \boldsymbol{\gamma}, \phi) - \mathbf{x}_i' \boldsymbol{\beta} \leq \alpha F_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) - \mathbf{x}_i' \boldsymbol{\beta})/\partial y_i = \partial \Phi_1(\alpha F_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) - \mathbf{x}_i' \boldsymbol{\beta})/\partial y_i$. Unlike the density (11) under the spatial beta-Bernoulli mixture model, the regression coefficients $\boldsymbol{\beta}$ under the spatial generalized Tobit model appear in both the distribution of zero/one values and the proportions. For a positive β_j , there is more probability mass at 1 and less at 0 as x_{ij} increases with other covariates hold constant, while the effect of x_{ij} varies for $y_i \in (0, 1)$.

The maximum likelihood estimation for model (22) is again not feasible for relatively large sample sizes and thus, we take a composite likelihood approach. Similar to Section 2, for a pair of sites (i, i') , the pair of responses $(Y_i, Y_{i'})$ have three possible scenarios. When both of them are integers (0 or 1), the bivariate density is the same as the spatial Tobit model. That is, parameterized by $\alpha, \boldsymbol{\beta}, \sigma^2, \rho$ but not $\boldsymbol{\gamma}, \phi$,

$$\begin{aligned} g_{\text{Bern2}}(0, 0; \alpha, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 0, Y_{i'} = 0) = \Phi_2(-\mathbf{x}_i' \boldsymbol{\beta}, -\mathbf{x}_{i'}' \boldsymbol{\beta}; \tilde{\rho}_{ii'}) \\ g_{\text{Bern2}}(0, 1; \alpha, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 0, Y_{i'} = 1) = \Phi_1(-\mathbf{x}_i' \boldsymbol{\beta}) - \Phi_2(-\mathbf{x}_i' \boldsymbol{\beta}, \alpha - \mathbf{x}_{i'}' \boldsymbol{\beta}; \tilde{\rho}_{ii'}) \\ g_{\text{Bern2}}(1, 0; \alpha, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 1, Y_{i'} = 0) = \Phi_1(-\mathbf{x}_{i'}' \boldsymbol{\beta}) - \Phi_2(\alpha - \mathbf{x}_i' \boldsymbol{\beta}, -\mathbf{x}_{i'}' \boldsymbol{\beta}; \tilde{\rho}_{ii'}) \\ g_{\text{Bern2}}(1, 1; \alpha, \boldsymbol{\beta}, \sigma^2, \rho) &= P(Y_i = 1, Y_{i'} = 1) = \Phi_2(\mathbf{x}_i' \boldsymbol{\beta} - \alpha, \mathbf{x}_{i'}' \boldsymbol{\beta} - \alpha; \tilde{\rho}_{ii'}), \end{aligned}$$

where $\tilde{\rho}_{ii'} = \sigma^2 \rho^{d_{ii'}}$.

When one response is binary and the other is a proportion (e.g., $y_i = 0$ and $y_{i'} \in (0, 1)$), the bivariate density is parameterized by $\boldsymbol{\theta}$ as,

$$\begin{aligned} g_{\text{Bern,Beta}}(0, y_{i'}; \boldsymbol{\theta}) &= \partial P(Y_i = 0, Y_{i'} \leq y_{i'}) / \partial y_{i'} \\ &= \alpha \phi_1(v(y_i; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi)) f_{\text{Beta}}(y_{i'}; \boldsymbol{\gamma}, \phi) \Phi_1((\mathbf{x}'_{i'} \boldsymbol{\beta} - \alpha + \tilde{\rho}_{ii'} v(y_i; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi))(1 - \tilde{\rho}_{ii'}^2)^{-1/2}), \end{aligned}$$

where $v(\cdot; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi) = \alpha F_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) - \mathbf{x}'_i \boldsymbol{\beta}$ and $f_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi)$ is a beta probability density function. Unlike $u(\cdot; \boldsymbol{\gamma}, \phi)$ in the bivariate density (15) of the spatial beta-Bernoulli mixture model, $v(\cdot; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi)$ here cannot be further simplified and thus contains both $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$.

When both responses are proportions, the bivariate density is parameterized by $\boldsymbol{\theta}$,

$$\begin{aligned} g_{\text{Beta2}}(y_i, y_{i'}; \boldsymbol{\theta}) &= \partial P(Y_i \leq y_i, Y_{i'} \leq y_{i'}) / \partial y_i \partial y_{i'} \\ &= \alpha^2 \phi_2(v(y_i; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi), v(y_{i'}; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi); \tilde{\rho}_{ii'}) f_{\text{Beta}}(y_i; \boldsymbol{\gamma}, \phi) f_{\text{Beta}}(y_{i'}; \boldsymbol{\gamma}, \phi), \end{aligned}$$

where recall $v(\cdot; \boldsymbol{\beta}, \boldsymbol{\gamma}, \phi) = \alpha F_{\text{Beta}}(\cdot; \boldsymbol{\gamma}, \phi) - \mathbf{x}'_i \boldsymbol{\beta}$. In summary, the bivariate density is

$$g(y_i, y_{i'}; \boldsymbol{\theta}) = \begin{cases} g_{\text{Bern2}}(y_i, y_{i'}; \alpha, \boldsymbol{\beta}, \sigma^2, \rho) & \text{if } y_i, y_{i'} \in \{0, 1\} \\ g_{\text{Bern,Beta}}(y_i, y_{i'}; \boldsymbol{\theta}) & \text{if } y_i \in \{0, 1\}, y_{i'} \in (0, 1) \\ g_{\text{Beta,Bern}}(y_i, y_{i'}; \boldsymbol{\theta}) & \text{if } y_i \in (0, 1), y_{i'} \in \{0, 1\} \\ g_{\text{Beta2}}(y_i, y_{i'}; \boldsymbol{\theta}) & \text{if } y_i, y_{i'} \in (0, 1). \end{cases} \quad (24)$$

The computation of the MCLE, their standard errors, and CLIC is analogous to the spatial beta-Bernoulli mixture model in Section 3.2. The consistency and asymptotic normality of the maximum composite likelihood estimates $\hat{\boldsymbol{\theta}}$ can also be derived under suitable regularity conditions.

