The fusion process of interval opinions based on the dynamic bounded confidence

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Abstract: In this paper, we propose a novel opinion dynamics model that is based on bounded confidence and termed interval opinion dynamics with the dynamic bounded confidence. In this opinion dynamics model, the agents express their opinions in numerical intervals, and the bounded confidences vary in a specified interval as time varies (i.e., dynamic bounded confidence). Based on several theoretical analyses of the proposed opinion dynamics, we propose conditions that are sufficient to form a consensus or fragmentations among the agents. Moreover, we also design several simulation experiments to investigate the effects of the dynamic bounded confidence and interval widths on the proposed opinion dynamics and to illustrate the differences between the proposed model and the original opinion dynamics with bounded confidence.

Keywords: Opinion dynamics, Fusion process, Interval opinions, Dynamic bounded confidences, Consensus

1. Introduction

Opinion dynamics is a fusion process of individual opinions that can be defined as a group of interacting agents who continuously update their opinions regarding the same issue based on the established fusion rules and reach a consensus (or fragmentation) in the final stage.

In opinion dynamics, the establishment of the fusion rule is the core problem [1]. Recently, some research results regarding the fusion rule have been reported, such as the voter model [2], the persuasiveness and supportiveness model [3], the bounded confidence model [4,5] and the Alexford model [6]. Among these existing models, the use of the bounded confidence model as the fusion rule has become a topic of intensive research in recent years.

The bounded confidence model assumes that the agents only communicate with the peers

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who hold similar opinions and tend to ignore the peers with sufficiently different opinions [7]. The earliest bounded confidence models were introduced independently by Hegselmann and Krause (HK model) [4] and Deffuant and Weisbuch (DW model) [5]. In the HK model, the agents synchronously update their opinions by averaging all of the opinions in their confidence set [4], and in the DW model, the agents adhere to pairwise-sequential updating mechanisms [5]. Based on the original HK and DW models, the following different types of bounded confidence models have been proposed: (i) The agent-based homogeneous models [8-13], (ii) The agent-based heterogeneous models [14-16], (iii) The density-based homogeneous models [17-20], and (iv) The density-based heterogeneous models [9, 15, 21].

Previous studies have significantly advanced the bounded confidence model. In this study, we propose the interval opinion dynamics with dynamic bounded confidence model. The study is motivated by the following two aspects:

(1) In the existing studies, each agent uses a crisp number to express his/her opinion, i.e., a crisp opinion. However, in the practical processes of opinion dynamics, the opinions of the agents often exhibit uncertainty. Thus, it is necessary to investigate the effects of the uncertain opinions on opinion dynamics. Generally, interval opinions are the most basic formats of uncertain opinions [22-24]. Thus, in the present study, the fusion of interval opinions will provide a foundation for investigating the effects of uncertain opinions on opinion dynamics.

(2) Bounded confidence is the basic assumption of the opinion dynamics problem. In the existing studies, there are two types of bounded confidences, i.e., homogeneous and heterogeneous bounded confidences. The former refers to cases in which all of the agents have a uniform bounded confidence. The latter refers to cases in which each agent has his/her own bounded confidence. However, in the practical processes of opinion dynamics, the bounded confidence will vary dynamically. For example, with deep interactions, the trust between the agents may be strengthened. In such situations, their bounded confidences will increase over time, whereas with increases in conflicts of opinions, the agents' bounded confidences decrease over time. Therefore, it is necessary to study the opinion dynamics with the assumption of dynamic bounded confidence.

The aim of this paper is to discuss the fusion process of interval opinions with dynamic bounded confidence. The remainder of this paper is arranged as follows. In Section 2, we introduce the HK model. Next, in Section 3, we propose the interval opinion dynamics model with dynamic bounded confidence. In section 4, we conduct the theoretical analyses of the proposed model. In the theoretical analyses, we provide the conditions that are sufficient for the formation of a consensus or fragmentations among the agents. Furthermore, in Section 5

we design some simulation experiments to investigate the effects of the dynamic bounded confidence and the interval widths on the proposed opinion dynamics and illustrate the differences between the proposed model and the original HK model. Finally, concluding remarks are presented in Section 6.

2. Preliminary: the HK model

In this section, we introduce the original HK model, which will provide a foundation for this study.

The fundamental difference between the DW and HK models is the number of agents that communicate, which is labelled as the communication regime. In this paper, without loss of the generality, we adopt the original HK model as the foundation model. We could propose a similar study that involved the adoption of the original DW model.

The HK model [4] is briefly introduced as follows:

Consider a set of agents, $A = \{A_1, A_2, \dots, A_N\}$ and a discrete time t , $t = 0, 1, 2, \dots$. The crisp opinion of an agent $A_i \in A$ at time t is represented by $x_i(t) \in [0, 1]$. Let $X(t) = (x_1(t), x_2(t), \dots, x_N(t))^T$ be the vector of the opinions of all of the agents at time t that is called the opinion profile. Let ε_i be the bounded confidence of agent A_i . Agent A_i only considers the opinions that differ from his/her own opinion by not more than ε_i . When $\varepsilon_i = \varepsilon_j$ for $i, j = 1, 2, \dots, N$, the model is termed the HK model with the homogeneous bounded confidence; otherwise, the model is termed the HK model with the heterogeneous bounded confidence.

The process of the HK model includes three steps:

(1) Determinations of the confidence sets.

Let $I(A_i, x_i(t))$ be the confidence set of agent A_i at time t , and $I(A_i, x_i(t))$ is determined as:

$$I(A_i, x_i(t)) = \{A_j \mid |x_i(t) - x_j(t)| \leq \varepsilon_i, j = 1, 2, \dots, N\}, \quad i = 1, 2, \dots, N. \quad (1)$$

(2) Calculations of the weights.

