Global large solutions for a coupled compressible Navier–Stokes/Allen–Cahn system with initial vacuum

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Abstract

In this paper, we investigate an initial boundary value problem for 1D compressible Navier–Stokes/Allen–Cahn system, which describes the motion of a mixture of two viscous compressible fluids. We establish the global existence and uniqueness of strong and classical solutions of Navier–Stokes/Allen–Cahn system by using the energy estimates, structure of the equations and the properties of one dimension. Here, we emphasize that the initial vacuum is allowed here.

Keywords: compressible Navier–Stokes/Allen–Cahn system; vacuum; strong and classical solutions

1 Introduction

In this paper, we study the “Navier–Stokes/Allen–Cahn” system, a combination of the compressible Navier–Stokes equations with an Allen–Cahn phase field description. Under some suitable assumptions, mathematically, the model in one dimension takes following form in the Eulerian coordinates [7] (c.f. [3]),

$$\rho_t + (\rho u)_x = 0, \quad (1.1)$$

$$\rho u_t + \rho u u_x + (a\rho^\gamma)_x = \nu u_{xx} - \frac{\delta}{2}(\chi_x^2)_x, \quad (1.2)$$

$$\rho \chi_t + \rho u \chi_x = -\mu, \quad (1.3)$$

$$\rho \mu = -\delta \chi_{xx} + \frac{\rho}{\delta}(\chi^3 - \chi), \quad (1.4)$$

for $(t, x) \in (0, +\infty) \times (0, 1)$. Here $\rho \geq 0$ denotes the total density, u represents the mean velocity of the fluid mixture, χ is the concentration difference of the two components, μ denotes the chemical potential, the viscous coefficient $\nu > 0$, the constant $\delta > 0$ and $\sqrt{\delta}$ represents the thickness of the interfacial region. We supplement (1.1)–(1.4) with the initial value conditions

$$(\rho, u, \chi)(0, x) = (\rho_0, u_0, \chi_0)(x), \quad x \in (0, 1), \quad (1.5)$$

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and the boundary conditions for viscous fluids and the concentration difference

$$(u, \chi)(t, 0) = (u, \chi)(t, 1) = (0, 0), \quad t \geq 0. \quad (1.6)$$

This model describes two-phase patterns in a flowing liquid including phase transformations. It is well-known that fluid-mechanical theory for two-phase mixtures of fluids faces a big mathematical difficulty: the movement of the interfaces is naturally amenable to a Lagrangian description, while the bulk fluid flow is usually considered in the Eulerian framework. Fortunately, the phase-field methods overcome this problem by postulating the existence of a “diffuse” interface spread over a possibly narrow region covering the “real” sharp interface boundary. A phase variable χ is introduced to demarcate the two species and to indicate the location of the interface. Moreover, this model also allows phases to grow or shrink due to changes of densities and incorporates their transport with the current. As pointed out in [3], the model should be viewed as a first step towards incorporating transport mechanism into the description of phase-formation processes.

Before stating our main results, we review some previous works related. The diffuse interface models for two-phase flows of incompressible fluids have been extensively studied. Boyer deeply studied the coupled Navier–Stokes/Cahn–Hilliard system in [4], including the existence of global weak/strong solutions in 2D, the existence of unique strong solution in 3D and the stability of the stationary solutions. For this type of models, the results of the well-posedness, the asymptotic behavior and the attractor were considered in [1, 16, 18] and the references cited therein. For the coupled Navier–Stokes/Allen–Cahn system, Zhao et al. [23] investigated the vanishing viscosity limit, where they proved that the solutions of the Navier–Stokes/Allen–Cahn system converge to that of the Euler/Allen–Cahn system in a suitable small time. Also, Xu et al. discussed the 3D axisymmetric solutions in [22]. And, Gal and Grasselli [17] discussed the existence of the trajectory attractor for both incompressible Navier–Stokes/Allen–Cahn and Navier–Stokes/Cahn–Hilliard systems.

However, there are much less theoretical results concerning with the compressible models. For the multidimensional compressible Navier–Stokes/Cahn–Hilliard system, Abels et al. [2] derived the global existence of weak solutions, where the density ρ is a renormalized solution. After that, Kotschote et al. [19] and Kotschote [20] established the local existence of unique strong solution without vacuum. For the Navier–Stokes/Allen–Cahn system, Feireisl et al. [15] proved the existence of weak solutions in 3D, where the density ρ is a measurable function. And a different compressible Navier–Stokes/Allen–Cahn system arising from the biological material change in the process of stem cell differentiation had been studied in [21]. Very recently, Ding et al. [7] studied the existence and uniqueness of global strong/classical solutions and the existence of weak solutions for Navier–Stokes/Allen–Cahn system in one dimension without vacuum. We also emphasize for a different two-phase model, Evje et al. obtained a series results in [10–14] and references therein.

Now, we introduce the notions and conventions used in the following paper. For simplicity, we denote

$$\int f dx = \int_I f dx,$$

where $I = (0, 1)$. We also denote $Q_T = [0, T] \times I$ for any $T > 0$. For $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(I), \quad W^{k,r} = \left\{ f \in L^r : \nabla^k f \in L^r \right\},$$

$$H^k = W^{k,2}, \quad H_0^k = W_0^{k,2}.$$

For any points $P_1(t_1, x_1), P_2(t_2, x_2) \in Q_T$, define the parabolic distance between them as

$$d(P_1, P_2) = (|x_1 - x_2|^2 + |t_1 - t_2|)^{\frac{1}{2}}.$$

Let $v(t, x)$ be a function on Q_T . For $0 < \alpha < 1$, define

$$[v]_{C^{\alpha, \alpha/2}(Q_T)} = \sup_{P_1, P_2 \in Q_T, P_1 \neq P_2} \frac{|v(P_1) - v(P_2)|}{d^\alpha(P_1, P_2)},$$

which is a semi-norm, and denote by $C^{\alpha, \alpha/2}(Q_T)$ the set of all functions on Q_T such that $[v]_{C^{\alpha, \alpha/2}(Q_T)} < +\infty$, endowed with norm

$$\|v\|_{C^{\alpha, \alpha/2}(Q_T)} = \|v\|_{C^0(Q_T)} + [v]_{C^{\alpha, \alpha/2}(Q_T)},$$

where $\|v\|_{C^0(Q_T)}$ is the maximum norm of $v(t, x)$ on Q_T .

Now, we state our main results in the paper.

Theorem 1.1 *Assume that $\rho_0 \geq 0$, $\rho_0 \in H^1$, $\mu_0 \in L^2$, $u_0 \in H_0^1$ and $\chi_0 \in H_0^1 \cap H^2$, and the initial data satisfying the following compatibility condition*

$$\nu u_{0xx} - (A\rho_0^\gamma)_x - \frac{1}{\delta} (\chi_{0x}^2)_x = \rho_0 f(x), \quad x \in [0, 1], \quad (1.7)$$

for a given function $f \in L^2$. Then there exists a global strong solution (ρ, u, χ) to the initial boundary value problem (1.1)–(1.6) such that for all $T \in (0, \infty)$,

$$\begin{cases} \rho \in L^\infty([0, T]; H^1), & u \in L^\infty([0, T]; H_0^1) \cap L^2(0, T; H^2), & \rho_t \in L^\infty([0, T]; L^2), \\ u_t \in L^2(0, T; H_0^1), & ((\rho u)_t, \rho \chi_t, u_{xx}) \in L^2([0, T]; L^2), & \chi \in L^\infty([0, T]; H_0^1 \cap H^2). \end{cases}$$

If the initial datum (ρ_0, u_0, χ_0) satisfy some additional regularity and compatibility conditions, then the global strong solutions obtained in Theorem 1.1 becomes classical ones.

Theorem 1.2 *Assume that $\rho_0 \geq 0$, $\rho_0 \in H^2$, $\mu_0 \in L^2$, $\rho_0^\gamma \in H^2$, $u_0 \in H_0^1 \cap H^3$, $\chi_0 \in H_0^1 \cap H^3$ and that the initial data satisfying the compatibility conditions*

$$\begin{cases} \nu u_{0xx}(x) - (A\rho_0^\gamma)_x - \frac{1}{\delta} (\chi_{0x}^2)_x(x) = \rho_0 g(x), & x \in [0, 1], \\ \mu_0(x) = \rho_0 h(x), \end{cases} \quad (1.8)$$

for given functions $g \in H^1$ and $h_{xx} \in L^2$. Then for any $T > 0$ there exists a unique global classical solution (ρ, u, χ) to (1.1)–(1.6) satisfying

$$\begin{cases} (\rho, \rho^\gamma) \in C([0, T]; H^2), & (\rho_t, (\rho^\gamma)_t) \in C([0, T]; H^1), & \rho_{tt} \in C([0, T]; L^2), \\ \rho \geq 0, & (\rho^\gamma)_{tt} \in L^\infty([0, T]; L^2), & (\rho u)_t \in C([0, T]; H^1), & (\rho \chi)_t \in C([0, T]; H^1), \\ (u, \chi) \in C([0, T]; H_0^1 \cap H^3), & (u_t, \chi_t) \in L^\infty([0, T]; H_0^1) \cap L^2([0, T]; H^2). \end{cases}$$

It is well-known that the possible presence of vacuum is one of the major difficulties in dealing with the compressible fluids, our main results improve the previous results in [7] with vacuum. The main difficulties come from the strong nonlinearity terms $\rho(\chi^3 - \chi)$ and $\frac{1}{2}(\chi_x)^2$, especially in obtaining the desired a priori estimate $\sup_{0 \leq t \leq T} \|\chi_{xx}\|_{L^2}$ and uniqueness of the main results, which are mainly depending on the lower bound of the

density ρ in [7]. Compared with the previous paper [7], we supplement the boundary condition on the concentration as $\chi(t, 0) = \chi(t, 1) = 0$ in (1.6) (c.f. [3]) instead of $\chi_x(t, 0) = \chi_x(t, 1) = 0$ for technical reason. For example, the key a priori estimates obtained in Lemma 2.5, the term $\int \rho^2 u_x \chi_t^2 dx$ in (2.11) is really difficult to get under the previous boundary condition, see also the a priori estimates in (2.55). Finally, we manage to obtain all the desired a priori estimates under the boundary condition in (1.6) and the advantage of the one space dimension.

Here, we emphasize that we borrow some ideas from [8] where they discussed the global large classical solutions to the compressible Navier–Stokes equations. Compared with [8], our results are more complicated due to the presence of the concentration difference of two components and the strong nonlinearity terms mentioned above. We also emphasize that the compatibility conditions in (1.7) and (1.8), which are similar as in [5, 6] for proving the existence and uniqueness of strong and classical solutions to compressible Navier–Stokes equations. And we emphasize that, in order to prove the global strong solutions, we only assume the compatibility (1.7) on the momentum equation.

Since the constants A , ν and δ do not play any role in the analysis, we assume henceforth that $A = \nu = \delta = 1$. Here, we emphasize that the viscosity ν depends on χ is much more realistic, we restrict ourself to the constant viscosity due to the technical reason.

The paper is organized as follows. In the next section, we deduce the desired uniform estimates globally in time for the proof of the main theorems. Due to the a priori estimates obtained, we finish the proof of Theorem 1.1 and 1.2 in the third section.

Before ending this section, we state the following local well-posedness result that we will need later.

