Hybrid Almost Output Regulation of Linear Impulsive Systems with Average Dwell-Time[☆]

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Abstract

This paper deals with the hybrid almost output regulation problem for a class of linear systems with average dwell-time impulses. The proposed hybrid output regulator is constructed as a linear impulsive system that undergoes synchronous impulses with the controlled plant. Lyapunov-based sufficient conditions of the output regulability and weighted $\mathcal{L}_2$ performance for the linear impulsive systems are first derived. Based on the analysis results, the hybrid synthesis problem is formulated in terms of linear matrix equations plus a set of linear matrix inequalities (LMIs). With this hybrid synthesis scheme, both flow and jump dynamics of the hybrid regulator can be jointly designed by solving a convex optimization problem in minimizing the weighted $\mathcal{L}_2$ gain from the perturbation signal to the error output. A numerical example is used to demonstrate the proposed approach.

Keywords: Linear impulsive systems; hybrid control; output regulation; average dwell-time; convex optimization.

1. Introduction

The problem of output regulation, which concerns controlling a given plant such that its output tracks references or rejects disturbances generated by an exosystem, has been extensively studied over the past decades for various dynamical systems (see, [1, 2, 3, 4, 5] and the references therein). In particular, necessary and sufficient solvability conditions for the output regulation of linear time-invariant (LTI) systems were elegantly established as a set of linear algebraic equations in [1], which paves the foundations for the subsequent research on, for instance, optimal output regulation [6]; output regulation with saturations [2, 7]; output regulator synthesis with multiple performance objectives [8, 9], and also applications in many practical problems [10, 11]. The output regulation problem was also addressed later from an external model perspective using perfect disturbance decoupling techniques, see, e.g., [12] based on frequency-domain method, and [13] using time-domain method. It is not until the nineties that the output regulation problem for general nonlinear systems was first pursued by the pioneering works [14] and [3]. Different from the LTI case, the solvability conditions of output regulation problem for nonlinear systems are typically expressed in terms of mixed partial differential and nonlinear algebraic equations, which are difficult to solve. This motivates the alternative solution of using gain-scheduling control approach for nonlinear output regulation [15, 4]. Recent research on the output regulation problem has shifted to hybrid dynamical systems that combine continuous-time and discrete-time behaviors, such as [16, 17, 5, 18, 19].

In this paper, we will study the hybrid output regulation problem for a class of linear systems subject to impulse effects. This type of systems are classified as linear impulsive dynamical systems that are typically modeled by the combination of ordinary differential equations and instantaneous state jumps or resets (also referred to as impulses) [20, 21]. As a special case of hybrid systems, linear impulsive systems have been widely considered in the research

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of control community. They are useful in real-world applications including spacecraft guidance and control, sampled-data control systems, etc (see, e.g., [20, 22]). Stability properties of such systems have been extensively investigated, see, for instance, [23, 20, 24, 21]. In spite of rich literature on the analysis and control design of hybrid linear systems, the output regulation problem of linear impulsive systems has not been addressed adequately. One of the main difficulties relates to the hybrid nature of linear impulsive systems. Due to the impulsive dynamics induced by the state jumps, most previous results for output regulation of continuous linear/nonlinear systems are not directly applicable in this case. It is worth to mention the latest contribution [5] in which new concepts and tools were developed for hybrid output regulation of single-input single-output (SISO) linear systems with periodic state jumps. This work was subsequently applied in [18] to design hybrid regulators with different structures for minimum-phase SISO linear systems with relative degree greater than one. Some limitations encountered in [18] were nicely overcome by [25] and [26, 27]. All these results, however, rely heavily on the periodicity of the impulse sequences, which could limit their applications to practical engineering problems. The most recent work by Carnevale et. al., in [28] has successfully extended the aforementioned results to hybrid systems with aperiodic jumps using full information feedback control, while the problem of error feedback control is yet to be solved. Another difficulty is of a technical nature and lies in how to formulate the associated hybrid output regulation synthesis problem as computationally tractable conditions. Therefore, it is desirable to develop an effective control design approach for hybrid output regulation of linear impulsive systems, which motivates our current work.

The above-mentioned difficulties on hybrid output regulation of linear impulsive systems will be overcome by using the average dwell-time (ADT) technique [29, 37, 21] incorporating with a new Lyapunov-like function. Specifically, the impulse time instants of the linear impulsive system under consideration are allowed to occur non-periodically, but with a frequency constraint, which is formalized in terms of an ADT condition. On the other hand, the so-called Lyapunov-like function for hybrid impulsive system analysis differs from classical Lyapunov functions in the sense that it is allowed to have discontinuities at each impulse instant with bounded jumps, but required to be continuous and decrease between impulses with certain convergence rate. The proposed hybrid output regulator is constructed as a linear impulsive system that undergoes synchronous impulses with the controlled plant. The resulting hybrid output regulation synthesis problem with guaranteed weighted $\mathcal{L}_2$ performance (from [30]) is formulated as computationally tractable conditions consisting of linear matrix equations and linear matrix inequalities (LMIs), which can be solved effectively using existing techniques [31]. As a result, the hybrid controller/regulator gain matrices with respect to both flow dynamics and discrete state jumps can be designed simultaneously by solving a convex LMI-based optimization.

The rest of the paper is organized as follows. The problem statement and some basic definitions are given in Section 2. Section 3 provides Lyapunov-based conditions for stabilization, almost output regulation and weighted $\mathcal{L}_2$ gain performance of linear impulsive systems. The hybrid synthesis and controller construction algorithms are derived in Section 4. Section 5 presents the simulation results of the proposed hybrid control approach on a simple example, and Section 6 concludes the paper.

Notation. $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}_+$ for the set of positive real numbers. The set of positive integers is denoted by $\mathbb{N}_+$. $\mathbb{R}^{m \times n}$ is the set of real $m \times n$ matrices. The transpose of a real matrix M is denoted by M^T . The hermitian operator $He\{\cdot\}$ is defined as $He\{M\} = M + M^T$ for real matrices. The identity matrix of any dimension is denoted by I . $\mathbb{S}^n$ and $\mathbb{S}_+^n$ are used to denote the set of real symmetric $n \times n$ matrices and positive definite matrices, respectively. A block diagonal matrix with matrices $X_1, X_2, \dots, X_p$ on its main diagonal is denoted by $diag\{X_1, X_2, \dots, X_p\}$. Furthermore, we use the symbol $\star$ in LMIs to denote entries that follow from symmetry. For $x \in \mathbb{R}^n$, its norm is defined as $\|x\| := (x^T x)^{1/2}$. The space of square integrable functions is denoted by $\mathcal{L}_2$, that is, for any $u \in \mathcal{L}_2$, $\|u\|_2 := [\int_0^\infty u^T(t)u(t)dt]^{1/2}$ is finite. For two integers $k_1 < k_2$, we denote $\mathbf{I}[k_1, k_2] = \{k_1, k_1 + 1, \dots, k_2\}$.