Let $w_{ij}(t)$ be the weight that agent A_i assigns to agent A_j at time t . Using the obtained set $I(A_i, x_i(t))$, we can calculate the weight $w_{ij}(t)$ as:

$$w_{ij}(t) = \begin{cases} 0, & A_j \notin I(A_i, x_i(t)) \\ 1/|I(A_i, x_i(t))|, & A_j \in I(A_i, x_i(t)) \end{cases}, \quad i = 1, 2, \dots, N, \quad (2)$$

where $|\cdot|$ denotes the cardinality of $I(A_i, x_i(t))$. Clearly, $w_{ij}(t) \geq 0$ and $\sum_{j=1}^N w_{ij}(t) = 1$.

(3) Evolutions of the opinions.

The evolutions of the opinions in the HK model are modelled as the weighted arithmetic means of the opinions in the confidence sets, i.e.,

$$x_i(t+1) = w_{i1}(t)x_1(t) + w_{i2}(t)x_2(t) + \dots + w_{iN}(t)x_N(t), \quad i=1,2,\dots,N. \quad (3)$$

3. The proposed model

In this section, we propose the interval opinion dynamics model with the dynamic bounded confidence. In our proposal, agent $A_i \in A$ expresses his/her opinion by a numerical interval $x_i(t) = [x_i^L(t), x_i^U(t)]$, where $x_i(t) \subseteq [0,1]$. Let $X^L(t) = (x_1^L(t), x_2^L(t), \dots, x_N^L(t))^T$ and $X^U(t) = (x_1^U(t), x_2^U(t), \dots, x_N^U(t))^T$ be the lower and upper bounds of the opinion profile, respectively. Let $\varepsilon_i(t)$ be the dynamic bounded confidence of agent A_i at time t , where $0 \leq \varepsilon_i(t) \leq \alpha$, and α is the maximum threshold of the dynamic bounded confidence.

Inspired by the HK model, we define the new confidence set $\tilde{I}(A_i, x_i(t))$ as:

$$\tilde{I}(A_i, x_i(t)) = \{A_j \mid d(x_i(t), x_j(t)) \leq \varepsilon_i(t), j=1,2,\dots,N\}, \quad i=1, 2, \dots, N, \quad (4)$$

where $d(x_i(t), x_j(t))$ denotes the distance between the interval opinions $x_i(t)$ and $x_j(t)$.

The distance $d(x_i(t), x_j(t))$ can be calculated using various distance measures (e.g., Manhattan or Euclidean).

When adopting the Manhattan distance measure, the distance $d(x_i(t), x_j(t))$ is given by

$$d(x_i(t), x_j(t)) = \frac{|x_i^L(t) - x_j^L(t)| + |x_i^U(t) - x_j^U(t)|}{2}. \quad (5)$$

When adopting the Euclidean distance measure, the distance $d(x_i(t), x_j(t))$ is given by

$$d(x_i(t), x_j(t)) = \sqrt{\frac{1}{2}[(x_i^L(t) - x_j^L(t))^2 + (x_i^U(t) - x_j^U(t))^2]}. \quad (6)$$

Regardless of which of the above distance measures we use, similar results are obtained. Without loss of generality, we adopt the Euclidean distance measure in the present study.

The evolutions of the interval opinions in the proposed model are modelled as the weighted arithmetic means of the upper and lower bounds of the interval opinions in the confidence sets, i.e.,

$$x_i^L(t+1) = w_{i1}(t)x_1^L(t) + w_{i2}(t)x_2^L(t) + \dots + w_{iN}(t)x_N^L(t), \quad i=1,2,\dots,N, \quad (7)$$

and

$$x_i^U(t+1) = w_{i1}(t)x_1^U(t) + w_{i2}(t)x_2^U(t) + \dots + w_{iN}(t)x_N^U(t), \quad i=1,2,\dots,N, \quad (8)$$

respectively, where the weight $w_{ij}(t)$ is given by

$$w_{ij}(t) = \begin{cases} 0, & A_j \notin \tilde{I}(A_i, x_i(t)) \\ 1/|\tilde{I}(A_i, x_i(t))|, & A_j \in \tilde{I}(A_i, x_i(t)) \end{cases}, \quad i=1,2,\dots,N. \quad (9)$$

Let $W(t) = (w_{ij}(t))_{N \times N}$. Then, based on Eqs. (5) and (6), the lower and upper bounds of opinion profile are given by

$$x^L(t+1) = W(t)x^L(t). \quad (10)$$

$$x^U(t+1) = W(t)x^U(t). \quad (11)$$

The opinion profiles are further determined as

$$x^L(t+1) = W(t)W(t-1)\dots W(0)x^L(0). \quad (12)$$

$$x^U(t+1) = W(t)W(t-1)\dots W(0)x^U(0). \quad (13)$$

The proposed model shares a strong linkage to the HK model. If $\varepsilon_i(t) = \varepsilon_i(t+1)$ and $\varepsilon_i(t) = \varepsilon_j(t)$ are satisfied for $i, j=1, 2, \dots, N$ and $t=0,1,\dots$, then the proposed model is reduced to the HK model with the homogeneous bounded confidence. If only $\varepsilon_i(t) = \varepsilon_i(t+1)$ is satisfied for $i=1, 2, \dots, N$ and $t=0,1,\dots$, then the proposed model is reduced to the HK model with the heterogeneous bounded confidence.

4. Theoretical analysis

In this section, we propose some theorems on the proposed model to investigate the conditions to form the stabilized result (i.e., consensus and fragmentations).

Before proposing these theorems, we introduce several definitions regarding the consensus, fragmentation, confidence chain, and accumulated weight matrix.

Definition 1: Let $x_i(t) = [x_i^L(t), x_i^U(t)]$ and $x_j(t) = [x_j^L(t), x_j^U(t)]$ be as before. If $x_i(t) = x_j(t)$ (i.e., $x_i^L(t) = x_j^L(t)$ and $x_i^U(t) = x_j^U(t)$), then we call that a consensus among the agents is reached at time t ; otherwise, if $x_i(t) \neq x_j(t)$ and $d(x_i(t), x_j(t)) > \alpha$, then we call that fragmentations among the agents are formed.

