Consensus building in multiperson decision making with heterogeneous preference representation structures: a perspective based on prospect theory

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Abstract—Multiperson decision making (MPDM) models with heterogeneous preference representation structures have initiated by Chiclana et al. In this study, we propose a novel framework for MPDM problems with heterogeneous preference representation structures. This framework takes the decision makers' psychological behaviors into consideration and is based on the prospect theory, which is one of the most influential psychological behavior decision theories. In this framework, the heterogeneous preference representation structures are all transformed into preference orderings. The preference-approval structures are introduced to determine the reference points in prospect theory, and then the prospect value function is used to obtain individual and collective prospect values. Finally, based on the prospect values and preference-approval structures, we propose the two-step feedback adjustment rules, which will adjust both the preference evaluations and the preference-approval information, to help the decision makers reach higher consensus degree. Further, we find that there are two significant observations in this framework: 1) the uses of different reference points will yield different outputs of the selection process in MPDM; 2) compared with only adjusting the preference evaluations in the existing studies, the pace of consensus reaching will be accelerated in adjusting both the preference evaluations and the preference-approval information.

Keywords: multiperson decision making; prospect theory; preference-approval structures; reference points; consensus; heterogeneous.

1. Introduction

Multiperson decision making (MPDM) (or group decision making) can be defined as a situation where a group of decision makers participate in a decision process, provide their opinions regarding the given alternatives, and select the best alternative(s) by the aggregation of their opinions.

In MPDM problems, heterogeneous preference representation structures are generally

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used due to that different decision makers have different education background, experience, cultures and even cognitive degrees. Chiclana et al. [1-4], Dong et al. [5], Herrera et al. [6], and Herrera-Viedma et al. [7] initiated and developed the MPDM models with heterogeneous preference representation structures represented by preference orderings, utility functions, additive preference relations (also called fuzzy preference relations), multiplicative preference relations. Dong et al. [8] proposed a linguistic MPDM model, where the preference information is represented by means of linguistic preference relations, integrating fuzzy preference relations, different types of multiplicative preference relations and multi-granular linguistic preference relations. Further, Dong and Zhang [9] presented a direct consensus framework for the MPDM with heterogeneous preference structures, and the framework satisfies two desire properties: 1) it can avoid internal inconsistency issue and 2) it satisfies Pareto principle of social choice theory.

Driven by trust between experts in a social network environment, Wu et al. [10] proposed a trust based consensus model for the MPDM with incomplete linguistic information.

Similarly, several MPDM models with multi-granularity linguistic evaluation information have been successfully constructed (see [11-17]). Moreover, Fan et al. [18, 19], Ma et al. [20], and Fan and Zhang [21] initiated several optimization-based models to integrate heterogeneous preference representation structures.

The existing methods have made significant contributions to the MPDM problems with heterogeneous preference representation structures. However, the decision makers' psychological behaviors are seldom considered in these methods. A lot of empirical evidences [22-24] have shown that the decision maker's psychological behavior would play an important role in decision analysis. The prospect theory proposed by Kahneman and Tversky [23, 24] is one of the most influential psychological behavior theories and is founded on a large number of empirical and experimental studies. And it incorporates three behavioral principles: reference dependence, loss aversion and diminishing sensitivity. Following that, several scholars have extended prospect theory into multi-attribute decision-making context [25-31] and MPDM context [32, 33], and these studies have shown that decision making based on prospect theory is more consistent with decision maker's behavior.

However, to our knowledge, there exists no research on extending the prospect theory to reaching consensus in MPDM problems. The aim of this study is to propose a novel consensus framework based on prospect theory to solve the MPDM problems with heterogeneous preference representation structures. This framework takes the decision makers' psychological behaviors into consideration. In the framework of this study, all the heterogeneous preference representation structures are transformed into preference

orderings. And each decision maker's reference point over positional order is determined by his/her providing preference-approval information [34]. Then, the gains and losses over alternatives for different decision makers are calculated. Further, by substituting these gains and losses into the prospect value function proposed by Kahneman and Tversky [23], the individual and collective prospect values are obtained. Finally, we propose the two-step feedback adjustment rules, which will adjust both the preference evaluations and the preference-approval information to help the decision makers reach the higher consensus degree. Further, we find that there are two significant observations in this framework: 1) the uses of different reference points will yield different outputs of the selection process in MPDM; 2) compared with only adjusting the preference evaluations in the existing studies, the pace of consensus reaching will be accelerated in adjusting both the preference evaluations and the preference-approval information.

The rest of this study is arranged as follows. Section 2 introduces the preliminaries regarding four kinds of heterogeneous preference representation structures (preference ordering, utility functions, additive preference relation and multiplicative preference relation), *OWA* operator, prospect theory and preference-approval structures. Section 3 presents a consensus framework based on prospect theory for MPDM with heterogeneous preference representation structures. In Sections 4 and 5, the selection process and the consensus process are proposed based on prospect theory, respectively. The illustrative examples and discussions are presented in Section 6. Finally, conclusions are given in Section 7.

2. Preliminaries

This section introduces the basic knowledge associated with four kinds of preference representation structures, *OWA* operator, prospect theory and preference-approval structures.

2.1. Preference representation structures

Let $E = \{e_1, e_2, \dots, e_m\}, (m \geq 2)$ be the set of the decision makers, $X = \{x_1, x_2, \dots, x_n\}, (n \geq 2)$ be a finite set of alternatives, where e_k and x_i denote the k th decision maker and i th alternative. These alternatives have to be classified from best to worst, according to the preference information given by the set of the decision makers.

In this study, the decision maker's preference information on the set of X is assumed to be represented in one of the following four formats.

(1) Preference orderings of [35]. The decision maker e_k directly provides his/her individual preference information by means of a preference ordering $O^k = (o_1^k, o_2^k, \dots, o_n^k)$,

where o_i^k indicates the positional order of alternative x_i in X .

(2) Utility functions [36]. The decision maker e_k provides his/her individual preference information by means of a utility function vector $U^k = (u_1^k, u_2^k, \dots, u_n^k)$, where $u_i^k \in [0,1]$ denotes the utility evaluation value given by decision maker e_k to the alternative x_i .

(3) Additive preference relations (also called fuzzy preference relations) [37, 38]. The decision maker e_k 's individual preference information is described by a matrix $P^k = (p_{ij}^k)_{n \times n}$, where $p_{ij}^k \in [0,1]$ denotes the preference degree of the alternative x_i over x_j . An additive preference relation usually assumed to be the additive reciprocal, i.e. $p_{ij}^k + p_{ji}^k = 1, \forall i, j$.

(4) Multiplicative preference relations [39]. The decision maker e_k expresses his/her individual preference information as a matrix $A^k = (a_{ij}^k)_{n \times n}$, where a_{ij}^k indicates a ratio of the preference intensity of alternative x_i with respect to x_j . A multiplicative preference relation usually assumed to be the multiplicative reciprocal, i.e. $a_{ij}^k \times a_{ji}^k = 1, \forall i, j$.

2.2. Ordered weighted averaging (OWA) operator

Let $\psi = \{a_1, a_2, \dots, a_n\}$ be a set of values to aggregate. An OWA operator of dimension n is a mapping $OWA: \psi^n \rightarrow \psi$ that has an associated weighting vector

$W = (w_1, w_2, \dots, w_n)^T$ of dimension n with $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, such that

$$OWA(a_1, a_2, \dots, a_n) = \sum_{i=1}^n w_i b_i \quad (1)$$

where b_i is the i th largest value in $\{a_1, a_2, \dots, a_n\}$.

As seen from Eq. (1), the associated weight vector has the importance influence on the rationale of aggregated result. Therefore, determining the appropriate associated weight vector is necessary in implementing OWA operator. To do this, Yager [40] proposed an effective way to calculate the value of w_i as the following expression:

$$w_i = Q(i/n) - Q((i-1)/n) \quad (2)$$

where $Q(r)$ can be represented as the expression [41]:

$$Q(r) = \begin{cases} 0, & r < a \\ \frac{r-a}{b-a}, & a \leq r \leq b \\ 1, & r > b \end{cases} \quad (3)$$

with $a, b, r \in [0, 1]$.

The most representative linguistic quantifiers such as most, at least half and as many as possible can be also defined by the coefficients (a, b) , which correspond to $(0.3, 0.8)$, $(0, 0.5)$, and $(0.5, 1)$, respectively. When a linguistic quantifier Q is used to calculate the weights of OWA operator, it is symbolized by OWA_Q .

2.3. Prospect theory

The prospect theory has been initiated by Kahneman and Tversky [23], and it is a descriptive theory for forecasting individual actual decision behavior under risk [26]. Since the appearance of prospect theory, it has been regarded as one of the most influential behavioral decision theories.