2. Problem Statement and Basic Definitions

Consider a linear impulsive plant

$$\left. \begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + A_w w(t) + B_{p1} d(t) + B_{p2} u(t) \\ \dot{w}(t) &= A_e w(t) + Q d(t) \\ x_p^+(t) &= J_p x_p(t) + J_w w(t) \\ w^+(t) &= J_e w(t) \end{aligned} \right\}, \quad t \neq t_k, k \in \mathbb{N}_+ \quad (1)$$

$$\begin{aligned} e(t) &= C_{p1} x_p(t) + C_{w1} w(t) + D_{p11} d(t) + D_{p12} u(t) \\ y(t) &= C_{p2} x_p(t) + C_{w2} w(t) + D_{p21} d(t) \end{aligned}$$

where $\{t_1, t_2, \dots\}$ is a strictly increasing sequence of impulse time instants in (t_0, ∞) for some initial time t_0 , and contains a finite number of elements on any finite time interval. At impulse instants, the plant state $x_p \in \mathbb{R}^n$ (exosystem state $w \in \mathbb{R}^{n_w}$) will have sudden changes specified by the jump matrices $J_p \in \mathbb{R}^{n \times n}$ and $J_w \in \mathbb{R}^{n_w \times n_w}$ ($J_e \in \mathbb{R}^{n_w \times n_w}$). Both $x_p(t)$ and $w(t)$ are absolutely continuous between impulses and assumed to be right-continuous with $(\cdot)^+(t) := \lim_{s \downarrow t} (\cdot)(s)$. $d \in \mathbb{R}^{n_d}$ is the unknown perturbation input that injects into both plant and exosystem dynamics, $y \in \mathbb{R}^{n_y}$ is the measurement output available for feedback control use. $u \in \mathbb{R}^{n_u}$ is the control input, and $e \in \mathbb{R}^{n_e}$ is the error output to be regulated. As typically adopted in the output regulation literature (see, e.g., [2, 26, 27]), we assume $n_u \geq n_e$ to ensure solvability of the output regulation equations. All the system matrices are constant matrices of compatible dimensions and assumed to be known for controller design. We further have the following assumptions:

A1. A_e is anti-Hurwitz (i.e., all eigenvalues of A_e are in the closed right half-plane).

A2. The pair (A_p, B_{p2}) is stabilizable.

A3. The pair $\left(\begin{bmatrix} A_p & -A_w \\ 0 & A_e \end{bmatrix}, \begin{bmatrix} C_{p2} & -C_{w2} \end{bmatrix} \right)$ is detectable.

Following the same discussions as presented in [7], it is clear that the above three assumptions do not cause any loss of generality. Specifically, Assumption A1 is made to avoid trivialities, taking into account that we are dealing with the case of tracking possibly unbounded reference trajectories. Clearly, with this assumption, the flow dynamics of the exosystem is allowed to have unstable and/or marginally stable poles. This implies that the tracking reference trajectories generated by the exosystem could be unbounded regardless of the boundedness of perturbation d . Similar situations have been considered and illustrated in most existing literature (see, e.g., [4, 19]). Assumptions A2 and A3 are necessary to guarantee the existence of a feedback stabilizing hybrid output regulator from y to u , which will be further demonstrated in Section 4 (see Remark 2).

We will focus our study on system (1) subject to a class of impulse time sequences with average dwell-time (ADT).

Definition 1. Given an impulse time sequence $\{t_k\}_{k=1}^\infty$, it is said that the linear impulsive system (1) possesses the property of ADT impulse, if there exist two positive numbers N_0 and τ_a such that

$$N(t, s) \leq N_0 + \frac{t - s}{\tau_a}, \quad \forall t \geq s \geq t_0 \quad (2)$$

where $N(t, s)$ denotes the number of impulses in the semi-open interval $(s, t]$, N_0 is the chatter bound, and τ_a is called the “average dwell-time”.

Remark 1. The concept of ADT impulse has also been used in [21] for stability analysis of impulsive systems, it corresponds to the concept of average dwell-time for switched systems introduced in [29]. The ADT constraint (2) essentially provides an upper bound on the frequency of impulses occurred during a finite interval $(s, t]$. The special case of $N_0 = 1$ reduces to a dwell-time condition in which any two consecutive impulses must be separated by at least τ_a units of time. Also, as an extreme case, $\tau_a \rightarrow 0$ implies that the constraint on the impulse frequency is almost eliminated and the resulting impulse is allowed to occur arbitrarily fast.

We would like to design a hybrid dynamic output-feedback control law in the form of

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c y(t), & t \neq t_k, k \in \mathbb{N}_+ \\ x_c^+(t) = J_c x_c(t), & t = t_k, k \in \mathbb{N}_+ \\ u(t) = C_c x_c(t) + D_c y(t) \end{cases} \quad (3)$$

where $x_c \in \mathbb{R}^{n_c}$ is the continuous-time state of the controller that undergoes instantaneous jumps at each impulse instant t_k . The impulse time sequence is synchronized between the impulsive plant and the controller. Matrices A_c, B_c, C_c, D_c and J_c are controller coefficients to be designed.

The closed-loop system formed by the interconnection of the impulsive plant (1) and the hybrid controller (3) can be expressed in a general linear hybrid system as below

$$\begin{cases} \dot{x}_{cl}(t) = A_{cl} x_{cl}(t) + A_{cl,w} w(t) + B_{cl} d(t) \\ \dot{w}(t) = A_e w(t) + Q d(t) \\ x_{cl}^+(t) = A_I x_{cl}(t) + A_{I,w} w(t) \\ w^+(t) = J_e w(t) \end{cases}, \quad t \neq t_k, k \in \mathbb{N}_+ \quad (4)$$

$$e(t) = C_{cl} x_{cl}(t) + C_{cl,w} w(t) + D_{cl} d(t)$$

with $x_{cl} := \begin{bmatrix} x_p^T & x_c^T \end{bmatrix}^T$ and

$$\begin{bmatrix} A_{cl} & A_{cl,w} & B_{cl} \\ C_{cl} & C_{cl,w} & D_{cl} \end{bmatrix} = \begin{bmatrix} A_p + B_{p2} D_c C_{p2} & B_{p2} C_c & A_w + B_{p2} D_c C_{w2} & B_{p1} + B_{p2} D_c D_{p21} \\ -\frac{B_c C_{p2}}{C_{p1} + D_{p12} D_c C_{p2}} & -\frac{A_c}{D_{p12} C_c} & -\frac{B_c C_{w2}}{C_{w1} + D_{p12} D_c C_{w2}} & -\frac{B_c D_{p21}}{D_{p11} + D_{p12} D_c D_{p21}} \end{bmatrix},$$

$$A_I = \begin{bmatrix} J_p & 0 \\ 0 & J_c \end{bmatrix}, \quad A_{I,w} = \begin{bmatrix} J_w \\ 0 \end{bmatrix}.$$

We also have the following basic definitions regarding the stability, output regulation, and trajectory tracking/disturbance rejection properties for the hybrid impulsive system (4).

Definition 2. Given an impulsive time sequence $\{t_k\}_{k=1}^\infty$, the linear hybrid system (4) with $w(t) \equiv 0$ and $d(t) \equiv 0$ is globally uniformly asymptotically stable (GUAS) if there exists a class $\mathcal{KL}$ function β such that for all $t \geq t_0$ and $x_{cl}(t_0)$, the corresponding solution to (4) exists globally and satisfies $\|x_{cl}(t)\| \leq \beta(\|x_{cl}(t_0)\|, t - t_0)$.

Definition 3. Given an impulsive time sequence $\{t_k\}_{k=1}^\infty$, the linear hybrid system (4) is said to achieve output regulation if the error output $e(t) \rightarrow 0$ as $t \rightarrow \infty$ for any $\begin{bmatrix} x_{cl}(t_0) \\ w(t_0) \end{bmatrix}$ and $d(t) \equiv 0$.

Definition 4. Given an impulsive time sequence $\{t_k\}_{k=1}^\infty$, the linear hybrid system (4) is said to be GUAS with a weighted $\mathcal{L}_2$ gain γ from d to e , if under zero initial conditions, the system (4) is GUAS and satisfies

$$\int_{t_0}^\infty e^{-\lambda(t-t_0)} \|e(t)\|^2 dt < \gamma^2 \int_{t_0}^\infty \|d(t)\|^2 dt \quad (5)$$

for some $\lambda > 0$ and any $d \in \mathcal{L}_2$.