Proposition 1.3 *Assume that $\rho_0 \in C^{1,\alpha}(I)$ satisfies $0 < C_0^{-1} \leq \rho_0 \leq C_0$ for some constant $\alpha \in (0, 1)$ and $C_0 > 0$, $u_0, \chi_0 \in C^{2,\alpha}(I)$ with $u_0(0) = u_0(1) = \chi_0(0) = \chi_0(1) = 0$. Then there exists a small time $T^* > 0$ depending only on (ρ_0, u_0, χ_0) such that the initial boundary value problem (1.1)–(1.6) admits a unique classical solution $(\rho, u, \chi) : [0, T^*) \times I \rightarrow \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ satisfying that*

$$(\rho_x, \rho_t) \in C^{\alpha, \frac{\alpha}{2}}(Q_{T^*}), \quad 0 < C^{-1} \leq \rho \leq C \text{ on } Q_{T^*}, \quad (u, \chi) \in C^{2+\alpha, \frac{2+\alpha}{2}}(Q_{T^*}).$$

The result is similar as Theorem 2.1 in [7], and the method in [7] is still valid here and thus we omit the details here.

2 A priori estimates

In this section, we establish some necessary uniform (in time) a priori estimates of the solutions to (1.1)–(1.6) to extend the local solution, guaranteed by the Proposition 1.3, to a global one. Thus, let $T > 0$ be a fixed time and (ρ, u, χ) is a smooth solution to (1.1)–(1.6) on $(0, T) \times I$ with smooth initial data. We emphasize the in the following of this section, C denotes some positive constant, which may depend on the initial data (ρ_0, u_0, χ_0) and T , without the lower bound of the initial density ρ_0 . First, we have the following basic energy estimates.

Lemma 2.1 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} \int \left(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + \frac{\rho(\chi^2 - 1)^2}{4} + \frac{\chi_x^2}{2} \right) dx + \int_0^T \int (u_x^2 + \mu^2) dx dt \leq C. \quad (2.1)$$

Proof. Multiplying (1.2) by u , then integrating it over I , using integration by parts, ones easily deduce

$$\frac{d}{dt} \int \left(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} \right) dx + \int u_x^2 dx = \frac{1}{2} \int \chi_x^2 u_x dx. \quad (2.2)$$

Next, multiplying (1.3) by μ and integrating the resultant equation over I , together with the help of (1.1) and (1.4), we obtain

$$\frac{d}{dt} \int \left(\rho \left(\frac{\chi^4}{4} - \frac{\chi^2}{2} \right) + \frac{\chi_x^2}{2} \right) dx + \int \mu^2 dx = -\frac{1}{2} \int \chi_x^2 u_x dx. \quad (2.3)$$

Finally, adding (2.2) to (2.3), and integrating the resulting equation over $(0, t)$, we can show (2.1). Thus, we complete the proof of Lemma 2.1. $\square$

Due to the basic energy inequality, we now deduce the upper bound of the concentration difference χ .

Lemma 2.2 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} \|\chi\|_{L^\infty} \leq C. \quad (2.4)$$

Proof. The proof of this lemma is similar as the one in [7] (c.f. Lemma 3.2), we sketch it here for completeness. From the equation (1.1), we have

$$\int \rho(t, \xi) d\xi = \int \rho_0(\xi) d\xi.$$

Without loss of generality, we assume that $\rho_0 \not\equiv 0$ (otherwise the problem is trivial: $\rho \equiv 0$) and $\int \rho_0(\xi) d\xi = 1$ for simplicity throughout this paper. Then, it follows from (2.1), that

$$\int \rho(\chi^2 - 1)^2 dx \leq C,$$

which implies that

$$\int \rho \chi^4 dx \leq 2 \int \rho \chi^2 dx - \int \rho dx + C \leq \frac{1}{4} \int \rho \chi^4 dx + C.$$

Therefore, we have

$$\int \rho \chi^4 dx \leq C,$$

and

$$\int \rho \chi dx \leq \int \rho \chi^4 dx + \int \rho dx \leq C.$$

Observing that $\int \rho dx = 1$ and $\rho \geq 0$, for any $(t, x) \in (0, T) \times I$, we deduce

$$|\chi(t, x)| = \left| \chi(t, x) \int \rho(t, y) dy \right|$$

$$\begin{aligned}
&\leq \left| \int \rho(t, y) (\chi(t, x) - \chi(t, y)) dy \right| + \left| \int \rho(t, y) \chi(t, y) dy \right| \\
&\leq \left| \int \rho(t, y) \left(\int_y^x \chi_\xi(t, y) d\xi \right) dy \right| + C \\
&\leq \int |\chi_x| \left(\int \rho(t, y) dy \right) dx + C \\
&\leq \left(\int \chi_x^2 dx \right)^{\frac{1}{2}} + C \leq C,
\end{aligned}$$

where we have used (2.1) and Hölder inequality. Therefore, we complete the proof of Lemma 2.2. $\square$

Next, we proceed to derive a uniform (in time) upper bound for the density ρ .

Lemma 2.3 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} \|\rho\|_{L^\infty} \leq C. \quad (2.5)$$

Proof. For any $t \in [0, T]$, denote

$$w(t, x) = \int_0^t \left(u_x - \frac{\chi_x^2}{2} - \rho u^2 - \rho^\gamma \right) + \int_0^x \rho_0 u_0(\xi) d\xi,$$

which implies

$$w_x = \rho u.$$

This, together with (2.1) and Cauchy inequality shows

$$\int |w_x| dx \leq C.$$

On the other hand, one can easily obtain

$$\int |w| dx \leq C.$$

Since $W^{1,1} \hookrightarrow L^\infty$, one deduces

$$\sup_{t \in [0, T]} \|w\|_{L^\infty} \leq C. \quad (2.6)$$

Since $\rho \geq 0$, it suffices to prove $\rho(s, y) \leq C$. For any $(s, y) \in (0, T) \times I$, let $x(t, y)$ solve the following problem

$$\begin{cases} \frac{dx(t, y)}{dt} = u(t, x(t, y)), & 0 \leq t < s, \\ x(s, y) = y, & 0 \leq y \leq 1. \end{cases}$$

Let us denote

$$F(t, x) = \exp \left\{ \int_1^{\rho(t, x)} \frac{1}{\xi} d\xi + w(t, x) \right\}.$$

Then, due to (1.1), we have

$$\frac{dF(t, x(t, y))}{dt} = F_t + u F_x$$

$$\begin{aligned}
&= F \left(\frac{\rho_t}{\rho} + u_x - \frac{\chi_x^2}{2} - \rho u^2 - \rho^\gamma + \frac{u\rho_x}{\rho} + \rho u^2 \right) \\
&= - \left(\frac{\chi_x^2}{2} + \rho^\gamma \right) F \leq 0.
\end{aligned}$$

Then integrating the above inequality over $(0, t)$, we have

$$\rho(t, x) + \exp \{w(t, x)\} \leq \rho_0 + \exp \left\{ \int \rho_0 u_0 dx \right\} \leq C,$$

which together with (2.6) shows (2.5). Hence, we complete the proof of Lemma 2.3. $\square$

To proceed, we derive the L^2 -estimate on χ_{xx} .

Lemma 2.4 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\|\chi_{xx}\|_{L^2(0, T; L^2)} \leq C. \quad (2.7)$$

Proof. It follows from (1.4), that

$$\chi_{xx} = \rho (\chi^3 - \chi) - \rho \mu,$$

then

$$\begin{aligned}
\|\chi_{xx}\|_{L^2}^2 &\leq C \|\rho (\chi^3 - \chi)\|_{L^2}^2 + C \|\rho \mu\|_{L^2}^2 \\
&\leq C + C \|\mu\|_{L^2}^2,
\end{aligned}$$

where we have used (2.4) and (2.5). This together with (2.1), we obtain the desired estimate. Hence, we complete the proof of Lemma 2.4. $\square$

Next, we focus on L^2 -estimates about $\rho \chi_t$ and u_x , which are the key estimates for the proof of the main theorems.

Lemma 2.5 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} (\|\rho \chi_t\|_{L^2}^2 + \|u_x\|_{L^2}^2) + \int_0^T (\|\chi_{xt}\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2) dt \leq C. \quad (2.8)$$

Proof. We split the proof of this lemma into three steps.

Step 1 First, let us rewrite (1.3) and (1.4) as follows

$$\rho^2 \chi_t + \rho^2 u \chi_x = \chi_{xx} - \rho (\chi^3 - \chi). \quad (2.9)$$

Differentiate (2.9) with respect to t , we deduce that

$$\rho^2 \chi_{tt} + (\rho^2)_t \chi_t + \rho^2 u \chi_{xt} = \chi_{xxt} - (\rho^2)_t u \chi_x - \rho^2 u_t \chi_x - \rho_t (\chi^3 - \chi) - \rho (3\chi^2 - 1) \chi_t. \quad (2.10)$$

Then, multiplying the above equality by χ_t and integrating it over I with respect to x , one has

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int \rho^2 \chi_t^2 dx + \int \chi_{xt}^2 dx \\
&= \frac{1}{2} \int \rho^2 u_x \chi_t^2 dx - 2 \int \rho^2 u \chi_t \chi_{xt} dx - \int \rho^2 u u_x \chi_x \chi_t dx - \int \rho^2 u^2 \chi_{xx} \chi_t dx
\end{aligned} \quad (2.11)$$

$$\begin{aligned}
& - \int \rho^2 u^2 \chi_x \chi_{xt} dx - \int \rho^2 u_t \chi_x \chi_t dx - \int \rho u (3\chi^2 - 1) \chi_x \chi_t dx \\
& - \int \rho u (\chi^3 - \chi) \chi_{xt} dx - \int \rho (3\chi^2 - 1) \chi_t^2 dx \\
& = \sum_{i=1}^9 I_i.
\end{aligned}$$

Now, we estimate each term on the right-hand side of (2.11). First, due to Sobolev inequality, Hölder inequality and Young inequality

$$\begin{aligned}
I_1 & \leq C \|\rho\|_{L^\infty} \|\chi_t\|_{L^\infty} \|u_x\|_{L^2} \|\rho \chi_t\|_{L^2} \\
& \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2,
\end{aligned}$$

where we have used (2.5) and the boundary condition (1.6). Similarly, we can also estimate I_2 and I_3 as follows:

$$I_2 \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2,$$

and

$$I_3 \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C \|u_x\|_{L^2}^4.$$

It follows from (2.9), we can rewrite I_4 as follows

$$I_4 = - \int \rho^4 u^2 \chi_t^2 dx - \int \rho^4 u^3 \chi_x \chi_t dx - \int \rho^3 u^2 (\chi^3 - \chi) \chi_t dx = \sum_{i=1}^3 I_{4i}. \quad (2.12)$$

Due to (2.1), (2.4) and (2.5), we can estimate each term on the right-hand side of (2.12) as follow

$$\begin{aligned}
I_{41} & \leq C \|\rho\|_{L^\infty}^2 \|u\|_{L^\infty}^2 \|\rho \chi_t\|_{L^2}^2 \leq C \|u_x\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2, \\
I_{42} & \leq C \|\rho\|_{L^\infty}^3 \|u\|_{L^\infty}^3 \|\chi_x\|_{L^2} \|\rho \chi_t\|_{L^2} \\
& \leq C \|u_x\|_{L^2}^4 + C \|u_x\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2, \\
I_{43} & \leq C \|\rho\|_{L^\infty}^2 \|\chi^3 - \chi\|_{L^\infty} \|\chi_t\|_{L^\infty} \|\sqrt{\rho} u\|_{L^2}^2 \\
& \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C.
\end{aligned}$$

Then, substituting the above estimates into (2.12), we finally deduce that

$$I_4 \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2 + C \|u_x\|_{L^2}^4 + C \|\chi_{xx}\|_{L^2}^2 + C.$$

