Interior local null controllability for multi-dimensional compressible flow near a constant state

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Abstract

We consider compressible flow with periodic boundary conditions. In a neighborhood of a state of constant density and nonzero velocity, we prove exact null controllability of the system with a control localized in space and acting only on the momentum equation.

Keywords: null controllability; compressible flow; parabolic-hyperbolic system

1. Introduction

Let us consider the domain Ω in $\mathbb{R}^d$, for $d = 2, 3$, where

$$\Omega \simeq \prod_{j=1}^d (0, L_j),$$

the d -dimensional torus. In this paper we study null controllability of the compressible Navier-Stokes system with periodic boundary condition in Ω , in a neighbourhood of a constant steady state solution. We seek a localized control g , acting only on velocity, to bring the density $\hat{\rho}$ and velocity $\hat{u}$ of a compressible fluid satisfying

$$\begin{aligned} \hat{\rho}_t + \operatorname{div}(\hat{\rho}\hat{u}) &= 0 \quad \text{in } \Omega \times (0, T), \\ \hat{\rho}(\hat{u}_t + (\hat{u} \cdot \nabla)\hat{u}) - \mu \Delta \hat{u} - (\mu_0 + \mu) \nabla[\operatorname{div} \hat{u}] + \nabla p(\hat{\rho}) &= g \quad \text{in } \Omega \times (0, T), \\ p(\hat{\rho}) &= a \hat{\rho}^\lambda, \quad \text{for } a > 0, \lambda > 0, \\ \hat{\rho}(\cdot, 0) &= \hat{\rho}_0(\cdot), \quad \hat{u}(\cdot, 0) = \hat{u}_0(\cdot) \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

to a constant steady state (ρ_s, u_s) , at sufficiently large time T , for nearby initial conditions. Here we consider $\rho_s > 0$ and u_s , nonzero. From the density equation of (1.1), we get

$$\int_{\Omega} \hat{\rho}(x, t) dx = \int_{\Omega} \hat{\rho}_0(x) dx, \quad \forall t \in [0, T].$$

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Since we are looking for local null controllability of (1.1) around (ρ_s, u_s) , we consider the initial condition $(\hat{\rho}_0, \hat{u}_0)$ with

$$\frac{1}{|\Omega|} \int_{\Omega} \hat{\rho}_0(x) dx = \rho_s, \quad \min_{\Omega} \hat{\rho}_0 > 0. \quad (1.2)$$

We denote by $H_{per}^s(\Omega)$ the Sobolev space of periodic functions and by $\dot{H}_{per}^s(\Omega)$ the subspace of the functions belonging to $H_{per}^s(\Omega)$ with mean value zero.

Our main result is the following: We consider any constants $\rho_s > 0$ and $u_s \in \mathbb{R}^d \setminus \{0\}$. Without loss of generality, we assume that $(u_s)_1 \neq 0$, where $(u_s)_1$ is the first component of the constant velocity vector u_s .

Let $(\hat{l}, \hat{r})$ be any subinterval of $(0, L_1)$ and $\hat{\mathcal{O}}$ be a subdomain of Ω in the form

$$\hat{\mathcal{O}} = (\hat{l}, \hat{r}) \times \prod_{j=2}^d (0, L_j). \quad (1.3)$$

Let us set

$$T_0 = \frac{L_1}{|(u_s)_1|}, \quad (1.4)$$

where L_1 is the periodicity of the first component of the space variable and $(u_s)_1$ is the first component of the constant velocity vector u_s .

Theorem 1.1. *We assume that $\rho_s > 0$, $u_s \in \mathbb{R}^d \setminus \{0\}$ such that $(u_s)_1 \neq 0$, where $(u_s)_1$ is the first component of the constant velocity vector u_s . Let $\hat{\mathcal{O}}$ be a subdomain of Ω in the form (1.3) for any subinterval $(\hat{l}, \hat{r})$ in $(0, L_1)$ and T_0 be as defined in (1.4). For any $T > T_0$, there exists a positive constant $\hat{\kappa}$, depending on T , Ω , $\hat{\mathcal{O}}$, such that for any $(\hat{\rho}_0, \hat{u}_0) \in H_{per}^2(\Omega) \times (H_{per}^2(\Omega))^d$, where $\hat{\rho}_0$ satisfies (1.2) and $(\hat{\rho}_0, \hat{u}_0)$ obeys*

$$\|(\hat{\rho}_0 - \rho_s, \hat{u}_0 - u_s)\|_{\dot{H}_{per}^2(\Omega) \times H_{per}^2(\Omega)} \leq \hat{\kappa}, \quad (1.5)$$

there exists a control g in $(L^2(0, T; H_0^1(\hat{\mathcal{O}})))^d$ for which (1.1) has a solution $(\hat{\rho}, \hat{u})$ in $C([0, T]; H_{per}^2(\Omega)) \times (L^2(0, T; H_{per}^3(\Omega)) \cap H^1(0, T; H_{per}^1(\Omega)))^d$ satisfying

$$(\hat{\rho}(\cdot, T), \hat{u}(\cdot, T)) = (\rho_s, u_s) \quad \text{in } \Omega.$$

We may choose $\rho_s = 1$ without loss of generality. We seek solutions in the neighborhood of the constant state (ρ_s, u_s) . Let us consider (ρ, u) , defined by $\rho = \hat{\rho} - 1$, $u = \hat{u} - u_s$. The system satisfied by (ρ, u) on $\Omega \times (0, T)$ is

$$\begin{aligned} \rho_t + (u + u_s) \cdot \nabla \rho + \rho \operatorname{div} u + \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0, T), \\ u_t + (u_s \cdot \nabla) u - \mu \Delta u - (\mu_0 + \mu) \nabla \operatorname{div} u + p'(1) \nabla \rho &= g - \rho[u_t \\ + ((u + u_s) \cdot \nabla) u] - (u \cdot \nabla) u - [\nabla p(\rho + 1) - p'(1) \nabla \rho] &\quad \text{in } \Omega \times (0, T), \\ \rho(\cdot, 0) = \rho_0 = \hat{\rho}_0 - 1, \quad u(\cdot, 0) = u_0 = \hat{u}_0 - u_s &\quad \text{in } \Omega, \quad \int_{\Omega} \rho_0(x) dx = 0. \end{aligned} \quad (1.6)$$

Our method of proving Theorem 1.1 is based on an iterative construction in which a linear control problem is solving at each step of the iteration. Since the density equation is hyperbolic, its solution does not involve any gain of regularity which would compensate for the loss of a derivative in the term $(u \cdot \nabla)\rho$. In this term, we therefore replace u by the iterate from the previous iteration, while keeping ρ the new iterate (an alternative is to use a moving coordinate system; this approach is taken in [6, 14, 10]). Hence, we need to solve a variable coefficient problem consisting of a heat and transport equation. As an intermediate step, we first solve a problem with controls in both density and momentum equations and then, in a subsequent step, use an observability inequality to eliminate the control in the density equation.