Similar to the confidence chain presented in [25], we define the confidence chain for the proposed model in **Definition 2**.

Definition 2: Let s and t ($s < t$) be any two times, and let $A_{i_0}, A_{i_1}, \dots, A_{i_{t-s}}$ be a sequence of the $t-s+1$ agents. If $A_{i_l} \in \tilde{I}(A_{i_{l-1}}, x_{i_l}(t-l))$ for $l=1,2,\dots,t-s$, then we call that

there exists a confidence chain from A_i to A_k between the times s and t .

Similar to the accumulated weight presented in [4], we define the accumulated weight matrix for the proposed model in **Definition 3**.

Definition 3: Let s and t ($s < t$) be any two times, and $B(t, s) = (b_{ij}(t, s))_{N \times N}$ be the accumulated weight matrix between times s and t where $B(t, s) = W(t-1) \cdots W(s)$.

Clearly, the accumulated weight matrix $B(t, s)$ is a nonnegative row-stochastic matrix, i.e., $b_{ij}(t, s) \geq 0$ and $\sum_{j=1}^N b_{ij}(t, s) = 1$ for $i, j = 1, 2, \dots, N$.

Let $z_v = [z_v^L, z_v^U]$ ($v = 1, 2, \dots, p$) be the set of numerical intervals, and let $C(z_1, z_2, \dots, z_p)$ be the set of all of the convex combinations of these numerical intervals where

$$C(z_1, z_2, \dots, z_p) = \left\{ \sum_{v=1}^p \eta_v z_v \mid \eta_v \geq 0 \text{ and } \sum_{v=1}^p \eta_v = 1 \right\}. \text{ Let } F(z_1, z_2, \dots, z_p) = \max_{u, r \in C(z_1, z_2, \dots, z_p)} d(u, r).$$

Lemma 1: Let $x_1(t), x_2(t), \dots, x_N(t)$ be as before. Let $\underline{x}^L(t) = \min\{x_i^L(t) \mid i = 1, 2, \dots, N\}$, $\underline{x}^U(t) = \min\{x_i^U(t) \mid i = 1, 2, \dots, N\}$, $\bar{x}^L(t) = \max\{x_i^L(t) \mid i = 1, 2, \dots, N\}$ and $\bar{x}^U(t) = \max\{x_i^U(t) \mid i = 1, 2, \dots, N\}$. Then,

$$F(x_1(t), \dots, x_N(t)) \leq (1 - \min_{1 \leq i, j \leq N} \sum_{k=1}^N \{w_{ik}(t), w_{jk}(t)\}) d(\underline{x}(t), \bar{x}(t))$$

The proof of **Lemma 1** is provided in the Appendix.

Theorem 1: A consensus will be reached if one of the following three conditions is satisfied:

(1) Let $t_0, t_1, t_2, \dots$ be a sequence of times and $\delta_1, \delta_2, \dots$ be a sequence of positive numbers where $0 = t_0 < t_1 < t_2 < \dots$, $\delta_1, \delta_2, \dots \in [0, 1]$ and $\lim_{r \rightarrow \infty} \sum_{m=1}^r \delta_m = \infty$. Then,

$$\sum_{k=1}^N \min\{b_{ik}(t_m, t_{m-1}), b_{jk}(t_m, t_{m-1})\} \geq \delta_m \text{ for } m = 1, 2, \dots, \infty \text{ and } i, j = 1, 2, \dots, n;$$

(2) Let A_i and A_j be as before, i.e., $i, j = 1, 2, \dots, n$. Then the two confidence chains from A_i to A_k and A_j to A_k between the times $(m-1)h$ and mh exist because $A_k \in A$, $m = 1, 2, \dots$ and $h \geq 1$;

(3) Let $\Delta \varepsilon_i(t) = \varepsilon_i(t+1) - \varepsilon_i(t)$. Then, $\Delta \varepsilon_i(t) \geq \max_{j=1, 2, \dots, N} d(x_i(t), x_j(t)) - \varepsilon_i(t)$, for $i = 1, 2, \dots, N$.

The proof of **Theorem 1** is provided in the Appendix.

Theorem 1 provides some conditions that are sufficient to reach a consensus. We show that a consensus will be reached in the following conditions: the sum of the minimum

accumulated weights between any two agents is a positive number, and the confidence chain among any two agents exists in the same time interval or the new bounded confidence of each agent is larger than the maximum distance between the current interval opinions.

Lemma 2: Let $B(t_0, t_1)$ be the accumulated weight matrix between the times t_0 and t_1 ($t_0 < t_1$), and let $\mu(B(t_0, t_1))$ be the smallest positive entry of $B(t_0, t_1)$. Then $\mu(B(t_0, t_1)) \geq (1/N)^{N^2-N+2}$ given that $\varepsilon_i(t) = \varepsilon_j(t)$ for $i, j = 1, 2, \dots, n$ and $t = 0, 1, 2, \dots$.

The proof of **Lemma 2** is provided in the Appendix.