Similarly, we can estimate I_5 – I_9 as follow

$$\begin{aligned}
I_5 & \leq C \|\rho\|_{L^\infty}^2 \|u\|_{L^\infty}^2 \|\chi_x\|_{L^2} \|\chi_{xt}\|_{L^2} \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C \|u_x\|_{L^2}^4, \\
I_6 & \leq C \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\chi_x\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} \|\rho \chi_t\|_{L^2} \\
& \leq \delta \|\sqrt{\rho} u_t\|_{L^2}^2 + C(\delta) (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) \|\rho \chi_t\|_{L^2}^2, \\
I_7 & \leq C \|u\|_{L^\infty} \|3\chi^2 - 1\|_{L^\infty} \|\chi_x\|_{L^2} \|\rho \chi_t\|_{L^2} \leq C \|u_x\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2 + C,
\end{aligned}$$

$$I_8 \leq C \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\chi^3 - \chi\|_{L^\infty} \|\sqrt{\rho}u\|_{L^2} \|\chi_{xt}\|_{L^2} \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C,$$

$$I_9 \leq C \|3\chi^2 - 1\|_{L^\infty} \|\rho\chi_t\|_{L^2} \|\chi_t\|_{L^2} \leq \frac{1}{16} \|\chi_{xt}\|_{L^2}^2 + C \|\rho\chi_t\|_{L^2}^2.$$

Then, substituting all the above estimates into (2.11), and then integrating over (τ, t) with respect to time $t \in (0, T)$, where $\tau \in (0, t)$, one obtains

$$\begin{aligned} & \int \rho^2 \chi_t^2(t) dx + \int_\tau^t \int \chi_{xt}^2 dx ds \\ & \leq C \int_\tau^t \|u_x\|_{L^2}^2 \|\rho\chi_t\|_{L^2}^2 ds + C \int_\tau^t \|u_x\|_{L^2}^4 ds + \delta \int_\tau^t \|\sqrt{\rho}u_t\|_{L^2}^2 ds \\ & \quad + C \int_\tau^t \|\chi_{xx}\|_{L^2}^2 \|\rho\chi_t\|_{L^2}^2 ds + \int \rho^2 \chi_t^2(\tau) dx + C \int_\tau^t \|\chi_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 ds + C. \end{aligned} \quad (2.13)$$

On the other hand, multiplying (2.9) by χ_t , we can easily obtain

$$\begin{aligned} \int \rho^2 \chi_t^2(\tau) dx & \leq C \int \rho^2 u^2 \chi_x^2(\tau) dx + C \int \rho^{-2} \chi_{xx}^2(\tau) dx + C \\ & \leq C \|u_x(\tau)\|_{L^2}^2 \|\chi_x(\tau)\|_{L^2}^2 + C \|\mu(\tau)\|_{L^2}^2 + C, \end{aligned}$$

where we have used (1.4). Substituting the above inequality into (2.13), then letting $\tau \rightarrow 0^+$, we conclude that

$$\begin{aligned} & \int \rho^2 \chi_t^2 dx + \int_0^t \int \chi_{xt}^2 dx ds \\ & \leq C \int_0^t \|u_x\|_{L^2}^2 \|\rho\chi_t\|_{L^2}^2 ds + C \int_0^t \|u_x\|_{L^2}^4 ds + \delta \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2 ds \\ & \quad + C \int_0^t \|\chi_{xx}\|_{L^2}^2 \|\rho\chi_t\|_{L^2}^2 ds + C \int_0^t \|\chi_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 ds + C. \end{aligned} \quad (2.14)$$

Step 2 Multiplying (1.2) by u_t , integrating it over I with respect to x , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int u_x^2 dx + \int \rho u_t^2 dx \\ & = \frac{d}{dt} \int \rho^\gamma u_x dx + \frac{1}{2} \frac{d}{dt} \int \chi_x^2 u_x dx - \int \chi_x \chi_{xt} u_x dx \\ & \quad - \int (\rho^\gamma)_t u_x dx - \int \rho u u_t u_x dx \\ & = \frac{d}{dt} \int \rho^\gamma u_x dx + \frac{1}{2} \frac{d}{dt} \int \chi_x^2 u_x dx + \sum_{i=1}^3 J_i. \end{aligned} \quad (2.15)$$

Now, we first estimate J_1, J_2 and J_3 . First of all, we can estimate J_1 and J_3 as follow.

$$\begin{aligned} |J_1| & \leq C \|\chi_x\|_{L^\infty} \|\chi_{xt}\|_{L^2} \|u_x\|_{L^2} \\ & \leq \varepsilon \|\chi_{xt}\|_{L^2}^2 + C(\varepsilon) (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) \|u_x\|_{L^2}^2, \end{aligned} \quad (2.16)$$

and

$$|J_3| \leq \frac{1}{4} \|\sqrt{\rho}u_t\|_{L^2}^2 + C \int \rho u^2 u_x^2 dx \quad (2.17)$$

$$\leq \frac{1}{4} \|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u_x\|_{L^2}^4.$$

As for J_2 , using (1.1) one easily obtain

$$(\rho^\gamma)_t + (\rho^\gamma u)_x + (\gamma - 1)\rho^\gamma u_x = 0. \quad (2.18)$$

Due to (2.18), we can rewrite J_2 as follow

$$J_2 = \int (\rho^\gamma u)_x u_x dx + (\gamma - 1) \int \rho^\gamma u_x^2 dx \leq - \int \rho^\gamma u u_{xx} dx + C \|\rho\|_{L^\infty}^{\gamma-1} \|u_x\|_{L^2}^2. \quad (2.19)$$

In order to estimate the first term on the right-hand side of (2.19), it follows from (1.2), we can rewrite it as

$$\begin{aligned} - \int \rho^\gamma u u_{xx} dx &= - \int \rho^{\gamma+1} u u_t dx - \int \rho^{\gamma+1} u^2 u_x dx \\ &\quad - \int \rho^\gamma u (\rho^\gamma)_x dx - \frac{1}{2} \int \rho^\gamma u (\chi_x^2)_x dx \\ &= \sum_{i=1}^4 J_{2i}. \end{aligned} \quad (2.20)$$

For each term on the right-hand side of (2.20) can be estimated as follows:

$$\begin{aligned} J_{21} &\leq C \|\rho\|_{L^\infty}^\gamma \|\sqrt{\rho}u\|_{L^2} \|\sqrt{\rho}u_t\|_{L^2} \leq \frac{1}{4} \|\sqrt{\rho}u_t\|_{L^2}^2 + C, \\ J_{22} &\leq C \|\rho\|_{L^\infty}^{\gamma+\frac{1}{2}} \|u\|_{L^\infty} \|\sqrt{\rho}u\|_{L^2} \|u_x\|_{L^2} \leq C \|\rho\|_{L^\infty}^{\gamma+\frac{1}{2}} \|\sqrt{\rho}u\|_{L^2} \|u_x\|_{L^2}^2, \\ J_{23} &= \frac{1}{2} \int \rho^{2\gamma} u_x dx \leq C \|u_x\|_{L^2}^2 + C, \\ J_{24} &\leq C \|\rho\|_{L^\infty}^\gamma \|u\|_{L^\infty} \|\chi_x\|_{L^2} \|\chi_{xx}\|_{L^2} \leq C (\|\chi_{xx}\|_{L^2}^2 + 1), \end{aligned}$$

where we have used (2.1), (2.5), Sobolev inequality, Hölder inequality and Young inequality. Then, substituting all the above estimates into (2.20), we have

$$J_2 \leq \frac{1}{4} \|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u_x\|_{L^2}^2 + C\|\chi_{xx}\|_{L^2}^2 + C. \quad (2.21)$$

Then, inserting (2.16), (2.17) and (2.21) into (2.15), we deduce that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int u_x^2 dx + \frac{1}{2} \int \rho u_t^2 dx \\ &\leq \frac{d}{dt} \int \rho^\gamma u_x dx + \frac{1}{2} \frac{d}{dt} \int \chi_x^2 u_x dx + C \|\chi_{xx}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \\ &\quad + C \|u_x\|_{L^2}^4 + \varepsilon \|\chi_{xt}\|_{L^2}^2 + C \|\chi_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 + C. \end{aligned}$$

Integrating the above inequality over $(0, t)$ with respect to time $t \in (0, T)$, we deduce that

$$\begin{aligned} &\|u_x\|_{L^2}^2 + \int_0^t \int \rho u_t^2 dx ds \\ &\leq 2 \int \rho^\gamma u_x dx + \int \chi_x^2 u_x dx + C \int_0^t \|u_x\|_{L^2}^4 ds \end{aligned} \quad (2.22)$$

$$\begin{aligned}
& + \varepsilon \int_0^t \|\chi_{xt}\|_{L^2}^2 ds + C \int_0^t \|\chi_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 ds + C \\
& \leq \frac{1}{2} \|u_x\|_{L^2}^2 + C \|\chi_x\|_{L^2}^2 \|\chi_{xx}\|_{L^2}^2 + C \int_0^t \|u_x\|_{L^2}^4 ds \\
& + \varepsilon \int_0^t \|\chi_{xt}\|_{L^2}^2 ds + C \int_0^t \|\chi_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 ds + C,
\end{aligned}$$

where we have used (2.1), (2.5), Hölder inequality, Young inequality and Sobolev embedding inequality.

It follows from (2.9), one deduces that

$$\begin{aligned}
\|\chi_{xx}\|_{L^2} & \leq C \|\rho \chi_t\|_{L^2} + C \|\chi_x\|_{L^\infty} \|\sqrt{\rho} u\|_{L^2} + C \\
& \leq C \|\rho \chi_t\|_{L^2} + C \|\chi_x\|_{L^2}^{\frac{1}{2}} \|\chi_{xx}\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho} u\|_{L^2} + C \\
& \leq \frac{1}{2} \|\chi_{xx}\|_{L^2}^2 + C \|\rho \chi_t\|_{L^2}^2 + C \|\chi_x\|_{L^2}^2 \|\sqrt{\rho} u\|_{L^2}^2 + C,
\end{aligned}$$

which implies

$$\|\chi_{xx}\|_{L^2} \leq C \|\rho \chi_t\|_{L^2} + C. \quad (2.23)$$

Then, substituting (2.23) into (2.22), we have

$$\begin{aligned}
& \|u_x\|_{L^2}^2 + \int_0^t \int \rho u_t^2 dx ds \\
& \leq C_1 \|\rho \chi_t\|_{L^2}^2 + C \int_0^t \|u_x\|_{L^2}^4 ds \\
& + \varepsilon \int_0^t \|\chi_{xt}\|_{L^2}^2 ds + C \int_0^t \|\chi_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 ds + C.
\end{aligned} \quad (2.24)$$

Step 3 Multiplying (2.14) by $1 + C_1$, then adding the resulting inequality into (2.22), and choosing δ and ε sufficiently small, we have

$$\begin{aligned}
& \|\rho \chi_t\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \int_0^t (\|\chi_{xt}\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2) ds \\
& \leq C \int_0^t \|u_x\|_{L^2}^4 ds + C \int_0^t \|\chi_{xx}\|_{L^2}^2 \|u_x\|_{L^2}^2 ds + C \int_0^t \|u_x\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2 ds \\
& + C \int_0^t \|u_x\|_{L^2}^4 ds + C \int_0^t \|\chi_{xx}\|_{L^2}^2 \|\rho \chi_t\|_{L^2}^2 ds + C,
\end{aligned}$$

which together with Gronwall inequality, (2.1), (2.7) and (2.8) shows (2.8). Therefore, we complete the proof of Lemma 2.5. $\square$

Next, we will derive important L^2 -estimates on the spatial and time derivative of the smooth density ρ .