We refer to [3, 13, 5, 7, 8] for earlier results on linearized systems. A partial result on the null controllability of the nonlinear system in one space dimension is given in [1]. Null controllability of compressible one-dimensional flows is established in [9] and [15]. In [9], the authors establish a result on boundary controllability for Dirichlet boundary conditions, using a technique of proof which is actually based on using a distributed control located outside the domain of physical interest. In [15], periodic boundary conditions are considered, and a distributed control is added to the momentum equation only. Both approaches are based on combining Carleman estimates for the heat equation with the characteristics method for the transport equation. An important difference in the method is in how the decoupling of heat and transport equation is achieved. In [9], this is done by introducing a new type of Carleman weight which “follows characteristics” to deal with the transport equation. They then take advantage of the Carleman weight to accomplish the decoupling of heat and transport equation, choosing one of the parameters sufficiently large. In contrast, the argument in [15] is, roughly speaking, based on the intuition that controllability should not be affected by compact perturbations as long as approximate controllability holds. A transformation of variables reduces the differential order of the coupling, basically making it weak for high Fourier modes. This allows controllability “outside a finite dimensional space.” The finite dimensional correction is then achieved using a backward solution combined with an approximate controllability result.

The result of [9] has recently been extended to higher dimensions [10]. The purpose of the present paper is to extend the result of [15]. Our method of analysis rather closely follows that of [15], but there are some differences. Due to the different behavior of Sobolev embeddings, we need to work in spaces of higher regularity. Also, the second order derivative term in the momentum equation is not simply the Laplacian; we deal with this by using a decomposition of the velocity into a gradient and a divergence free part. As in [15], the step of eliminating the density control involves the solution of a hyperviscous Stokes problem; in the present paper, we allow for the controlled domain to be slightly enlarged in this step; as a result of this, we only need interior estimates for the hyperviscous Stokes problem and do not need to consider corner singularities. We could have used this approach also in the one-dimensional case.

The paper is organized as follows. In Section 2, we give an outline of our

solution procedure. Section 3 provides the required preliminary results for the controllability of parabolic and transport equation separately. In Section 4, we provide the estimates for the various stages of iteration involved in solving the linear control problem. Section 5 deals with the absorption of the control in the density equation. The convergence of the nonlinear iteration is given in Section 6.

2. Outline of the iterative process

Our system (1.6) is a coupled system between density and velocity variables and it is nonlinear. To deal with this, we first formulate an “outer” iteration which reduces the nonlinear problem to a sequence of linear problems. For this coupled linear system, we first consider an intermediate control problem, with controls in both equations. In a subsequent step, we eliminate the control in the transport equation. This section will focus on the linear problem with two controls. In this section, we give a rough outline; Section 4 will concern details of the estimates.

First, using a decomposition of the velocity variable (see (2.7)), we make the coupling of density and velocity similar to a coupling between a transport and a heat equation. The other parts of the velocity are decoupled. The null control problems for the decoupled parts of the velocity can be solved using the results for the heat equation and by solving an ordinary differential equation. For the coupling part, we construct a null control by an iteration which alternates between solving a heat and transport problem.

2.1. Formulation of system for iteration

Since we consider our system in the d -dimensional torus

$$\Omega \simeq \prod_{j=1}^d (0, L_j),$$

with respect to space variable, we equip the spaces for all $s \in \mathbb{R}$,

$$\begin{aligned} H_{per}^s(\Omega) &= \left\{ \varphi : \varphi = \sum_{k \in \mathbb{Z}^d} c_k e^{i2\pi \left(\sum_{j=1}^d \frac{k_j x_j}{L_j} \right)}, \quad \sum_{k \in \mathbb{Z}^d} |k|^{2s} |c_k|^2 < \infty \right\}, \\ \dot{H}_{per}^s(\Omega) &= \left\{ \varphi \in H_{per}^s(\Omega) : \int_{\Omega} \varphi(x) dx = 0 \right\}, \end{aligned}$$

where $x = (x_1, \dots, x_d) \in \Omega$ and $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, with the norms

$$\|\varphi\|_{H_{per}^s(\Omega)} = \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^{2s}) |c_k|^2 \right)^{\frac{1}{2}}, \quad \|\varphi\|_{\dot{H}_{per}^s(\Omega)} = \left(\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{2s} |c_k|^2 \right)^{\frac{1}{2}}.$$

We recall system (1.6):

$$\begin{aligned}
\rho_t + (u + u_s) \cdot \nabla \rho + \rho \operatorname{div} u + \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0, T), \\
u_t + (u_s \cdot \nabla) u - \mu \Delta u - (\mu_0 + \mu) \nabla \operatorname{div} u + p'(1) \nabla \rho &= g - \rho(u_t + (u_s \cdot \nabla) u) \\
- (u \cdot \nabla) u - (\nabla p(\rho + 1) - p'(1) \nabla \rho) &\quad \text{in } \Omega \times (0, T), \\
\rho(\cdot, 0) = \rho_0, \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega, \quad \int_{\Omega} \rho_0(x) dx &= 0.
\end{aligned} \tag{2.1}$$

We denote the nonlinear term by

$$G(\rho, \nabla \rho, u, \nabla u, u_t) := -\rho(u_t + (u_s \cdot \nabla) u) - (u \cdot \nabla) u - (\nabla p(\rho + 1) - p'(1) \nabla \rho). \tag{2.2}$$