Generally, the prospect theory is comprised of two modules: the editing phase and the evaluation phase. In the editing phase, outcomes of alternatives are coded as gains or losses relative to a reference point. If the value of the alternative is larger than the reference point, the part in excess can be regarded as ‘gain’; otherwise, the corresponding part can be regarded as ‘loss’. In the evaluation phase, the prospect values are evaluated by a prospect value function and a weighting function, and then the alternative with the highest prospect value is selected. The prospect theory involves three important principles [23]:

(1) Reference dependence. The decision makers perceive the gains and losses according to a reference point. Thus, the prospect value function can be divided into the gain domain and the loss domain relative to the reference point.

(2) Diminishing sensitivity. The decision makers exhibit risk-averse tendency for gains and risk-seeking tendency for losses. According to the principle of diminishing sensitivity, the prospect value function is concave in the loss domain and convex in the gain domain, i.e., the marginal value of both gains and losses is decreasing with the size.

(3) Loss aversion. The decision makers are more sensitive to losses than to equal gains [42]. In accordance with the principle of loss aversion, the prospect value function is steeper in the loss domain than in the gain domain.

In accordance with the above three principles, an S-shaped value function has been proposed in the prospect theory (see Fig.1.). In Fig.1, the original point is the reference point, x denotes the gain ($x \geq 0$) or the loss ($x < 0$) of the outcome compared with the reference point. The prospect value function $v(x)$ is given by Kahneman and Tversky

[23], and $v(x)$ denotes the function value for gain ($x \geq 0$) or loss ($x < 0$):

$$v(x) = \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda(-x)^\beta, & x < 0 \end{cases} \quad (4)$$

Fig.1. The prospect value function

In this value function, α and β are the two parameters which determine the concavity and convexity of the function, respectively, and $0 \leq \alpha, \beta \leq 1$. λ is the coefficient of loss aversion, $\lambda > 1$.

2.4. The preference-approval structures

In order to determine the reference points, the preference-approval structures proposed by Brams and Sanver [34] are used in our study. Some collective decision making models [43-45] have assumed that the decision makers take a common language when they evaluate alternatives, and these models aggregate labels such as approved and disapproved. Brams and Sanver [34] suggested a framework that could be considered as a compromise between standard and non-standard models by combining the information of ranking and approval in a hybrid system which they call preference-approval. The preference-approval structure can be expressed as the following example:

$$a \ b \mid c \ d$$

The alternatives on the left of the cut-off line are approved by the decision maker, and on the right are disapproved. That is to say, a and b are approved, and c and d are disapproved in this structure.

3. Proposed framework

In this section, we propose a novel MPDM resolution framework with heterogeneous preference representation structures based on prospect theory.

Let X and E be as earlier. Let E^O , E^U , E^P and E^A be four subsets of E , representing decision makers whose preference information on X is expressed as preference orderings (O), utility functions (U), additive preference relations (P) and

multiplicative preference relations (A), respectively, thus $E^O \cup E^U \cup E^P \cup E^A = E$. Without loss of generality, $E^O = \{e_1, e_2, \dots, e_{l_1}\}$, $E^U = \{e_{l_1+1}, e_{l_1+2}, \dots, e_{l_2}\}$, $E^P = \{e_{l_2+1}, e_{l_2+2}, \dots, e_{l_3}\}$ and $E^A = \{e_{l_3+1}, e_{l_3+2}, \dots, e_m\}$.

Solving MPDM problems follows a common resolution scheme composed of two different processes (or models) [7], [15]: 1) selection process and 2) consensus process. The implementation of the novel MPDM framework based on prospect theory also deals with this two-process procedure. The details of this framework are described in Fig. 2.

Fig.2. The MPDM framework based prospect theory

(1) The selection process

The use of the selection process yields the collective preference ordering. The selection process in our framework consists of a four-step procedure.

Step 1: Obtaining the individual preference orderings

In this step, the individual selection methods, associated with heterogeneous preference structures, are used to obtain individual preference orderings.

Step 2: Determining the reference points

Considering the individual preference-approval information provided by the decision makers, and combining the obtained individual preference orderings, each decision maker's reference point over the positional order can be determined.

Step 3: Calculating the gains and losses

In this step, the gains and losses for the positional orders of each alternative over the reference points are calculated. And a gain matrix and a loss matrix can be further obtained.

Step 4: Calculating the prospect values of the alternatives and obtaining the selection outcome

By substituting the gains and losses into the prospect value function developed by Tversky and Kahneman [24], the individual prospect values of different alternatives are calculated. Further, the collective prospect values of each alternative are obtained by using simple additive weighting method. Finally, a collective ordering is obtained.

The details of this selection process are introduced in Section 4.

(2) The consensus process

The purpose of the consensus process is to improve the consensus degree among all the decision makers. The consensus process consists of consensus measure and feedback adjustment.

Step 1: Consensus measure

In this step, a consensus measure method based on the prospect values is proposed to measure the consensus degree among the decision makers.

Step 2: Feedback adjustment

When the consensus degree among the decision makers does not reach the acceptable level, the opinions of them have to be adjusted. Feedback adjustments rules are introduced to provide the support for the decision makers adjusting their preference evaluations and preference-approval information.

The details of the consensus process are designed in Section 5.

Note 1: The consistency test of the preference relations guarantees that the preference relations are neither random nor illogic [39] [46] [48]. The consistency should be sought in the consensus reaching process, and Chiclana et al. [77] and Dong et al. [63] discussed this issue in detail.

4. The selection process

This section designs the selection process based on prospect theory and preference-approval structures.

4.1. Obtaining the individual preference orderings

Let $O^k = (o_1^k, o_2^k, \dots, o_n^k)$ ($k = 1, 2, \dots, m$) be individual preference orderings, associated the decision maker e_k . In order to obtain the preference orderings O^k , four cases are considered.

Case 1: $e_k \in E^U$

In this case, the decision maker e_k provides his/her preference information on X

by means of the utility function vector $U^k = (u_1^k, u_2^k, \dots, u_n^k)$. Considering that larger value of u_i^k indicates the smaller positional order o_i^k . Therefore, the preference ordering O^k of the alternatives from best to worst is obtained by using utility function.

Case 2: $e_k \in E^O$

In this case, the decision maker e_k provides his/her preference information on X by means of preference ordering. Therefore, the preference orderings O^k is a direct result of decision maker e_k 's preference information over X .

Case 3: $e_k \in E^P$

In this case, the decision maker e_k provides his/her preference information on X by means of the additive preference relations $P^k = (p_{ij}^k)_{n \times n}$. For the alternative x_i , the quantifier-guided dominance degree (QGDD) [1], is used to quantify the dominance that the alternative x_i has over all the others in an linguistic majority sense Q^k as follows:

$$QGDD_i^k = OWA_{Q^k}(p_{i1}^k, p_{i2}^k, \dots, p_{in}^k), \quad k = l_2 + 1, l_2 + 2, \dots, l, \quad i = 1, 2, \dots, n. \quad (5)$$

In Eq. (5), the larger value of $QGDD_i^k$ indicates the smaller positional order o_i^k . Therefore, we can obtain the preference ordering O^k according to $QGDD_i^k$.

Case 4: $e_k \in E^A$

In this case, the decision maker provides his/her preference information on X by the multiplicative preference relations $A^k = (a_{ij}^k)_{n \times n}$. Here, we use the row geometric mean method (RGMM) proposed by Crawford and Williams [46] to obtain the individual preference vector $w^k = (w_1^k, w_2^k, \dots, w_n^k)^T$. The RGMM method can be described as the following model:

$$\begin{aligned} \min & \sum_{i=1}^n \sum_{j>i}^n [\ln(a_{ij}^k) - (\ln(w_i^k) + \ln(w_j^k))]^2 \\ \text{s.t.} & \begin{cases} \sum_{i=1}^n w_i^k = 1 \\ w_i^k \geq 0 \end{cases} \end{aligned} \quad (6)$$

The unique optimal solution of model (6) can be simply found as the geometric means of the rows of matrix A^k :

$$w_i^k = \frac{(\prod_{j=1}^n a_{ij}^k)^{1/n}}{\sum_{i=1}^n (\prod_{j=1}^n a_{ij}^k)^{1/n}} \quad k = l_3 + 1, l_3 + 2, \dots, l, \quad i = 1, 2, \dots, n. \quad (7)$$

In Eq. (7), the larger value of w_i^k indicates the smaller positional order o_i^k .

Therefore, we can obtain the preference ordering O^k according to w_i^k .