Lemma 2.6 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} (\|\rho_x\|_{L^2}^2 + \|\rho_t\|_{L^2}^2) \leq C. \quad (2.25)$$

Proof. Differentiating (1.1) with respect to x , multiplying the resultant equation by ρ_x , and using integration by parts, we finally obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int \rho_x^2 dx &= -\frac{3}{2} \int \rho_x^2 u_x dx - \int \rho^2 \rho_x u_t dx - \int \rho^2 \rho_x u u_x dx \\
&\quad - \int \rho \rho_x (\rho^\gamma)_x dx + \frac{1}{2} \int \rho \rho_x (\chi_x^2)_x dx \\
&\leq C \|u_x\|_{L^\infty} \|\rho_x\|_{L^2}^2 + C \|\rho\|_{L^\infty}^{\frac{3}{2}} \|\sqrt{\rho} u_t\|_{L^2} \|\rho_x\|_{L^2} \\
&\quad + C \|\rho\|_{L^\infty}^2 \|\rho_x\|_{L^2} \|u_x\|_{L^2}^2 + C \|\rho\|_{L^\infty}^\gamma \|\rho_x\|_{L^2}^2 \\
&\quad + C \|\rho\|_{L^\infty} \|\chi_x\|_{L^\infty} \|\rho_x\|_{L^2} \|\chi_{xx}\|_{L^2} \\
&\leq C (\|u_x\|_{L^\infty} + 1) \|\rho_x\|_{L^2}^2 + C (\|\chi_{xx}\|_{L^2} + 1) \|\rho_x\|_{L^2},
\end{aligned} \tag{2.26}$$

where we have used (1.2), (2.4), (2.5) and (2.8). To proceed, we see that

$$\begin{aligned}
\|u_x\|_{L^\infty} &\leq C \|u_x - \rho^\gamma\|_{L^\infty} + C \\
&\leq C \int [|u_x - \rho^\gamma| + |(u_x - \rho^\gamma)_x|] dx + C \\
&\leq C \int |\rho u_t + \rho u u_x + \chi_x \chi_{xx}| dx + C \\
&\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|\chi_{xx}\|_{L^2} + C,
\end{aligned} \tag{2.27}$$

where we have used (2.4), (2.5), (2.8), Hölder inequality and Sobolev inequality. Substituting the above inequality into (2.26), using (2.7), (2.8) and Gronwall's inequality, one deduces

$$\sup_{t \in [0, T]} \|\rho_x\|_{L^2}^2 \leq C.$$

This together with (1.1) and (2.8) shows

$$\sup_{t \in [0, T]} \|\rho_t\|_{L^2}^2 \leq C.$$

Hence, we complete the proof of Lemma 2.6. $\square$

Next, we focus on the second and third spatial derivative of the concentration difference χ .

Lemma 2.7 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} \|\chi_{xx}\|_{L^2}^2 + \int_0^T \|\chi_{xxx}\|_{L^2}^2 dt \leq C. \tag{2.28}$$

Proof. It directly follows from (2.9) that

$$\|\chi_{xx}\|_{L^2}^2 \leq C \|\rho \chi_t\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \|\chi_x\|_{L^2}^2 + C,$$

which together with (2.8) and (2.1) shows the first part of (2.28). It still follows from (2.9),

$$\chi_{xxx} = 2\rho \rho_x \chi_t + \rho^2 \chi_{xt} + 2\rho \rho_x u \chi_x + \rho^2 u_x \chi_x + \rho^2 u \chi_{xx} + \rho_x (\chi^3 - \chi) + \rho (3\chi^2 - 1) \chi_x, \tag{2.29}$$

from which, one deduces that

$$\begin{aligned} \|\chi_{xxx}\|_{L^2}^2 &\leq C\|\chi_{xt}\|_{L^2}^2\|\rho_x\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2 + C\|\rho_x\|_{L^2}^2\|u_x\|_{L^2}^2 (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) \\ &\quad + C\|u_x\|_{L^2}^2 (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) + C\|u_x\|_{L^2}^2\|\chi_{xx}\|_{L^2}^2 + C\|\rho_x\|_{L^2}^2 + C\|\chi_x\|_{L^2}^2, \end{aligned} \quad (2.30)$$

where we have used (2.1), (2.4), (2.5), (2.7), (2.8), (2.25) and Sobolev inequality. From which together with (2.25), (2.1) and (2.8), one easily obtains the second part of (2.28). Therefore, we complete the proof of Lemma 2.7. $\square$

Next, with the a priori estimates at hands, we derive the L^2 -estimate on u_{xx} .

Lemma 2.8 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\int_0^T \|u_{xx}\|_{L^2}^2 dt \leq C. \quad (2.31)$$

Proof. It follows from (1.2), (2.5), (2.8), (2.25) and (2.28) that

$$\|u_{xx}\|_{L^2}^2 \leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u_x\|_{L^2}^4 + C\|\rho_x\|_{L^2}^2 + C\|\chi_x\|_{L^2}\|\chi_{xx}\|_{L^2}^3, \quad (2.32)$$

which together with (2.8), (2.25) and (2.28) shows that (2.31). This completes the proof of Lemma 2.8. $\square$

The following lemma shows the estimates on spatial and time derivatives of the velocity, which are crucial for deducing the higher order estimates of the solution.

Lemma 2.9 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} \|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T \|u_{xt}\|_{L^2}^2 dt \leq C. \quad (2.33)$$

Proof. Differentiating (1.2) with respect to t , we have

$$\rho u_{tt} + \rho_t u_t + \rho_t u u_x + \rho u_t u_x + \rho u u_{xt} + (\rho^\gamma)_{xt} = u_{xxt} - \frac{1}{2}(\chi_x^2)_{xt}. \quad (2.34)$$

Multiplying (2.34) by u_t , integrating over I and using integration by parts, one obtains

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \rho u_t^2 dx + \int u_{xt}^2 dx \\ &= -2 \int \rho u u_t u_{xt} dx - \int \rho_t u u_t u_x dx - \int \rho u_t^2 u_x dx \\ &\quad + \gamma \int \rho^{\gamma-1} \rho_t u_{xt} dx + \int \chi_x \chi_{xt} u_{xt} dx \\ &\leq C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|u_{xt}\|_{L^2} + C\|u\|_{L^\infty}\|u_t\|_{L^\infty}\|\rho_t\|_{L^2}\|u_x\|_{L^2} \\ &\quad + C\|u_x\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho\|_{L^\infty}^{\gamma-1}\|\rho_t\|_{L^2}\|u_{xt}\|_{L^2} + C\|\chi_x\|_{L^\infty}\|\chi_{xt}\|_{L^2}\|u_{xt}\|_{L^2} \\ &\leq C\|u_x\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|u_{xt}\|_{L^2} + C\|u_x\|_{L^2}^2\|\rho_t\|_{L^2}\|u_{xt}\|_{L^2} + C\|u_x\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}^2 \\ &\quad + C\|\rho_t\|_{L^2}\|u_{xt}\|_{L^2} + C\|\chi_x\|_{L^2}^{\frac{1}{2}}\|\chi_{xx}\|_{L^2}^{\frac{1}{2}}\|\chi_{xt}\|_{L^2}\|u_{xt}\|_{L^2}, \end{aligned}$$

which, together with Young inequality, shows that

$$\frac{d}{dt} \int \rho u_t^2 dx + \int u_{xt}^2 dx \leq C(1 + \|u_{xx}\|_{L^2})\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2 + C, \quad (2.35)$$

where we have used (2.7), (2.8) and (2.25).

Due to (1.7), we see that

$$u_t(0, x) = \frac{1}{\rho_0} (u_{0x} - \chi_{0x}^2 - \rho_0^\gamma)_x(x) - u_0 u_{0x} = g(x) - u_0 u_{0x} \in L^2. \quad (2.36)$$

Integrating (2.35) over $(0, t)$, and using (2.8), (2.27), (2.7), (2.36) and Gronwall inequality, we show that (2.33). Therefore, we complete the proof of Lemma 2.9. $\square$

From now on, we devote ourselves to the higher estimates on the smooth solution (ρ, u, χ) that are needed to guarantee the extension of a local classical solution to a global one.

Lemma 2.10 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} \|u_{xx}\|_{L^2}^2 dt \leq C. \quad (2.37)$$

Proof. We firstly claim that

$$\sup_{t \in [0, T]} \|u_x\|_{L^\infty} \leq C, \quad (2.38)$$

which comes from (2.27) together with the estimates (2.28) and (2.33).

Next, due to (2.5), (2.8), (2.38), (2.25) and (2.28), one easily deduces the desired estimate (2.37) from (2.32). $\square$

Lemma 2.11 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\begin{aligned} & \sup_{t \in [0, T]} (\|\rho_{xx}\|_{L^2}^2 + \|\rho_{xt}\|_{L^2}^2 + \|(\rho^\gamma)_{xx}\|_{L^2}^2 + \|(\rho^\gamma)_{xt}\|_{L^2}^2) \\ & + \int_0^T (\|\rho_{tt}\|_{L^2}^2 + \|(\rho^\gamma)_{tt}\|_{L^2}^2) dt \leq C. \end{aligned} \quad (2.39)$$

Proof. Differentiating twice (1.1) with respect to x , we obtain

$$\rho_{xxt} = -\rho_{xxx}u - 3\rho_{xx}u_x - 3\rho_x u_{xx} - \rho u_{xxx}. \quad (2.40)$$

Multiplying (2.40) by ρ_{xx} , integrating the resulting equation over I with respect to x , and using integration by parts, we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \rho_{xx}^2 dx & \leq C \|u_x\|_{L^\infty} \|\rho_{xx}\|_{L^2}^2 + C \|\rho_x\|_{L^\infty} \|\rho_{xx}\|_{L^2} \|u_{xx}\|_{L^2} \\ & + C \|\rho_{xx}\|_{L^2} \|u_{xxx}\|_{L^2} \\ & \leq C \|\rho_{xx}\|_{L^2}^2 + C (\|\rho_x\|_{L^2} + \|\rho_{xx}\|_{L^2}) \|\rho_{xx}\|_{L^2} + C \|u_{xxx}\|_{L^2}^2 \\ & \leq C \|\rho_{xx}\|_{L^2}^2 + C \|u_{xxx}\|_{L^2}^2 + C, \end{aligned} \quad (2.41)$$

where we have used (2.38), (2.25), Sobolev inequality and Young inequality. Similarly, differentiating (2.18) twice with respect to x , and then multiplying the resulting equation by $(\rho^\gamma)_{xx}$, integrating it over I , using the integration by parts, one can also obtain

$$\frac{1}{2} \frac{d}{dt} \int (\rho^\gamma)_{xx}^2 dx \leq C \|(\rho^\gamma)_{xx}\|_{L^2}^2 + C \|u_{xxx}\|_{L^2}^2 + C,$$

which together with (2.41), we have

$$\frac{1}{2} \frac{d}{dt} \int (\rho_{xx}^2 + (\rho^\gamma)_{xx}^2) dx \leq C (\|\rho_{xx}\|_{L^2}^2 + \|(\rho^\gamma)_{xx}\|_{L^2}^2) + C \|u_{xxx}\|_{L^2}^2 + C. \quad (2.42)$$