First, we study the null controllability of the following system for each $n \in \mathbb{N}$, to be solved in each iteration

$$\begin{aligned}
\rho_t^{n+1} + u_s \cdot \nabla \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} + (\operatorname{div} u^n) \rho^{n+1} + \operatorname{div} u^{n+1} &= 0 \quad \text{in } \Omega \times (0, T), \\
u_t^{n+1} + (u_s \cdot \nabla) u^{n+1} - \mu \Delta u^{n+1} - (\mu_0 + \mu) \nabla \operatorname{div} u^{n+1} + p'(1) \nabla \rho^{n+1} \\
&= g^{n+1} + G(\rho^n, \nabla \rho^n, u^n, \nabla u^n, u_t^n) \quad \text{in } \Omega \times (0, T), \\
\rho^{n+1}(\cdot, 0) = \rho_0, \quad u^{n+1}(\cdot, 0) = u_0 \quad \text{in } \Omega, \quad \int_{\Omega} \rho_0(x) dx &= 0.
\end{aligned} \tag{2.3}$$

Let us set $\Psi = u^n$. To study the null controllability of system (2.3), we first consider the system

$$\begin{aligned}
\varrho_t + u_s \cdot \nabla \varrho + \Psi \cdot \nabla \varrho + (\operatorname{div} \Psi) \varrho + \operatorname{div} v &= f \quad \text{in } \Omega \times (0, T), \\
v_t + (u_s \cdot \nabla) v - \mu \Delta v - (\mu_0 + \mu) \nabla \operatorname{div} v + p'(1) \nabla \varrho &= g + G \quad \text{in } \Omega \times (0, T), \\
\varrho(\cdot, 0) = \rho_0, \quad v(\cdot, 0) = u_0 \quad \text{in } \Omega, \quad \int_{\Omega} \rho_0(x) dx &= 0,
\end{aligned} \tag{2.4}$$

with two controls f and g in both density and velocity equations and the average of f is zero. Then finally, we absorb the control f by changing the control g and show that system (2.3) is null controllable by a control acting only in the velocity equation. While doing it in Section 5, we shall see that the new control in velocity will have a bigger support than the support of f , control in density. We handle it by considering the controls in (2.4) with support in a slightly smaller domain than what we need to have in Theorem 1.1, as follows. Let us recall that $\widehat{\mathcal{O}}$, introduced in (1.3), is a subdomain of Ω , where $(\widehat{l}, \widehat{r})$ be any subinterval of $(0, L_1)$. In our main result Theorem 1.1, we need to show the existence of a control in velocity equation with support in $\widehat{\mathcal{O}} \times (0, T)$. Now we consider a subinterval (l, r) of $(0, L_1)$ strictly contained in $(\widehat{l}, \widehat{r})$ and we set

$$\mathcal{O} = (l, r) \times \prod_{j=2}^d (0, L_j). \tag{2.5}$$

We shall study the null controllability of (2.4) with controls f and g having support for the space variable inside $\mathcal{O}$ instead of $\widehat{\mathcal{O}}$. Further the support of f for the time variable should be considered inside $(\delta, T_1 + \delta_1)$ for any $\delta > 0$ and for some $T_1 + \delta_1 \in (T_0, T)$, and because of the coupling it forces us to consider the support of g for the time variable inside (δ, T) .

In the following, we shall use the function $\chi_0(x)$, which will be introduced in the next section; here we just mention that χ_0 is assumed to be C^∞ with compact support in a proper subset (will be specified in the next section where our Carleman weight assumes its constant minimum) of $\mathcal{O}$ and have unit integral. We shall denote the mean value of a function f by $\langle f \rangle$.

Let us denote

$$\begin{aligned} v_m(\cdot, t) &= \langle v \rangle(t) \chi_0 = \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx \, \chi_0, \quad \forall t \in (0, T), \\ G_m(\cdot, t) &= \langle G \rangle(t) \chi_0 = \frac{1}{|\Omega|} \int_{\Omega} G(x, t) \, dx \, \chi_0, \quad \forall t \in (0, T). \end{aligned} \quad (2.6)$$

To get rid of the term involving $\nabla \operatorname{div}$ in (2.4)₂, we shall decompose the velocity and forcing term in the velocity equation

$$\begin{aligned} u_0(x) &= \nabla w_0(x) + \nabla \times a_0(x) + \langle u_0 \rangle \chi_0, \quad \operatorname{div} a_0 = 0, \quad \forall x \in \Omega, \\ v(x, t) &= \nabla w(x, t) + \nabla \times a(x, t) + v_m(x, t), \quad \operatorname{div} a(x, t) = 0, \\ G(x, t) &= \nabla G_1(x, t) + \nabla \times G_2(x, t) + G_m(x, t), \quad \operatorname{div} G_2(x, t) = 0, \\ \forall (x, t) &\in \Omega \times (0, T), \end{aligned} \quad (2.7)$$

where w and G_1 are scalar valued and a and G_2 are vectors in $\mathbb{R}^d$. The advantage is that w and a satisfy heat type equations in Ω with forcing terms G_1 and G_2 and initial conditions w_0 and a_0 . In view of the next lemma, we can decompose any periodic function u_0 uniquely as in (2.7) where w_0 and a_0 are also periodic.

Lemma 2.1. *Let $u_0 \in H_{per}^2(\Omega)$ be given. Let us consider the equations*

$$-\Delta w_0 = -\operatorname{div} \left(u_0 - \langle u_0 \rangle \chi_0 \right) \quad \text{in } \Omega, \quad (2.8)$$

$$-\Delta a_0 = -\nabla \times \left(u_0 - \langle u_0 \rangle \chi_0 \right), \quad \operatorname{div} a_0 = 0 \quad \text{in } \Omega. \quad (2.9)$$

Then we have that (2.8) and (2.9) admit unique solutions (up to constants) w_0 and a_0 in $H_{per}^3(\Omega)$. Further, we can write u_0 as

$$u_0(x) = \nabla w_0(x) + \nabla \times a_0(x) + \langle u_0 \rangle \chi_0(x), \quad \operatorname{div} a_0 = 0, \quad \forall x \in \Omega,$$

where w_0 and a_0 satisfy (2.8) and (2.9).