Theorem 2: Let $L(x_{\max}^{(0)}) = \max\{x_i^U(0) - x_i^L(0), i = 1, 2, \dots, n\}$ and $L(x_{\min}^{(0)}) = \min\{x_i^U(0) - x_i^L(0), i = 1, 2, \dots, n\}$. If the proposed model reaches a consensus at time t , denoted as $x^c = [x^{cL}, x^{cU}]$, then the upper bound of the width of $L(x^c)$ is given as follows:

(1) If $t < N^2 - N + 2$, then

$$L(x^c) \leq [1 - (N-1)(1/N)^t] L(x_{\max}^{(0)}) + (N-1)(1/N)^t L(x_{\min}^{(0)}).$$

(2) If $t \geq N^2 - N + 2$ and $\varepsilon_1(t) = \varepsilon_2(t) = \dots = \varepsilon_N(t)$, then

$$L(x^c) \leq [1 - (N-1)(1/N)^{N^2-N+2}] L(x_{\max}^{(0)}) + (1/N)^{N^2-N+2} L(x_{\min}^{(0)}).$$

(3) If $t \geq N^2 - N + 2$ and $\varepsilon_i(t) \neq \varepsilon_k(t)$ for $\exists i, k \in \{1, 2, \dots, N\}$, then

$$L(x^c) \leq [1 - (N-1)(1/N)^t] L(x_{\max}^{(0)}) + (N-1)(1/N)^t L(x_{\min}^{(0)}).$$

The proof of **Theorem 2** is provided in the Appendix.

Theorem 2 provides the upper bounds of the widths of the consensus interval opinion.

Theorem 3: Let α be defined as before. We assume there exists $f(t)$ interval opinions $\bar{x}^1(t), \bar{x}^2(t), \dots, \bar{x}^{f(t)}(t)$ at time t , $f(t) \geq 2$. Fragmentations are formed if one of the following conditions is satisfied:

(1) Let $\varepsilon_i(t)$ and $\varepsilon_i(\tilde{t})$ be the dynamic bounded confidence at times t and $\tilde{t}$ ($\tilde{t} \geq t$), respectively. Then $\varepsilon_i(\tilde{t}) \leq \varepsilon_i(t)$, for $i = 1, 2, \dots, N$.

(2) $d(\bar{x}^g(t), \bar{x}^h(t)) > \alpha$, for $g, h = 1, 2, \dots, f(t)$.

The proof of **Theorem 3** is provided in the Appendix.

Theorem 3 provides the conditions that are sufficient to form fragmentations. We have shown that fragmentations will be formed if the dynamic bounded confidence is decreasing or the distance between any two interval opinions is larger than α .

5. Simulation analysis

In this section, we propose the simulations to explore the following three issues: the

effects of the dynamic bounded confidence on the opinion dynamics, the effects of the widths of the interval opinions on the opinion dynamics, and the comparisons between our proposal and the original HK model.

5.1 The effects of dynamic bounded confidence

We investigated the effects of the dynamic bounded confidence from two aspects: the stabilized results (consensus or fragmentation), and the consensus time.

Let SR be the flag of the stabilized result, where $SR=1$ denotes a consensus, and $SR=0$ denotes fragmentations. Let t^* be the time at which a consensus is reached, i.e., the consensus time. Then, we present **Simulation method I** to determine the SR and t^* values.

Simulation method I:

Input: N and α .

Output: SR and t^* .

Step 1: Let $t=0$. Interval opinions $x_i(t)=[x_i^L(t), x_i^U(t)]$ ($i=1,2,\dots,N$) are uniformly and randomly selected from $[0,1]$.

Step 2: Dynamic bounded confidences $\varepsilon_i(t)$ ($i=1,2,\dots,N$) are uniformly and randomly selected from $[0, \alpha]$.

Step 3: Using Eqs. (7) and (8) obtains the interval opinion $x_i(t+1)$ at time $t+1$. If the conditions regarding stabilized results (see Definition 1) are satisfied, then go to **Step 4**; otherwise, let $t=t+1$, then go to **Step 2**.

Step 4: If $x_i(t+1)=x_j(t+1)$ for $i,j=1,2,\dots,N$, then $SR=1$ and $t^*=t+1$; otherwise, $SR=0$.

Step 5: Output SR and t^* .

Note 1 On average, the increases of the α values often cause the increases of the dynamic bounded confidences. So, on average, through changing the α values, the effects of dynamic bounded confidences on opinion dynamics can be discussed.

Let $N=1000$. We set different α values, and run the **Simulation method I** 1000 times, obtaining the consensus ratio values, which reflect the ratio of reaching a consensus in the proposed model. Fig. 1 shows the consensus ratio values for different α values.

Fig. 1. The average SR values for different α values

From Fig. 1, we obtain the following observation: the average SR values increase as the α values increase. This observation implies that it is easier to reach a consensus as the ranges of the dynamic bounded confidences increase. Moreover, from Fig. 1, we find that a consensus will be reached when $\alpha \geq 0.3$.

Furthermore, we set different N and α values, and run the **Simulation method I** 1000 times, obtaining the average t^* values, which are shown in Fig. 2.

Fig. 2. The average t^* values for different α values

From Fig. 2, we obtain the following observations:

(1) The average t^* values decrease as the α values increase. This relation implies that the time required to reach a consensus will decrease as the ranges of the dynamic bounded confidences increase.

(2) Less time will be required to reach a consensus for cases with smaller N values.

The above observations based on Figs. 1-2 are attributed to the following reason. The

agents have more opportunities to increase their interactions with increases in the ranges of the dynamic bounded confidences. Consequently, the opinions of the agents can reach a consensus in a shorter time.

5.2 The effects of the widths of interval opinions

We investigate the effects of the widths of interval opinions on the average number of fragmentations and the convergence times (i.e., the time to form the stabilized results).

Let NF be the number of fragmentations, and t' be the convergence time. Let $\lambda = \max\{x_i^U(0) - x_i^L(0), i=1,2,\dots,N\}$ be the maximum width of all the initial interval opinions. Then, we present the **Simulation method II** to determine the NF and t' values.

Simulation method II

Input: N , λ and α .

Output: NF and t' .

Step 1: Let $t=0$, and crisp opinions $y_i(0)$ ($i=1,2,\dots,N$) are uniformly and randomly selected from $[0, 1]$.

Step 2: Let $x_i(0)=[x_i^L(0), x_i^U(0)]$ ($i=1,2,\dots,N$), where $x_i^L(0) = \max\{0, y_i(0) - \delta_i/2\}$, $x_i^U(0) = \min\{y_i(0) + \delta_i/2, 1\}$, and δ_i is uniformly and randomly selected from $[0, \lambda]$.