5. Simulation Study

To examine the finite-sample properties of the methods developed in Section 2–4, we conducted a simulation study on an $L \times L$ square lattice. First, data were generated under the spatial beta-Bernoulli mixture model (10), where $L = 30$ or 40 , corresponding to sample size $N = 900$ or 1600 . We let the mixture probability be $\eta = 0.1, 0.5$, and 0.9 , corresponding to a smaller, moderate, and larger amount of zero/one values, respectively. For a single covariate X_1 that follows the standard Gaussian distribution $N(0, 1)$, we let the regression coefficients in the Bernoulli component be $\beta = (\beta_0, \beta_1)' = (1, -1)'$, and let the regression coefficients in the beta component be $\gamma = (\gamma_0, \gamma_1)' = (1, -1)'$. The choice that β_1 and γ_1 have the same sign and magnitude allows comparison of the estimation accuracy of β and γ for different values of η . Let the precision parameter be $\phi = 4$. We also let $\sigma^2 = 0.8$ and $\rho = 0.6$, corresponding to a relatively strong spatial dependence. Next, data were generated under the spatial generalized Tobit model (22), where the cutoff point is set to $\alpha = 3$, and the remaining simulation setup is the same.

Table 1 reports the properties of the parameter estimates for the spatial beta-Bernoulli mixture model by the mean estimates, relative biases, nominal standard deviations, mean standard errors, and mean squared errors, for different sample sizes N and mixture probabilities η . For a given mixture probability η , as the sample size N increases, the parameter estimates $\hat{\theta}$ become more accurate, and the standard error estimates are closer to the nominal standard deviations. This indicates that $\hat{\theta}$ and $\text{var}(\hat{\theta})$ are consistent estimates of θ and $\text{var}(\hat{\theta})$, respectively. For a given sample size N , as the mixture probability η increases, the data contain more zero/one values. In this case, the nominal standard deviations and standard errors of $\hat{\beta}$ in the Bernoulli component become smaller, reflecting an increasing amount of information at zero/one, whereas those of $\hat{\gamma}$ and $\hat{\phi}$ in the beta component become larger. Overall, the simulation results indicate that the composite likelihood approach is able to recover the regression coefficients for both the Bernoulli and beta component, as well as the precision parameter and the dependence parameters, even when the sample size is relatively moderate. Similarly, for the spatial generalized Tobit model, Table 2 provides the mean estimates, relative biases, nominal standard deviations, mean standard errors,

and mean squared errors for $N = 900$ and $N = 1600$. As the sample size N increases, the parameter estimates $\hat{\theta}$ become more accurate, and the standard errors are closer to the nominal standard deviations, which indicates the consistency of $\hat{\theta}$ and $\text{var}(\hat{\theta})$ as before.

To compare the spatial beta-Bernoulli mixture model developed in Section 2 and the spatial generalized Tobit model in Section 4, we revisited the simulated data above. In particular, we considered the data generated from a spatial beta-Bernoulli model with $N = 1600$, $\eta = 0.1, 0.5$, or 0.9 , and the remaining parameters $(\beta_0, \beta_1, \gamma_0, \gamma_1, \phi, \sigma^2, \rho)' = (1, -1, 1, -1, 4, 0.8, 0.6)'$. Then we fitted these data by a misspecified spatial generalized Tobit model with $(\alpha, \beta_0, \beta_1, \gamma_0, \gamma_1, \phi, \sigma^2, \rho)'$ and a misspecified spatial Tobit model with parameters $(\alpha, \beta_0, \beta_1, \sigma^2, \rho)'$. The parameter estimation results under these three models are given in Table 3. The magnitudes of the correlation parameter estimate, $\hat{\rho}$, in the two misspecified models are close to the true value in the spatial beta-Bernoulli mixture model, while the variance component estimates ($\hat{\sigma}^2$) underestimate σ^2 . The parameter estimates under $\eta = 0.9$ generally have less bias for all three models. The CLIC values obtained from fitting the correct spatial beta-Bernoulli model are smaller than the two misspecified models in all 100 simulations, while the discrepancy tends to be smaller for the spatial generalized Tobit model, indicating superiority over the existing spatial Tobit model.

For further comparison with the spatial generalized Tobit model and spatial Tobit model, we took one data set simulated from the spatial beta-Bernoulli mixture model, each with $N = 1600$ and $\eta = 0.1, 0.5$ or 0.9 . Figure 2 gives the density plot of the actual simulated data from the spatial beta-Bernoulli mixture model, one density plot each for a data set simulated from the fitted correct model, the misspecified spatial generalized Tobit model, and the misspecified spatial Tobit model, for $\eta = 0.1$ (upper left), $\eta = 0.5$ (upper right), and $\eta = 0.9$ (lower left). The density of the actual data has two peaks at both ends and the middle part is relatively flat. When η increases from 0.1 to 0.5 and then to 0.9, the density curves from the three fitted models become closer. This is expected, because in the extreme case of $\eta = 1$, all three models are reduced to a spatial probit model for a binary response variable. On the other hand, when η is relatively small, the spatial Tobit model does not adequately capture the density of data in the range between 0 and 1. This is also expected,

because the threshold mechanism in a spatial Tobit model will only transform the upper tail of the scaled Gaussian latent variable to one and the lower tail to zero, while the unimodal shape remains in the middle range between 0 and 1. In contrast, the density curves from the fitted spatial beta-Bernoulli mixture model and the fitted spatial generalized Tobit model are much closer to the true curve. This example illustrates the greater flexibility of the new models proposed here that capture a wider variety of shapes of the data distributions.