In this paper, we will solve the **hybrid almost output regulation problem** such that the hybrid closed-loop system (4) with the ADT impulse constraint achieves

1. Globally uniformly asymptotic stability and output regulation when $d(t) \equiv 0$;
2. A guaranteed weighted $\mathcal{L}_2$ gain γ from d to e for any $d \in \mathcal{L}_2$,

by constructing a hybrid output-feedback law in the form of (3).

3. Output Regulability and Weighted $\mathcal{L}_2$ -Gain Analysis

In this section, with the ADT impulse constraint, sufficient conditions for almost output regulation with weighted $\mathcal{L}_2$ performance of the hybrid system (4) will be established by using a Lyapunov-like function [32]. Note that different from classical Lyapunov functions, the Lyapunov-like function adopted in this paper for hybrid impulsive system analysis is allowed to have discontinuities at each impulse instant with bounded jumps, but required to be continuous and decrease between impulses with a convergence rate.

Theorem 1. *Given two scalars $\lambda_0 \in \mathbb{R}_+$ and $\mu > 1$ for the hybrid system (4), if there exist a positive definite matrix $P \in \mathbb{S}_+^{n+n_c}$, a rectangular matrix $\Pi \in \mathbb{R}^{(n+n_c) \times n_w}$ and a positive scalar $\gamma \in \mathbb{R}_+$ such that the following conditions hold.*

$$A_{cl}\Pi - \Pi A_e + A_{cl,w} = 0, \quad (6)$$

$$C_{cl}\Pi + C_{cl,w} = 0, \quad (7)$$

$$A_I\Pi - \Pi J_e + A_{I,w} = 0, \quad (8)$$

$$\begin{bmatrix} He\{PA_{cl}\} + \lambda_0 P & \star & \star \\ B_{cl}^T P - Q^T \Pi^T P & -\gamma I & \star \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0, \quad (9)$$

$$\begin{bmatrix} \mu P & \star \\ PA_I & P \end{bmatrix} \geq 0. \quad (10)$$

Then, almost output regulation with weighted $\mathcal{L}_2$ performance γ can be achieved for every impulse time sequence $\{t_k\}_{k=1}^\infty$ with average dwell-time

$$\tau_a > \frac{\ln(\mu)}{\lambda_0}. \quad (11)$$

PROOF. Applying the following coordinate transformation

$$\tilde{x}_{cl} = x_{cl} - \Pi w \quad (12)$$

to the system state in (4), we obtain an equivalent hybrid system as

$$\begin{cases} \dot{\tilde{x}}_{cl}(t) = A_{cl}\tilde{x}_{cl}(t) + (A_{cl}\Pi - \Pi A_e + A_{cl,w})w(t) + (B_{cl} - \Pi Q)d(t), & t \neq t_k, k \in \mathbb{N}_+ \\ \tilde{x}_{cl}^+(t) = A_I\tilde{x}_{cl}(t) + (A_I\Pi - \Pi J_e + A_{I,w})w(t), & t = t_k, k \in \mathbb{N}_+ \\ e(t) = C_{cl}\tilde{x}_{cl}(t) + (C_{cl}\Pi + C_{cl,w})w(t) + D_{cl}d(t) \end{cases} \quad (13)$$

Furthermore, with conditions (6)–(8), the transformed system (13) reduces to

$$\begin{cases} \dot{\tilde{x}}_{cl}(t) = A_{cl}\tilde{x}_{cl}(t) + (B_{cl} - \Pi Q)d(t), & t \neq t_k, k \in \mathbb{N}_+ \\ \tilde{x}_{cl}^+(t) = A_I\tilde{x}_{cl}(t), & t = t_k, k \in \mathbb{N}_+ \\ e(t) = C_{cl}\tilde{x}_{cl}(t) + D_{cl}d(t) \end{cases} \quad (14)$$

Since $e(t)$ does not explicitly relate to the exosystem state $w(t)$ and D_{cl} is a constant matrix, according to the Definition 3, it suffices to prove that the above transformed system (14) is GUAS in achieving output regulation.

Consider the Lyapunov-like function $V(\tilde{x}_{cl}) = \tilde{x}_{cl}^T P \tilde{x}_{cl}$ for the system (14) with $d(t) \equiv 0$. From the (1, 1) element of the matrix in condition (9), we have

$$\dot{V}(\tilde{x}_{cl}) = \tilde{x}_{cl}^T (A_{cl}^T P + P A_{cl}) \tilde{x}_{cl} < -\lambda_0 \tilde{x}_{cl}^T P \tilde{x}_{cl} = -\lambda_0 V(\tilde{x}_{cl}) \quad (15)$$

for all $\tilde{x}_{cl} \in \mathbb{R}^{n+n_c}$. On the other hand, by Schur complement, condition (10) is equivalent to

$$\mu P - A_I^T P A_I \geq 0, \quad (16)$$

which implies

$$V(\tilde{x}_{cl}^+) \leq \mu V(\tilde{x}_{cl}), \quad \forall \tilde{x}_{cl} \in \mathbb{R}^{n+n_c}. \quad (17)$$

Condition (15) combined with (17) yields that for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_+$,

$$\begin{aligned}
V(\tilde{x}_{cl}(t)) &< e^{-\lambda_0(t-t_k)} V(\tilde{x}_{cl}^+(t_k)) \\
&\leq \mu e^{-\lambda_0(t-t_k)} V(\tilde{x}_{cl}(t_k)) \\
&< \mu^2 e^{-\lambda_0(t-t_{k-1})} V(\tilde{x}_{cl}(t_{k-1})) \\
&< \dots \\
&< \mu^{N(t,t_0)} e^{-\lambda_0(t-t_0)} V(\tilde{x}_{cl}(t_0)) \\
&= e^{N(t,t_0) \ln(\mu) - \lambda_0(t-t_0)} V(\tilde{x}_{cl}(t_0)) \\
&\leq e^{(\frac{1}{\tau_a} \ln(\mu) - \lambda_0)(t-t_0)} V(\tilde{x}_{cl}(t_0)).
\end{aligned}$$

where the last inequality is obtained by using $N(t, t_0) \leq \frac{t-t_0}{\tau_a}$. Note that $N(t, t_0)$ has been used to denote the number of impulses in the semi-open time interval $(t_0, t]$. Throughout the proof, in order to simplify the expressions, we consider a typical case with zero chatter bound, i.e., $N_0 = 0$. The proof for the general case with $N_0 \neq 0$ can be accomplished similarly. Thus, if the average dwell-time τ_a satisfies constraint (11), we conclude that $V(\tilde{x}_{cl})$ converges to zero as $t \rightarrow \infty$. Thereby, system (14) is GUAS since P is a positive definite matrix, and $e(t)$ is asymptotically regulatable.