Step 3 : Dynamic bounded confidences $\varepsilon_i(t)$ ($i=1,2,\dots,N$) are uniformly and randomly selected from $[0, \alpha]$.

Step 4: Using Eqs. (7) and (8) obtains the interval opinion $x_i(t+1)$ at time $t+1$. If the conditions regarding stabilized results (see Definition 1) are satisfied, then let $t'=t+1$, go to **Step 5**; otherwise, let $t=t+1$, then go to **Step 3**.

Step 5: Output t' and NF .

Note 2: Step 2 guarantees that the width of interval opinion $x_i(0)$ is smaller than λ , and $x_i(0) \subseteq [0,1]$, for $i=1,2,\dots,N$.

We set different N , λ and α values, and run the **Simulation method II** 1000 times, obtaining the average values for NF and t' , which are shown in Figs. 3 and 4.

Fig. 3. The average NF values for different α values

Fig. 4. The average t' values for different α values

From Figs. 3 and 4, we obtain the following three observations:

(1) Larger λ values yield more fragmentations. This observation implies that it will be easier to form fragmentations as the widths of the interval opinions increase. This observation is related to the following reason. In the simulation, we find that with the increases of the ranges of the interval opinions, the differences among the initial interval opinions of the agents will increase. Consequently, fragmentations between the agents are formed.

(2) With increases in the λ values, the average t' values first decrease and then increase again.

(3) More time will be required to form stable results for cases with the larger N values. Moreover, larger N values will result in greater numbers of fragmentations (NF).

5.3 Comparisons of the proposed model and the HK model

In this subsection, we compare the proposed model with the HK model. In these comparisons, two aspects were investigated: the sequence consistency, and the conditions that result in the formation of fragmentations.

(1) Sequence consistency:

Blondel et al. [11] proved that the HK model with homogeneous bounded confidence satisfies sequence consistency. In the following, we discuss the finding that the proposed model with dynamic bounded confidence does not satisfy sequence consistency.

Let t' be as before. If $x_i(0) \leq x_j(0)$ and $x_i(t') \leq x_j(t')$, then in this paper, we consider A_i and A_j to be a pair of agents that satisfy sequence consistency.

Note 3: There exist several methods for comparing two intervals [26-29]. In our study, the method presented in [27, 29] is used to compare two intervals. Specifically, let $x_i = [x_i^L, x_i^U]$ and $x_j = [x_j^L, x_j^U]$ be two interval numbers. The comparison of x_i and x_j is defined as follows: (a) $x_i > x_j$ if $P(x_i > x_j) > 0.5$, and (b) $x_i = x_j$ if $P(x_i > x_j) = 0.5$, where $P(x_i \geq x_j)$ is the degree of preference for x_i over x_j , i.e.,

$$P(x_i \geq x_j) = \frac{\max(0, x_i^U - x_j^L) - \max(0, x_i^L - x_j^U)}{x_i^U - x_i^L + x_j^U - x_j^L}.$$

Let PI be the number of pairs of agents that satisfy sequence consistency. Let $p = PI/C_N^2$; we then used the ratio p to investigate whether the proposed model satisfies sequence consistency. When $p=1$, the proposed model satisfied sequence consistency. The large p value indicates that the proposed model exhibits a higher ratio to satisfy sequence consistency.

Then, we present **Simulation method III** to calculate the ratio p .

Simulation method III:

Input: N and α .

Output: p .

Step 1: Let $t=0$ and $PI=0$. Interval opinions $x_i(0)$ ($i=1,2,\dots,N$) are uniformly and randomly selected from $[0, 1]$.

Step 2: Dynamic bounded confidences $\varepsilon_i(t)$ ($i=1,2,\dots,N$) are uniformly and randomly selected from $[0, \alpha]$.

Step 3: Using Eqs. (7) and (8) obtains the interval opinion $x_i(t+1)$ at time $t+1$. If the conditions regarding stabilized results (see Definition 1) are satisfied, then let $t'=t$, go to **Step 4**; otherwise, let $t=t+1$, go to **Step 2**.

Step 4: For two agents A_i and A_j ($i, j=1,2,\dots,N$), if $x_i(0) \leq x_j(0)$ and $x_i(t') \leq x_j(t')$, then $PI = PI + 1$.

Step 5: Let $p = PI/C_N^2$, and output the p value.

We set different N and α values, and run the **Simulation method III** 200 times, obtaining the average p values, which are shown in Fig. 5.

Fig. 5. The average p values for different α values

From Fig. 5, we obtain two observations:

- (i) The proposed model does not satisfy the sequence consistency.
- (ii) The average p values in the proposed model ranged from 0.45 to 0.52, which implies that there exist many agents who violate sequence consistency.

The observations in Fig. 5 are attributable to two phenomena: (a) the variations in the upper and lower bounds of the interval opinions are not uniform during the process of opinion dynamics; and (b) the existence of dynamic bounded confidence causes differences in the confidence sets of the agents.

(2) Condition required to form fragmentations

In the HK model [4], the condition for the formation of fragmentations is that all of the opinions at any two neighbouring times are completely identical, and there exist at least two different opinions.

In the following, we propose a counterexample to illustrate that it is possible that the proposed model will not form fragmentations in this condition. In this example, there are 50 agents, their initial interval opinions are given by

$$x_i(0) = \begin{cases} [0, 0.2], & i = 1, 2, \dots, 10 \\ [0.3, 0.35], & i = 11, 12, \dots, 20 \\ [0.45, 0.6], & i = 21, 22, \dots, 30 \\ [0.7, 0.75], & i = 31, 32, \dots, 40 \\ [0.95, 1], & i = 41, 42, \dots, 50 \end{cases}$$

and their initial dynamic bounded confidences are given by

$$\varepsilon_i(0) = \begin{cases} 0.15, & i = 1, 2, \dots, 10 \\ 0.17, & i = 11, 12, \dots, 20 \\ 0.2, & i = 21, 22, \dots, 30 \\ 0.21, & i = 31, 32, \dots, 40 \\ 0.24, & i = 41, 42, \dots, 50 \end{cases}$$

Based on Eqs. (7) and (8), we find that $x_i(0) = x_i(1)$ for $i = 1, 2, \dots, 50$, and there exist five interval opinions at time $t = 1$.