We also considered the data simulated from the spatial generalized Tobit model and fitted them by the spatial beta-Bernoulli mixture model and spatial Tobit model as two misspecified models. For sample size $N = 1600$, we set the true parameters to be $(\alpha, \beta_0, \beta_1, \gamma_0, \gamma_1, \phi, \sigma^2, \rho)' = (3, 1, -1, 1, -1, 4, 0.8, 0.6)'$. The parameter estimation results under these three models are given in Table 4. The proportion of zero/one values in the simulated data is about 32%, which is reflected in $\hat{\eta} = 0.316$ in the spatial beta-Bernoulli mixture model. Although the interpretation of the regression parameters $\hat{\beta}_1$ and $\hat{\gamma}_1$ differs from the mixture model, the signs are the same as those of the true parameters. As before, the variance component estimate $\hat{\sigma}^2$ underestimates σ^2 and the intercept estimate $\hat{\beta}_0$ is quite different from the true parameter. Since the spatial Tobit model is only a special case of the spatial generalized Tobit model, the estimates have relatively larger biases compared to those under the correct model. The CLIC values obtained from the correct spatial generalized Tobit model are smaller than the two misspecified models in all 100 simulations.

To further compare with the spatial beta-Bernoulli mixture model and the spatial Tobit model, we took one simulated data set from the spatial generalized Tobit model. The lower right figure in Figure 2 gives the density curve of the actual simulated data from the spatial generalized Tobit model, one density plot each for a simulated dataset from the fitted correct model and the fitted misspecified models. It shows that the fitted correct model is closest to the density curve of the actual data. The fitted spatial beta-Bernoulli mixture model has a density curve that is much closer to the truth than the spatial Tobit model, again demonstrating the flexibility of the models developed here.

Finally, we considered two covariates X_1 and X_2 from the standard Gaussian distribution. For

both the spatial beta-Bernoulli mixture and the spatial generalized Tobit models, we set the percentage of zero/one values close to the land cover data example. In the spatial beta-Bernoulli mixture model, the vector of regression coefficients for the beta component is set as $\beta = (\beta_0, \beta_1, \beta_2)' = (2, 0, 1)'$, and the vector of regression coefficients for the Bernoulli component is $\gamma = (\gamma_0, \gamma_1, \gamma_2)' = (1, -0.5, 0)'$. Under this setup, only X_2 affects the beta component, and only X_1 affects the Bernoulli component, representing two distinct different processes in the two mixture components. We set $\eta = 0.1$ so that the percentage of zero/one values is exactly 10%. The precision parameter is $\phi = 1$, and the dependence parameters are $(\sigma^2, \rho)' = (0.8, 0.6)'$. In the spatial generalized Tobit model, the parameters are set as $\alpha = 4$, $\beta = (\beta_0, \beta_1, \beta_2)' = (1.5, -0.5, 0)'$, $\gamma = (\gamma_0, \gamma_1, \gamma_2)' = (0.5, 0, 0)'$, $\phi = 1$, and $(\sigma^2, \rho)' = (0.8, 0.6)'$. Under these parameter choices, the percentage of zero/one values is around 10%, and the inverse beta CDF transformation is $\text{Beta}(1/(1 + \exp(-0.5)), 1)$ across all sampling sites. Similar to the case of a single covariate, the results in Tables 1 and 2 show that both the parameter estimates and the variance estimates are consistent. When the data have a relatively low (or high) proportions of zero/one values, the standard errors of $\hat{\beta}$ in the Bernoulli component (or $\hat{\gamma}$ in the beta component) can be elevated. In contrast, for the spatial generalized Tobit model, the standard deviations and standard errors of $\hat{\beta}$ and $\hat{\gamma}$ are comparable, although the standard errors of $\hat{\gamma}$ may also be large when the amount of proportions on $(0, 1)$ low.

6. Data Example

6.1. Study Background

In the motivating data example, the spatial unit of analysis is a quarter section (or, 1/36th of a township, 160 acres $\approx$ 65 ha). There are a total of $N = 1429$ quarter sections in the study area in northern Wisconsin, USA (Figure 1). Forest cover types were derived from the Wisconsin Land Economic Inventory, a land survey that represents land covers in 1930. Here we consider an early successional forest cover, known as the aspen-paper birch (APB) (Fu *et al.*, 2013; Feng *et al.*, 2014). The response variable is the proportion of APB in a quarter section, varying from 0

(no APB) to 1 (all APB). The covariates are about land-ownership characteristics derived from historical plat maps in 1915, as land ownership offers a spatially specific, quantifiable way to assess the influence of anthropogenic and historical factors on landscape structure. In this study, we consider the binary variable *reservation* that indicates whether the quarter section is on an Indian reservation; the *number of parcels* in a given quarter section which is a measure of parcel density; the *proportion of the average parcel size* as a measure of parcel size within a quarter section; the *proportion of the largest parcel* which reports the largest parcel polygon as a quarter section proportion; the *average size of all parcels* as a measure of all parcels that intersect with a quarter section; and *total ownership area* which shows the total property area throughout the project study area of the dominant owner of the quarter section. We took a natural logarithm transformation of *total ownership area* since the values of total area are skewed to the right. All the covariates except the binary variable *reservation* were centered and scaled to have sample mean 0 and sample variance 1.

6.2. Model Fitting and Selection

To assess the influence of land-ownership characteristics in 1915 to APB in 1930, we applied the spatial beta-Bernoulli mixture model and the spatial generalized Tobit model to analyze the data. We also fitted the spatial Tobit model for comparison. The weight radius is set to be $r = 5$ (Feng *et al.*, 2014). Maximum likelihood estimation assuming independence provided the initial values for composite likelihood estimation. Table 5 presents the parameter estimates and their standard errors, CLIC values, and computational time under each model. By a backward elimination, the final spatial generalized Tobit model has slightly lower CLIC (-0.688) than the final mixture model (-0.676). The common covariates retained in the two final models are *reservation* and *average size of all parcels* (Figure 1). After we excluded the insignificant covariate *number of parcels* and refitted the spatial generalized Tobit model, the CLIC is -0.681 , which is comparable to the mixture model.