To show the weighted $\mathcal{L}_2$ performance from $d(t)$ to $e(t)$, we obtain from condition (9) via Schur complement that

$$\dot{V}(\tilde{x}_{cl}) < -\lambda_0 V(\tilde{x}_{cl}) - \frac{1}{\gamma} e^T e + \gamma d^T d \quad (18)$$

for all $\tilde{x}_{cl} \in \mathbb{R}^{n+n_c}$ and $d \in \mathbb{R}^{n_d}$. Thus, based on the conditions (17) and (18), for any $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned}
V(\tilde{x}_{cl}(t)) &< V(\tilde{x}_{cl}^+(t_k)) e^{-\lambda_0(t-t_k)} - \int_{t_k}^t e^{-\lambda_0(t-\tau)} \phi(\tau) d\tau \\
&\leq \mu V(\tilde{x}_{cl}(t_k)) e^{-\lambda_0(t-t_k)} - \int_{t_k}^t e^{-\lambda_0(t-\tau)} \phi(\tau) d\tau \\
&< \mu \left[V(\tilde{x}_{cl}^+(t_{k-1})) e^{-\lambda_0(t_k-t_{k-1})} - \int_{t_{k-1}}^{t_k} e^{-\lambda_0(t_k-\tau)} \phi(\tau) d\tau \right] e^{-\lambda_0(t-t_k)} - \int_{t_k}^t e^{-\lambda_0(t-\tau)} \phi(\tau) d\tau \\
&\quad \vdots \\
&< \mu^k e^{-\lambda_0(t-t_0)} V(\tilde{x}_{cl}(t_0)) - \mu^k \int_{t_0}^{t_1} e^{-\lambda_0(t-\tau)} \phi(\tau) d\tau - \mu^{k-1} \int_{t_1}^{t_2} e^{-\lambda_0(t-\tau)} \phi(\tau) d\tau \\
&\quad - \dots - \mu^0 \int_{t_k}^t e^{-\lambda_0(t-\tau)} \phi(\tau) d\tau \\
&= \mu^{N(t,t_0)} e^{-\lambda_0(t-t_0)} V(\tilde{x}_{cl}(t_0)) - \int_{t_0}^t \mu^{N(t,\tau)} e^{-\lambda_0(t-\tau)} \phi(\tau) d\tau \\
&= e^{-\lambda_0(t-t_0) + N(t,t_0) \ln(\mu)} V(\tilde{x}_{cl}(t_0)) - \int_{t_0}^t e^{-\lambda_0(t-\tau) + N(t,\tau) \ln(\mu)} \phi(\tau) d\tau,
\end{aligned}$$

where $\phi(\tau) := \frac{1}{\gamma} e^T(\tau) e(\tau) - \gamma d^T(\tau) d(\tau)$, and the property (17) was invoked at the impulse instants in some of the inequalities.

Assuming zero initial condition and based on the fact that $V(\tilde{x}_{cl}(t_0)) = 0$, $V(\tilde{x}_{cl}(t)) \geq 0$, we obtain

$$\int_{t_0}^t e^{-\lambda_0(t-\tau) + N(t,\tau) \ln(\mu)} e^T(\tau) e(\tau) d\tau < \gamma^2 \int_{t_0}^t e^{-\lambda_0(t-\tau) + N(t,\tau) \ln(\mu)} d^T(\tau) d(\tau) d\tau.$$

Multiplying both sides of the above inequality by $e^{-N(t,t_0) \ln(\mu)}$, it yields

$$\int_{t_0}^t e^{-\lambda_0(t-\tau) - N(\tau,t_0) \ln(\mu)} e^T(\tau) e(\tau) d\tau < \gamma^2 \int_{t_0}^t e^{-\lambda_0(t-\tau) - N(\tau,t_0) \ln(\mu)} d^T(\tau) d(\tau) d\tau. \quad (19)$$

Then, according to the average dwell-time constraint (11), we may choose a positive scalar $0 < \lambda < \lambda_0$ such that $N(\tau, t_0) \ln(\mu) \leq \lambda(\tau - t_0) < \lambda_0(\tau - t_0)$ holds for any $\tau > t_0$. Therefore, from inequality (19), we have

$$\int_{t_0}^t e^{-\lambda_0(t-\tau)-\lambda(\tau-t_0)} e^T(\tau) e(\tau) d\tau < \gamma^2 \int_{t_0}^t e^{-\lambda_0(t-\tau)} d^T(\tau) d(\tau) d\tau.$$

Consequently, we integrate the above inequality from $t = t_0$ to ∞ to get

$$\begin{aligned} & \int_{t_0}^{\infty} \left(\int_{t_0}^t e^{-\lambda_0(t-\tau)-\lambda(\tau-t_0)} e^T(\tau) e(\tau) d\tau \right) dt \\ &= \int_{t_0}^{\infty} e^{-\lambda(\tau-t_0)} e^T(\tau) e(\tau) \left(\int_{\tau}^{\infty} e^{-\lambda_0(t-\tau)} dt \right) d\tau = \frac{1}{\lambda_0} \int_{t_0}^{\infty} e^{-\lambda(\tau-t_0)} e^T(\tau) e(\tau) d\tau \\ &< \int_{t_0}^{\infty} \left(\gamma^2 \int_{t_0}^t e^{-\lambda_0(t-\tau)} d^T(\tau) d(\tau) d\tau \right) dt \\ &= \gamma^2 \int_{t_0}^{\infty} d^T(\tau) d(\tau) \left(\int_{\tau}^{\infty} e^{-\lambda_0(t-\tau)} dt \right) d\tau = \gamma^2 \frac{1}{\lambda_0} \int_{t_0}^{\infty} d^T(\tau) d(\tau) d\tau. \end{aligned}$$

Thus,

$$\int_{t_0}^{\infty} e^{-\lambda(\tau-t_0)} e^T(\tau) e(\tau) d\tau < \gamma^2 \int_{t_0}^{\infty} d^T(\tau) d(\tau) d\tau$$

holds for any $d \in \mathcal{L}_2$, which completes the proof.

Three linear matrix equations (6)–(8) in Theorem 1 are referred to as the hybrid output regulation equations. In particular, conditions (6) and (7) are used to guarantee the flow invariance of manifold $x_{cl} = \Pi w$, in turns, the output regulability with respect to the flow dynamics of the hybrid system (4). While the jump invariance of $x_{cl} = \Pi w$ is ensured by condition (8). It is easy to see that if $A_I = I$, $A_{I,w} = 0$ and $J_e = I$, that is if the plant and exosystem are purely LTI systems, all the above discussions would degenerate to the classical regulation problem with LTI settings [1]. In this case, condition (8) would be automatically satisfied, and conditions (6)–(7) are essentially the well-known Francis equations [1]. As can be seen, the hybrid output regulation equations are all linear algebraic, which can be solved effectively. Similar results are also obtained and discussed in detail in [25], which however require to solve a set of time-varying (differential) hybrid output regulation equations. On the other hand, condition (9) corresponds to the well-known bounded real lemma (BRL) condition for LTI systems [33], it is used to guarantee the $\mathcal{L}_2$ performance of the closed loop over the flowing time intervals. Condition (10) is termed as the boundary condition that restricts the possible increase of the system energy caused by the abrupt state jumps at each impulse instant. We stress here that three hybrid output regulation equations (6)–(8) can be solved independently from the other two inequality conditions (9)–(10).

4. Hybrid Controller Synthesis

Based on the analysis results derived in Theorem 1, this section provides a control design approach for the hybrid output regulator.

Motivated from the classical results for LTI systems [1, 34, 35], we partition the hybrid regulator state x_c in (3) as $x_c = \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix}$ with $x_{c1} \in \mathbb{R}^{n_w}$, $x_{c2} \in \mathbb{R}^{n_c-n_w}$, and parameterize the controller coefficient matrices as

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} A_e - B_{c1}(C_{w2} + C_{p2}\Pi_p) & A_{c12} & B_{c1} \\ -B_{c2}(C_{w2} + C_{p2}\Pi_p) & A_{c22} & B_{c2} \\ \Gamma - \bar{D}_c(C_{w2} + C_{p2}\Pi_p) & \bar{C}_{c2} & \bar{D}_c \end{bmatrix}, \quad J_c = \begin{bmatrix} J_e & J_{c12} \\ 0 & J_{c22} \end{bmatrix} \quad (20)$$

where $A_{c12} \in \mathbb{R}^{n_w \times (n_c-n_w)}$, $A_{c22} \in \mathbb{R}^{(n_c-n_w) \times (n_c-n_w)}$, $B_{c1} \in \mathbb{R}^{n_w \times n_y}$, $B_{c2} \in \mathbb{R}^{(n_c-n_w) \times n_y}$, $C_{c2} \in \mathbb{R}^{n_u \times (n_c-n_w)}$, $D_c \in \mathbb{R}^{n_u \times n_y}$, which together with $\Pi_p \in \mathbb{R}^{n \times n_w}$, $\Gamma \in \mathbb{R}^{n_u \times n_w}$ and $J_{c12} \in \mathbb{R}^{n_w \times (n_c-n_w)}$, $J_{c22} \in \mathbb{R}^{(n_c-n_w) \times (n_c-n_w)}$ are free variables subject to design.