Now from (2.4), we derive the following system for $(\varrho, w, a, v_m, \langle v \rangle)$

$$\begin{aligned}
& \varrho_t + u_s \cdot \nabla \varrho + \Psi \cdot \nabla \varrho + (\operatorname{div} \Psi) \varrho + \Delta w + \operatorname{div} v_m = f \quad \text{in } \Omega \times (0, T), \\
& w_t + u_s \cdot \nabla w - \nu \Delta w + p'(1) \varrho = g_1 + G_1 \quad \text{in } \Omega \times (0, T), \\
& a_t + (u_s \cdot \nabla) a - \mu \Delta a = g_2 + G_2 \quad \text{in } \Omega \times (0, T), \\
& \partial_t v_m = g_m + G_m \\
& -\langle v \rangle(t) \left(u_s \cdot \nabla \chi_0 - \mu \Delta \chi_0 \right) + \nu_0 \nabla \operatorname{div} \left(\langle v \rangle(t) \chi_0 \right) \quad \text{in } \Omega \times (0, T), \\
& \partial_t \langle v \rangle(t) = g_3(t) + \langle G \rangle(t) \quad \text{in } (0, T), \\
& \varrho(\cdot, 0) = \rho_0, \quad (w, a)(\cdot, 0) = (w_0, a_0) \quad \text{in } \Omega \quad \text{in } \Omega, \\
& v_m(\cdot, 0) = \langle u_0 \rangle \chi_0, \quad \langle v \rangle(0) = \langle u_0 \rangle \quad \text{in } \Omega,
\end{aligned} \tag{2.10}$$

where $\nu = \mu_0 + 2\mu$ and $\nu_0 = \mu_0 + \mu$.

To study the null controllability of (2.10) for $(\varrho, w, a, v_m, \langle v \rangle)$, note that (2.10)₅ for $\langle v \rangle$ is decoupled and once the null control problem for $\langle v \rangle$ is solved, the control problem (2.10)₄ for v_m is also solved, see Proposition 3.9 in the next section. Further, (2.10)₃, the equation for a , is a decoupled heat type equation and the control problem for a can be solved separately by Theorem 3.8 given in the next section. Therefore, we only need to consider the (ϱ, w) equations, i.e., (2.10)₁ – (2.10)₂, since it is a coupling between a transport equation and a heat equation. How to handle the coupling and to obtain the null controllability of the system will be discussed in the next subsection.

2.2. Analysis for each iterative step

To solve the null control problem for (2.10)₁ – (2.10)₂, our strategy is to introduce a new variable γ , to reduce the differential order of the coupling terms. This allows us to decompose our equations into a problem where the coupling is indeed small and a finite dimensional perturbation. This will be discussed in Step 1. We then solve the problem with small coupling. The finite dimensional problem allows a backward solution. The outcome is a solution that is controlled to zero, but from an initial conditions which differs from the given one in a finite dimensional space. In Step 2, we need to reconstruct ρ from γ . This is slightly technical because we need to get ϱ having average zero. In Step 3, we correct for the fact that our initial condition is wrong by a finite dimensional term. For this, we use approximate controllability of the linearized system. The outcome is a system which is controlled to zero from the correct initial conditions, but with inhomogeneous forcing terms which are different from the original ones. However, we shall obtain estimates showing that the norm of this error in the forcing terms is small relative to the norm of the original forcing terms. We can then use this estimate as a basis for a contraction argument. In the sequel, we shall refer to this contraction argument to solve the linear problem as the “inner” iteration, while the solution of the nonlinear problem by a sequence of linear problems is the “outer” iteration.

2.2.1. Step 1

Let us recall that $\Omega \simeq \prod_{j=1}^d (0, L_j)$, the d -dimensional torus. Using the Fourier basis, any function f in $L^2(\Omega)$ can be written as follows

$$f(x) = \sum_{k \in \mathbb{Z}^d} f_k e^{i2\pi \left(\sum_{j=1}^d \frac{k_j x_j}{L_j} \right)},$$

where $x = (x_1, \dots, x_d) \in \Omega$ and $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$. We consider the orthogonal decomposition of $L^2(\Omega)$ as

$$L^2(\Omega) = H_{fin} \oplus H_{inf},$$

where

$$\begin{aligned} H_{fin} &= \text{span} \left\{ e^{i2\pi \left(\sum_{j=1}^d \frac{k_j x_j}{L_j} \right)} \mid k \in \mathbb{Z}^d, \quad |k| \leq N \right\}, \\ H_{inf} &= \text{span} \left\{ e^{i2\pi \left(\sum_{j=1}^d \frac{k_j x_j}{L_j} \right)} \mid k \in \mathbb{Z}^d, \quad |k| > N \right\}. \end{aligned} \quad (2.11)$$

Note that for any $N \in \mathbb{N}$, H_{fin} is a finite dimensional subspace of $L^2(\Omega)$. This N will be determined later. We denote by Π the projection of $L^2(\Omega)$ onto H_{fin} .

Here we assume that v_m is a known function by solving (2.10)₄. Next, to handle the coupling between (2.10)₁ – (2.10)₂, it is advantageous to reduce the differential order of the coupling term in (2.10)₁. We do so by setting a new variable $\gamma = \varrho + w/\nu$, and then we can write (2.10)₁ – (2.10)₂ as

$$\begin{aligned} \gamma_t + u_s \cdot \nabla \gamma + \Psi \cdot \nabla \gamma + (\text{div } \Psi) \gamma + \frac{p'(1)}{\nu} \left(\gamma - \frac{w}{\nu} \right) - (\text{div } \Psi) (w/\nu) - \Psi \cdot \nabla (w/\nu) \\ + \text{div } v_m = f + g_1/\nu + G_1/\nu \quad \text{in } \Omega \times (0, T), \\ w_t + u_s \cdot \nabla w - \nu \Delta w + p'(1)(\gamma - w/\nu) = g_1 + G_1 \quad \text{in } \Omega \times (0, T), \\ \gamma(\cdot, 0) = \gamma_0 = \varrho_0 + w_0/\nu, \quad w(\cdot, 0) = w_0 \quad \text{in } \Omega. \end{aligned} \quad (2.12)$$

To decouple (2.12) into a transport and heat equation and then to obtain the controllability of (2.12) using a fixed point argument, it would be convenient to have the coefficient of w in (2.12) small enough in a suitable norm. However, $p'(1)/\nu^2$ is constant and not necessarily small. We can, however, decompose w into two parts, of which one will indeed be tractable as a small perturbation, and the other will be finite dimensional and handled separately later. To this end, we define a projection as follows:

$$(\Pi_* w)(x, t) = \begin{cases} 0, & \text{if } t > T_*, \\ \Pi w(x, t), & \text{if } t < T_*. \end{cases} \quad (2.13)$$

Here T_* is sufficiently close to T , to be determined later.