4.2. Determining the reference points

Let the preference-approval structure provided by e_k be expressed as (8):

$$A^k(X) = x_{\sigma(1)}^k, x_{\sigma(2)}^k, \dots, x_{\sigma(l)}^k \mid x_{\sigma(l+1)}^k, x_{\sigma(l+2)}^k, \dots, x_{\sigma(n)}^k \quad (8)$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$, and $x_{\sigma(i)}^k \in X$, $x_{\sigma(i)}^k \succ x_{\sigma(i+1)}^k$, i.e., if x_i is the j th largest based on the preference ordering of e_k , then $x_{\sigma(j)}^k = x_i^k$.

Then, based on the preference-approval structure $A^k(X)$, we can determine the reference point of the decision maker e_k . Under the situation of considering the decision maker e_k 's reference dependence behavior, if one alternative is approved, then decision maker e_k regards this alternative as gain; Otherwise, this alternative is thought as loss by decision maker e_k .

In Eq. (8), the last approved alternative is $x_{\sigma(l)}^k$ and the first disapproved alternative is $x_{\sigma(l+1)}^k$. Let o^{k*} be decision maker e_k 's reference point on preference ordering. In this study, o^{k*} is given by

$$o^{k*} = \frac{2l+1}{2}, \quad k=1, 2, \dots, m. \quad (9)$$

4.3. Calculating the gains and losses

Let G_i^k and L_i^k be the gain and loss of the alternative x_i associated with the decision maker e_k , i.e.,

$$G_i^k = -(o_i^k - o^{k*}), \quad o_i^k < o^{k*} \quad (10)$$

$$L_i^k = -(o_i^k - o^{k*}), \quad o_i^k \geq o^{k*} \quad (11)$$

In Eqs. (10)-(11), if $o_i^k < o^{k*}$, then the decision maker e_k 's gain on alternative x_i can be regarded as $o^{k*} - o_i^k$; if $o_i^k \geq o^{k*}$, then the decision maker e_k 's loss on alternative x_i can be regarded as $o_i^k - o^{k*}$. Let $G = [G_i^k]_{m \times n}$ and $L = [L_i^k]_{m \times n}$ be the gain and loss matrices.

4.4. Calculating the prospect values and obtaining the selection outcome

In this subsection, we propose a novel selection outcome according to prospect

values.

Let v_i^k be the prospect value of the positional order of alternative x_i over the reference point o^{k*} . Based on the idea of classical prospect theory, v_i^k is calculated by

$$v_i^k = \begin{cases} (G_i^k)^{\alpha_k} & o_i^k < o^{k*} \\ -\lambda_k (L_i^k)^{\beta_k} & o_i^k \geq o^{k*} \end{cases} \quad k=1, 2, \dots, m, \quad i=1, 2, \dots, n \quad (12)$$

where the parameter α_k ($0 \leq \alpha_k \leq 1$) represents decision maker e_k 's sensitive degree on gain, the parameter β_k ($0 \leq \beta_k \leq 1$) represents decision maker e_k 's sensitive degree on loss and the parameter $\lambda_k > 1$ represents decision maker e_k 's loss aversion degree. As for determining the three parameter values, some scholars have conducted the experiments (see in [42, 48-52]). Some available parameters are then determined. It is worth pointing out that the most influential and widely used available parameter values is the result determined by Tversky and Kahneman [24]. They suggest $\alpha_k = \beta_k = 0.88$ and $\lambda_k = 2.25$.

On the basis of obtaining the prospect values $v_i^1, v_i^2, \dots, v_i^m$, the collective prospect value v_i^c of the alternative x_i can be determined by

$$v_i^c = \frac{1}{m} \sum_{k=1}^m v_i^k, \quad i=1, 2, \dots, n \quad (13)$$

Obviously, the higher value of v_i^c is, the alternative x_i is ranking higher. Therefore, a ranking order of all alternatives is obtained and a collective prospect values vector is obtained.

Due to the special structure of the prospect value function, the approved alternatives' prospect values are always non-negative while the disapproved alternatives' are negative. Based on this rule, the collective preference-approval structure is easily obtained according to the collective prospect values. For example, the collective prospect values for a set of six alternatives are $v_1^c = -0.5$, $v_2^c = 1$, $v_3^c = 3$, $v_4^c = -3$, $v_5^c = 2$, $v_6^c = -1$, then the collective preference ordering is $x_3 \succ x_5 \succ x_2 \succ x_1 \succ x_6 \succ x_4$, and the collective preference-approval structure is $A^c(X) = x_3, x_5, x_2 | x_1, x_6, x_4$.

5. The consensus process

Classically, consensus is defined as the full and unanimous agreement of all decision makers regarding all possible alternatives. However, this definition is inconvenient and complete agreement is not always necessary in real life. This has led to the use of "soft" consensus (i.e. consensus measures) [53-55].

A lot of scholars [56-64] have investigated the consensus process for MPDM

problems, based on the use of “soft” consensus. More recent contributions can be found to analyze consensus from different approaches (see in [65-74]). Cabrierzo et al. [75], Herrera-Viedma et al. [66] and Palomares et al. [76] provided excellent surveys of “soft” consensus models.

This section proposes the consensus process in the MPDM framework based on prospect theory. The proposed consensus process deals with consensus measure and feedback adjustment.

5.1. Consensus measure

In general, the consensus degree for MPDM problems is calculated as the distance between individual preference values and collective preference values [77][78].

In this study, we define the prospect value consensus degree (*PVCD*) by measuring the distance between the individual prospect values and the collective prospect values, as Definition 1.

Definition 1. Let v_i^c and v_i^k be as earlier. The consensus degree of the decision maker e_k is defined by

$$PVCD(e^k) = \sqrt{\frac{1}{n} \sum_{i=1}^n \left(\frac{v_i^k - v_i^c}{v(n) - v(-n)} \right)^2} \quad k = 1, 2, \dots, m, \quad (14)$$

The consensus degree among all decision makers is

$$PVCD = 1 - \frac{1}{m} \sum_{k=1}^m PVCD(e_k) \quad (15)$$

The larger *PVCD* value shows the higher consensus degree among all the decision makers. If *PVCD*=1, that means the full and unanimous agreement of all decision makers.

5.2. Feedback adjustment

When the consensus degree *PVCD* does not reach the acceptable level, the opinions of the decision makers must be adjusted. The feedback adjustment aims at giving adjustment suggestions to help the decision makers improve the consensus degree. In this study, we propose a two-step feedback adjustment process which will adjust both the preference evaluations and the preference-approval information. The rules of this feedback adjustment can be described as follows.

(1) Adjustment of the preference evaluations

Similar to Herrera-videma et al. [7], the opinions will be changed using the following

three rules:

R.1. If $v_i^k > v_i^c$, then the decision maker e_k should decrease the preference evaluation of x_i ;

R.2. If $v_i^k = v_i^c$, then the decision maker e_k should not change the preference evaluation of x_i ;

R.3. If $v_i^k < v_i^c$, then the decision maker e_k should increase the preference evaluation of x_i ;

Particularly, according to different formats of preference representation structures, four cases need to be considered.

Case 1: $e_k \in E^U$

Let $U^{k*} = (u_1^{k*}, u_2^{k*}, \dots, u_n^{k*})$ be the utility function vector provided by decision maker e_k at the new round. For any one alternative x_i , the following conditions are concerned.

- (a) If $v_i^k > v_i^c$, then $u_i^{k*} < u_i^k$;
- (b) If $v_i^k = v_i^c$, then $u_i^{k*} = u_i^k$;
- (c) If $v_i^k < v_i^c$, then $u_i^{k*} > u_i^k$.

Case 2: $e_k \in E^O$

Let $O^{k*} = (o_1^{k*}, o_2^{k*}, \dots, o_n^{k*})$ be the preference ordering vector provided by decision maker e_k at the new round. For any one alternative x_i , the following conditions are concerned.

- (a) If $v_i^k > v_i^c$, then $o_i^{k*} > o_i^k$;
- (b) If $v_i^k = v_i^c$, then $o_i^{k*} = o_i^k$;
- (c) If $v_i^k < v_i^c$, then $o_i^{k*} < o_i^k$.

Case 3: $e_k \in E^P$

Let $P^{k*} = (p_{ij}^{k*})_{n \times n}$ be the additive preference relations provided by decision maker e_k at the new round. For any one alternative x_i , the following conditions are concerned.

- (a) If $v_i^k > v_i^c$, then $p_{ij}^{k*} \leq p_{ij}^k$, $j = i+1, i+2, \dots, n$, $p_{ji}^{k*} = 1 - p_{ij}^k$ and $\exists l \in \{i+1, i+2, \dots, n\}$, such that $p_{il}^{k*} < p_{il}^k$;
- (b) If $v_i^k = v_i^c$, then $p_{ij}^{k*} = p_{ij}^k$, $j = i+1, i+2, \dots, n$.