For the spatial beta-Bernoulli mixture model, the mixture probability is estimated to be $\hat{\eta} = 0.12$, which agrees well with the observed 12% of quarter sections with no or all APB. For the

regression coefficients in the Bernoulli component, the estimates are not significant except $\hat{\beta}_1$ for the binary variable *reservation*. This may be because we only have a relatively small amount of zero/one values, and the standard errors can be inflated as shown in the simulation results Table 1. For the regression coefficients in the beta component, there is an additional significant covariate *average size of all parcels* ($\hat{\gamma}_5$) besides *reservation* ($\hat{\gamma}_1$). The first-order correlation $\sigma^2\rho$ is estimated to be $0.904 \times 0.502 = 0.454$. The results provide evidence that after accounting for the other covariates, the quarter sections on the reservation have larger proportions, including 100%, of APB than off the reservation, supporting previous studies (Steen-Adams *et al.*, 2007, 2011). In addition, the off-reservation quarter sections with higher average parcel size tend to have larger proportions of APB.

For the spatial generalized Tobit model, both *reservation* ($\hat{\beta}_1$) and *average size of all parcels* ($\hat{\beta}_5$) in the Gaussian latent variable model are significant. The results are in agreement with those from the spatial beta-Bernoulli mixture model. However, the interpretation is in terms of the latent variables instead of the original response variables. As for the inverse beta CDF transformation, the estimates are not significant except for the intercept $\hat{\gamma}_0$. That is, the inverse beta CDF transformation of the spatial latent variable does not depend on covariates. The first-order correlation $\sigma^2\rho$ is $0.855 \times 0.641 = 0.548$, which is higher than that of the spatial beta-Bernoulli model. Compared with the two proposed models, the spatial Tobit model has an additional significant covariate *total ownership area* ($\hat{\beta}_6$), although the small standard error estimate may be misleading as indicated in the simulation study. The first-order correlation is about 0.533, which is close to the spatial generalized Tobit model, as demonstrated in the simulation example.

6.3. Prediction and Model Comparison

We now compare the models by prediction. First, we split the data into a training set (70%) and a test set (30%). Let Y_0 denote the response at say s_0 in the test set. We predicted Y_0 by a conditional expectation

$$E(Y_0|\mathbf{Y}_T) = P(Y_0 = 1|\mathbf{Y}_T) + \int_0^1 y_0 \partial P(Y_0 \leq y_0|\mathbf{Y}_T) / \partial y_0 dy_0, \quad (25)$$

where $\mathbf{Y}_T$ denotes the vector of the training data. However, the computation of (25) is not straightforward. Let N_R denote the number of proportional responses within $R = 10$ units and $\mathbf{Y}_{N_R}$ denote the corresponding training data. The conditional expectation (25) was approximated by replacing $P(Y_0 = 1|\mathbf{Y}_T)$ with $P(Y_0 = 1|\mathbf{Y}_{N_R}) = \eta \int_{-\mathbf{x}'_0\boldsymbol{\beta}}^{+\infty} \phi_{N_R+1}(y_0^* - \mathbf{x}'_0\boldsymbol{\beta}, \mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R+1}) dy_0^* / \phi_{N_R}(\mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R})$ and replacing $\partial P(Y_0 \leq y_0|\mathbf{Y}_T)/\partial y_0$ with

$$\begin{aligned} \frac{\partial}{\partial y_0} P(Y_0 \leq y_0|\mathbf{Y}_{N_R}) &= (1 - \eta) \phi_{N_R+1}(\Phi_1^{-1}(F_{\text{Beta}}(y_0; \boldsymbol{\gamma}, \phi)), \mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R+1}) F_{\text{Beta}}(y_0; \boldsymbol{\gamma}, \phi) \\ &\quad \times \left\{ \phi_{N_R}(\mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R}) \phi_1(\Phi_1^{-1}(F_{\text{Beta}}(y_0; \boldsymbol{\gamma}, \phi))) \right\}^{-1}, \end{aligned}$$

where $\mathbf{t}_{N_R} = (\Phi_1^{-1}(F_{\text{Beta}}(Y_1; \boldsymbol{\gamma}, \phi)), \dots, \Phi_1^{-1}(F_{\text{Beta}}(Y_{N_R}; \boldsymbol{\gamma}, \phi)))'$, $\boldsymbol{\Sigma}_{N_R}$ is an $N_R \times N_R$ the covariance matrix of $\mathbf{t}_{N_R}$, and $\boldsymbol{\Sigma}_{N_R+1}$ is an $(N_R + 1) \times (N_R + 1)$ covariance matrix of $(y_0^*, \mathbf{t}_{N_R}')'$. Further ϕ_{N_R} and ϕ_{N_R+1} denote the N_R -variate and $N_R + 1$ -variate Gaussian density functions.

Similarly, the conditional expectation under the spatial generalized Tobit model was approximated by

$$E(Y_0|\mathbf{Y}_{N_R}) = P(Y_0 = 1|\mathbf{Y}_{N_R}) + \int_0^1 y_0 \partial P(Y_0 \leq y_0|\mathbf{Y}_{N_R}) / \partial y_0 dy_0,$$

where $P(Y_0 = 1|\mathbf{Y}_{N_R}) = \int_{\alpha - \mathbf{x}'_0\boldsymbol{\beta}}^{+\infty} \phi_{N_R+1}(y_0^* - \mathbf{x}'_0\boldsymbol{\beta}, \mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R+1}) dy_0^* / \phi_{N_R}(\mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R})$, $\mathbf{t}_{N_R} = (\alpha F_{\text{Beta}}(Y_1; \boldsymbol{\gamma}, \phi), \dots, \alpha F_{\text{Beta}}(Y_{N_R}; \boldsymbol{\gamma}, \phi))'$ - $\mathbf{X}_{N_R}\boldsymbol{\beta}$, $\mathbf{X}_{N_R}$ is the design matrix corresponding to $\mathbf{Y}_{N_R}$, and $\frac{\partial}{\partial y_0} P(Y_0 \leq y_0|\mathbf{Y}_{N_R}) = \alpha f_{\text{Beta}}(y_0; \boldsymbol{\gamma}, \phi) \phi_{N_R+1}(\alpha F_{\text{Beta}}(y_0; \boldsymbol{\gamma}, \phi) - \mathbf{x}'_0\boldsymbol{\beta}, \mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R+1}) / \phi_{N_R}(\mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R})$.