At the first step, we determine (γ^1, w^1) to satisfy the system

$$\begin{aligned}
& \gamma_t^1 + u_s \cdot \nabla \gamma^1 + \Psi \cdot \nabla \gamma^1 + (\operatorname{div} \Psi) \gamma^1 + \frac{p'(1)}{\nu} \gamma^1 - \frac{p'(1)}{\nu^2} (I - \Pi_*) w^1 \\
& - \Psi \cdot \nabla (w^1 / \nu) - (\operatorname{div} \Psi) (w^1 / \nu) + \operatorname{div} v_m = f^1 + g_1^1 / \nu + G_1 / \nu \quad \text{in } \Omega \times (0, T), \\
& w_t^1 + u_s \cdot \nabla w^1 - \nu \Delta w^1 + p'(1)(\gamma^1 - w^1 / \nu) = g_1^1 + G_1 \quad \text{in } \Omega \times (0, T), \\
& \gamma^1(\cdot, 0) = \gamma_0 = \varrho_0 + w_0 / \nu, \quad w^1(\cdot, 0) = w_0 \quad \text{in } \Omega.
\end{aligned} \tag{2.14}$$

The localized controls f^1 and g_1^1 are determined in such a way that the solution is controlled to zero at $t = T$. The coupling terms $(I - \Pi_*) w^1$, $\Psi \cdot \nabla (w^1 / \nu)$ and $(\operatorname{div} \Psi) (w^1 / \nu)$ turn out to be weak because of the choice of T_* in the definition of Π_* and the smallness of Ψ . Exploiting this, we can solve (2.14) iteratively, in a fashion which alternates between a heat and transport problem.

Next we solve (γ^2, w^2) by solving the system backward in time

$$\begin{aligned}
& \gamma_t^2 + u_s \cdot \nabla \gamma^2 + \frac{p'(1)}{\nu} (\gamma^2 - \frac{w^2}{\nu}) - \frac{p'(1)}{\nu^2} \Pi_* w^1 = 0 \quad \text{in } \Omega \times (0, T), \\
& w_t^2 + u_s \cdot \nabla w^2 - \nu \Delta w^2 + p'(1)(\gamma^2 - w^2 / \nu) = 0 \quad \text{in } \Omega \times (0, T), \\
& \gamma^2(\cdot, T) = 0, \quad w^2(\cdot, T) = 0 \quad \text{in } \Omega.
\end{aligned} \tag{2.15}$$

Since in the backward system (2.15), the terminal condition is zero and the nonzero forcing term in the first equation belongs to the finite dimensional space H_{fin} , the system is wellposed in the finite dimensional space H_{fin} . Hence the backward system is solvable and the solution (γ^2, w^2) takes value in H_{fin} .

Setting $\gamma^3 = \gamma^1 + \gamma^2$ and $w^3 = w^1 + w^2$, we obtain that (γ^3, w^3) satisfies

$$\begin{aligned}
& \gamma_t^3 + u_s \cdot \nabla \gamma^3 + \Psi \cdot \nabla \gamma^3 + (\operatorname{div} \Psi) \gamma^3 + \frac{p'(1)}{\nu} (\gamma^3 - \frac{w^3}{\nu}) - \Psi \cdot \nabla (w^3 / \nu) \\
& - (\operatorname{div} \Psi) (w^3 / \nu) + \operatorname{div} v_m = f^1 + g_1^1 / \nu + G_1 / \nu + \Psi \cdot \nabla \gamma^2 \\
& + (\operatorname{div} \Psi) \gamma^2 - \Psi \cdot \nabla (w^2 / \nu) - (\operatorname{div} \Psi) (w^2 / \nu) \quad \text{in } \Omega \times (0, T), \\
& w_t^3 + u_s \cdot \nabla w^3 - \nu \Delta w^3 + p'(1)(\gamma^3 - w^3 / \nu) = g_1^1 + G_1 \quad \text{in } \Omega \times (0, T), \\
& \gamma_0^3 := \gamma^3(\cdot, 0) = \gamma_0 + \gamma^2(\cdot, 0), \quad w_0^3 := w^3(\cdot, 0) = w_0 + w^2(\cdot, 0) \quad \text{in } \Omega.
\end{aligned} \tag{2.16}$$

2.2.2. Step 2

To be able to reconstruct a density ϱ in $\Omega \times (0, T)$ satisfying $\int_{\Omega} \varrho(x, t) dx = 0$ for all $t \in (0, T)$ from γ^3 and w^3 , we need to impose the conditions that $(\gamma^3 - w^3 / \nu)$ and the control f^1 have zero average. To accomplish this, we define

for all $(x, t) \in \Omega \times (0, T)$,

$$\begin{aligned}
s_1(t) &= \langle \gamma^3 - \frac{w^3}{\nu} \rangle(t), \quad s_1(0) = \langle \rho_0 + \gamma^2(\cdot, 0) - \frac{w^2(\cdot, 0)}{\nu} \rangle \\
\tilde{\gamma}^3(x, t) &= \gamma^3(x, t) - s_1(t)\chi_0(x), \\
\tilde{g}_1^1(x, t) &= g_1^1(x, t) - p'(1)s_1(t)\chi_0(x), \\
\tilde{f}^1(x, t) &= f^1(x, t) - s_1'(t)\chi_0(x) - u_s \cdot \nabla (s_1(t)\chi_0) - \Psi \cdot \nabla (s_1(t)\chi_0) \\
&\quad - (\operatorname{div} \Psi) s_1(t)\chi_0.
\end{aligned} \tag{2.17}$$

From (2.16), we can derive that $s_1'(t) = \langle f^1 \rangle(t)$ and from (2.17), we note that $\langle \tilde{f}^1 \rangle(t) = 0$. Further, we also note that if f^1 vanishes after T_1 for some $T_1 < T$, $\tilde{f}^1$ inherits this property.