Figure 1: The hybrid closed-loop structure.

It is observed that the hybrid regulator includes the exosystem dynamics quantified by A_e and J_e as internal models. In fact, as shown in Figure 1, with such a specific structure, the resulting hybrid closed-loop system can be viewed as an interconnection of an auxiliary hybrid controller

$$\begin{cases} \dot{x}_{c2}(t) = A_{c22}x_{c2}(t) - B_{c2}(C_{w2} + C_{p2}\Pi_p)x_{c1}(t) + B_{c2}y(t), & t \neq t_k, k \in \mathbb{N}_+ \\ x_{c2}^+(t) = J_{c22}x_{c2}(t), & t = t_k, k \in \mathbb{N}_+ \\ v_1(t) = C_{c2}x_{c2}(t) - D_c(C_{w2} + C_{p2}\Pi_p)x_{c1}(t) + D_cy(t) \\ v_2(t) = A_{c12}x_{c2}(t) - B_{c1}(C_{w2} + C_{p2}\Pi_p)x_{c1}(t) + B_{c1}y(t) \end{cases} \quad (21)$$

and the extended plant (as denoted by the dashed box $\mathcal{P}_e$ in Figure 1)

$$\begin{cases} \dot{x}_p(t) = A_px_p(t) + B_{p2}\Gamma x_{c1}(t) + A_w w(t) + B_{p1}d(t) + B_{p2}v_1(t) \\ \dot{x}_{c1}(t) = A_ex_{c1}(t) + v_2(t) \\ \dot{w}(t) = A_w w(t) + Qd(t) \\ x_p^+(t) = J_px_p(t) + J_w w(t) \\ x_{c1}^+(t) = J_ex_{c1}(t) + J_{c12}x_{c2}(t) \\ w^+(t) = J_e w(t) \end{cases}, \quad t \neq t_k, k \in \mathbb{N}_+ \\
 \begin{cases} x_p^+(t) = J_px_p(t) + J_w w(t) \\ x_{c1}^+(t) = J_ex_{c1}(t) + J_{c12}x_{c2}(t) \\ w^+(t) = J_e w(t) \end{cases}, \quad t = t_k, k \in \mathbb{N}_+ \\
 e(t) = C_{p1}x_p(t) + D_{p12}\Gamma x_{c1}(t) + C_{w1}w(t) + D_{p11}d(t) + D_{p12}v_1(t) \\
 y(t) = C_{p2}x_p(t) + C_{w2}w(t) + D_{p21}d(t). \end{cases} \quad (22)$$

As can be seen from Figure 1, the proposed hybrid output regulator structure (20) can be decomposed into two portions with different functions. The first portion consisting of state x_{c1} with a similar dynamics as of the exosystem, serves as an estimator/observer to mimic the dynamic behaviors of the exosystem. Whereas the second portion associated with state x_{c2} in (21) is in charge of the stabilization of the overall closed-loop system. The idea of parameterizing the controller gains in such a specific structure is not new, similar design schemes can also be found in the literature (see, e.g., [1, 34, 35, 8, 9, 36, 19]). The adoption of this hybrid output regulator parametrization is motivated from the well-known internal model principle for LTI systems in [1, 34] and linear hybrid systems in [5]. It will be shown in the sequel that the hybrid output regulation conditions (6)–(8) in Theorem 1 is readily satisfied under such a controller parametrization. Furthermore, the hybrid regulation problem will reduce to a stabilization problem of designing a controller in the form of (21) that renders the overall closed-loop system GUAS.

In addition, the closed-loop system (4) with the controller parametrization (20) can be rewritten as following by partitioning $x_{cl} = [x_p^T \ x_{c1}^T \ x_{c2}^T]^T$

$$\begin{aligned}
A_{cl} &= \begin{bmatrix} A_p + B_{p2}D_cC_{p2} & B_{p2}[\Gamma - D_c(C_{w2} + C_{p2}\Pi_p) \ C_{c2}] \\ \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} C_{p2} & \begin{bmatrix} A_e & A_{c12} \\ 0 & A_{c22} \end{bmatrix} + \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} [-(C_{w2} + C_{p2}\Pi_p) \ 0] \end{bmatrix}, \\
B_{cl} &= \begin{bmatrix} B_{p1} + B_{p2}D_cD_{p21} \\ \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} D_{p21} \end{bmatrix}, \\
C_{cl} &= \begin{bmatrix} C_{p1} + D_{p12}D_cC_{p2} & D_{p12}[\Gamma - D_c(C_{w2} + C_{p2}\Pi_p) \ C_{c2}] \end{bmatrix}, \\
D_{cl} &= D_{p11} + D_{p12}D_cD_{p21}, \\
A_I &= \begin{bmatrix} J_p & 0 & 0 \\ 0 & J_e & J_{c12} \\ 0 & 0 & J_{c22} \end{bmatrix}, \quad A_{I,w} = \begin{bmatrix} J_w \\ 0 \\ 0 \end{bmatrix}.
\end{aligned} \tag{23}$$

The following theorem summarizes the synthesis conditions of the hybrid impulsive controllers that possess the special architecture and achieve almost output regulation with guaranteed weighted $\mathcal{H}_\infty$ performance for the closed-loop system (4) with (23). The hybrid controller construction algorithm is also provided.

Theorem 2. *Given two scalars $\lambda_0 \in \mathbb{R}_+$ and $\mu > 1$ for the hybrid closed-loop system (4), if there exist positive definite matrices $R_1 \in \mathbb{S}_+^n, \hat{S} \in \mathbb{S}_+^{n+n_w}$, rectangular matrices $\hat{A}_{c2} \in \mathbb{R}^{(n+n_w) \times n}, \hat{B}_c \in \mathbb{R}^{(n+n_w) \times n_y}, \hat{C}_{c2} \in \mathbb{R}^{n_u \times n}, \hat{D}_c \in \mathbb{R}^{n_u \times n_y}, \hat{J}_{c2} \in \mathbb{R}^{(n+n_w) \times n}, \Pi_p \in \mathbb{R}^{n \times n_w}, \Gamma \in \mathbb{R}^{n_u \times n_w}$ and a positive scalar $\gamma \in \mathbb{R}_+$ such that the conditions (24)–(29) hold.*

$$\Pi_p A_e - A_w = A_p \Pi_p + B_{p2} \Gamma, \tag{24}$$

$$-C_{w1} = C_{p1} \Pi_p + D_{p12} \Gamma, \tag{25}$$

$$\Pi_p J_e - J_w = J_p \Pi_p, \tag{26}$$

$$\begin{bmatrix} R_1 & \star \\ \begin{bmatrix} I \\ \Pi_p^T \end{bmatrix} & \hat{S} \end{bmatrix} > 0, \tag{27}$$