To proceed, differentiating (1.2) with respect to x , one obtains

$$u_{xxx} = \rho_x u_t + \rho u_{xt} + \rho_x u u_x + \rho u_x^2 + \rho u u_{xx} + (\rho^\gamma)_{xx} + \chi_{xx}^2 + \chi_x \chi_{xxx}, \quad (2.43)$$

which combining (2.5), (2.7), (2.8), (2.25), (2.28), (2.33), (2.37) and Sobolev inequality, we have

$$\begin{aligned} \|u_{xxx}\|_{L^2}^2 &\leq C \|u_{xt}\|_{L^2}^2 \|\rho_x\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 + C (\|\rho_x\|_{L^2}^2 + \|\rho_{xx}\|_{L^2}^2) \|u_x\|_{L^2}^4 \\ &\quad + C \|u_x\|_{L^2}^3 \|u_{xx}\|_{L^2} + C \|u_x\|_{L^2}^2 \|u_{xx}\|_{L^2}^2 + C \|(\rho^\gamma)_{xx}\|_{L^2}^2 \\ &\quad + C \|\chi_{xx}\|_{L^2}^3 \|\chi_{xxx}\|_{L^2} + C (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) \|\chi_{xxx}\|_{L^2}^2 \\ &\leq C \|\chi_{xxx}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 + C \|(\rho^\gamma)_{xx}\|_{L^2}^2 + C \|\rho_{xx}\|_{L^2}^2 + C. \end{aligned} \quad (2.44)$$

Combining (2.42) and (2.44), we deduce

$$\frac{d}{dt} \int (\rho_{xx}^2 + (\rho^\gamma)_{xx}^2) dx \leq C \|\rho_{xx}\|_{L^2}^2 + C \|(\rho^\gamma)_{xx}\|_{L^2}^2 + C \|\chi_{xxx}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 + C.$$

Due to (2.28) and (2.33) together with Gronwall inequality, we get

$$\sup_{t \in [0, T]} (\|\rho_{xx}\|_{L^2}^2 + \|(\rho^\gamma)_{xx}\|_{L^2}^2) \leq C. \quad (2.45)$$

Next, it follows from (1.1), that

$$\rho_{xt} = -\rho_{xx} u - 2\rho_x u_x - \rho u_{xx},$$

which follows from (2.8), (2.45), (2.38) and (2.37), one obtains

$$\|\rho_{xt}\|_{L^2}^2 \leq C \|u_x\|_{L^2}^2 \|\rho_{xx}\|_{L^2}^2 + C \|u_x\|_{L^\infty}^2 \|\rho_x\|_{L^2}^2 + C \|u_{xx}\|_{L^2}^2 \leq C. \quad (2.46)$$

And similarly, it follows from (2.18), we show

$$\sup_{t \in [0, T]} \|(\rho^\gamma)_{xt}\|_{L^2}^2 \leq C,$$

and

$$\sup_{t \in [0, T]} (\|\rho_{xt}\|_{L^2}^2 + \|(\rho^\gamma)_{xt}\|_{L^2}^2) \leq C. \quad (2.47)$$

Moreover, it follows from (1.1), that

$$\rho_{tt} = -\rho_{xt} u + \rho_x u_t + \rho_t u_x - \rho u_{xt},$$

from which we have

$$\|\rho_{tt}\|_{L^2}^2 \leq C \|u_x\|_{L^2}^2 \|\rho_{xt}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 \|\rho_x\|_{L^2}^2 + C \|u_x\|_{L^\infty}^2 \|\rho_t\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2. \quad (2.48)$$

Similarly, one obtains

$$\|(\rho^\gamma)_{tt}\|_{L^2}^2 \leq C \|u_x\|_{L^2}^2 \|(\rho^\gamma)_{xt}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 \|\rho_x\|_{L^2}^2 + C \|u_x\|_{L^\infty}^2 \|\rho_t\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2, \quad (2.49)$$

which together with (2.48), (2.38), (2.47) and (2.33), shows that

$$\int_0^T (\|\rho_{tt}\|_{L^2}^2 + \|(\rho^\gamma)_{tt}\|_{L^2}^2) dt \leq C.$$

Combining this together with (2.47) and (2.45), one deduces (2.39). Therefore, we complete the proof of Lemma 2.11. $\square$

From (2.44), (2.33) and (2.39), we can immediately obtain the following result.

Lemma 2.12 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\int_0^T \|u_{xxx}\|_{L^2}^2 dt \leq C. \quad (2.50)$$

The next lemma plays the key role to prove the existence of global classical solutions in Theorem 1.2.

Lemma 2.13 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} (\|u_{xt}\|_{L^2}^2 + \|\chi_{xt}\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + \|\rho\chi_{tt}\|_{L^2}^2) dt \leq C. \quad (2.51)$$

Proof. Multiplying (2.10) by χ_{tt} , integrating it over I , and using integration by parts, we deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \chi_{xt}^2 dx + \int \rho^2 \chi_{tt}^2 dx \\ &= -\frac{d}{dt} \int \rho_t (\chi^3 - \chi) \chi_t dx - \int \rho^2 u \chi_{xt} \chi_{tt} dx - 2 \int \rho \rho_t u \chi_x \chi_{tt} dx \\ & \quad - \int \rho^2 u_t \chi_x \chi_{tt} dx - 2 \int \rho \rho_t \chi_t \chi_{tt} dx - \int \rho (3\chi^2 - 1) \chi_t \chi_{tt} dx \\ & \quad + \int \rho_{tt} (\chi^3 - \chi) \chi_t dx + \int \rho_t (3\chi^2 - 1) \chi_t^2 dx \\ &= -\frac{d}{dt} \int \rho_t (\chi^3 - \chi) \chi_t dx + \sum_{i=1}^7 L_i. \end{aligned} \quad (2.52)$$

Now, we estimate each term on the right-hand side of (2.40). It follows from (2.4), (2.5), Hölder inequality, Sobolev inequality and Young inequality that

$$\begin{aligned} L_1 &\leq C \|\rho\|_{L^\infty} \|u\|_{L^\infty} \|\rho\chi_{tt}\|_{L^2} \|\chi_{xt}\|_{L^2} \leq \frac{1}{10} \|\rho\chi_{tt}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \|\chi_{xt}\|_{L^2}^2; \\ L_2 &\leq C \|u\|_{L^\infty} \|\chi_x\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho\chi_{tt}\|_{L^2} \\ &\leq \frac{1}{10} \|\rho\chi_{tt}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) \|\rho_t\|_{L^2}^2; \\ L_3 &\leq C \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\chi_x\|_{L^\infty} \|\sqrt{\rho}u_t\|_{L^2} \|\rho\chi_{tt}\|_{L^2} \\ &\leq \frac{1}{10} \|\rho\chi_{tt}\|_{L^2}^2 + C (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) \|\sqrt{\rho}u_t\|_{L^2}^2; \\ L_4 &\leq C \|\chi_t\|_{L^\infty} \|\rho\chi_{tt}\|_{L^2} \|\rho_t\|_{L^2} \leq \frac{1}{10} \|\rho\chi_{tt}\|_{L^2}^2 + C \|\chi_{xt}\|_{L^2}^2 \|\rho_t\|_{L^2}^2; \\ L_5 &\leq C \|(3\chi^2 - 1)\|_{L^\infty} \|\chi_t\|_{L^2} \|\rho\chi_{tt}\|_{L^2} \leq \frac{1}{10} \|\rho\chi_{tt}\|_{L^2}^2 + C \|\chi_{xt}\|_{L^2}^2; \end{aligned}$$

$$\begin{aligned}
L_6 &\leq C\|(\chi^3 - \chi)\|_{L^\infty}\|\chi_t\|_{L^\infty}\|\rho_{tt}\|_{L^2} \leq C\|\chi_{xt}\|_{L^2}^2 + C\|\rho_{tt}\|_{L^2}^2; \\
L_7 &\leq C\|(3\chi^2 - 1)\|_{L^\infty}\|\chi_t\|_{L^\infty}^2\|\rho_t\|_{L^2} \leq C\|\chi_{xt}\|_{L^2}^2 + C\|\rho_{tt}\|_{L^2}^2.
\end{aligned}$$

Substituting all the above estimates into (2.52), then integrating over $(0, t)$, one deduces

$$\begin{aligned}
&\|\chi_{xt}(t)\|_{L^2}^2 + \int_0^t \|\rho\chi_{tt}\|_{L^2}^2 dt \\
&\leq C\|\chi_{xt}(0)\|_{L^2}^2 + C \int_0^t \|\chi_{xt}\|_{L^2}^2 dt + C \int \rho_t(\chi^3 - \chi)\chi_t(t) dx \\
&\quad + C \int \rho_t(\chi^3 - \chi)\chi_t(0) dx + C. \\
&\leq \frac{1}{2}\|\chi_{xt}(t)\|_{L^2}^2 + C \int_0^t \|\chi_{xt}\|_{L^2}^2 dt + C,
\end{aligned} \tag{2.53}$$

where we have used (2.1), (2.4), (2.5), (2.8), (2.25), (2.28) and the compatibility condition (1.8). Thus, it follows from (2.33) and Gronwall inequality, we obtain that

$$\sup_{t \in [0, T]} \|\chi_{xt}\|_{L^2}^2 + \int_0^T \|\rho\chi_{tt}\|_{L^2}^2 dt \leq C. \tag{2.54}$$

Next, differentiating (2.10) with respect to t , then multiplying the resultant equation by χ_{tt} , integrating it over I , and using integration by parts, we deduce

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int \rho^2 \chi_{tt}^2 dx + \int \chi_{xtt}^2 dx \\
&= -\frac{1}{2} \int \rho^2 u_x \chi_{tt}^2 dx - 4 \int \rho \rho_t \chi_{tt}^2 dx - 2 \int \rho_t^2 \chi_t \chi_{tt} dx \\
&\quad - 2 \int \rho \rho_{tt} \chi_t \chi_{tt} dx - 4 \int \rho \rho_t u \chi_{xt} \chi_{tt} dx - \int \rho^2 u_t \chi_{xt} \chi_{tt} dx - 2 \int \rho_t^2 u \chi_x \chi_{tt} dx \\
&\quad - 2 \int \rho \rho_{tt} u \chi_x \chi_{tt} dx - 4 \int \rho \rho_t u_t \chi_x \chi_{tt} dx - 2 \int \rho^2 u_{tt} \chi_x \chi_{tt} dx \\
&\quad - 2 \int \rho^2 u_t \chi_{xt} \chi_{tt} dx - \int \rho_{tt} (\chi^3 - \chi) \chi_{tt} dx - 2 \int \rho_t (3\chi^2 - 1) \chi_t \chi_{tt} dx \\
&\quad - 6 \int \rho \chi \chi_t^2 \chi_{tt} dx - \int \rho (3\chi^2 - 1) \chi_{tt}^2 dx = \sum_{i=1}^{15} K_i.
\end{aligned} \tag{2.55}$$

Now, we estimate each term on the right-hand side of (2.55).