We now have the equation for $(\tilde{\gamma}^3, w^3)$

$$\begin{aligned}
&\tilde{\gamma}_t^3 + u_s \cdot \nabla \tilde{\gamma}^3 + \Psi \cdot \nabla \tilde{\gamma}^3 + (\operatorname{div} \Psi) \tilde{\gamma}^3 + \frac{p'(1)}{\nu} (\tilde{\gamma}^3 - \frac{w^3}{\nu}) - \Psi \cdot \nabla (w^3/\nu) \\
&- (\operatorname{div} \Psi) (w^3/\nu) + \operatorname{div} v_m = \tilde{f}^1 + \tilde{g}_1^1/\nu + G_1/\nu + \left(\Psi \cdot \nabla \gamma^2 + (\operatorname{div} \Psi) \gamma^2 \right) \\
&\quad - \left(\Psi \cdot \nabla (w^2/\nu) + (\operatorname{div} \Psi) (w^2/\nu) \right) \quad \text{in } \Omega \times (0, T), \\
&w_t^3 + u_s \cdot \nabla w^3 - \nu \Delta w^3 + p'(1)(\tilde{\gamma}^3 - w^3/\nu) = \tilde{g}_1^1 + G_1 \quad \text{in } \Omega \times (0, T), \\
&\tilde{\gamma}_0^3 := \tilde{\gamma}^3(\cdot, 0) = \gamma_0 + \gamma^2(\cdot, 0) - s_1(0)\chi_0, \quad w_0^3 := w^3(\cdot, 0) = w_0 + w^2(\cdot, 0) \quad \text{in } \Omega.
\end{aligned}$$

For all $(x, t) \in \Omega \times (0, T)$, setting

$$\begin{aligned}
\varrho^3(x, t) &= \tilde{\gamma}^3(x, t) - \frac{w^3(x, t)}{\nu}, \quad F^3(x, t) = \Psi \cdot \nabla (\gamma^2 - w^2/\nu) \\
&\quad + (\operatorname{div} \Psi) (\gamma^2 - w^2/\nu),
\end{aligned} \tag{2.18}$$

we get that

$$\begin{aligned}
&\varrho_t^3 + u_s \cdot \nabla \varrho^3 + \Psi \cdot \nabla \varrho^3 + \Delta w^3 + (\operatorname{div} \Psi) \varrho^3 + \operatorname{div} v_m = \tilde{f}^1 + F^3, \\
&w_t^3 + u_s \cdot \nabla w^3 - \nu \Delta w^3 + p'(1)\varrho^3 = \tilde{g}_1^1 + G_1 \quad \text{in } \Omega \times (0, T), \\
&\varrho^3(\cdot, 0) = \varrho_0^3 := \rho_0 + \gamma^2(\cdot, 0) - \frac{w^2(\cdot, 0)}{\nu} - s_1(0)\chi_0, \\
&w^3(\cdot, 0) = w_0^3 = w_0 + w^2(\cdot, 0) \quad \text{in } \Omega, \\
&\int_{\Omega} \varrho_0^3(x) dx = 0.
\end{aligned} \tag{2.19}$$

By our construction, ϱ^3 and w^3 vanish at $t = T$. However, we have not kept any control over their initial conditions.

Remark 2.2. We note that γ^2 and v^2 take values in the finite dimensional space H_{fin} , and therefore so do their initial values. It follows that $\tilde{\gamma}^3 - \gamma^1$ takes values in the linear span of H_{fin} and χ_0 . We find that the difference between ρ_0 and the initial value of ϱ^3 takes values in the span of H_{fin} and χ_0 , i.e. in a space of C^∞ functions. Similarly, the difference between w_0 and w_0^3 is in a space of C^∞ functions.

Taking advantage of this, in the next subsection, we adjust the initial conditions considering a suitable linear control problem.

2.2.3. Step 3

Let us recall T_0 introduced in (1.4) and $T_0 < T$. For the final step to manage the initial conditions, we choose $\epsilon > 0$ such that $T_0 + \epsilon < T$ and we set $\tilde{\varrho} = \varrho^3 + \varrho^4$, $\tilde{w} = w^3 + w^4$, where ϱ^4 and w^4 satisfy a linear control problem on $\Omega \times (0, T_0 + \frac{\epsilon}{2})$ and then a linear system with forcing term on $\Omega \times (T_0 + \frac{\epsilon}{2}, T)$ in such a way that ϱ^4 and w^4 become zero on the time interval $[T_0 + \epsilon, T]$. On $\Omega \times (0, T_0 + \frac{\epsilon}{2})$, we consider

$$\begin{aligned} \varrho_t^4 + u_s \cdot \nabla \varrho^4 + \triangle w^4 &= 0 \quad \text{in } \Omega \times (0, T_0 + \frac{\epsilon}{2}), \\ w_t^4 + u_s \cdot \nabla w^4 - \nu \triangle w^4 + p'(1)\varrho^4 &= g_1^4 \quad \text{in } \Omega \times (0, T_0 + \frac{\epsilon}{2}), \\ \varrho^4(\cdot, 0) = \varrho_0^4 := \rho_0 - \varrho_0^3, \quad w^4(\cdot, 0) = w_0^4 := w_0 - w_0^3 &\quad \text{in } \Omega \quad \int_{\Omega} \varrho_0^4(x) dx = 0, \end{aligned} \quad (2.20)$$

where g_1^4 is a control localized in $\mathcal{O} \times (0, T_0 + \frac{\epsilon}{2})$. By proving the approximate controllability of (2.20) in $\dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega)$, we shall show that for any small $\tilde{\epsilon} > 0$, there exists a control $g_1^4 \in L^2(0, T_0 + \frac{\epsilon}{2}; H_0^3(\mathcal{O}))$ such that (ϱ^4, w^4) , the solution of (2.20) belongs to $C([0, T_0 + \frac{\epsilon}{2}]; \dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega))$ satisfying

$$\|(\varrho_f^4, w_f^4)\|_{\dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega)} \leq \tilde{\epsilon} \|(\varrho^4(\cdot, 0), w^4(\cdot, 0))\|_{\dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega)}. \quad (2.21)$$

Finally, we take a smooth function which interpolates between (ϱ_f^4, v_f^4) at time $t = T_0 + \epsilon/2$ and $(0, 0)$ at time $t = T_0 + \epsilon$, i.e.,

$$\varrho^4(x, t) = \zeta(t)\varrho_f^4(x), \quad w^4(x, t) = \zeta(t)w_f^4(x), \quad \forall (x, t) \in \Omega \times (T_0 + \frac{\epsilon}{2}, T), \quad (2.22)$$

where ζ is a smooth cut-off function on $(T_0 + \frac{\epsilon}{2}, T)$ taking value 1 in a neighborhood of $t = T_0 + \frac{\epsilon}{2}$ and taking value 0 on $[T_0 + \epsilon, T]$. Thus on $\Omega \times (T_0 + \frac{\epsilon}{2}, T)$, (ϱ^4, w^4) satisfies