$$\begin{bmatrix} He \{ A_p R_1 + B_{p2} \hat{C}_{c2} \} + \lambda_0 R_1 & \star & \star & \star \\ \left(\begin{array}{c} \hat{A}_{c2} + \begin{bmatrix} A_p & -A_w \\ 0 & A_e \end{bmatrix}^T \begin{bmatrix} I \\ \Pi_p^T \end{bmatrix} \\ + \begin{bmatrix} C_{p2}^T \\ -\hat{C}_{c2}^T \end{bmatrix} \hat{D}_c^T B_{p2}^T + \lambda_0 \begin{bmatrix} I \\ \Pi_p^T \end{bmatrix} \end{array} \right) & He \left\{ \begin{array}{c} \hat{S} \begin{bmatrix} A_p & -A_w \\ 0 & A_e \end{bmatrix} \\ + \hat{B}_c \begin{bmatrix} C_{p2} & -C_{w2} \end{bmatrix} \end{array} \right\} + \lambda_0 \hat{S} & \star & \star \\ B_{p1}^T + D_{p21}^T \hat{D}_c^T B_{p2}^T - Q^T \Pi_p^T & D_{p21}^T \hat{B}_c^T + \begin{bmatrix} B_{p1} \\ -Q \end{bmatrix}^T \hat{S} & -\gamma I & \star \\ C_{p1} R_1 + D_{p12} \hat{C}_{c2} & \begin{bmatrix} C_{p1} & -C_{w1} \end{bmatrix} + D_{p12} \hat{D}_c \begin{bmatrix} C_{p2} & -C_{w2} \end{bmatrix} & D_{p11} + D_{p12} \hat{D}_c D_{p21} & -\gamma I \end{bmatrix} < 0, \tag{28}$$

$$\begin{bmatrix} \mu R_1 & \star & \star & \star \\ \mu \begin{bmatrix} I \\ \Pi_p^T \end{bmatrix} & \mu \hat{S} & \star & \star \\ J_p R_1 & J_p \begin{bmatrix} I & \Pi_p \end{bmatrix} & R_1 & \star \\ \hat{J}_{c2} & \hat{S} \begin{bmatrix} J_p & -J_w \\ 0 & J_e \end{bmatrix} & \begin{bmatrix} I \\ \Pi_p^T \end{bmatrix} & \hat{S} \end{bmatrix} \geq 0. \tag{29}$$

Then, there exists a hybrid controller of the form (3) with order $n_c = n + n_w$ that solves the hybrid almost output regulation problem for every impulse time sequence $\{t_k\}_{k=1}^\infty$ with average dwell-time $\tau_a > \frac{\ln(\mu)}{\lambda_0}$. Moreover, the hybrid controller gain matrices are readily obtained through the following algorithm:

- Let $W = \begin{bmatrix} I & \Pi_p \\ 0 & I \end{bmatrix}$ to obtain $S = W^{-T} \hat{S} W^{-1}$. Partition $S = \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix}$, $S^{-1} := \begin{bmatrix} \hat{S}_1 & \hat{S}_2 \\ \hat{S}_2^T & \hat{S}_3 \end{bmatrix}$ with $S_1, \hat{S}_1 \in \mathbb{S}_+^n$, $S_2, \hat{S}_2 \in \mathbb{R}^{n \times n_w}$, $S_3, \hat{S}_3 \in \mathbb{S}_+^{n_w}$. Then, for any given $R_2 \in \mathbb{R}^{n \times n_w}$, compute

$$R_3 = \hat{S}_3 + (\hat{S}_2 - R_2)^T (R_1 - \hat{S}_1)^{-1} (\hat{S}_2 - R_2).$$

Thus, we obtain $R := \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} > 0$. Note that the matrix $R_1 - \hat{S}_1$ is invertible [37].

- Solve $M \in \mathbb{R}^{(n+n_w) \times n_w}$ through the factorization $R - S^{-1} = MUM^T$, where $U \in \mathbb{S}_+^{n_w}$, and define $N := -SMU$ so that M, N satisfy the identity $RS + MN^T = I$. Furthermore, we partition M, N as $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, $N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ so that $M_1, N_1 \in \mathbb{R}^{n \times n}$ are invertible and $M_2, N_2 \in \mathbb{R}^{n_w \times n}$.
- Compute the coefficient matrices $A_{c12}, A_{c22}, B_{c1}, B_{c2}, C_{c2}, D_c$ and J_{c12}, J_{c22} with

$$\begin{aligned} D_c &= \hat{D}_c, \\ C_{c2} &= (\hat{C}_{c2} + D_c(C_{w2} + C_{p2}\Pi_p)R_2^T - \Gamma R_2^T - D_c C_{p2}R_1) M_1^{-T}, \\ \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} &= \begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix}^{-1} \left(W^{-T} \hat{B}_c - \begin{bmatrix} S_1 \\ S_2^T \end{bmatrix} B_{p2} D_c \right), \\ \begin{bmatrix} A_{c12} \\ A_{c22} \end{bmatrix} &= \begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix}^{-1} \left(W^{-T} \hat{A}_{c2} - \begin{bmatrix} S_1 \\ S_2^T \end{bmatrix} B_{p2} C_{c2} M_1^T \right. \\ &\quad \left. - \begin{bmatrix} S_1 \\ S_2^T \end{bmatrix} \left[A_p + B_{p2} D_c C_{p2} \quad B_{p2} (\Gamma - D_c (C_{w2} + C_{p2}\Pi_p)) \right] \begin{bmatrix} R_1 \\ R_2^T \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix} \left(\begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \begin{bmatrix} C_{p2} & -(C_{w2} + C_{p2}\Pi_p) \end{bmatrix} + \begin{bmatrix} 0 & A_e \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} R_1 \\ R_2^T \end{bmatrix} \right) M_1^{-T}, \\ \begin{bmatrix} J_{c12} \\ J_{c22} \end{bmatrix} &= \left(\begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix}^{-1} W^{-T} \left(\hat{J}_{c2} - \hat{S} \begin{bmatrix} J_p R_1 \\ 0 \end{bmatrix} \right) - \begin{bmatrix} J_e \\ 0 \end{bmatrix} R_2^T \right) M_1^{-T}. \end{aligned} \tag{30}$$

- Obtain the realization of the hybrid impulsive controller in the form of (3) via (20).

PROOF. Consider the hybrid impulsive closed-loop system (4) with (23). Based on the analysis results in Theorem 1, the hybrid output regulation conditions (6)–(8) can be verified easily by using the controller parametrization (20) and specifying $\Pi = \begin{bmatrix} \Pi_p^T & I & 0 \end{bmatrix}^T$.

For conditions (9)–(10), we partition the Lyapunov matrix

$$P = \begin{bmatrix} S & N \\ N^T & ? \end{bmatrix} := \begin{bmatrix} S_1 & S_2 & N_1 \\ S_2^T & S_3 & N_2 \\ N_1^T & N_2^T & ? \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} R & M \\ M^T & ? \end{bmatrix} := \begin{bmatrix} R_1 & R_2 & M_1 \\ R_2^T & R_3 & M_2 \\ M_1^T & M_2^T & ? \end{bmatrix}$$

in which $MN^T = I - RS$, and $S_1, R_1 \in \mathbb{S}_+^n$, $S_3, R_3 \in \mathbb{S}_+^{n_w}$, $N_1, M_1 \in \mathbb{R}^{n \times n}$ are invertible, $N_2, M_2 \in \mathbb{R}^{n_w \times n}$, “?” means the elements we don’t care. Furthermore, let

$$Z_1 = \begin{bmatrix} R_1 & I & 0 \\ R_2^T & 0 & I \\ M_1^T & 0 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} I & S_1 & S_2 \\ 0 & S_2^T & S_3 \\ 0 & N_1^T & N_2^T \end{bmatrix}$$

such that $PZ_1 = Z_2$. Note that since M_1 is nonsingular, Z_1 is also nonsingular, which thereby implies the invertibility of Z_2 as P is positive definite. Then, we specify

$$W = \begin{bmatrix} I & \Pi_p \\ 0 & I \end{bmatrix}, \quad \tilde{Z}_1 = \begin{bmatrix} \begin{bmatrix} R_1 \\ R_2^T \\ M_1^T \end{bmatrix} & W \\ 0 & 0 \end{bmatrix}, \quad \tilde{Z}_2 = \begin{bmatrix} I & SW \\ 0 & N^T W \end{bmatrix},$$