- (c) If $v_i^k < v_i^c$, then $p_{ij}^{k*} \geq p_{ij}^k$, $j = i+1, i+2, \dots, n$, $p_{ji}^{k*} = 1 - p_{ij}^k$ and $\exists l \in \{i+1, i+2, \dots, n\}$, such that $p_{il}^{k*} > p_{il}^k$.

Case 4: $e_k \in E^A$

Let $A^* = (a_{ij}^*)_{n \times n}$ be the multiplicative preference relations provided by decision maker e_k at the new round, $t \in N$ and $t \geq 1$.

For any one alternative x_i , the following conditions are concerned.

- (a) If $v_i^k > v_i^c$, then $a_{ij}^{k*} \leq a_{ij}^k$, $j = i+1, i+2, \dots, n$, $a_{ji}^{k*} = 1/a_{ij}^{k*}$ and $\exists l \in \{i+1, i+2, \dots, n\}$, such that $a_{il}^{k*} < a_{il}^k$;
- (b) If $v_i^k = v_i^c$, then $a_{ij}^{k*} = a_{ij}^k$, $j = i+1, i+2, \dots, n$.
- (c) If $v_i^k < v_i^c$, then $a_{ij}^{k*} \geq a_{ij}^k$, $j = i+1, i+2, \dots, n$, $a_{ji}^{k*} = 1/a_{ij}^{k*}$ and $\exists l \in \{i+1, i+2, \dots, n\}$, such that $a_{il}^{k*} > a_{il}^k$;

(2) Adjustment of the preference-approval information

Secondly, the decision makers need to adjust their preference-approval information to update the reference points. The rules are easy to apply:

R.1. If $v_i^c \geq 0$, then put the alternative x_i into the approved set;

R.2. If $v_i^c < 0$, then put the alternative x_i into the disapproved set;

After the adjustments for preference evaluations and preference-approval, we use the selection process in Section 4 to obtain the new preference orderings. The processes will carry on until the consensus degree reaches the acceptable level.

6. Numerical examples

In this section, we use two examples to demonstrate the proposed novel consensus framework. Further, we propose some discussions by two experiments.

6.1. Example 1

Let $E = \{e_1, e_2, \dots, e_6\}$ be the set of six decision makers, and let $X = \{x_1, x_2, \dots, x_4\}$ be the set of four alternatives. Specifically, the decision makers e_1 and e_2 provide their preference information by means of preference orderings O^1 and O^2 . The decision makers e_3 and e_4 provide their preference information by means of utility values U^3 and U^4 . The decision maker e_5 provides his preference information by means of additive preference relations P^5 . The decision maker e_6 provides his preference

information by means of multiplicative preference relations A^6 . Their preferences are as follows:

$$e_1: O^1 = (3, 1, 4, 2),$$

$$e_2: O^2 = (3, 2, 1, 4)$$

$$e_3: U^3 = (0.5, 0.7, 1, 0.1)$$

$$e_4: U^4 = (0.7, 0.9, 0.6, 0.3)$$

$$e_5: P^5 = \begin{bmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.7 & 0.4 \\ 0.4 & 0.3 & 0.5 & 0.9 \\ 0.3 & 0.6 & 0.1 & 0.5 \end{bmatrix}$$

$$e_6: A^6 = \begin{bmatrix} 1 & 5 & 1/3 & 4 \\ 1/5 & 1 & 2 & 6 \\ 3 & 1/2 & 1 & 1/8 \\ 1/4 & 1/6 & 8 & 1 \end{bmatrix}$$

(1) Selection process

Step 1: Obtaining individual preference orderings

Unify the heterogeneous preference structures into preference orderings by the individual selection methods.

Since $e_3, e_4 \in E^U$, the preference orderings for e_3 and e_4 are

$$e_3: O^3 = (3, 2, 1, 4), \quad e_4: O^4 = (2, 1, 3, 4)$$

Since $e_5 \in E^P$, based on Eqs. (1)-(3) and (5) with linguistic quantifier “as many as possible” (i.e., $Q^k = (0.5, 1)$), the values of $QGDD_i^5$ can be then calculated as follows.

$$QGDD^5 = (0.3, 0.45, 0.35, 0.2)$$

Therefore, the preference orderings for e_5 is

$$e_5: O^5 = (3, 1, 2, 4)$$

Since $e_6 \in E^A$, according to Eq. (7), the weighting vectors $(w_1^6, w_2^6, w_3^6, w_4^6)$ can be calculated. By comparing the values of $(w_1^6, w_2^6, w_3^6, w_4^6)$, the individual preference orderings for e_6 is obtained as:

$$e_6: O^6 = (1, 2, 4, 3)$$

Step 2: Determining the reference points

Based on the obtained individual preference orderings, the decision-makers provide their preference-approval structures as follows:

$$e_1: A^1(X) = x_2 | x_4, x_1, x_3 \quad e_2: A^2(X) = x_3, x_2 | x_1, x_4$$

$$e_3: A^3(X) = x_3, x_2, x_1 | x_4, \quad e_4: A^4(X) = x_2, x_1 | x_3, x_4$$

$$e_5: A^5(X) = x_2, x_3, x_1 | x_4, \quad e_6: A(X^6) = x_1, x_2, x_4 | x_3$$

Based on the Eq. (9), each decision maker's reference point over positional order is

obtained as:

$$o^{1*} = 1.5, \quad o^{2*} = 2.5, \quad o^{3*} = 3.5, \quad o^{4*} = 2.5, \quad o^{5*} = 3.5, \quad o^{6*} = 3.5$$

Step 3: Calculating the gains and losses

After obtaining the reference points, according to the Eqs. (10)-(11), the gain matrix $G = [G_i^k]_{6 \times 4}$ and loss matrix $L = [L_i^k]_{6 \times 4}$ are obtained as follows:

$$G = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1.5 & 0 \\ 0.5 & 1.5 & 2.5 & 0 \\ 0.5 & 1.5 & 0 & 0 \\ 0.5 & 2.5 & 1.5 & 0 \\ 2.5 & 1.5 & 0 & 0.5 \end{bmatrix} \quad L = \begin{bmatrix} -1.5 & 0 & -2.5 & -0.5 \\ -0.5 & 0 & 0 & -1.5 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & -0.5 & -1.5 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & -0.5 & 0 \end{bmatrix}$$

Step 4: Calculating the prospect values and ranking the alternatives

Assume all the decision makers set the parameters α , β and λ as the experiment result conducted by Kahneman and Tversky [23], i.e., $\alpha = \beta = 0.88$, $\lambda = 2.25$.

Then, based on the Eq. (12), let $V = (v_i^k)_{6 \times 4}$ be the prospect values matrix:

$$V = \begin{bmatrix} -3.21472 & 0.543367 & -5.03929 & -1.22258 \\ -1.22258 & 0.543367 & 1.428763 & -3.21472 \\ 0.543367 & 1.428763 & 2.239686 & -1.22258 \\ 0.543367 & 1.428763 & -1.22258 & -3.21472 \\ 0.543367 & 2.239686 & 1.428763 & -1.22258 \\ 2.239686 & 1.428763 & -1.22258 & 0.543367 \end{bmatrix}$$

Further, by Eq. (13), the collective prospect values of the alternatives x_1, x_2, x_3, x_4 are obtained, respectively.

$$v_1^c = -0.094585 \quad v_2^c = 1.26878 \quad v_3^c = -0.39789 \quad v_4^c = -1.5923$$

Therefore, the collective preference ordering of the four alternatives is obtained:

$$x_2 \succ x_1 \succ x_3 \succ x_4$$

Then, the collective preference-approval structure is obtained:

$$A^c(X) = x_2 | x_1, x_3, x_4$$

(2) Consensus process

The consensus process deals with a two-step procedure: consensus measure and feedback adjustment.

Step 1: Consensus measure

By Eqs. (14)-(15), the consensus degree among all the decision makers are:

$$PVCD=0.8944$$

Without loss of generality, we set that the acceptable consensus degree is 0.93. Because $PVCD=0.8944 < 0.93$, we use the feedback adjustment to improve the consensus level.