$$\begin{aligned} \varrho_t^4 + u_s \cdot \nabla \varrho^4 + \triangle w^4 &= F^4 \quad \text{in } \Omega \times (T_0 + \frac{\epsilon}{2}, T), \\ w_t^4 + u_s \cdot \nabla w^4 - \nu \triangle w^4 + p'(1)\varrho^4 &= G_1^4 \quad \text{in } \Omega \times (T_0 + \frac{\epsilon}{2}, T), \\ \varrho^4(\cdot, T_0 + \frac{\epsilon}{2}) &= \varrho_f^4(\cdot), \quad w^4(\cdot, T_0 + \frac{\epsilon}{2}) = w_f^4(\cdot) \quad \text{in } \Omega, \end{aligned} \quad (2.23)$$

where for all $(x, t) \in \Omega \times (T_0 + \frac{\epsilon}{2}, T)$,

$$\begin{aligned} F^4(x, t) &= \zeta'(t)\varrho_f^4(x) + \zeta(t)u_s \cdot \nabla \varrho_f^4(x) + \zeta(t)\triangle w_f^4(x), \\ G_1^4(x, t) &= \zeta'(t)w_f^4(x) + \zeta(t)u_s \cdot \nabla w_f^4 - \nu\zeta(t)\triangle w_f^4(x) + p'(1)\zeta(t)\varrho_f^4(x). \end{aligned} \quad (2.24)$$

Note that as $\zeta = 0$ on $[T_0 + \epsilon, T]$, F^4 and G_1^4 are zero on $\Omega \times [T_0 + \epsilon, T]$. From the construction (2.22), it is clear that $(\varrho^4, v^4)(\cdot, t) = (0, 0)$, for all $t \in$

$[T_0 + \epsilon, T)$. Moreover, since (ϱ_f^4, w_f^4) is in $\dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega)$, F^4 and G_1^4 belong to $L^2(0, T; \dot{H}_{per}^2(\Omega))$ and $L^2(0, T; H_{per}^2(\Omega))$.

Combining (2.20)-(2.23) and setting F^4 and G_1^4 equal to zero on the interval $(0, T_0 + \epsilon/2)$, we finally derive the system on $\Omega \times (0, T)$

$$\begin{aligned} \varrho_t^4 + u_s \cdot \nabla \varrho^4 + \triangle w^4 &= F^4 \quad \text{in } \Omega \times (0, T), \\ w_t^4 + u_s \cdot \nabla w^4 - \nu \triangle w^4 + p'(1) \varrho^4 &= g_1^4 + G_1^4 \quad \text{in } \Omega \times (0, T), \\ \varrho^4(\cdot, 0) = \varrho_0^4 &:= \rho_0 - \varrho_0^3, \quad w^4(\cdot, 0) = w_0^4 := w_0 - w^3(\cdot, 0) \quad \text{in } \Omega, \\ \int_{\Omega} \varrho_0^4(x) dx &= 0, \end{aligned} \tag{2.25}$$

We finally end up with an approximation of $(2.10)_1 - (2.10)_2$ satisfying the equations on $\Omega \times (0, T)$

$$\begin{aligned} \check{\varrho}_t + u_s \cdot \nabla \check{\varrho} + \Psi \cdot \nabla \check{\varrho} + (\operatorname{div} \Psi) \check{\varrho} + \triangle \check{w} + \operatorname{div} v_m \\ = \tilde{f}^1 + F^3 + F^4 + \Psi \cdot \nabla \varrho^4 + (\operatorname{div} \Psi) \varrho^4 \quad \text{in } \Omega \times (0, T), \\ \check{w}_t + u_s \cdot \nabla \check{w} - \nu \triangle \check{w} + p'(1) \check{\varrho} = \tilde{g}_1^1 + g_1^4 + G_1 + G_1^4 \quad \text{in } \Omega \times (0, T), \\ \check{\varrho}(\cdot, 0) = \rho_0, \quad \check{w}(\cdot, 0) = w_0 \quad \text{in } \Omega. \end{aligned} \tag{2.26}$$

Here we see that the above system satisfies a new problem of the same type as $(2.10)_1 - (2.10)_2$, but with initial and inhomogeneous terms whose norms are smaller than those of the original terms. This can then be used as the basis of a contraction argument and hence we can get the result for our original system $(2.10)_1 - (2.10)_2$.

3. Preliminary

In view of our strategy to solve the null control problem discussed in Section 2, we need to use the null controllability results of heat and transport equations. So, in this section, we discuss the null controllability and some required estimates for the solutions of the null control problems for heat and transport equations with controls having support inside $\mathcal{O}$, introduced in (2.5).

3.1. Null controllability of the heat equation

Let us first consider

$$w_t + u_s \cdot \nabla w - \nu \triangle w + \nu_1 w = g_1 + G_1, \quad w(x, 0) = w_0(x), \tag{3.1}$$

with lower order terms and a forcing term G_1 , satisfying $w(\cdot, T) = 0$. Let us assume that $w_0 \in H_{per}^3(\Omega)$ and $G_1 \in L^2(0, T; H_{per}^2(\Omega))$. We study the null controllability of (3.1) with localized interior control $g_1 \in L^2(0, T; H_0^2(\mathcal{O}))$. This will be done by using the Carleman inequalities available in the literature, for example [12]. The lines of the proof of the null controllability of (3.1) will follow the arguments done in Section 2.1 in [15] with the proper adaptation to the multidimensional case.

Let us consider $\mathcal{O}$, a proper subset of Ω and $\mathcal{O}_1$ contained in $\mathcal{O}$. We consider χ_1 , a smooth cut-off function with support inside $\mathcal{O}$ such that $0 \leq \chi_1 \leq 1$ and

$$\chi_1 \equiv 1 \quad \text{on} \quad \mathcal{O}_1. \quad (3.2)$$

The smooth cut-off function χ_1 is introduced to get a control g_1 whose support is inside $\mathcal{O}$ and traces on $\partial\mathcal{O}$ vanish.

Now, let us consider $\mathcal{O}'$, a proper subset of $\mathcal{O}_1$. We recall the weight function ψ from Lemma 2.1 in [12] where $\psi \in C_{per}^\infty(\Omega; \mathbb{R})$ satisfying $|\nabla\psi| > 0$ outside of $\mathcal{O}'$ and $1 < \psi \leq 2$.