The conditional expectation under the spatial Tobit model was approximated by

$$E(Y_0|\mathbf{Y}_{N_R}) = P(Y_0 = 1|\mathbf{Y}_{N_R}) + \int_0^1 y_0 \partial P(Y_0 \leq y_0|\mathbf{Y}_{N_R}) / \partial y_0 dy_0, \quad (26)$$

where

$$\begin{aligned}
P(Y_0 = 1 | \mathbf{Y}_{N_R}) &= \alpha^{N_R} \int_{\alpha}^{+\infty} f(y_0^* | \alpha \mathbf{Y}_{N_R}) dy_0^* \\
&= \alpha^{N_R} \int_{\alpha}^{+\infty} f(y_0^*, \alpha \mathbf{Y}_{N_R}) dy_0^* / \phi_{N_R}(\mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R}) \\
&= \alpha^{N_R} \int_{\alpha - \mathbf{x}'_0 \boldsymbol{\beta}}^{+\infty} \phi_{N_R+1}(y_0^* - \mathbf{x}'_0 \boldsymbol{\beta}, \mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R+1}) dy_0^* / \phi_{N_R}(\mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R}), \\
\frac{\partial}{\partial y_0} P(Y_0 \leq y_0 | \mathbf{Y}_{N_R}) &= \alpha^{N_R+1} \phi_{N_R+1}(y_0 - \mathbf{x}'_0 \boldsymbol{\beta}, \mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R+1}) / \phi_{N_R}(\mathbf{t}_{N_R}; \boldsymbol{\Sigma}_{N_R}) \text{ for } y_0 \in (0, 1),
\end{aligned}$$

and $\mathbf{t}_{N_R} = \alpha \mathbf{Y}_{N_R} - \mathbf{X}_{N_R} \boldsymbol{\beta}$.

The mean absolute error (MAE) and the root mean squared error (RMSE) were computed to evaluate the prediction performance. The MAE is defined as $n^{-1} \sum_{i=1}^n |y_i - \hat{y}_i|$, where n is the sample size of the training or test data, y_i is the observation, and $\hat{y}_i$ is the prediction based on conditional expectation. The MAE is an average of absolute differences and is not sensitive to unusually large errors. In contrast, RMSE, defined as $\{n^{-1} \sum_{i=1}^n (y_i - \hat{y}_i)^2\}^{1/2}$, is based on an average of squared differences and is more sensitive to large errors.

Figures 3 and 4 present the prediction results for the spatial beta-Bernoulli mixture model, spatial generalized Tobit model, and the spatial Tobit model. The scatter plots in Figure 3 show that the two new models slightly overfit zero/one values, while the spatial Tobit model underfit them. Our additional simulation results indicate that the overfitting in the data example was more due to random chances (results not shown). In Figure 4, the three models have similar patterns of predictions across sampling sites. Compared with the two new models, the spatial Tobit model tends to have more smoothed predictions, especially at sites with zero/one values, as observed in Figure 3. For the training data, the smallest RMSE and MAE are obtained for the spatial generalized Tobit model, with 0.107 and 0.071, respectively. In addition, the RMSE of the spatial beta-Bernoulli mixture model is about 12% higher than the spatial Tobit model, while the MAEs are about the same. For the test data, the spatial Tobit model has the smallest RMSE and the spatial generalized Tobit model has the smallest MAE (Table 6).

Figure 5 presents the density plot of the observed APB, as well as the density curves based on 100 simulated datasets from each of the fitted spatial beta-Bernoulli mixture model, fitted spatial generalized Tobit model, fitted spatial Tobit model, and the final fitted spatial generalized Tobit model. For the spatial Tobit model, the unimodal shape in the open interval $(0, 1)$ is restrictive, while for all the other models, the density curves from the simulated datasets are quite close to the that from the observed data.

In terms of the computational time, the spatial Tobit model took about 1 minute on a 64 bit Linux operating system with 48–64 GB of RAM, and the spatial beta-Bernoulli mixture and generalized spatial Tobit model took about 30 minutes, which is a moderately high price paid for the model flexibility. The additional computational cost in the latter two models can be attributed largely to the numerical score functions in the optimization procedure instead of the closed-form score function as in the spatial Tobit model.

7. Conclusions and Discussion

In this paper, we have developed a beta-Bernoulli mixture model and a spatial generalized Tobit model for spatial proportional data with zero/one values. With a composite likelihood approach, numerical examples have shown that both the regression coefficients and the dependence parameters can be estimated consistently. Moreover, the composite likelihood estimation developed here is relatively efficient to compute and thus, is applicable for spatial data with a relatively large sample size. Both the simulation study and the data example have shown that the proposed spatial beta-Bernoulli mixture model and spatial generalized Tobit model are more flexible than the existing spatial Tobit model, while the cost for this added flexibility is primarily computational.

In both new models, the Gaussian latent variables provide a way to account for the spatial dependence among observations. The spatial beta-Bernoulli mixture model assumes that the data are generated from two distributions, one for the zero/one values with mean ν_i in relation to covariates via β and the other one for the proportions with mean μ_i in relation to covariates via γ . The expectation of the response variable is given by $E(Y_i) = \eta\nu_i + (1 - \eta)\mu_i$ and the interpretation

is straightforward. With the spatial generalized Tobit model, the data are governed by the spatial Gaussian latent variables with model flexibility enhanced by an inverse beta CDF transformation. The response variable does not follow a known distribution and the expectation of the response variable $E(Y_i)$ is not straightforward to derive. However, the expectation of the response variable increases as the mean of the spatial latent variable $\mathbf{x}_i'\boldsymbol{\beta}$ and the mean of beta random variable μ_i increase and thus, the interpretation is also intuitive.

The numerical examples show that the spatial generalized Tobit model appears to be as flexible as the spatial beta-Bernoulli mixture model, and the computational cost for the two proposed models is comparable. In practice, if the zero/one values are due to thresholding such as a detection limit, a spatial Tobit model can be applied first. When there remains a large discrepancy between the fitted and the actual observations, a spatial generalized Tobit model can be considered. If the zero/one values and proportions are associated with different factors, one could consider a two-stage approach where the first stage determines whether the response is zero/one or a proportion, and the second stage models the proportional data on $(0, 1)$ with a beta distribution excluding the zero/one values. The mixture model has the advantage of accounting for the two processes in a single model and stage.