Step 2: Feedback adjustment

(1) Adjustments of preference evaluations

The decision makers adjust their preference evaluations according to the adjustment rules proposed in Section 5.2. Their new preference representation structures are as follows:

$$\begin{aligned} e_1: O^1 &= (2, 1, 3, 4), & e_2: O^2 &= (2, 1, 4, 3) \\ e_3: U^3 &= (0.4, 0.7, 0.6, 0.1) & e_4: U^4 &= (0.7, 0.8, 0.6, 0.4) \\ e_5: P^5 &= \begin{bmatrix} 0.5 & 0.2 & 0.8 & 0.7 \\ 0.8 & 0.5 & 0.8 & 0.3 \\ 0.2 & 0.2 & 0.5 & 0.8 \\ 0.3 & 0.7 & 0.2 & 0.5 \end{bmatrix} & e_6: A^6 &= \begin{bmatrix} 1 & 1/3 & 1/6 & 4 \\ 3 & 1 & 1 & 7 \\ 6 & 1 & 1 & 1/2 \\ 1/4 & 1/7 & 2 & 1 \end{bmatrix} \end{aligned}$$

The new individual orderings of each decision maker are calculated as follows.

$$\begin{aligned} e_1: O^1 &= (2, 1, 3, 4), & e_2: O^2 &= (2, 1, 4, 3) \\ e_3: O^3 &= (3, 1, 2, 4), & e_4: O^4 &= (2, 1, 3, 4) \\ e_5: O^5 &= (2, 1, 4, 3), & e_6: O^6 &= (3, 1, 2, 4) \end{aligned}$$

(2) Adjustments of the preference-approval structures

According to the rules proposed in Section 5.2, the decision makers adjust their information to obtain new preference-approval structures as:

$$\begin{aligned} e_1: A^1(X) &= x_2 | x_1, x_3, x_4, & e_2: A^2(X) &= x_2 | x_1, x_4, x_3 \\ e_3: A^3(X) &= x_2 | x_3, x_1, x_4, & e_4: A^4(X) &= x_2 | x_1, x_3, x_4 \\ e_5: A^5(X) &= x_2 | x_1, x_4, x_3, & e_6: A^6(X) &= x_2 | x_3, x_1, x_4 \end{aligned}$$

Based on the Eq. (9), the updated reference points for all the decision makers are as follows.

$$o^{1*} = 1.5, \quad o^{2*} = 1.5, \quad o^{3*} = 1.5, \quad o^{4*} = 1.5, \quad o^{5*} = 1.5, \quad o^{6*} = 1.5$$

According to the Eqs. (10)-(11), the new gain matrix $G^* = (g_i^k)_{6 \times 4}$ and loss matrix $L^* = (l_i^k)_{6 \times 4}$ are obtained as follows:

$$G^* = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix} \quad L^* = \begin{bmatrix} -0.5 & 0 & -1.5 & -2.5 \\ -1.5 & 0 & -2.5 & -1.5 \\ -1.5 & 0 & -0.5 & -2.5 \\ -0.5 & 0 & -1.5 & -2.5 \\ -0.5 & 0 & -2.5 & -1.5 \\ -1.5 & 0 & -0.5 & -2.5 \end{bmatrix}$$

By Eqs. (14)-(15), the new consensus degree among all the decision makers is obtained as follows:

$$PVCD = 0.9332$$

Comparing with the previous consensus degree 0.8944, the consensus degree among the decision makers is obviously improved. The final ordering of the four alternatives is $x_2 \succ x_1 \succ x_3 \succ x_4$.

6.2. Example 2

Let us consider a more complex example presented in [7]. In this example, there are eight decision makers $E = \{e_1, e_2, \dots, e_8\}$ and six alternatives $X = \{x_1, x_2, \dots, x_6\}$.

Specifically, the decision makers e_1 and e_2 provide their opinions by means of preference orderings O^1 and O^2 . The decision makers e_3 and e_4 provide their preference information by means of utility functions U^3 and U^4 . The decision makers e_5 and e_6 provide their preference information by means of additive preference relations P^5 and P^6 ; The decision makers e_7 and e_8 provide their preference information by means of multiplicative preference relations A^7 and A^8 . Their preferences are as follows:

$$e_1: O^1 = (2, 1, 3, 6, 4, 5),$$

$$e_2: O^2 = (1, 3, 4, 2, 6, 5)$$

$$e_3: U^3 = (0.3, 0.2, 0.8, 0.6, 0.4, 0.1)$$

$$e_4: U^4 = (0.3, 0.9, 0.4, 0.2, 0.7, 0.5)$$

$$e_5: P^5 = \begin{bmatrix} 0.5 & 0.55 & 0.45 & 0.25 & 0.7 & 0.3 \\ 0.45 & 0.5 & 0.7 & 0.85 & 0.4 & 0.8 \\ 0.55 & 0.3 & 0.5 & 0.65 & 0.7 & 0.6 \\ 0.75 & 0.15 & 0.35 & 0.5 & 0.95 & 0.6 \\ 0.3 & 0.6 & 0.3 & 0.05 & 0.5 & 0.85 \\ 0.7 & 0.2 & 0.4 & 0.4 & 0.15 & 0.5 \end{bmatrix}$$

$$e_6: P^6 = \begin{bmatrix} 0.5 & 0.7 & 0.75 & 0.95 & 0.6 & 0.85 \\ 0.3 & 0.5 & 0.55 & 0.8 & 0.4 & 0.65 \\ 0.25 & 0.45 & 0.5 & 0.7 & 0.6 & 0.45 \\ 0.05 & 0.2 & 0.3 & 0.5 & 0.85 & 0.4 \\ 0.4 & 0.6 & 0.4 & 0.15 & 0.5 & 0.75 \\ 0.15 & 0.35 & 0.55 & 0.6 & 0.25 & 0.5 \end{bmatrix}$$

$$e_7: A^7 = \begin{bmatrix} 1 & 1/2 & 1/3 & 4 & 3 & 5 \\ 2 & 1 & 1/3 & 1/4 & 4 & 6 \\ 3 & 3 & 1 & 7 & 6 & 9 \\ 1/4 & 4 & 1/7 & 1 & 1/2 & 3 \\ 1/3 & 1/4 & 1/6 & 2 & 1 & 4 \\ 1/5 & 1/6 & 1/9 & 1/3 & 1/4 & 1 \end{bmatrix} \quad e_8: A^8 = \begin{bmatrix} 1 & 1/5 & 1/4 & 1/2 & 3 & 1/6 \\ 5 & 1 & 2 & 4 & 6 & 1/3 \\ 4 & 1/2 & 1 & 3 & 5 & 4 \\ 2 & 1/4 & 1/3 & 1 & 3 & 6 \\ 1/3 & 1/6 & 1/5 & 1/3 & 1 & 8 \\ 6 & 3 & 1/4 & 1/6 & 1/8 & 1 \end{bmatrix}$$

(1) Selection process

Step 1: Obtaining individual preference orderings

Unify the heterogeneous preference structures into preference orderings by the individual selection methods.

Since $e_3, e_4 \in E^U$, the preference orderings for e_3 and e_4 are

$$e_3: O^3 = (4, 5, 1, 2, 3, 6), \quad e_4: O^4 = (5, 1, 4, 6, 2, 3)$$

Since $e_5, e_6 \in E^P$, based on Eqs. (1)-(3) and (5) with linguistic quantifier “as many as possible” (i.e., $Q^k = (0.5, 1)$), the values of $QGDD_i^5$, $QGDD_i^6$ can be then calculated.

The preference orderings for e_5 and e_6 are

$$e_5: O^5 = (4, 1, 2, 3, 5, 6), \quad e_6: O^6 = (1, 2, 3, 6, 4, 5)$$

Since $e_7, e_8 \in E^A$, according to Eq. (7), the preference orderings for e_7 and e_8 are

$$e_7: O^7 = (2, 3, 1, 4, 5, 6), \quad e_8: O^8 = (6, 2, 1, 3, 5, 4)$$

Step 2: Determining the reference points

Based on the obtained individual preference orderings, the decision-makers provide their preference-approval structures as follows:

$$\begin{aligned} e_1: A^1(X) &= x_2, x_1, x_3 | x_5, x_6, x_4, & e_2: A^2(X) &= x_1, x_4, x_2, x_3 | x_6, x_5 \\ e_3: A^3(X) &= x_3, x_4 | x_5, x_1, x_2, x_6, & e_4: A^4(X) &= x_2, x_5 | x_6, x_3, x_1, x_4 \\ e_5: A^5(X) &= x_2, x_3, x_4 | x_1, x_5, x_6, & e_6: A^6(X) &= x_1, x_2, x_3, x_5, x_6 | x_4 \\ e_7: A^7(X) &= x_3, x_1, x_2, x_4 | x_5, x_6, & e_8: A^8(X) &= x_3, x_2, x_4 | x_6, x_5, x_1 \end{aligned}$$

Based on the Eq. (9), each decision maker's reference point on positional order is obtained as:

$$\begin{aligned} o^{1*} &= 3.5, & o^{2*} &= 4.5, & o^{3*} &= 2.5, & o^{4*} &= 2.5 \\ o^{5*} &= 3.5, & o^{6*} &= 5.5, & o^{7*} &= 4.5, & o^{8*} &= 3.5 \end{aligned}$$