$$\begin{aligned}
K_1 &\leq C\|u_x\|_{L^\infty}\|\rho\chi_{tt}\|_{L^2}^2; \\
K_2 &\leq C\|\chi_{tt}\|_{L^\infty}\|\rho_t\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \leq \frac{1}{14}\|\chi_{xtt}\|_{L^2}^2 + C\|\rho\chi_{tt}\|_{L^2}^2; \\
K_3 &\leq C\|\chi_t\|_{L^\infty}\|\chi_{tt}\|_{L^\infty}\|\rho_t\|_{L^2}^2 \leq \frac{1}{14}\|\chi_{xtt}\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2; \\
K_4 &\leq C\|\chi_t\|_{L^\infty}\|\rho_{tt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \leq C\|\rho_{tt}\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2\|\rho\chi_{tt}\|_{L^2}^2; \\
K_5 &\leq C\|u\|_{L^\infty}\|\chi_{tt}\|_{L^\infty}\|\rho_t\|_{L^2}\|\chi_{xt}\|_{L^2} \leq \frac{1}{14}\|\chi_{xtt}\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2; \\
K_6 &\leq C\|u_t\|_{L^\infty}\|\chi_{xt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \leq C\|u_{xt}\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2\|\rho\chi_{tt}\|_{L^2}^2;
\end{aligned}$$

$$\begin{aligned}
K_7 &\leq C\|u\|_{L^\infty}\|\chi_{tt}\|_{L^\infty}\|\chi_x\|_{L^\infty}\|\rho_t\|_{L^2}^2 \leq \frac{1}{14}\|\chi_{xtt}\|_{L^2}^2 + C; \\
K_8 &\leq C\|u\|_{L^\infty}\|\chi_x\|_{L^\infty}\|\rho_{tt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \leq C\|\rho_{tt}\|_{L^2}^2 + C\|\rho\chi_{tt}\|_{L^2}^2; \\
K_9 &\leq C\|u_t\|_{L^\infty}\|\chi_x\|_{L^\infty}\|\rho_t\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \leq C\|\rho\chi_{tt}\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2; \\
K_{10} &\leq C\|\chi_x\|_{L^\infty}\|\sqrt{\rho}u_{tt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \leq \varepsilon\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\|\rho\chi_{tt}\|_{L^2}^2; \\
K_{11} &\leq C\|u_t\|_{L^\infty}\|\chi_{xt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \leq C\|u_{xt}\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2\|\rho\chi_{tt}\|_{L^2}^2; \\
K_{12} &\leq C\|\chi_{tt}\|_{L^2}\|\chi^3 - \chi\|_{L^\infty}\|\rho_{tt}\|_{L^2} \leq \frac{1}{14}\|\chi_{xtt}\|_{L^2}^2 + C\|\rho_{tt}\|_{L^2}^2; \\
K_{13} &\leq C\|\chi_{tt}\|_{L^\infty}\|\chi_t\|_{L^\infty}\|3\chi^2 - 1\|_{L^\infty}\|\rho_t\|_{L^2} \leq \frac{1}{14}\|\chi_{xtt}\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2; \\
K_{14} &\leq C\|\chi_t\|_{L^\infty}^2\|\chi\|_{L^\infty}\|\rho\chi_{tt}\|_{L^2} \leq C\|\chi_{xt}\|_{L^2}^4 + C\|\rho\chi_{tt}\|_{L^2}^2; \\
K_{15} &\leq C\|\chi_{tt}\|_{L^\infty}\|3\chi^2 - 1\|_{L^\infty}\|\rho\chi_{tt}\|_{L^2} \leq \frac{1}{14}\|\chi_{xtt}\|_{L^2}^2 + C\|\rho\chi_{tt}\|_{L^2}^2.
\end{aligned}$$

Then substituting all the above inequality into (2.55) and integrating it over $(0, t)$, we get,

$$\|\rho^2\chi_{tt}\|_{L^2}^2 dx + \int_0^t \|\chi_{xtt}\|_{L^2}^2 ds \leq \int_0^t \varepsilon \|\sqrt{\rho}u_{tt}\|_{L^2}^2 ds + C \int_0^t \|u_{xt}\|_{L^2}^2 ds + C, \quad (2.56)$$

where we have used (2.10), (2.39), (2.54) and $h_{xx} \in L^2$.

Next, multiplying (2.34) by u_{tt} , integrating it over I with respect to x , and using integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int u_{xt}^2 dx + \int \rho u_{tt}^2 dx \\
&= -\frac{1}{2} \frac{d}{dt} \int \rho_t u_t^2 dx + \int \rho_{tt} u_t^2 dx - \frac{d}{dt} \int \rho_t u u_x u_t dx + \int \rho_{tt} u u_x u_t dx \\
&\quad + \int \rho_t u_t^2 u_x dx + \int \rho_t u u_{xt} u_t dx - \int \rho u_{xt} u_{tt} dx + \frac{d}{dt} \int (\rho^\gamma)_t u_{xt} dx \\
&\quad - \int (\rho^\gamma)_{tt} u_{xt} dx - \int \rho u_t u_x u_{tt} dx - \int \rho u u_{xt} u_{tt} dx \\
&\quad + \frac{1}{2} \frac{d}{dt} \int (\chi_x^2)_t u_{xt} dx - \int \chi_{xt}^2 u_{xt} dx - \int \chi_x \chi_{xtt} u_{xt} dx \\
&= -\frac{1}{2} \frac{d}{dt} \int \rho_t u_t^2 dx - \frac{d}{dt} \int \rho_t u u_x u_t dx + \frac{d}{dt} \int (\rho^\gamma)_t u_{xt} dx + \frac{1}{2} \frac{d}{dt} \int (\chi_x^2)_t u_{xt} dx + \sum_{i=1}^{10} M_i.
\end{aligned} \quad (2.57)$$

Now, we estimate each term on the right-hand side of (2.57). It follows from Sobolev inequality, we deduce

$$\begin{aligned}
M_1 &\leq C\|\rho_{tt}\|_{L^2}\|u_t\|_{L^\infty}^2 \leq C\|u_{xt}\|_{L^2}^4 + C\|\rho_{tt}\|_{L^2}^2; \\
M_2 &\leq C\|u\|_{L^\infty}\|u_t\|_{L^\infty}\|u_x\|_{L^2}\|\rho_{tt}\|_{L^2} \leq C\|u_{xt}\|_{L^2}^2 + C\|\rho_{tt}\|_{L^2}^2\|u_x\|_{L^2}^4; \\
M_3 &\leq C\|u_t\|_{L^\infty}^2\|\rho_t\|_{L^2}\|u_x\|_{L^2} \leq C\|u_{xt}\|_{L^2}^4 + C\|\rho_t\|_{L^2}^2\|u_x\|_{L^2}^2; \\
M_4 &\leq C\|u\|_{L^\infty}\|u_t\|_{L^\infty}\|\rho_t\|_{L^2}\|u_{xt}\|_{L^2} \leq C\|u_{xt}\|_{L^2}^4 + C\|\rho_t\|_{L^2}^2\|u_x\|_{L^2}^2; \\
M_5 &\leq C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}u_{tt}\|_{L^2}\|u_{xt}\|_{L^2} \leq \frac{1}{6}\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\|\rho\|_{L^\infty}\|u_{xt}\|_{L^2}^2; \\
M_6 &\leq C\|(\rho^\gamma)_{tt}\|_{L^2}\|u_{xt}\|_{L^2} \leq C\|u_{xt}\|_{L^2}^2 + C\|(\rho^\gamma)_{tt}\|_{L^2}^2; \\
M_7 &\leq C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u_t\|_{L^\infty}\|\sqrt{\rho}u_{tt}\|_{L^2}\|u_x\|_{L^2} \leq \frac{1}{6}\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\|\rho\|_{L^\infty}\|u_{xt}\|_{L^2}^2\|u_x\|_{L^2}^2;
\end{aligned}$$

$$M_8 \leq C \|\rho\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^\infty} \|\sqrt{\rho} u_{tt}\|_{L^2} \|u_{xt}\|_{L^2} \leq \frac{1}{6} \|\sqrt{\rho} u_{tt}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \|u_{xt}\|_{L^2}^2.$$

For M_9 , we need the following estimate, which is deduced from (2.10), that

$$\begin{aligned} \|\chi_{xxt}\|_{L^2}^2 &\leq C \|\rho\|_{L^\infty}^2 \|\rho \chi_{tt}\|_{L^2}^2 + C \|\rho_t\|_{L^\infty}^2 \|\rho \chi_t\|_{L^2}^2 + C \|\rho\|_{L^\infty}^4 \|u_x\|_{L^2}^2 \|\chi_{xt}\|_{L^2}^2 \\ &\quad + C \|\rho\|_{L^\infty}^2 \|u_x\|_{L^2}^2 \|\chi_{xx}\|_{L^2}^2 \|\rho_t\|_{L^2}^2 + C \|\rho\|_{L^\infty} \|\chi_{xx}\|_{L^2}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 \\ &\quad + C \|(\chi^3 - \chi)\|_{L^\infty}^2 \|\rho_t\|_{L^2}^2 + C \|3\chi^2 - 1\|_{L^\infty}^2 \|\rho \chi_t\|_{L^2}^2 \\ &\leq C \|\rho \chi_{tt}\|_{L^2}^2 + C, \end{aligned} \quad (2.58)$$

where we have used (2.1), (2.4), (2.5), (2.8), (2.25), (2.28) and (2.33). Therefore, we have

$$\begin{aligned} M_9 &\leq C \|\chi_{xt}\|_{L^\infty} \|\chi_{xt}\|_{L^2} \|u_{xt}\|_{L^2} \leq C \|\chi_{xxt}\|_{L^2}^2 + C \|\chi_{xt}\|_{L^2}^6 \|u_{xt}\|_{L^2}^4 \\ &\leq C_2 \|\rho \chi_{tt}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^4 + C. \end{aligned}$$

Finally, we consider the last term M_{10} , that

$$\begin{aligned} M_{10} &\leq C \|\chi_x\|_{L^\infty} \|\chi_{xtt}\|_{L^2} \|u_{xt}\|_{L^2} \\ &\leq \delta \|\chi_{xtt}\|_{L^2}^2 + C (1 + \|\chi_{xx}\|_{L^2}^2) \|u_{xt}\|_{L^2}^2 + C. \end{aligned}$$

Then, substituting all the above estimates M_1 – M_{10} into (2.57) and integrating over $(0, t)$, one obtains

$$\begin{aligned} &\int u_{xt}^2(t) dx + \int_0^t \int \rho u_{tt}^2 dx ds \\ &\leq C_2 \int_0^t \|\rho \chi_{tt}\|_{L^2}^2 ds + \delta \int_0^t \|\chi_{xtt}\|_{L^2}^2 ds \\ &\quad + C \int_0^t (1 + \|\chi_{xx}\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2) \|u_{xt}\|_{L^2}^2 ds + C, \end{aligned} \quad (2.59)$$

where we have used (1.8).

Finally, multiplying (2.56) by $C_2 + 1$, and then adding it to (2.59), and choosing ε and δ small enough, one obtains

$$\begin{aligned} &\|\rho^2 \chi_{tt}\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2 + \int_0^t (\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|\chi_{xtt}\|_{L^2}^2) ds \\ &\leq C \int_0^t (1 + \|u_{xt}\|_{L^2}^2) (\|u_{xt}\|_{L^2}^2 + \|\chi_{xt}\|_{L^2}^2) ds + C, \end{aligned}$$

which together with (2.33), (2.54) and Gronwall inequality, we get (2.51). Therefore, we finish the proof of Lemma 2.13. $\square$

Lemma 2.14 *Let (ρ, u, χ) be a smooth solution of (1.1)–(1.6) on $(0, T) \times I$. Then one has*

$$\sup_{t \in [0, T]} (\|u_{xxx}\|_{L^2}^2 + \|\chi_{xxx}\|_{L^2}^2) \leq C, \quad (2.60)$$

$$\int_0^T (\|u_{xxt}\|_{L^2}^2 + \|\chi_{xxt}\|_{L^2}^2) dt \leq C, \quad (2.61)$$

and

$$\sup_{t \in [0, T]} (\|\rho_{tt}\|_{L^2}^2 + \|(\rho^\gamma)_{tt}\|_{L^2}^2) \leq C. \quad (2.62)$$

Proof. Due to (2.30), we can obtain

$$\sup_{t \in [0, T]} \|\chi_{xxx}\|_{L^2}^2 \leq C, \quad (2.63)$$

where we have used (2.25), (2.51), (2.1) and (2.28). Then, it follows from (2.44) with the estimates (2.63), (2.51) and (2.39), that

$$\sup_{t \in [0, T]} \|\chi_{xxx}\|_{L^2}^2 \leq C,$$

which together with (2.63), one shows (2.60).