which gives $P\tilde{Z}_1 = \tilde{Z}_2$. Applying the congruence transformation on condition (9) with matrices $\text{diag}\{\tilde{Z}_1, I, I\}$, we have

$$\begin{aligned}
\tilde{Z}_1^T P \tilde{Z}_1 &= \begin{bmatrix} R_1 & \begin{bmatrix} I & \Pi_p \end{bmatrix} \\ \begin{bmatrix} I \\ \Pi_p^T \end{bmatrix} & \hat{S} \end{bmatrix}, \\
\tilde{Z}_1^T P A_{cl} \tilde{Z}_1 &= \tilde{Z}_2^T A_{cl} \tilde{Z}_1 \\
&= \begin{bmatrix} A_p R_1 + B_{p2} \hat{C}_{c2} & \begin{bmatrix} I & \Pi_p \end{bmatrix} \begin{bmatrix} A_p & -A_w \\ 0 & A_e \end{bmatrix} + B_{p2} \hat{D}_c \begin{bmatrix} C_{p2} & -C_{w2} \end{bmatrix} \\ \hat{A}_{c2} & \hat{S} \begin{bmatrix} A_p & -A_w \\ 0 & A_e \end{bmatrix} + \hat{B}_c \begin{bmatrix} C_{p2} & -C_{w2} \end{bmatrix} \end{bmatrix}, \\
(B_{cl}^T - Q^T \Pi^T) P \tilde{Z}_1 &= (B_{cl}^T - Q^T \Pi^T) \tilde{Z}_2 \\
&= \begin{bmatrix} B_{p1}^T + D_{p21}^T \hat{D}_c^T B_{p2}^T - Q^T \Pi_p^T & D_{p21}^T \hat{B}_c^T + \begin{bmatrix} B_{p1} \\ -Q \end{bmatrix}^T \hat{S}^T \\ C_{cl} \tilde{Z}_1 = \begin{bmatrix} C_{p1} R_1 + D_{p12} \hat{C}_{c2} & \begin{bmatrix} C_{p1} & -C_{w1} \end{bmatrix} + D_{p12} \hat{D}_c \begin{bmatrix} C_{p2} & -C_{w2} \end{bmatrix} \end{bmatrix}.
\end{aligned}$$

where $\hat{S} = W^T S W$ and

$$\begin{aligned}
\hat{D}_c &= D_c, \\
\hat{C}_{c2} &= D_c C_{p2} R_1 + (\Gamma - D_c(C_{w2} + C_{p2} \Pi_p)) R_2^T + C_{c2} M_1^T, \\
\hat{B}_c &= W^T \left(\begin{bmatrix} S_1 \\ S_2^T \end{bmatrix} B_{p2} D_c + \begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix} \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \right), \\
\hat{A}_{c2} &= W^T \begin{bmatrix} S_1 \\ S_2^T \end{bmatrix} B_{p2} C_{c2} M_1^T + W^T \begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix} \begin{bmatrix} A_{c12} \\ A_{c22} \end{bmatrix} M_1^T \\
&\quad + W^T \begin{bmatrix} S_1 \\ S_2^T \end{bmatrix} \begin{bmatrix} A_p + B_{p2} D_c C_{p2} & B_{p2} (\Gamma - D_c(C_{w2} + C_{p2} \Pi_p)) \end{bmatrix} \begin{bmatrix} R_1 \\ R_2^T \end{bmatrix} \\
&\quad + W^T \begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix} \left(\begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \begin{bmatrix} C_{p2} & -(C_{w2} + C_{p2} \Pi_p) \end{bmatrix} + \begin{bmatrix} 0 & A_e \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} R_1 \\ R_2^T \end{bmatrix}.
\end{aligned} \tag{31}$$

Thus, condition (27) confirms that P is positive definite, and the results obtained above verify the equivalence between conditions (9) and (28).

To verify the boundary condition (10), we perform again the congruence transformation on (10) with another matrix $\text{diag}\{\tilde{Z}_1, \tilde{Z}_1\}$ to obtain

$$\tilde{Z}_1^T P A_I \tilde{Z}_1 = \tilde{Z}_2^T A_I \tilde{Z}_1 = \begin{bmatrix} J_p R_1 & J_p \begin{bmatrix} I & \Pi_p \end{bmatrix} \\ \hat{J}_{c2} & \hat{S} \begin{bmatrix} J_p & -J_w \\ 0 & J_e \end{bmatrix} \end{bmatrix},$$

where

$$\hat{J}_{c2} = \hat{S} \begin{bmatrix} J_p R_1 \\ 0 \end{bmatrix} + W^T \begin{bmatrix} S_2 & N_1 \\ S_3 & N_2 \end{bmatrix} \left(\begin{bmatrix} J_e \\ 0 \end{bmatrix} R_2^T + \begin{bmatrix} J_{c12} \\ J_{c22} \end{bmatrix} M_1^T \right). \tag{32}$$

Consequently, condition (29) can be deduced from condition (10). Furthermore, the hybrid controller formulas (30) can also be established by inverting the relations (31) and (32).

Note that all the design variables appear linearly in the synthesis conditions (24)–(29). The results in Theorem 2 can be posed as an optimization problem of minimizing the weighted $\mathcal{L}_2$ gain γ , which provides a new solution to the hybrid synthesis problem of jointly designing the hybrid controller gain matrices with respect to both flow dynamics and discrete state jumps. Given the average dwell-time parameters (λ_0, μ) , the hybrid control synthesis can

be formulated as the following LMI optimization problem

$$\begin{aligned} & \min_{R_1, \hat{S}, \hat{A}_{c2}, \hat{B}_c, \hat{C}_{c2}, \hat{D}_c, \hat{J}_{c2}, \Pi_p, \Gamma} \gamma \\ & \text{s.t.} \quad (24)–(29). \end{aligned} \quad (33)$$

It should be pointed out that the hybrid output regulation equations (24)–(26) are linear algebraic equations, which are easy to solve. However, they may involve more equation constraints than unknown variables. Thus, in order to guarantee solvability of the hybrid output regulation problem, it may be necessary to impose some structural constraint on the system matrices. Similar issues and arguments can also be seen in [18, 28].

Remark 2. Regarding the necessity of Assumptions A2 and A3 in the Problem Statement section, it is now clear from condition (28) that the stabilizability and detectability requirements in these two assumptions are indeed adequate for solvability of the hybrid output regulation problem.

Remark 3. As mentioned in the introduction section, one of the difficulties in output regulation of hybrid linear impulsive systems is to formulate the associated control synthesis problem in computationally tractable conditions. It is observed that this mainly rises from the boundary condition (10) induced by the impulsive jump dynamics. Therefore, a direct extension of the existing LTI output regulation results (e.g., [1, 12, 8]) to linear impulsive systems would lead to non-convex synthesis conditions, which could be difficult to solve. Such a difficulty has been effectively overcome by introducing an impulsive dynamics into the output regulator structure, which undergoes synchronous jumps with the controlled plant. As a result, a convex solution to the output regulation of linear impulsive systems is derived and expressed in terms of linear matrix equations plus a set of LMIs (as shown in Theorem 2), which can be solved effectively using existing algorithms [31].

Remark 4. Theorem 2 improves the solution in [36] for hybrid control synthesis subject to regulation constraints of linear impulsive systems by adopting a reduced-order controller. Specifically, the hybrid impulsive controller (3) is designed to have its state order equal to the order of the plant plus that of the exosystem, i.e., $n_c = n + n_w$; whereas the controller designed in [36] will have higher order (n_w more states). Owing to the reduction of the controller parameters, the above optimization problem can be solved more efficiently with reduced computational cost especially when the order of the exosystem (n_w) is large. Another important enhancement of Theorem 2 resides in more general settings about both controlled plant and exosystem involving impulsive dynamics. Furthermore, the output regulability issues for both flow and jump dynamics are addressed in the present design framework, whereas the method of [36] only concerns about that of the flow dynamics. Specifically, the regulation condition (26) ensuring jump invariance of the manifold $x_{cl} = \Pi w$ is not considered in [36]. The effect of this important condition will be illustrated in the example section.