Let $\alpha = 0.25$. Based on the proposed model, we obtain the evolutions of the interval opinions, which are shown in Fig. 6. The red and blue lines in Fig. 6 represent the evolutions of the lower and upper bounds of interval opinions, respectively.

Fig. 6. The evolutions of the interval opinions in the proposed model

From Fig. 6, we obtain the following observation: the evolutions of the interval opinions continue after the time $t = 1$, and a consensus is finally reached in opinion dynamics. This implies that the proposed model does not satisfy the conditions for forming fragmentations in the HK model.

The observation in Fig. 6 can be explained by the dynamic bounded confidence. Changes in dynamic bounded confidences often cause changes of opinions. Consequently, the conditions for the formation of fragmentations of the HK model and the proposed model are not identical.

6. Conclusions

In the present study, we propose a novel model, termed the interval opinion dynamics with the dynamic bounded confidence. The primary contributions of this study are as follows:

(1) In contrast to the original HK model, in the proposed model, the agents express interval opinions rather than crisp number opinions and employ dynamic bounded confidences in specified intervals.

(2) We propose some conditions that are sufficient for the formation of consensus or fragmentations in the proposed opinion dynamics.

(3) Through the simulation experiments, we investigate the effects of dynamic bounded confidence on opinion dynamics and the effects of the widths of the interval opinions on opinion dynamics, and also compared our proposal with the original HK model.

In our view, the opinion dynamics models are related to the consensus-reaching processes of group decision-making problems [30-41]. In particular, with the rapid development of information technology and social economic, proposals related to and studies of large-scale group decision-making have increasingly been performed [42]. We argue that opinion dynamics models will potentially be tool that support consensus-reaching in large-scale group decision-making problems. This issue is now under investigation.

Acknowledgments

This work was supported in part by the NSF of China via grant 71171160 and in part by Sichuan University via grant skqx201308.

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Appendix

A1. Proof of Lemma 1.

Proof. According to Eqs. (5) and (6), $x_i(t+1) \in \text{conv}\{x_1(t), \dots, x_N(t)\}$, $i = 1, 2, \dots, N$.

Thus, the distance $d(x_i(t+1), x_j(t+1))$ is given as follows:

$$\begin{aligned}
 d(x_i(t+1), x_j(t+1)) &= \left[\frac{1}{2} \left[\left(\sum_{k=1}^N (w_{ik}(t) - w_{jk}(t)) x_k^L(t) \right)^2 + \left(\sum_{k=1}^N (w_{ik}(t) - w_{jk}(t)) x_k^U(t) \right)^2 \right] \right]^{1/2} \\
 &= \left[\frac{1}{2} \left[\left(\sum_{k=1}^N (w_{ik}(t) - \min\{w_{ik}(t), w_{jk}(t)\}) x_k^L(t) - (w_{jk}(t) - \min\{w_{ik}(t), w_{jk}(t)\}) x_k^L(t) \right)^2 + \right. \right. \\
 &\quad \left. \left. \left(\sum_{k=1}^N (w_{ik}(t) - \min\{w_{ik}(t), w_{jk}(t)\}) x_k^U(t) - (w_{jk}(t) - \min\{w_{ik}(t), w_{jk}(t)\}) x_k^U(t) \right)^2 \right] \right]^{1/2} \\
 &\leq \left[\frac{1}{2} \left[\left(1 - \sum_{k=1}^N \min\{w_{ik}(t), w_{jk}(t)\} \right)^2 (\underline{x}^L(t) - \bar{x}^L(t))^2 + \left(1 - \sum_{k=1}^N \min\{w_{ik}(t), w_{jk}(t)\} \right)^2 (\underline{x}^U(t) - \bar{x}^U(t))^2 \right] \right]^{1/2} \\
 &= \left[\left(1 - \sum_{k=1}^N \min\{w_{ik}(t), w_{jk}(t)\} \right)^2 \frac{1}{2} [(\underline{x}^L(t) - \bar{x}^L(t))^2 + (\underline{x}^U(t) - \bar{x}^U(t))^2] \right]^{1/2} \\
 &= \left(1 - \min_{1 \leq i, j \leq N} \sum_{k=1}^N \{w_{ik}(t), w_{jk}(t)\} \right) \sqrt{\frac{(\underline{x}^L(t) - \bar{x}^L(t))^2 + (\underline{x}^U(t) - \bar{x}^U(t))^2}{2}} \\
 &= \left(1 - \min_{1 \leq i, j \leq N} \sum_{k=1}^N \{w_{ik}(t), w_{jk}(t)\} \right) d(\underline{x}(t), \bar{x}(t))
 \end{aligned}$$

Due to the arbitrariness of $x_i(t+1)$ and $x_j(t+1)$, we can further obtain

$$F(\text{conv}\{x_1(t), \dots, x_N(t)\}) \leq \left(1 - \min_{1 \leq i, j \leq N} \sum_{k=1}^N \{w_{ik}(t), w_{jk}(t)\} \right) d(\underline{x}(t), \bar{x}(t)).$$

This completes the proof of **Lemma 1**. $\square$

A2. Proof of Theorem 1.