Next, it follows from (2.58) and (2.51), that

$$\int_0^T \|\chi_{xxt}\|_{L^2}^2 dt \leq C. \quad (2.64)$$

And it follows from (2.34), that we see

$$\begin{aligned} & \int_0^T \|u_{xxt}\|_{L^2}^2 dt \\ & \leq C \|\rho\|_{L^\infty} \|\sqrt{\rho} u_{tt}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 \|\rho_t\|_{L^2}^2 + C \|u_x\|_{L^2}^2 (\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2) \|\rho_t\|_{L^2}^2 \\ & \quad + C \|\rho\|_{L^\infty}^2 \|u_{xt}\|_{L^2}^2 \|u_x\|_{L^2}^2 + C \|(\rho^\gamma)_{xt}\|_{L^2}^2 + C (\|\chi_x\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2) \|\chi_{xxt}\|_{L^2}^2 \\ & \quad + C \|\chi_{xt}\|_{L^2}^2 (\|\chi_{xx}\|_{L^2}^2 + \|\chi_{xxx}\|_{L^2}^2), \end{aligned}$$

which together with (2.1), (2.5), (2.7), (2.8), (2.25), (2.28), (2.37), (2.39) and (2.51), one have

$$\int_0^T \|u_{xxt}\|_{L^2}^2 dt \leq C \int_0^T (\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|\chi_{xxx}\|_{L^2}^2) dt + C.$$

Thus, from which together with (2.51) and (2.64), we deduce (2.61).

In order to prove (2.62), one can be obtained directly from (2.48) and (2.49) with the estimate (2.38), (2.8), (2.25) and (2.51). Therefore, we complete the proof of Lemma 2.14. $\square$

3 Proof of the main theorems

Based on the a priori estimates derived in the last section, we can extend the local classical solution obtained in Proposition 1.3 to the global one. First, we prove Theorem 1.1, i.e. due to Proposition 1.3, we can obtain global strong solution to the initial boundary value problems (1.1)–(1.6).

Proof of Theorem 1.1 To prove Theorem 1.1, we construct a sequence of approximate solutions by giving the initial density a positive lower bound. Let $j_\delta(x)$ be a standard mollifier with width δ . Define the approximate initial density $\rho_0^\delta = j_\delta * \rho_0 + \delta$, and $(u_0^\delta, \chi_0^\delta)$ is the unique smooth solution of the following elliptic problems for each $\delta > 0$:

$$\begin{cases} u_{0xx}^\delta - ((\rho_0^\delta)^\gamma)_x - \frac{1}{2}[(\chi_0^\delta)^2]_x = \rho_0^\delta g(x), \\ \chi_{0xx}^\delta + \rho_0^\delta \mu_0^\delta = \rho_0^\delta ((\chi_0^\delta)^3 - \chi_0^\delta), \end{cases} \quad x \in (0, 1), \quad (3.1)$$

and

$$u_0^\delta(0) = u_0^\delta(1) = \chi_0^\delta(0) = \chi_0^\delta(1) = 0. \quad (3.2)$$

Since $\rho_0 \in H^1$, one easily deduces

$$\rho_0^\delta \rightarrow \rho_0, \quad \text{in } H^1. \quad (3.3)$$

It follows from (3.1)–(3.3) and the elliptic theory (c.f. [9]):

$$u_0^\delta \rightarrow u_0, \quad \chi_0^\delta \rightarrow \chi_0, \quad \text{in } H^2, \quad (3.4)$$

as $\delta \rightarrow 0$.

Consider (1.1)–(1.4) with initial and boundary conditions

$$(\rho^\delta, u^\delta, \chi^\delta)(0, x) = (\rho_0^\delta, u_0^\delta, \chi_0^\delta)(x), \quad \text{in } I, \quad (3.5)$$

and

$$u^\delta(0) = u^\delta(1) = \chi^\delta(0) = \chi^\delta(1) = 0, \quad \text{for } t \geq 0. \quad (3.6)$$

We can get a unique solution $(\rho^\delta, u^\delta, \chi^\delta)$ for each $\delta > 0$ by Proposition 1.3, and with the estimates shown in the following Proposition 3.1.

It follows from Lemma 2.1–2.9 step by step, one can get the following result.

Proposition 3.1 *Assume that $(\rho^\delta, u^\delta, \chi^\delta)$ be smooth solution of (1.1)–(1.4), (3.5) and (3.6). Then there exists a generic positive constant C depending on the initial data and T , such that*

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\|\rho^\delta\|_{H^1}^2 + \|\rho_t^\delta\|_{L^2}^2 + \|u^\delta\|_{H_0^1}^2 + \|\chi^\delta\|_{H_0^1 \cap H^2}^2 \right) \\ & + \int_0^T \left(\|u^\delta\|_{H^2}^2 + \|\chi^\delta\|_{H^3}^2 + \|\chi_t^\delta\|_{H^1}^2 + \|u_t^\delta\|_{H^1}^2 \right) dt \leq C. \end{aligned} \quad (3.7)$$

Based on Proposition 3.1, we can obtain a solution (ρ, u, χ) to (1.1)–(1.6) after taking limit $\delta \rightarrow 0$ (taking the subsequence if necessary),

$$\begin{cases} \rho \in L^\infty([0, T]; H^1), \quad \rho_t \in L^\infty([0, T]; L^2), \quad \rho \geq 0, \\ u \in L^\infty([0, T]; H_0^1) \cap L^2([0, T]; H^2), \quad u_t \in L^2([0, T]; H^1), \\ \chi \in L^\infty([0, T]; H_0^1 \cap H^2), \quad \chi_t \in L^2([0, T]; H^1). \end{cases} \quad (3.8)$$

Thus, we complete the proof of the existence of Theorem 1.1.

It remains for us to prove the uniqueness. Let (ρ_1, u_1, χ_1) and (ρ_2, u_2, χ_2) be two solutions to the problem (1.1)–(1.6) obtained as above. Denote

$$\bar{\rho} = \rho_1 - \rho_2, \quad \bar{u} = u_1 - u_2, \quad \bar{\chi} = \chi_1 - \chi_2.$$

Then it follows from (1.1)–(1.2) and (2.9) that

$$\bar{\rho}_t + (\bar{\rho}u_1)_x + (\rho_2\bar{u})_x = 0, \quad (3.9)$$

$$\rho_1\bar{u}_t - \bar{u}_{xx} = -\bar{\rho}u_{2t} - \rho_1u_1\bar{u}_x - \rho_1\bar{u}u_{2x} - \bar{\rho}u_2u_{2x} - (\rho_1^\gamma - \rho_2^\gamma)_x - \chi_{1x}\bar{\chi}_{xx} - \bar{\chi}_x\chi_{2xx}, \quad (3.10)$$

$$\begin{aligned} \rho_1^2\bar{\chi}_t - \bar{\chi}_{xx} &= -(\rho_1 + \rho_2)\bar{\rho}\chi_{2t} - \rho_1^2u_1\bar{\chi}_x - \rho_1^2\bar{u}\chi_{2x} \\ &\quad - (\rho_1 + \rho_2)\bar{\rho}u_2\chi_{2x} - \rho_1\bar{\chi}(\chi_1^2 + \chi_1\chi_2 + \chi_2^2 - 1) - \bar{\rho}(\chi_2^3 - \chi_2), \end{aligned} \quad (3.11)$$

for $(t, x) \in (0, +\infty) \times I$, subject to the initial boundary conditions

$$(\bar{\rho}, \bar{u}, \bar{\chi})(0, x) = (0, 0, 0), \quad \text{for } x \in I,$$

and

$$\bar{u}(t, 0) = \bar{u}(t, 1) = \bar{\chi}(t, 0) = \bar{\chi}(t, 1) = 0, \quad \text{for } t \geq 0.$$

Multiplying (3.9) by $\bar{\rho}$, integrating the resultant equation over I , and using integration by parts, one shows

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \bar{\rho}^2 dx &= -\frac{1}{2} \int \bar{\rho}^2 u_{1x} dx - \int (\rho_{2x} \bar{u} + \rho_2 \bar{u}_x) \bar{\rho} dx \\ &\leq C \|u_{1x}\|_{L^\infty} \|\bar{\rho}\|_{L^2}^2 + C \|\bar{u}\|_{L^\infty} \|\rho_{2x}\|_{L^2} \|\bar{\rho}\|_{L^2} + C \|\rho_2\|_{L^\infty} \|\bar{u}_x\|_{L^2} \|\bar{\rho}\|_{L^2} \\ &\leq C (1 + \|u_1\|_{H^2}) \|\bar{\rho}\|_{L^2}^2 + C_3 \|\bar{u}_x\|_{L^2}^2, \end{aligned} \quad (3.12)$$

where we have used Hölder inequality, Young inequality and Sobolev inequality and the regularities on ρ_1, ρ_2, u_1, u_2 showed in (3.7).

Then, multiplying (3.10) by $\bar{u}$, integrating the resultant equation over I , and using integration by parts, one deduces

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \rho_1 \bar{u}^2 dx + \int \bar{u}_x^2 dx \\ &= - \int \bar{\rho} \bar{u} u_{2t} dx - \int \rho_1 \bar{u}^2 u_{2x} dx - \int \bar{\rho} \bar{u} u_2 u_{2x} dx + \int (\rho_1^\gamma - \rho_2^\gamma) \bar{u}_x dx \\ &\quad + \int \chi_{1xx} \bar{\chi}_x \bar{u} dx + \int \chi_{1x} \bar{\chi}_x \bar{u}_x dx - \int \bar{\chi}_x \bar{u} \chi_{2xx} dx \\ &\leq C \|\bar{u}\|_{L^\infty} \|\bar{\rho}\|_{L^2} \|u_{2t}\|_{L^2} + C \|u_{2x}\|_{L^\infty} \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + C \|\bar{u}\|_{L^\infty} \|u_2\|_{L^\infty} \|\bar{\rho}\|_{L^2} \|u_{2x}\|_{L^2} \\ &\quad + C \|\bar{\rho}\|_{L^2} \|\bar{u}_x\|_{L^2} + C \|\bar{u}\|_{L^\infty} \|\chi_{1xx}\|_{L^2} \|\bar{\chi}_x\|_{L^2} + C \|\chi_{1x}\|_{L^\infty} \|\bar{\chi}_x\|_{L^2} \|\bar{u}_x\|_{L^2} \\ &\quad + C \|\bar{u}\|_{L^\infty} \|\bar{\chi}_x\|_{L^2} \|\chi_{2xx}\|_{L^2} \\ &\leq \frac{1}{2} \|\bar{u}_x\|_{L^2}^2 + C (\|u_{2xt}\|_{L^2}^2 + 1) \|\bar{\rho}\|_{L^2}^2 + C_4 \|\bar{\chi}_x\|_{L^2}^2 + C (1 + \|u_{2xx}\|_{L^2}^2) \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2. \end{aligned} \quad (3.13)$$