Common to both the mixture and Tobit type of models is the specification of a spatial latent variable. A practical concern is whether the parameter estimates change significantly if the latent distribution is not Gaussian. The robustness of latent variable models has been discussed for correlated binary data by Tan *et al.* (1999). Liu (2004) introduced the “robit regression” using a latent T-distribution as a more robust alternative to the traditional logit and probit models. It would be interesting to evaluate how robust the parameter estimates and predictions are in our models. In addition, for spatial compositional data with zero values, Leininger *et al.* (2013) proposed a Bayesian hierarchical model by a power scaling transformation to model the presence of zeros in compositional data, under the assumption that there exists at least one nonzero component for all observations. The composite likelihood approach may also be extended to models for spatial compositional data. We leave these for future research.

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Tables and Figures

Figure 1: Data example: maps of the proportions of aspen-paper birch (APB) (left), *reservation* (middle), and *average size of all parcels* (right) in $N = 1429$ quarter sections of a study area in northern Wisconsin.

Figure 2: Simulation study: density plots of observed and fitted data from one simulation under the true spatial beta-Bernoulli mixture model with the mixture probability $\eta = 0.1$ (upper left), 0.5 (upper right), 0.9 (lower left) and under the true spatial generalized Tobit model (lower right) with the cutoff point $\alpha = 3$. The sample size is $N = 1600$.

Figure 3: Data example: predictions versus observations under spatial beta-Bernoulli mixture model (row 1), spatial generalized Tobit model (row 2), and spatial Tobit model (row 3) for 70% training data (left column) and 30% testing data (right column).

Figure 4: Data example: observations (row 1) and predictions from spatial beta-Bernoulli mixture model (row 2), spatial generalized Tobit model (row 3), and spatial Tobit model (row 4) for 70% training data (left column) and 30% testing data (right column).

Figure 5: Data example: comparison between the density curves of the observed data and 100 simulated datasets from the fitted spatial beta-Bernoulli mixture model (upper left), fitted spatial generalized Tobit model (upper right), fitted spatial Tobit model (lower left), and final spatial generalized Tobit model (lower right).

Table 1: Simulation study: composite likelihood estimates for the parameters of a spatial beta-Bernoulli mixture model: mean estimates (Estimate), relative biases (RB), nominal standard deviations (SD), mean standard errors (SE), and mean squared errors (MSE), with sample size $N = 900$ or 1600 based on 100 simulations. $X_1, X_2 \sim N(0, 1)$.

N	Estimate		RB		SD		SE		MSE	
Parameter	900	1600	900	1600	900	1600	900	1600	900	1600
	$\eta = 0.1$, single covariate									
$\eta = 0.1$	0.100	0.100	-0.003	0.002	0.010	0.007	0.010	0.008	0.000	0.000
$\beta_0 = 1$	1.059	1.039	0.059	0.039	0.294	0.212	0.261	0.190	0.090	0.046
$\beta_1 = -1$	-1.063	-1.060	0.063	0.060	0.291	0.210	0.260	0.190	0.089	0.048
$\sigma^2 = 0.8$	0.864	0.826	0.080	0.033	0.100	0.094	0.131	0.094	0.014	0.009
$\rho = 0.6$	0.557	0.573	-0.072	-0.045	0.083	0.063	0.088	0.066	0.009	0.005
$\gamma_0 = 1$	1.026	0.997	0.026	-0.003	0.132	0.096	0.129	0.096	0.018	0.009
$\gamma_1 = -1$	-1.003	-0.973	0.003	-0.027	0.044	0.029	0.039	0.029	0.002	0.002
$\phi = 4$	4.125	4.173	0.031	0.043	0.393	0.284	0.391	0.291	0.170	0.110
	$\eta = 0.5$, single covariate									
$\eta = 0.5$	0.501	0.502	0.002	0.004	0.015	0.012	0.017	0.013	0.000	0.000
$\beta_0 = 1$	1.028	1.024	0.028	0.024	0.161	0.133	0.165	0.122	0.027	0.018
$\beta_1 = -1$	-1.022	-1.012	0.022	0.012	0.101	0.081	0.115	0.084	0.011	0.007
$\sigma^2 = 0.8$	0.860	0.826	0.075	0.032	0.113	0.111	0.151	0.107	0.016	0.013
$\rho = 0.6$	0.558	0.573	-0.069	-0.044	0.086	0.069	0.094	0.070	0.009	0.006
$\gamma_0 = 1$	1.033	1.014	0.033	0.014	0.134	0.097	0.134	0.102	0.019	0.010
$\gamma_1 = -1$	-1.002	-1.002	0.002	0.002	0.054	0.039	0.051	0.038	0.003	0.002
$\phi = 4$	4.147	4.130	0.037	0.033	0.439	0.310	0.435	0.320	0.214	0.113
	$\eta = 0.9$, single covariate									
$\eta = 0.9$	0.900	0.901	0.000	0.001	0.011	0.008	0.010	0.008	0.000	0.000
$\beta_0 = 1$	1.031	1.026	0.031	0.026	0.157	0.114	0.154	0.115	0.026	0.014
$\beta_1 = -1$	-1.015	-1.008	0.015	0.008	0.085	0.062	0.090	0.066	0.008	0.004
$\sigma^2 = 0.8$	0.851	0.817	0.064	0.022	0.135	0.133	0.190	0.137	0.021	0.018
$\rho = 0.6$	0.559	0.573	-0.068	-0.046	0.095	0.078	0.106	0.080	0.011	0.007
$\gamma_0 = 1$	1.029	1.007	0.029	0.007	0.161	0.112	0.167	0.124	0.027	0.013
$\gamma_1 = -1$	-0.992	-1.006	-0.008	0.006	0.124	0.091	0.112	0.084	0.015	0.008
$\phi = 4$	4.339	4.234	0.085	0.058	0.785	0.608	0.741	0.530	0.731	0.424
	$\eta = 0.1$, two covariates									
$\eta = 0.1$	0.100	0.101	0.004	0.009	0.010	0.007	0.010	0.008	0.000	0.000
$\beta_0 = 2$	2.411	2.230	0.206	0.115	1.015	0.413	0.776	0.398	1.199	0.224
$\beta_1 = 0$	-0.057	-0.030	—	—	0.435	0.194	0.387	0.216	0.193	0.038
$\beta_2 = 1$	1.163	1.135	0.163	0.135	0.594	0.307	0.566	0.313	0.380	0.112
$\sigma^2 = 0.8$	0.861	0.828	0.077	0.035	0.102	0.094	0.117	0.084	0.014	0.010
$\rho = 0.6$	0.557	0.573	-0.071	-0.045	0.082	0.063	0.078	0.059	0.009	0.005
$\gamma_0 = 1$	1.035	1.018	0.035	0.018	0.180	0.133	0.161	0.123	0.034	0.018
$\gamma_1 = -0.5$	-0.493	-0.493	-0.014	-0.015	0.048	0.033	0.040	0.030	0.002	0.001
$\gamma_2 = 0$	-0.011	0.000	—	—	0.040	0.036	0.034	0.025	0.002	0.001
$\phi = 1$	1.036	1.036	0.036	0.036	0.097	0.069	0.094	0.070	0.011	0.006