Proof. (1) Based on Lemma 1, we obtain

$$F(\text{conv}\{x_1(t), \dots, x_N(t)\}) \leq \left(1 - \min_{1 \leq i, j \leq N} \sum_{k=1}^N \{b_{ik}(t-s), b_{jk}(t-s)\} \right) d(\underline{x}(s), \bar{x}(s)), \quad t > s.$$

Because $\sum_{k=1}^N \min\{b_{ik}(t_m, t_{m-1}), b_{jk}(t_m, t_{m-1})\} \geq \delta_m$, for time $t > t_{m+1}$, we have

$$F(\text{conv}\{x_1(t), \dots, x_N(t)\}) \leq (1 - \delta_m)(1 - \delta_{m-1}) \cdots (1 - \delta_0) d(\underline{x}(0), \bar{x}(0))$$

Because $1 - \delta_m \leq e^{-\delta_m}$ for $\delta_m \geq 0$. Hence,

$$F(\text{conv}\{x_1(t), \dots, x_N(t)\}) \leq e^{-\delta_m} e^{-\delta_{m-1}} \cdots e^{-\delta_0} d(\underline{x}(0), \bar{x}(0)) = e^{-\sum_{r=1}^m \delta_r} d(\underline{x}(0), \bar{x}(0))$$

Because $\lim_{r \rightarrow \infty} \sum_{m=1}^r \delta_m = \infty$, $F(\text{conv}\{x_1(t), \dots, x_N(t)\}) \leq 0$.

We further obtain $F(\text{conv}\{x_1(t), \dots, x_N(t)\}) = 0$. According to the meaning of $F(\text{conv}\{x_1(t), \dots, x_N(t)\})$, a consensus between the agents is reached, i.e., $\lim_{t \rightarrow \infty} x_i(t) = x^*$ for $i = 1, 2, \dots, N$. $\square$

(2) Because confidence chains exist from A_i to A_k and A_j to A_k between the times $(m-1)h$ and mh , we can obtain

$$I_i(mh, x_i(mh)) \cap I_j(mh, x_j(mh)) = \emptyset, \text{ e.g. } A_k \in I_i(mh, x_i(mh)) \cap I_j(mh, x_j(mh)).$$

Furthermore, we have $\sum_{k=1}^N \min \{b_{ik}(mh, (m-1)h), b_{jk}(mh, (m-1)h)\} \geq (\frac{1}{N})^h$. Obviously, $(\frac{1}{N})^h > 0$, and when $m \rightarrow \infty$, $m(\frac{1}{N})^h \rightarrow \infty$.

According to **Lemma 1** and **Theorem 1**, we can obtain $\lim_{t \rightarrow \infty} d(x_i(t), x_j(t)) = 0$, which indicates that any pair of agents hold the same opinion; thus, a consensus is reached.

(3) When a consensus is not reached at time t , then there exists one agent whose confidence set does not include all of the agents.

Without a loss of generality, we assume $|I(i, x_i(t))| < N-1$, $d(x_i(t), x_k(t)) > \varepsilon_i(t)$, and $\max_{j=1,2,\dots,N} d(x_i(t), x_j(t)) \geq d(x_i(t), x_k(t))$.

If $\Delta \varepsilon_i(t) \geq \max_{j=1,2,\dots,N} d(x_i(t), x_j(t)) - \varepsilon_i(t)$, it implies that

$$\varepsilon_i(t+1) = \varepsilon_i(t) + \Delta \varepsilon_i(t) \geq \max_{j=1,2,\dots,N} d(x_i(t), x_j(t)).$$

Then $|I(i, x_i(t+1))| = N$ is satisfied at time $t+1$. Clearly, a consensus is reached at time $t+1$.

This completes the proof of **Theorem 1**. $\square$

A3. Proof of Lemma 2.

Proof. Let $n^* = t_0 - t_1$ because $\mu(B(t_0, t_1)) = \mu(A(t_1-1)A(t_1-2) \cdots A(t_0))$; thus, we obtain

$$\mu(B(t_0, t_1)) = \mu(A(t_1-1)A(t_1-2) \cdots A(t_0)) \geq \mu(A(t_1-1))\mu(A(t_1-2)) \cdots \mu(A(t_0)) \geq (1/N)^{n^*}.$$

If $n^* \leq N^2 - N + 2$, then $\mu(B(t_0, t_1)) \geq (1/N)^{N^2 - N + 2}$ naturally holds.

Because $\varepsilon_1(t) = \varepsilon_2(t) = \cdots = \varepsilon_N(t)$, then the weight matrix $W(t)$ at time t satisfies the following two conditions: (1) if $w_{ij}(t) = 0$, then $w_{ji}(t) = 0$; and (2) if $w_{ij}(t) > 0$, then

$$w_{ji}(t) < 0.$$

For any two matrices B and C , $B, C \in \mathbb{R}_{\geq 0}^{n \times n}$, proof of the following conclusion is needed:

$$\text{If } \mu(CB) < \mu(B), \text{ then } \exists i, j, (cb)_{ij} > 0 \text{ and } b_{ij} = 0. \quad (C1)$$

To prove this conclusion, we assume that the zero patterns of BC and B are the same and that $\mu(CB) \geq \mu(B)$.

According to this assumption, if $b_{kj} = 0$, then $(cb)_{kj} = 0$, and $(cb)_{kj}$ is rewritten as:

$$(cb)_{kj} = \sum_{l=1}^n c_{kl} b_{lj} = 0 \xrightarrow{b_{lj} > 0} c_{kl} = 0 \Rightarrow c_{ik} = 0$$

Let $N' = \{i \mid (cb)_{ij} > 0 \text{ and } c_{ij} > 0\}$ be the set of indices. Obviously, we have

$$\sum_{k \in N', c_{kj} > 0} b_{ik} = 1. \text{ Thus,}$$

$$(cb)_{ij} = \sum_{k \in N', b_{kj} > 0} c_{ik} b_{kj} \geq (\min_{k \in N', b_{kj} > 0} b_{kj}) (\sum_{k \in N', b_{kj} > 0} c_{ik}) \geq \min_{k \in N', b_{kj} > 0} b_{kj} \geq \mu(B).$$

Therefore, after one multiplication between the two accumulated weight matrices, at least one zero entry will lose. Because the diagonal elements of the accumulated weight matrix at any time are always nonzero, the number of zero elements in the accumulated weight matrix is equal to at most $N^2 - N + 1$. Therefore, $\mu(B(t_0, t_1)) \geq (1/N)^{N^2 - N + 2}$.