Step 3: Calculating the gains and losses

After obtaining the reference points, according to the Eqs. (10)-(11), the gain matrix $G = [G_i^k]_{8 \times 6}$ and loss matrix $L = [L_i^k]_{8 \times 6}$ are obtained as follows:

$$G = \begin{bmatrix} 1.5 & 2.5 & 0.5 & 0 & 0 & 0 \\ 3.5 & 1.5 & 0.5 & 2.5 & 0 & 0 \\ 0 & 0 & 1.5 & 0.5 & 0 & 0 \\ 0 & 1.5 & 0 & 0 & 0.5 & 0 \\ 0 & 2.5 & 1.5 & 0.5 & 0 & 0 \\ 4.5 & 3.5 & 2.5 & 0 & 1.5 & 0.5 \\ 2.5 & 1.5 & 3.5 & 0.5 & 0 & 0 \\ 0 & 1.5 & 2.5 & 0.5 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 & -2.5 & -0.5 & -1.5 \\ 0 & 0 & 0 & 0 & -1.5 & -0.5 \\ -1.5 & -2.5 & 0 & 0 & -0.5 & -3.5 \\ -2.5 & 0 & -1.5 & -3.5 & 0 & -0.5 \\ -0.5 & 0 & 0 & 0 & -1.5 & -2.5 \\ 0 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & -1.5 \\ -2.5 & 0 & 0 & 0 & -1.5 & -0.5 \end{bmatrix}$$

Step 4: Calculating the prospect values and ranking the alternatives

Assume all the decision makers set the parameters α , β and λ as the experiment result conducted by Kahneman and Tversky [23], i.e., $\alpha = \beta = 0.88$, $\lambda = 2.25$.

Then, based on the Eq. (12), let $V = (v_i^k)_{8 \times 6}$ be the prospect values matrix:

$$V = \begin{bmatrix} 1.428763372 & 2.239686 & 0.543367 & -5.03929 & -1.22258 & -3.21472 \\ 3.011479277 & 1.428763 & 0.543367 & 2.239686 & -3.21472 & -1.22258 \\ -3.214717586 & -5.03929 & 1.428763 & 0.543367 & -1.22258 & -6.77583 \\ -5.039294339 & 1.428763 & -3.21472 & -6.77583 & 0.543367 & -1.22258 \\ -1.22257672 & 2.239686 & 1.428763 & 0.543367 & -3.21472 & -5.03929 \\ 3.756877308 & 3.011479 & 2.239686 & -1.22258 & 1.428763 & 0.543367 \\ 2.239686373 & 1.428763 & 3.011479 & 0.543367 & -1.22258 & -3.21472 \\ -5.039294339 & 1.428763 & 2.239686 & 0.543367 & -3.21472 & -1.22258 \end{bmatrix}$$

Further, by Eq. (13), the collective prospect values of the alternatives are obtained as

$$v_1^c = -0.50988 \quad v_2^c = 1.020826 \quad v_3^c = 1.02755$$

$$v_4^c = -1.07807 \quad v_5^c = -1.41747 \quad v_6^c = -2.67112$$

Therefore, the collective preference ordering of the six alternatives is obtained:

$$x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_5 \succ x_6$$

Then, the collective preference-approval structure is obtained:

$$A^c(X) = x_3, x_2 | x_1, x_4, x_5, x_6 .$$

(2) Consensus process

The consensus process deals with a two-step procedure: consensus measure and feedback adjustment.

Step 1: Consensus measure

By Eqs. (14)-(15), the consensus degree among all the decision makers are:

$$PVCD = 0.8498.$$

Without loss of generality, we set that the acceptable consensus degree is 0.95.

Because $PVCD = 0.8944 < 0.95$, we use the feedback adjustment to improve the consensus level.

Step 2: Feedback adjustment

(1) Adjustment of preference evaluations

The decision makers adjust their preference evaluations according to the adjustment rules proposed in Section 5.2. Their new preference representation structures are as follows:

$$\begin{aligned}
 e_1: O^1 &= (3, 2, 1, 4, 5, 6), & e_2: O^2 &= (2, 3, 1, 4, 5, 6) \\
 e_3: U^3 &= (0.6, 0.9, 0.8, 0.5, 0.4, 0.3) & e_4: U^4 &= (0.5, 0.8, 0.6, 0.7, 0.4, 0.3) \\
 e_5: P^5 &= \begin{bmatrix} 0.5 & 0.6 & 0.55 & 0.4 & 0.8 & 0.5 \\ 0.4 & 0.5 & 0.6 & 0.8 & 0.3 & 0.6 \\ 0.45 & 0.4 & 0.5 & 0.5 & 0.6 & 0.55 \\ 0.6 & 0.2 & 0.5 & 0.5 & 0.7 & 0.4 \\ 0.2 & 0.7 & 0.4 & 0.3 & 0.5 & 0.95 \\ 0.5 & 0.4 & 0.45 & 0.6 & 0.05 & 0.5 \end{bmatrix} & e_6: P^6 &= \begin{bmatrix} 0.5 & 0.6 & 0.5 & 0.6 & 0.55 & 0.75 \\ 0.4 & 0.5 & 0.5 & 0.6 & 0.6 & 0.7 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.8 & 0.7 \\ 0.4 & 0.4 & 0.5 & 0.5 & 0.95 & 0.65 \\ 0.45 & 0.4 & 0.2 & 0.5 & 0.5 & 0.8 \\ 0.25 & 0.3 & 0.3 & 0.35 & 0.2 & 0.5 \end{bmatrix} \\
 e_7: A^7 &= \begin{bmatrix} 1 & 1/5 & 1/4 & 3 & 1/2 & 7 \\ 5 & 1 & 1/2 & 1/2 & 3 & 4 \\ 4 & 2 & 1 & 5 & 3 & 2 \\ 1/3 & 2 & 1/5 & 1 & 1/4 & 2 \\ 2 & 1/3 & 1/3 & 4 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix} & e_8: A^8 &= \begin{bmatrix} 1 & 1/2 & 1/2 & 2 & 5 & 2 \\ 2 & 1 & 3 & 6 & 3 & 1/2 \\ 2 & 1/3 & 1 & 2 & 3 & 5 \\ 1/2 & 1/6 & 1/2 & 1 & 1/2 & 5 \\ 1/5 & 1/3 & 1/3 & 1/2 & 1 & 9 \\ 1/2 & 2 & 1/5 & 1/5 & 1/9 & 1 \end{bmatrix}
 \end{aligned}$$

The new individual orderings of each decision maker are calculated as follows.

$$\begin{aligned}
 e_1: O^1 &= (3, 2, 1, 4, 6, 5), & e_2: O^2 &= (2, 3, 1, 4, 5, 6) \\
 e_3: O^3 &= (3, 1, 2, 4, 5, 6), & e_4: O^4 &= (4, 1, 3, 2, 5, 6) \\
 e_5: O^5 &= (1, 2, 3, 4, 6, 5), & e_6: O^6 &= (1, 2, 3, 4, 5, 6) \\
 e_7: O^7 &= (4, 2, 1, 5, 3, 6), & e_8: O^8 &= (3, 1, 2, 4, 5, 6)
 \end{aligned}$$

(2) Adjustment of preference-approval structures

According to the rules proposed in Section 5.2, the decision makers adjust their information to obtain new preference-approval structures as:

$$\begin{aligned}
 e_1: A^1(X) &= x_3, x_2 | x_1, x_4, x_6, x_5, & e_2: A^2(X) &= x_3, x_1, x_2 | x_4, x_5, x_6 \\
 e_3: A^3(X) &= x_2, x_3 | x_1, x_4, x_5, x_6, & e_4: A^4(X) &= x_2, x_4, x_3 | x_1, x_5, x_6 \\
 e_5: A^5(X) &= x_1, x_2, x_3 | x_4, x_6, x_5, & e_6: A^6(X) &= x_1, x_2, x_3 | x_4, x_5, x_6
 \end{aligned}$$

$$e_7: A^7(X) = x_3, x_2 | x_5, x_1, x_4, x_6, \quad e_8: A^8(X) = x_2, x_3 | x_1, x_4, x_5, x_6$$

Based on the Eq. (9), the updated reference points for all the decision makers are as follows:

$$\begin{aligned} o^{1*} &= 2.5, & o^{2*} &= 3.5, & o^{3*} &= 2.5, & o^{4*} &= 3.5 \\ o^{5*} &= 3.5, & o^{6*} &= 3.5, & o^{7*} &= 2.5, & o^{8*} &= 2.5 \end{aligned}$$