Next, multiplying (3.11) by $\bar{\chi}$, integrating the resulting it over I and using integration by parts, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \rho_1^2 \bar{\chi}^2 dx + \int \bar{\chi}_x^2 dx \\ &= \int \bar{\rho}^2 \bar{\chi} \chi_{2t} dx - 2 \int \rho_1 \bar{\rho} \bar{\chi} \chi_{2t} dx - \int \rho_1^2 u_1 \bar{\chi} \bar{\chi}_x dx - \int \rho_1^2 \bar{u} \bar{\chi} \chi_{2x} dx + \int \bar{\rho}^2 u_2 \bar{\chi} \chi_{2x} dx \\ &\quad - 2 \int \rho_1 \bar{\rho} u_2 \bar{\chi} \chi_{2x} dx - \int \rho_1 \bar{\chi}^2 (\chi_1^2 + \chi_1 \chi_2 + \chi_2^2 - 1) dx - \int \bar{\rho} \bar{\chi} (\chi_2^3 - \chi_2) dx \\ &\leq C \|\bar{\chi}\|_{L^\infty} \|\chi_{2t}\|_{L^\infty} \|\bar{\rho}\|_{L^2}^2 + C \|\chi_{2t}\|_{L^\infty} \|\bar{\rho}\|_{L^2} \|\bar{\chi}\|_{L^2} + C \|u_1\|_{L^\infty} \|\rho_1 \bar{\chi}\|_{L^2} \|\bar{\chi}_x\|_{L^2} \\ &\quad + C \|\chi_{2x}\|_{L^\infty} \|\sqrt{\rho_1} \bar{u}\|_{L^2} \|\rho_1 \bar{\chi}\|_{L^2} + C \|u_2\|_{L^\infty} \|\bar{\chi}\|_{L^\infty} \|\chi_{2x}\|_{L^\infty} \|\bar{\rho}\|_{L^2}^2 \\ &\quad + C \|u_2\|_{L^\infty} \|\chi_{2x}\|_{L^\infty} \|\rho_1 \bar{\chi}\|_{L^2} \|\bar{\rho}\|_{L^2} + C \|\chi_1^2 + \chi_1 \chi_2 + \chi_2^2 - 1\|_{L^\infty} \|\rho_1 \bar{\chi}\|_{L^2} \|\bar{\chi}\|_{L^2} \\ &\quad + C \|\chi_2^3 - \chi_2\|_{L^\infty} \|\bar{\rho}\|_{L^2} \|\bar{\chi}\|_{L^2} \\ &\leq \frac{1}{2} \|\bar{\chi}_x\|_{L^2}^2 + C (1 + \|\chi_{2xt}\|_{L^2}^2) \|\bar{\rho}\|_{L^2}^2 + C \|\rho_1 \bar{\chi}\|_{L^2}^2 + C \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2, \end{aligned} \quad (3.14)$$

where we have used the boundary conditions $\bar{\chi}(t, 0) = \bar{\chi}(t, 1) = 0$, Poincaré inequality and Young inequality.

Therefore, multiplying (3.13) and (3.14) by $2(C_3 + 1)$ and $4C_4(C_3 + 1) + 1$ respectively, and then adding the resulting inequalities to (3.12), and using the uniform estimates obtained in (3.8), one obtains

$$\frac{d}{dt} (\|\bar{\rho}\|_{L^2}^2 + \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + \|\rho_1 \bar{\chi}\|_{L^2}^2) + (\|\bar{u}_x\|_{L^2}^2 + \|\bar{\chi}_x\|_{L^2}^2)$$

$$\leq C \left(1 + \|(\chi_{2xt}, u_{2xt}, u_{1xx}, u_{2xx})\|_{L^2}^2 \right) (\|\bar{\rho}\|_{L^2}^2 + \|\sqrt{\rho_1}\bar{u}\|_{L^2}^2 + \|\rho_1\bar{\chi}\|_{L^2}^2),$$

from which together with Gronwall inequality shows

$$\sup_{t \in [0, T]} (\|\bar{\rho}\|_{L^2}^2 + \|\sqrt{\rho_1}\bar{u}\|_{L^2}^2 + \|\rho_1\bar{\chi}\|_{L^2}^2) \leq 0.$$

Therefore, we deduce $(\bar{\rho}, \bar{u}, \bar{\chi}) = 0$. Thus, we finish the proof of the Theorem 1.1. $\square$

Next, we devote ourselves to the global existence of classical solutions to the initial boundary value problems (1.1)–(1.6).

Proof of Theorem 1.2 To prove Theorem 1.2, we construct a sequence of approximate solutions to (1.1)–(1.6) by giving the initial density a positive lower bound δ , precisely, we denote $\rho_0^\delta = \rho_0 + \delta$, $(u_0^\delta, \chi_0^\delta)$ is the unique solution of the following elliptic problems for each $\delta > 0$:

$$\begin{cases} u_{0xx}^\delta - ((\rho_0^\delta)^\gamma)_x - \frac{1}{2}[(\chi_{0x}^\delta)^2]_x = \rho_0^\delta g(x), \\ \chi_{0xx}^\delta + \rho_0^\delta \mu_0^\delta = \rho_0^\delta ((\chi_0^\delta)^3 - \chi_0^\delta), \end{cases} \quad x \in (0, 1), \quad (3.15)$$

and

$$u_0^\delta(0) = u_0^\delta(1) = \chi_0^\delta(0) = \chi_0^\delta(1) = 0. \quad (3.16)$$

Since $\rho_0, \rho_0^\gamma \in H^2$, we have as $\delta \rightarrow 0$

$$\rho_0^\delta \rightarrow \rho_0, \quad \text{in } H^2, \quad (3.17)$$

and

$$(\rho_0^\delta)^\gamma \rightarrow (\rho_0)^\gamma, \quad \text{in } H^2. \quad (3.18)$$

It follows from (3.15)–(3.18) and the elliptic theory (c.f. [9]), we have

$$(u_0^\delta, \chi_0^\delta) \rightarrow (u_0, \chi_0), \quad \text{in } H^3 \quad \text{as } \delta \rightarrow 0. \quad (3.19)$$

Consider (1.1)–(1.3), (3.15) and with the initial boundary condition

$$(\rho^\delta, u^\delta, \chi^\delta) = (\rho_0^\delta, u_0^\delta, \chi_0^\delta), \quad \text{in } I, \quad (3.20)$$

and

$$u^\delta(0) = u^\delta(1) = \chi^\delta(0) = \chi^\delta(1) = 0. \quad (3.21)$$

We can get a unique classical solution $(\rho^\delta, u^\delta, \chi^\delta)$ for each $\delta > 0$ by Proposition 1.3, with the estimates shown in following Proposition 3.2.

It follows step by step from Lemma 2.1–2.14, we obtain

Proposition 3.2 *Assume that $(\rho^\delta, u^\delta, \chi^\delta)$ be smooth solution of (1.1)–(1.3), (3.20) and (3.21). Then there exists a generic positive constant C depending only on the initial data and T , such that*

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\|(\rho^\delta, (\rho^\delta)^\gamma)\|_{H^2} + \|(\rho_t^\delta, (\rho_t^\delta)^\gamma)\|_{H^1} + \|(\rho_{tt}^\delta, (\rho_{tt}^\delta)^\gamma)\|_{L^2} + \|(u^\delta, \chi^\delta)\|_{H^3} \right. \\ & \left. + \|(u_t^\delta, \chi_t^\delta)\|_{H^1} \right) + \int_0^T \left(\|(\sqrt{\rho^\delta} u_{tt}^\delta, \rho^\delta \chi_{tt}^\delta)\|_{L^2}^2 + \|(u_{xxt}^\delta, \chi_{xxt}^\delta)\|_{L^2}^2 \right) dt \leq C, \end{aligned} \quad (3.22)$$

where C is a positive constant, independent of δ .

Based on Proposition 3.2, we can get a solution (ρ, u, χ) to (1.1)–(1.6) after taking limit $\delta \rightarrow 0$ (take the subsequence if necessary):

$$\left\{ \begin{array}{l} \rho \in L^\infty([0, T]; H^2), \quad \rho_t \in L^\infty([0, T]; H^1), \quad \rho_{tt} \in L^\infty([0, T]; L^2), \quad \rho \geq 0, \\ (\rho^\gamma)_{xx} \in L^\infty([0, T]; L^2), \quad (\rho^\gamma)_{xt} \in L^\infty([0, T]; L^2), \quad (\rho^\gamma)_{tt} \in L^\infty([0, T]; L^2), \\ u \in L^\infty([0, T]; H^3 \cap H_0^1), \quad u_t \in L^\infty([0, T]; H_0^1) \cap L^2([0, T]; H^2), \\ \chi \in L^\infty([0, T]; H^3), \quad \chi_t \in L^\infty([0, T]; H^1) \cap L^2([0, T]; H^2). \end{array} \right.$$

Since $u \in L^\infty([0, T]; H^3)$ and $u_t \in L^\infty([0, T]; H_0^1)$, we have $u \in C([0, T]; H^2)$. This together with (1.1) and (2.18) (c.f. [5]) implies

$$\rho \in C([0, T]; H^2), \quad (3.23)$$

and

$$\rho^\gamma \in C([0, T]; H^2). \quad (3.24)$$

Then, since $\chi \in L^\infty([0, T]; H^3)$ and $\chi_t \in L^\infty([0, T]; H^1)$, we have

$$\chi \in C([0, T]; H^2).$$

It follows from (2.9), we see

$$\chi_{xx} = \rho^2 \chi_t + \rho^2 u \chi_x - \rho (\chi^3 - \chi) \in L^2([0, T]; H^2),$$

which together with the a priori estimate $\chi_{xxt} \in L^2([0, T]; L^2)$, shows

$$\chi_x \in C([0, T]; H^2).$$

From which we have

$$\chi \in C([0, T]; H^3). \quad (3.25)$$

Next, if we denote $G = (u_x - \rho^\gamma - \frac{1}{2}\chi_x^2)_x$, it follows from (1.2) and the regularities (3.22), we have

$$G = \rho u_t + \rho u u_x \in L^2([0, T]; H^2),$$

and

$$G_t = \left(u_x - \rho^\gamma - \frac{1}{2}\chi_x^2 \right)_{xt} \in L^2([0, T]; L^2).$$

Therefore, we have

$$\left(u_x - \rho^\gamma - \frac{1}{2}\chi_x^2 \right)_x \in C([0, T]; H^1).$$

This means

$$u_x - \rho^\gamma - \frac{1}{2}\chi_x^2 \in C([0, T]; H^2). \quad (3.26)$$

Combining (3.24), (3.25) and (3.26), we obtain

$$u_x \in C([0, T]; H^2),$$

which means

$$u \in C([0, T]; H^3). \quad (3.27)$$

Due to (1.1), (3.23) and (3.27), we have

$$\rho_t \in C([0, T]; H^1).$$

It then follows from (1.2), (3.23), (3.24) and (3.27) that

$$(\rho u)_t \in C([0, T]; H^1). \quad (3.28)$$

Differentiating (1.1) with respect to t , we have

$$\rho_{tt} = -(\rho u)_{xt}.$$

This combining (3.28) shows

$$\rho_{tt} \in C([0, T]; L^2).$$

Due to (2.18), (3.24) and (3.27), we obtain

$$(\rho^\gamma)_t \in C([0, T]; H^1).$$

Therefore, the existence of global classical solutions as in Theorem 1.2 is obtained. The uniqueness of solutions can be proved as in Theorem 1.1, we omit the details here. Thus, the proof of Theorem 1.2 is completed. $\square$

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