5. A Numerical Example

In this section, we will demonstrate the proposed hybrid output regulation approach for linear impulsive systems using the following example. The open loop system under consideration can be written in the form of (1) with the system matrices given by

$$\begin{aligned} A_p &= \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}, \quad A_w = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_{p2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad J_p = \begin{bmatrix} 1.1 & 0 \\ -1.4 & 1 \end{bmatrix}, \\ A_e &= \begin{bmatrix} 0 & -3 & 0 & -1 \\ 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad J_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ C_{p1} &= C_{p2} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_{w1} = \begin{bmatrix} 0 & 0 & -0.02 & -0.03 \end{bmatrix}, \quad D_{p12} = 0.01, \quad D_{p21} = 0.1, \end{aligned} \quad (34)$$

and all the other matrices are zeros with appropriate dimensions. The controlled plant is a second-order linear impulsive system with both states undergoing jumps at each impulse instant, while the exosystem is chosen as an LTI

system with all the eigenvalues located on the imaginary axis to satisfy Assumption A1. The reason of choosing such an LTI exosystem is that the hybrid output regulation approach proposed in our recent work [36] can be applied for comparison. The perturbation $d(t)$ provides a small disturbance to both the exosystem and the plant. The unperturbed exosystem generates a command signal $w(t)$ with multiple frequencies at 2.80rad/sec and 1.07rad/sec , as shown in Fig. 2 where only the two marginally stable states of the exosystem (i.e., $w_3(t)$ and $w_4(t)$) injecting into the plant are displayed.

Figure 2: Exosystem state $w(t)$ with $w(0) = [1, 1, 1, 1]^T$ and $d(t) \equiv 0$.

For time-domain simulations, the impulsive plant (34) is required to operate over the time interval $t \in [0, 50]\text{sec}$ with an impulse time sequence $\{5, 12, 20, 22, 25, 30, 40\}\text{sec}$. The impulse time sequence is deliberately chosen to impose denser impulses over the interval $[20, 30]\text{sec}$, so as to illustrate the effect of the average dwell time. Its associated average dwell-time index can be determined as $\tau_a = 50/7 = 7.1429\text{sec}$, where 7 is the number of impulses occurred during the entire time interval $[0, 50]\text{sec}$. We specify the constant parameters $\lambda_0 = 0.1$ and $\mu = 2$ in the optimization problem (33) for hybrid controller synthesis, such that the average dwell-time constraint (11) is satisfied, i.e., $\tau_a^* = \frac{\ln(\mu)}{\lambda_0} = 6.9315 < \tau_a$. By solving the optimization problem (33) and using the controller construction algorithm in Theorem 2, one can obtain the hybrid regulator in the form of (3). To demonstrate the effectiveness of the proposed control design scheme, results are also compared with those obtained by using the method of [36]. As can be seen in Table 1, the proposed hybrid output regulator in Theorem 2 not only improves the weighted $\mathcal{L}_2$ performance by 7.43%, but also significantly reduces the computational cost with less than one-third optimization variables. Note that

Table 1: Comparison of synthesis results.

	$\mathcal{L}_2$ gain γ	# of optimization variables
Method [36]	0.2194	232
Theorem 2	0.2031	70

in order to overcome the numerical issue associated with ill-conditioning matrices, we have constrained the closed-loop poles of the flow dynamics within a circle $|c + 20| \leq 20$ by using the method of [35]. The resulting hybrid output

regulator gains under the proposed design scheme are given by

$$\begin{aligned}\Pi_p &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0 & 2 & 3 \end{bmatrix}, \\ \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} &= \begin{bmatrix} 0 & -3 & 0 & -1 & 86.0352 & -82.7376 & -140.8864 \\ 3 & 0 & 0 & 0 & 147.5653 & -108.5009 & -255.5186 \\ 0 & 1 & 0 & 0 & -35.4911 & 34.5716 & 58.0928 \\ 0 & 0 & 3 & 0 & -79.4106 & 59.6139 & 137.0371 \\ 0 & 0 & 0 & 0 & -136.3490 & 13.2738 & 237.9372 \\ 0 & 0 & 0 & 0 & 15.9672 & -17.9093 & -22.9103 \\ 0 & 0 & 2 & 3 & -2.7901 & -175.9101 & 75.1105 \end{bmatrix}, \\ J_c &= \begin{bmatrix} 1 & 0 & 0 & 0 & -0.5141 & -1.0150 \\ 0 & 1 & 0 & 0 & -0.8714 & -1.7415 \\ 0 & 0 & 1 & 0 & 0.2120 & 0.4181 \\ 0 & 0 & 0 & 1 & 0.4703 & 0.9415 \\ 0 & 0 & 0 & 0 & 2.2140 & 2.4438 \\ 0 & 0 & 0 & 0 & -0.1455 & 0.7085 \end{bmatrix},\end{aligned}$$

With initial conditions $x_p(0) = 0$, $x_c(0) = 0$, $w(0) = [1 \ 1 \ 1 \ 1]^T$ and a vanishing perturbation input $d(t) = e^{-0.1t} \sin(t)$, we then carry out the time-domain simulations using both synthesized hybrid regulators. The simulation results including the error output $e(t)$ and control input $u(t)$ are plotted in Figure 3, where the red dashed curve corresponds to the method of [36], and the blue solid curve represents the present control scheme in Theorem 2. It is observed that despite the effects of the system state jumps, the persistent exo-signal $w(t)$ as well as the unknown perturbation $d(t)$, both hybrid impulsive regulators are indeed capable to stabilize the overall impulsive system and suppress these effects from the error output $e(t)$. Furthermore, it can be clearly seen that compared to the control scheme of [36], the new hybrid output regulator renders better controlled performance with smaller error output and less control effort. Under the present control scheme, when the perturbation signal $d(t)$ is vanished, the exo-signal $w(t)$ with multiple frequency components is completely rejected in the error output $e(t)$ (i.e., achieving exact output regulation). On the other hand, due to the impulse effects, the error output $e(t)$ with respect to the control method of [36] exhibits a spike phenomenon with abrupt jumps at each impulse time instant, while such a spike phenomenon does not occur under the present control scheme. This observation is consistent with the arguments in Remark 4 about the effect of including the jump invariance regulation condition (26) in the hybrid synthesis. Specifically, it is interesting to note that without taking into account the invariance issue associated with the jump dynamics (as in [36]), the system error output $e(t)$ might not be exactly regulated to zero on the hybrid time-domain axis. The aforementioned spike phenomenon would appear whenever the system experiences state jumps, although it does not affect the overall stability.

6. Conclusions

In this paper, an important problem of hybrid output regulation for linear impulsive systems has been solved by using an average dwell-time control technique. The regulation problem under consideration was generalized for a class of linear impulsive exosystems subject to unknown perturbations, and the hybrid regulator design was treated by considering both objectives of output regulation and weighted $\mathcal{L}_2$ performance. The hybrid regulator was constructed as a linear impulsive system that undergoes synchronous impulses with the controlled plant. Its synthesis conditions were formulated in terms of linear matrix equations plus a set of LMIs, which can be solved efficiently. The proposed hybrid output regulation approach establishes a systematic and unified framework for the joint design problem of hybrid regulator gains, it also provides a new synthesis condition of reduced-order hybrid controllers through convex optimizations.

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Figure 3: Almost output regulation of linear impulsive system.

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