Table 2: Simulation study: composite likelihood estimates for spatial generalized Tobit model: mean estimates (Estimate), relative biases (RB), nominal standard deviations (SD), mean standard errors (SE), and mean squared errors (MSE) with sample size $N = 900$ or 1600 based on 100 simulations. $X_1, X_2 \sim N(0, 1)$.

Parameter \ N	Estimate		RB		SD		SE		MSE	
	900	1600	900	1600	900	1600	900	1600	900	1600
single covariate										
$\alpha = 3$	3.033	3.044	0.011	0.015	0.169	0.118	0.188	0.142	0.030	0.016
$\beta_0 = 1$	1.033	1.029	0.033	0.029	0.149	0.111	0.160	0.123	0.023	0.013
$\beta_1 = -1$	-1.018	-1.019	0.018	0.019	0.067	0.048	0.075	0.056	0.005	0.003
$\sigma^2 = 0.8$	0.871	0.828	0.088	0.035	0.099	0.093	0.150	0.110	0.015	0.009
$\rho = 0.6$	0.555	0.573	-0.076	-0.045	0.082	0.063	0.102	0.080	0.009	0.005
$\gamma_0 = 1$	1.003	0.994	0.003	-0.006	0.040	0.030	0.040	0.029	0.002	0.001
$\gamma_1 = -1$	-0.999	-0.995	-0.001	-0.005	0.059	0.043	0.060	0.044	0.004	0.002
$\phi = 4$	4.042	4.018	0.011	0.005	0.307	0.210	0.307	0.223	0.096	0.045
two covariates										
$\alpha = 4$	4.060	4.056	0.015	0.014	0.251	0.179	0.264	0.198	0.067	0.035
$\beta_0 = 1.5$	1.541	1.533	0.028	0.022	0.177	0.120	0.175	0.134	0.033	0.015
$\beta_1 = -0.5$	-0.515	-0.508	0.029	0.015	0.074	0.050	0.069	0.051	0.006	0.003
$\beta_2 = 0$	-0.004	-0.002	—	—	0.062	0.048	0.056	0.041	0.004	0.002
$\sigma^2 = 0.8$	0.866	0.829	0.082	0.037	0.100	0.089	0.143	0.107	0.014	0.009
$\rho = 0.6$	0.555	0.572	-0.075	-0.046	0.083	0.061	0.101	0.079	0.009	0.004
$\gamma_0 = 0.5$	0.506	0.497	0.013	-0.007	0.074	0.050	0.073	0.055	0.006	0.002
$\gamma_1 = 0$	0.004	-0.002	—	—	0.062	0.047	0.061	0.045	0.004	0.002
$\gamma_2 = 0$	-0.008	0.001	—	—	0.049	0.040	0.052	0.039	0.002	0.002
$\phi = 1$	1.008	1.007	0.008	0.007	0.071	0.050	0.067	0.049	0.005	0.003

Table 3: Simulation study: composite likelihood estimates under the misspecified generalized spatial Tobit model (GTobit) and spatial Tobit model (Tobit). The true model is the spatial beta-Bernoulli mixture model (Mixture) with sample size $N = 1600$ based on 100 simulations.

Parameter	Estimate			SD			SE		
	Mixture	GTobit	Tobit	Mixture	GTobit	Tobit	Mixture	GTobit	Tobit
$\eta = 0.1$									
η	0.100	—	—	0.007	—	—	0.008	—	—
α	—	3.432	4.282	—	0.077	0.159	—	0.160	0.097
$\beta_0 = 1$	1.039	1.980	3.055	0.212	0.073	0.179	0.190	0.154	0.076
$\beta_1 = -1$	-1.060	-0.205	-0.811	0.210	0.036	0.038	0.190	0.040	0.035
$\gamma_0 = 1$	0.997	0.700	—	0.096	0.076	—	0.096	0.055	—
$\gamma_1 = -1$	-0.973	-0.730	—	0.029	0.037	—	0.029	0.041	—
$\phi = 4$	4.173	2.198	—	0.284	0.139	—	0.291	0.133	—
$\sigma^2 = 0.8$	0.826	0.586	0.549	0.094	0.082	0.087	0.094	0.120	0.081
$\rho = 0.6$	0.573	0.566	0.561	0.063	0.066	0.070	0.066	0.115	0.071
$\eta = 0.5$									
η	0.502	—	—	0.012	—	—	0.013	—	—
α	—	1.595	1.978	—	0.056	0.100	—	0.080	0.047
$\beta_0 = 1$	1.024	1.263	1.687	0.133	0.077	0.128	0.122	0.125	0.076
$\beta_1 = -1$	-1.012	-0.389	-0.607	0.081	0.033	0.034	0.084	0.033	0.035
$\gamma_0 = 1$	1.014	0.800	—	0.097	0.077	—	0.102	0.042	—
$\gamma_1 = -1$	-1.002	-0.823	—	0.039	0.040	—	0.038	0.037	—
$\phi = 4$	4.130	3.426	—	0.310	0.248	—	0.320	0.194	—
$\sigma^2 = 0.8$	0.826	0.326	0.329	0.111	0.080	0.082	0.107	0.142	0.079
$\rho = 0.6$	0.573	0.558	0.555	0.069	0.087	0.089	0.070	0.229	0.093
$\eta = 0.9$									
η	0.901	—	—	0.008	—	—	0.008	—	—
α	—	0.385	0.424	—	0.034	0.041	—	0.034	0.037
$\beta_0 = 1$	1.026	0.964	1.009	0.114	0.084	0.092	0.115	0.123	0.095
$\beta_1 = -1$	-1.008	-0.702	-0.729	0.062	0.046	0.045	0.066	0.047	0.042
$\gamma_0 = 1$	1.007	0.925	—	0.112	0.105	—	0.124	0.083	—
$\gamma_1 = -1$	-1.006	-0.931	—	0.091	0.093	—	0.084	0.079	—
$\phi = 4$	4.234	4.100	—	0.608	0.591	—	0.530	0.507	—
$\sigma^2 = 0.8$	0.817	0.459	0.470	0.133	0.141	0.141	0.137	0.187	0.124
$\rho = 0.6$	0.573	0.566	0.565	0.078	0.102	0.101	0.080	0.202	0.100