According to the Eqs. (10)-(11), the new gain matrix $G^* = (G_i^k)_{8 \times 6}$ and loss matrix $L^* = (L_i^k)_{8 \times 6}$ are obtained as follows:

$$G^* = \begin{bmatrix} 0 & 0.5 & 1.5 & 0 & 0 & 0 \\ 1.5 & 0.5 & 2.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0.5 & 0 & 0 & 0 \\ 0 & 2.5 & 0.5 & 1.5 & 0 & 0 \\ 2.5 & 1.5 & 0.5 & 0 & 0 & 0 \\ 2.5 & 1.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0.5 & 0 & 0 & 0 \end{bmatrix} \quad L^* = \begin{bmatrix} -0.5 & 0 & 0 & -1.5 & -3.5 & -2.5 \\ 0 & 0 & 0 & -0.5 & -1.5 & -2.5 \\ -0.5 & 0 & 0 & -1.5 & -2.5 & -3.5 \\ -0.5 & 0 & 0 & 0 & -1.5 & -2.5 \\ 0 & 0 & 0 & -0.5 & -2.5 & -1.5 \\ 0 & 0 & 0 & -0.5 & -1.5 & -2.5 \\ -1.5 & 0 & 0 & -2.5 & -0.5 & -3.5 \\ -0.5 & 0 & 0 & -1.5 & -2.5 & -3.5 \end{bmatrix}$$

By Eqs. (14)-(15), the new consensus degree among all the decision makers is obtained as follows:

$$PVCD = 0.9157.$$

Since the acceptable consensus degree is 0.95 and $PVCD = 0.9157 < 0.95$. Obviously, one more turn of feedback adjustment is needed.

Step 3 One more turn of Feedback adjustment

(1) Adjustments of preference evaluations

According to the preference evaluation rules proposed in 5.2, the new preference evaluations are represented as follows:

$$\begin{aligned} e_1: O^1 &= (4, 2, 1, 3, 5, 6), & e_2: O^2 &= (4, 2, 1, 3, 5, 6) \\ e_3: U^3 &= (0.6, 0.8, 0.9, 0.5, 0.4, 0.3) & e_4: U^4 &= (0.7, 0.8, 0.9, 0.5, 0.4, 0.3) \end{aligned}$$

$$e_5: P^5 = \begin{bmatrix} 0.5 & 0.3 & 0.4 & 0.9 & 0.8 & 0.4 \\ 0.7 & 0.5 & 0.2 & 0.7 & 0.3 & 0.6 \\ 0.6 & 0.8 & 0.5 & 0.5 & 0.6 & 0.55 \\ 0.1 & 0.3 & 0.5 & 0.5 & 0.7 & 0.6 \\ 0.2 & 0.7 & 0.4 & 0.3 & 0.5 & 0.95 \\ 0.6 & 0.4 & 0.45 & 0.4 & 0.05 & 0.5 \end{bmatrix} \quad e_6: P^6 = \begin{bmatrix} 0.5 & 0.4 & 0.2 & 0.6 & 0.55 & 0.75 \\ 0.6 & 0.5 & 0.5 & 0.6 & 0.6 & 0.7 \\ 0.8 & 0.5 & 0.5 & 0.5 & 0.8 & 0.7 \\ 0.4 & 0.4 & 0.5 & 0.5 & 0.95 & 0.65 \\ 0.45 & 0.4 & 0.2 & 0.5 & 0.5 & 0.8 \\ 0.25 & 0.3 & 0.3 & 0.35 & 0.2 & 0.5 \end{bmatrix}$$

$$e_7: A^7 = \begin{bmatrix} 1 & 1/5 & 1/4 & 3 & 1/2 & 7 \\ 5 & 1 & 1/2 & 1/2 & 3 & 4 \\ 4 & 2 & 1 & 5 & 3 & 2 \\ 1/3 & 2 & 1/5 & 1 & 4 & 2 \\ 2 & 1/3 & 1/3 & 1/4 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix} \quad e_8: A^8 = \begin{bmatrix} 1 & 1/2 & 1/2 & 2 & 5 & 2 \\ 2 & 1 & 1/3 & 6 & 3 & 1/2 \\ 2 & 3 & 1 & 2 & 3 & 5 \\ 1/2 & 1/6 & 1/2 & 1 & 1/2 & 5 \\ 1/5 & 1/3 & 1/3 & 1/2 & 1 & 9 \\ 1/2 & 2 & 1/5 & 1/5 & 1/9 & 1 \end{bmatrix}$$

The new individual orderings of each decision maker are updated as follows:

$$e_1: O^1 = (4, 2, 1, 3, 5, 6), \quad e_2: O^2 = (4, 2, 1, 3, 5, 6)$$

$$e_3: O^3 = (3, 2, 1, 4, 5, 6), \quad e_4: O^4 = (3, 2, 1, 4, 5, 6)$$

$$e_5: O^5 = (3, 1, 2, 4, 5, 6), \quad e_6: O^6 = (3, 2, 1, 4, 5, 6)$$

$$e_7: O^7 = (4, 2, 1, 3, 5, 6), \quad e_8: O^8 = (3, 2, 1, 4, 5, 6)$$

(2) Adjustments of the preference-approval structures

According to the rules proposed in Section 5.2, the decision makers adjust their information to obtain new preference-approval structures as:

$$e_1: A^1(X) = x_3, x_2 | x_4, x_1, x_5, x_6, \quad e_2: A^2(X) = x_3, x_2 | x_4, x_1, x_5, x_6$$

$$e_3: A^3(X) = x_3, x_2 | x_1, x_4, x_5, x_6, \quad e_4: A^4(X) = x_3, x_2 | x_1, x_4, x_5, x_6$$

$$e_5: A^5(X) = x_2, x_3 | x_1, x_4, x_5, x_6, \quad e_6: A^6(X) = x_3, x_2 | x_1, x_4, x_5, x_6$$

$$e_7: A^7(X) = x_3, x_2 | x_4, x_1, x_5, x_6, \quad e_8: A^8(X) = x_3, x_2 | x_1, x_4, x_5, x_6$$

Based on the Eq. (9), the updated reference points for all the decision makers are as follows:

$$o^{1*} = 2.5, \quad o^{2*} = 2.5, \quad o^{3*} = 2.5, \quad o^{4*} = 2.5$$

$$o^{5*} = 2.5, \quad o^{6*} = 2.5, \quad o^{7*} = 2.5, \quad o^{8*} = 2.5$$

According to the Eqs. (10)-(11), the new gain matrix $G^{**} = (G_i^k)_{8 \times 6}$ and loss matrix $L^{**} = (L_i^k)_{8 \times 6}$ are obtained as follows:

$$G^{**} = \begin{bmatrix} 0 & 0.5 & 1.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1.5 & 0 & 0 & 0 \end{bmatrix} \quad L^{**} = \begin{bmatrix} -1.5 & 0 & 0 & -0.5 & -2.5 & -3.5 \\ -1.5 & 0 & 0 & -0.5 & -2.5 & -3.5 \\ -0.5 & 0 & 0 & -1.5 & -2.5 & -3.5 \\ -0.5 & 0 & 0 & -1.5 & -2.5 & -3.5 \\ -0.5 & 0 & 0 & -1.5 & -2.5 & -3.5 \\ -0.5 & 0 & 0 & -1.5 & -2.5 & -3.5 \\ -1.5 & 0 & 0 & -0.5 & -2.5 & -3.5 \\ -0.5 & 0 & 0 & -1.5 & -2.5 & -3.5 \end{bmatrix}$$

By Eqs. (14)-(15), the new consensus degree among all the decision makers is obtained as follows:

$$PVCD=0.9640$$

Comparing with the previous consensus degree 0.9157, the consensus degree among the decision makers is obviously improved. The final ordering of the six alternatives is $x_2 \succ x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_6$.

6.3. Discussion

In this subsection, we propose Experiments I and II based on the two examples presented in Sections 6.1 and 6.2.

In Experiment I, we set different preference-approval structures (i.e. different reference points) in Examples 1 and 2, and then use the selection process to obtain the selection outcomes. The experimental results are listed in Tables 1 and 2.