This completes the proof of **Lemma 2**. $\square$

A4. Proof of Lemma 2.

Proof. Based on Eqs. (9) and (10), the interval opinions $x_i = [x_i^L(t+1), x_i^U(t+1)]$ can written as

$$x_i^L(t+1) = \sum_{j=1}^N b_{ij}(t, 0) x_j^L(0), \quad i = 1, 2, \dots, N$$

and

$$x_i^U(t+1) = \sum_{j=1}^N b_{ij}(t, 0) x_j^U(0), \quad i = 1, 2, \dots, N$$

where $b_{ij}(t, 0) > 0$ and $\sum_{j=1}^N b_{ij}(t, 0) = 1$.

Because a consensus is reached at time $t+1$,

$$x^c = [x^{cL}, x^{cU}] = [x_i^L(t+1), x_i^U(t+1)], \quad i = 1, 2, \dots, N.$$

Moreover, the width $L(x^c)$ can be written as

$$\begin{aligned}
L(x^c) &= x_i^U(t+1) - x_i^L(t+1) = \sum_{j=1}^N b_{ij}(t,0)(x_j^U(0) - x_j^L(0)) \\
&\leq \max_{j=1,2,\dots,N} \{b_{ij}(t,0)\} L(x_{\max}^{(0)}) + (1 - \max_{j=1,2,\dots,N} \{b_{ij}(t,0)\}) L(x_{\min}^{(0)})
\end{aligned}$$

(1) If $t < N^2 - N + 2$, the according to **Lemma 2**, we have $\min_{j=1,2,\dots,N} \{b_{ij}(t,0)\} \geq (1/N)^t$.

We can then determine the maximum accumulated weight as

$$\max_{j=1,2,\dots,N} \{b_{ij}(t,0)\} \leq 1 - (N-1) \min_{j=1,2,\dots,N} \{b_{ij}(t,0)\} = 1 - (N-1)(1/N)^t$$

Furthermore, we obtain

$$L(x^c) \leq [1 - (N-1)(1/N)^t] L(x_{\max}^{(0)}) + (N-1)(1/N)^t L(x_{\min}^{(0)})$$

(2) If $t \geq N^2 - N + 2$, and $\varepsilon_1(t) = \varepsilon_2(t) = \dots = \varepsilon_N(t)$, then according to Lemma 2, we have $\min_{j=1,2,\dots,N} \{b_{ij}(t,0)\} = (1/N)^{N^2-N+2}$.

Because $b_{ij}(t,0) > 0$, we can then determine the maximum accumulated weight as

$$\max_{j=1,2,\dots,N} \{b_{ij}(t,0)\} = 1 - (N-1) \min_{j=1,2,\dots,N} \{b_{ij}(t,0)\} = 1 - (N-1)(1/N)^{N^2-N+2}.$$

Furthermore, we obtain

$$L(x^c) \leq [1 - (N-1)(1/N)^{N^2-N+2}] L(x_{\max}^{(0)}) + (N-1)(1/N)^{N^2-N+2} L(x_{\min}^{(0)}).$$

(3) We can adopt a proof similar to (1).

This completes the proof of **Theorem 2**. $\square$

A5. Proof of Theorem 3.

Proof. Let $c_g(t)$ be the subset of agents who hold the interval opinion $\bar{x}^g(t)$ at time t ,

$$c_g(t) \subseteq A, \bigcup_{g=1}^{f(t)} c_g(t) = A, c_g(t) \cap c_h(t) = \emptyset, g, h = 1, 2, \dots, f(t).$$

(1) Let $A_k \in c_g(t)$ and $A_u \in c_h(t)$; thus, we can easily see that $d(\bar{x}^g(t), \bar{x}^h(t)) > \max\{\varepsilon_k(t), \varepsilon_u(t)\}$ always holds. Because $\varepsilon_k(\tilde{t}) \leq \varepsilon_k(t)$ and $\varepsilon_u(\tilde{t}) \leq \varepsilon_u(t)$, $d(\bar{x}^g(t), \bar{x}^h(t)) > \varepsilon_i(\tilde{t})$. Thus, the agents in the sets $c_g(t)$ and $c_h(t)$ do no merge at time $\tilde{t}$, $\tilde{t} \geq t$.

Let $A_l \in c_g(t)$. We obtain $d(x_k(t), x_l(t)) = 0$. Although $\varepsilon_k(\tilde{t}) \leq \varepsilon_k(t)$ and $\varepsilon_l(t') \leq \varepsilon_l(t)$, the agents A_i and A_j still mutually affect each other. Furthermore, the agents in the set $c_g(t)$ still hold the same opinions at time $\tilde{t}$. Thus, we further obtain $\bar{x}^g(t) = \bar{x}^g(\tilde{t})$, $g = 1, 2, \dots, f(t)$. All of the interval opinions at time t are stabilized, and fragmentations are formed.

(2) For any two subsets $c_g(t)$ and $c_h(t)$, if $d(\bar{x}^g(t), \bar{x}^h(t)) > \alpha$, then it is implied that every agent in the set $c_g(t)$ ($c_h(t)$) has no neighbours in the other subsets; thus, $\bar{x}^g(t) = \bar{x}^g(t+1)$ ($\bar{x}^h(t) = \bar{x}^h(t+1)$) for $t = 1, 2, \dots$. Clearly, all of the interval opinions at time t are stabilized, and fragmentations are formed.

This completes the proof of **Theorem 3**. $\square$