Table 4: Simulation study: composite likelihood estimates under the misspecified spatial beta-Bernoulli mixture model (Mixture) and spatial Tobit model (Tobit). The true model is the generalized spatial Tobit model (GTobit) with sample size $N = 1600$ based on 100 simulations.

Parameter	Estimate			SD			SE		
	Mixture	GTobit	Tobit	Mixture	GTobit	Tobit	Mixture	GTobit	Tobit
$\alpha = 3$	—	3.044	3.237	—	0.118	0.170	—	0.142	0.163
η	0.316	—	—	0.017	—	—	0.012	—	—
$\beta_0 = 1$	-1.096	1.029	1.700	0.256	0.111	0.176	0.232	0.123	0.062
$\beta_1 = -1$	-2.195	-1.019	-1.354	0.289	0.048	0.060	0.273	0.056	0.056
$\gamma_0 = 1$	0.821	0.994	—	0.056	0.030	—	0.067	0.029	—
$\gamma_1 = -1$	-1.622	-0.995	—	0.044	0.043	—	0.045	0.044	—
$\phi = 4$	6.627	4.018	—	0.408	0.210	—	0.371	0.223	—
$\sigma^2 = 0.8$	0.517	0.828	0.700	0.081	0.093	0.102	0.105	0.110	0.093
$\rho = 0.6$	0.559	0.573	0.565	0.078	0.063	0.068	0.092	0.080	0.068

Table 5: Data example: maximum composite likelihood estimates (Est) and standard errors (SE) of the regression coefficients for intercept (β_0, γ_0), reservation (β_1, γ_1), number of parcels (β_2, γ_2), proportion of the average parcel size (β_3, γ_3), quarter section proportion of the largest parcel (β_4, γ_4), average size of all parcels (β_5, γ_5), log transformed total ownership area (β_6, γ_6), and other parameters under the spatial beta-Bernoulli mixture model (Mixture), spatial generalized Tobit model (GTobit), spatial Tobit model (Tobit), and final models by backward elimination. Computing time is recorded in minutes.

Param	Mixture		Mixture Final		GTobit		GTobit Final		Tobit	
	Est	SE	Est	SE	Est	SE	Est	SE	Est	SE
α	—	—	—	—	3.537	0.205	3.532	0.200	2.889	0.203
η	0.121	0.009	0.121	0.009	—	—	—	—	—	—
β_0	-1.143	0.273	-1.113	0.238	1.245	0.152	1.242	0.152	1.169	0.119
β_1	2.976	0.398	2.977	0.380	1.202	0.281	1.201	0.275	1.314	0.202
β_2	0.024	0.265	—	—	-0.024	0.112	-0.012	0.065	-0.020	0.074
β_3	-0.100	0.604	—	—	0.028	0.173	—	—	-0.085	0.088
β_4	0.183	0.479	—	—	-0.050	0.138	—	—	0.121	0.088
β_5	0.842	0.482	0.673	0.373	0.265	0.116	0.271	0.093	0.181	0.072
β_6	-0.286	0.452	—	—	0.023	0.086	—	—	0.154	0.077
γ_0	-0.083	0.115	-0.061	0.114	0.322	0.089	0.345	0.087	—	—
γ_1	1.403	0.185	1.336	0.183	0.122	0.188	0.065	0.182	—	—
γ_2	-0.061	0.113	—	—	-0.012	0.122	-0.063	0.066	—	—
γ_3	-0.171	0.145	—	—	-0.123	0.182	—	—	—	—
γ_4	0.212	0.114	—	—	0.200	0.156	—	—	—	—
γ_5	0.204	0.078	0.285	0.061	-0.104	0.110	-0.020	0.069	—	—
γ_6	0.114	0.081	—	—	0.094	0.074	—	—	—	—
ϕ	1.113	0.057	1.103	0.057	0.663	0.033	0.658	0.034	—	—
σ^2	0.904	0.160	0.909	0.157	0.855	0.099	0.865	0.100	0.824	0.072
ρ	0.502	0.081	0.503	0.080	0.641	0.078	0.637	0.078	0.647	0.060
CLIC	-0.662		-0.676		-0.641		-0.688		1.827	
Time	27		15		28		20		1	

Table 6: Data example: mean absolute errors (MAE) and root mean squared errors (RMSE) computed for evaluating the prediction accuracy in training and test sets under the spatial beta-Bernoulli mixture model (Mixture), spatial generalized Tobit model (GTobit), and spatial Tobit model (Tobit).

Model	MAE		RMSE	
	Training	Test	Training	Test
Mixture	0.096	0.188	0.137	0.249
GTobit	0.071	0.184	0.107	0.250
Tobit	0.096	0.198	0.122	0.240