Table 1

The experimental results based on Example 1 in Experiments I

Preference-approval structures	Selection outcomes	PVCD
$e_1: A(X^1) = x_2 \mid x_4, x_1, x_3, \quad e_2: A(X^2) = x_3, x_2 \mid x_1, x_4$		
$e_3: A(X^3) = x_3, x_2, x_1 \mid x_4, \quad e_4: A(X^4) = x_2, x_1 \mid x_3, x_4$	$x_2 \succ x_1 \succ x_3 \succ x_4$	0.8944
$e_5: A(X^5) = x_2, x_3, x_1 \mid x_4, \quad e_6: A(X^6) = x_1, x_2, x_4 \mid x_3$		
$e_1: A(X^1) = x_2, x_4, x_1 \mid x_3, \quad e_2: A(X^2) = x_3 \mid x_2, x_1, x_4$		
$e_3: A(X^3) = x_3 \mid x_2, x_1, x_4, \quad e_4: A(X^4) = x_2, x_1, x_3 \mid x_4$	$x_2 \succ x_3 \succ x_1 \succ x_4$	0.8713
$e_5: A(X^5) = x_2, x_3 \mid x_1, x_4, \quad e_6: A(X^6) = x_1 \mid x_2, x_4, x_3$		

Table 2

The experimental results based on Example 2 in Experiments I

Preference-approval structures	Selection outcome	PVCD
--------------------------------	-------------------	------

$e_1: A(X^1) = x_2, x_1, x_3 \mid x_5, x_6, x_4, e_2: A(X^2) = x_1, x_4, x_2, x_3 \mid x_6, x_5$		
$e_3: A(X^3) = x_3, x_4 \mid x_5, x_1, x_2, x_6, e_4: A(X^4) = x_2, x_5 \mid x_6, x_3, x_1, x_4$	$x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_5 \succ x_6$	0.8498
$e_5: A(X^5) = x_2, x_3, x_4 \mid x_1, x_5, x_6, e_6: A(X^6) = x_1, x_2, x_3, x_5, x_6 \mid x_4$		
$e_7: A(X^7) = x_3, x_1, x_2, x_4 \mid x_5, x_6, e_8: A(X^8) = x_3, x_2, x_4 \mid x_6, x_5, x_1$		
$e_1: A(X^1) = x_2 \mid x_1, x_3, x_5, x_6, x_4, e_2: A(X^2) = x_1, x_4, x_2 \mid x_3, x_6, x_5$		
$e_3: A(X^3) = x_3, x_4, x_5, x_1, x_2 \mid x_6, e_4: A(X^4) = x_2 \mid x_5, x_6, x_3, x_1, x_4$	$x_2 \succ x_3 \succ x_1 \succ x_4 \succ x_5 \succ x_6$	0.8238
$e_5: A(X^5) = x_2 \mid x_3, x_4, x_1, x_5, x_6, e_6: A(X^6) = x_1, x_2 \mid x_3, x_5, x_6, x_4$		
$e_7: A(X^7) = x_3, x_1, x_2 \mid x_4, x_5, x_6, e_8: A(X^8) = x_3, x_2 \mid x_4, x_6, x_5, x_1$		

From Experiment I, we find that the uses of different reference points will yield different outputs of the selection process in MPDM.

In Experiment II, we consider that two cases:

Case A: In Case A, the decision makers adjust both the preference evaluations and the preference-approval information.

Case B: In Case B, the decision makers only adjust the preference evaluations.

The experimental results based on Examples 1 and 2 are listed in Tables 3 and Table 4, respectively.

Table 3

The experimental results based on Example 1 in Experiments II

	Adjusting both	PVCD	Adjusting only preference evaluations	PVCD
Round 1	$e_1: A(X^1) = x_2 \mid x_1, x_3, x_4,$	0.9332	$e_1: A(X^1) = x_2 \mid x_1, x_3, x_4$	0.9096
	$e_2: A(X^2) = x_2 \mid x_1, x_4, x_3$		$e_2: A(X^2) = x_2, x_1 \mid x_4, x_3$	
	$e_3: A(X^3) = x_2 \mid x_3, x_1, x_4$		$e_3: A(X^3) = x_2, x_3, x_1 \mid x_4,$	
	$e_4: A(X^4) = x_2 \mid x_1, x_3, x_4$		$e_4: A(X^4) = x_2, x_1 \mid x_3, x_4$	
	$e_5: A(X^5) = x_2 \mid x_1, x_4, x_3$		$e_5: A(X^5) = x_2, x_1, x_4 \mid x_3,$	
	$e_6: A(X^6) = x_2 \mid x_3, x_1, x_4$		$e_6: A(X^6) = x_2, x_3, x_1 \mid x_4$	

Table 4

The experimental results based on Example 2 in Experiments II

	Adjusting both	PVCD	Adjusting only preference evaluations	PVCD
Round 1	$e_1: A(X^1) = x_3, x_2 \mid x_1, x_4, x_6, x_5$	0.9157	$e_1: A(X^1) = x_3, x_2, x_1 \mid x_4, x_6, x_5$	0.9002

	$e_2 : A(X^2) = x_3, x_1, x_2 \mid x_4, x_5, x_6$		$e_2 : A(X^2) = x_3, x_1, x_2, x_4 \mid x_5, x_6$	
	$e_3 : A(X^3) = x_2, x_3 \mid x_1, x_4, x_5, x_6$		$e_3 : A(X^3) = x_2, x_3 \mid x_1, x_4, x_5, x_6$	
	$e_4 : A(X^4) = x_2, x_4, x_3 \mid x_1, x_5, x_6$		$e_4 : A(X^4) = x_2, x_4 \mid x_3, x_1, x_5, x_6$	
	$e_5 : A(X^5) = x_1, x_2, x_3 \mid x_4, x_6, x_5$		$e_5 : A(X^5) = x_1, x_2, x_3 \mid x_4, x_6, x_5$	
	$e_6 : A(X^6) = x_1, x_2, x_3 \mid x_4, x_5, x_6$		$e_6 : A(X^6) = x_1, x_2, x_3, x_4, x_5 \mid x_6$	
	$e_7 : A(X^7) = x_3, x_2 \mid x_5, x_1, x_4, x_6$		$e_7 : A(X^7) = x_3, x_2, x_5, x_1 \mid x_4, x_6$	
	$e_8 : A(X^8) = x_2, x_3 \mid x_1, x_4, x_5, x_6$		$e_8 : A(X^8) = x_2, x_3, x_1 \mid x_4, x_5, x_6$	
Round 2	$e_1 : A(X^1) = x_3, x_2 \mid x_4, x_1, x_5, x_6$	0.9640	$e_1 : A(X^1) = x_3, x_2, x_4 \mid x_1, x_5, x_6$	0.9185
	$e_2 : A(X^2) = x_3, x_2 \mid x_4, x_1, x_5, x_6$		$e_2 : A(X^2) = x_3, x_2, x_4, x_1 \mid x_5, x_6$	
	$e_3 : A(X^3) = x_3, x_2 \mid x_1, x_4, x_5, x_6$		$e_3 : A(X^3) = x_3, x_2 \mid x_1, x_4, x_5, x_6$	
	$e_4 : A(X^4) = x_3, x_2 \mid x_1, x_4, x_5, x_6$		$e_4 : A(X^4) = x_3, x_2 \mid x_1, x_4, x_5, x_6$	
	$e_5 : A(X^5) = x_2, x_3 \mid x_1, x_4, x_5, x_6$		$e_5 : A(X^5) = x_2, x_3, x_1 \mid x_4, x_5, x_6$	
	$e_6 : A(X^6) = x_3, x_2 \mid x_1, x_4, x_5, x_6$		$e_6 : A(X^6) = x_3, x_2, x_1, x_4, x_5 \mid x_6$	
	$e_7 : A(X^7) = x_3, x_2 \mid x_4, x_1, x_5, x_6$		$e_7 : A(X^7) = x_3, x_2, x_4, x_1 \mid x_5, x_6$	
	$e_8 : A(X^8) = x_3, x_2 \mid x_1, x_4, x_5, x_6$		$e_8 : A(X^8) = x_3, x_2, x_1 \mid x_4, x_5, x_6$	

As we can see from Tables 3 and 4, compared with only adjusting the preference evaluations, adjusting both the preference evaluations and the preference-approval information can accelerate the pace of consensus reaching. We also consider other examples in the experiments, and obtain the same observations. However, a mathematic and theoretic proof would be an interesting future research to completely validate these observations.

7. Conclusions

This study reinvestigates the MPDM problems with heterogeneous preference representation structures. The main contributions are as follows.

(1) We propose a novel framework for MPDM problems with heterogeneous preference representation structures. This framework takes the decision makers' psychological behaviors into consideration and is based on the prospect theory.

(2) The selection process is designed for the proposed framework. In this selection process, the preference-approval structures are introduced to determine the reference points in prospect theory, and then the prospect value function is used to rank alternatives.

(3) We design the consensus process for the proposed framework. In the consensus process, we propose the two-step feedback adjustment rules, which will adjust both the preference evaluations and the preference-approval information to help the decision makers reach the higher consensus degree.

Finally, based on some numerical analysis, we show that the uses of different reference points will yield different selection outputs, and the pace of consensus reaching will be accelerated in adjusting both the preference evaluations and the preference-approval information.

In the future, the experts-weights and dynamic environment in the consensus reaching process will be investigated based on prospect theory.

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Graphical Abstract