In addition, we choose a subdomain $\mathcal{O}_2$ contained in $\mathcal{O}'$ and assume that ψ satisfies

$$\psi(x) = \max_{x \in \Omega} \psi(x) = \psi_{max}, \quad \forall x \in \mathcal{O}_2.$$

Now in the same fashion as in [12] (see (2.3) and (2.4)), we define $\theta \in C^\infty(0, T)$, a function satisfying

$$0 \leq \theta(t) \leq T - t, \quad \theta(t) = \begin{cases} t, & t \in [0, \frac{T}{4}], \\ T - t, & t \in [\frac{3T}{4}, T]. \end{cases}$$

Furthermore, we assume that θ is monotonically increasing for $t \in (0, \frac{T}{2})$ and monotonically decreasing for $t \in (\frac{T}{2}, T)$. We introduce the functions

$$\tilde{\phi} := \tilde{\phi}_\lambda(x, t) = \frac{e^{\lambda\psi(x)}}{\theta(t)}, \quad \phi_\lambda(x) = e^{\lambda^2\psi_{max}} - e^{\lambda\psi(x)}, \quad \widehat{W}_\lambda(x, t) = e^{\frac{\phi_\lambda(x)}{\theta(t)}}, \quad (3.3)$$

where $\lambda > 1$ is a parameter and $\psi(x)$ and $\theta(t)$ are introduced above.

With this weight function, we recall the Carleman inequality from Lemma 2.2 in [12] in the following proposition. We note that introducing lower order terms in the right hand side of the heat equation does not change the estimate.

Proposition 3.1. *Let $T > 0$ and let $\mathcal{O} \subset \Omega$ be a subdomain properly containing $\mathcal{O}_1$. Let χ_1 be a smooth cut-off function with support inside $\mathcal{O}$ satisfying $0 \leq \chi_1 \leq 1$ and (3.2). Then there exists a constant $\widehat{\lambda} > 1$ such that for every $\lambda > \widehat{\lambda}$, there exists a $s_0(\lambda)$ such that for every $s \geq s_0(\lambda)$ and for any smooth function z in $\Omega \times (0, T)$, we have*

$$\begin{aligned} & s^3 \int_0^T \int_\Omega \tilde{\phi}^3 |\widehat{W}_\lambda(x, t)|^{-2s} |z(x, t)|^2 dx dt \\ & \quad + s \int_0^T \int_\Omega \tilde{\phi} |\widehat{W}_\lambda(x, t)|^{-2s} |\nabla z(x, t)|^2 dx dt \\ & \quad + \frac{1}{s} \int_0^T \int_\Omega \frac{|\widehat{W}_\lambda(x, t)|^{-2s}}{\tilde{\phi}} \left(\sum_{i,j=1}^d |\partial_{x_i x_j}^2 z(x, t)|^2 + |\partial_t z(x, t)|^2 \right) dx dt \\ & \leq C \left(\int_0^T \int_\Omega (\widehat{W}_\lambda(x, t))^{-2s} |\partial_t z + \nu \Delta z + u_s \cdot \nabla z - \nu_1 z|^2 dx dt \right) \\ & \quad + C s^3 \left(\int_0^T \int_\mathcal{O} \chi_1^2 \tilde{\phi}^3 |\widehat{W}_\lambda(x, t)|^{-2s} |z|^2 dx dt \right), \end{aligned} \quad (3.4)$$

where $C > 0$ is a constant independent of s and z .

Proof. Let the assumptions in Proposition 3.1 hold. We consider the weight functions as defined in (3.3). Using the argument similar to that of Lemma 2.2 in [12], we get

$$\begin{aligned}
& s^3 \int_0^T \int_{\Omega} \tilde{\phi}^3 |\widehat{W}_{\lambda}(x, t)|^{-2s} |z(x, t)|^2 dx dt \\
& \quad + s \int_0^T \int_{\Omega} \tilde{\phi} |\widehat{W}_{\lambda}(x, t)|^{-2s} |\nabla z(x, t)|^2 dx dt \\
& \quad + \frac{1}{s} \int_0^T \int_{\Omega} \frac{|\widehat{W}_{\lambda}(x, t)|^{-2s}}{\tilde{\phi}} \left(\sum_{i,j=1}^d |\partial_{x_i x_j}^2 z(x, t)|^2 + |\partial_t z(x, t)|^2 \right) dx dt \\
& \leq C \left(\int_0^T \int_{\Omega} (\widehat{W}_{\lambda}(x, t))^{-2s} |\partial_t z + \nu \Delta z + u_s \cdot \nabla z - \nu_1 z|^2 dx dt \right) \\
& \quad + C s^3 \left(\int_0^T \int_{\mathcal{O}_1} \tilde{\phi}^3 |\widehat{W}_{\lambda}(x, t)|^{-2s} |z|^2 dx dt \right),
\end{aligned}$$

for some positive constant C independent of s and z . Now in view of (3.2) and $\mathcal{O}_1 \subset \mathcal{O}$, we have

$$\int_0^T \int_{\mathcal{O}_1} \tilde{\phi}^3 |\widehat{W}_{\lambda}(x, t)|^{-2s} |z|^2 dx dt \leq \int_0^T \int_{\mathcal{O}} \chi_1^2 \tilde{\phi}^3 |\widehat{W}_{\lambda}(x, t)|^{-2s} |z|^2 dx dt,$$

and hence (3.4) holds. $\square$

To use the above Carleman inequality for proving null controllability of a parabolic equation, we need the initial condition to be zero and the forcing terms to tend to zero as t goes to zero. In view of this, we introduce $\tilde{w}$ satisfying

$$\begin{aligned}
& \tilde{w}_t + u_s \cdot \nabla \tilde{w} + \nu_1 \tilde{w} - \nu \Delta \tilde{w} = G_1 \quad \text{in } \Omega \times (0, T), \\
& \tilde{w}(x, 0) = w_0(x).
\end{aligned} \tag{3.5}$$

We set $\widehat{w}(x, t) = w(x, t) - \zeta(t)\tilde{w}(x, t)$, where ζ is a smooth cut-off function of t in $(0, T)$:

$$\zeta(t) = \begin{cases} 1, & \text{near } t = 0, \\ 0, & \text{near } t = T. \end{cases}$$

Then $\widehat{w}$ satisfies

$$\begin{aligned}
& \widehat{w}_t + u_s \cdot \nabla \widehat{w} + \nu_1 \widehat{w} - \nu \Delta \widehat{w} = -\zeta'(t)\tilde{w} + (1 - \zeta(t))G_1 + g_1 =: \widehat{G} + g_1, \\
& \widehat{w}(x, 0) = 0,
\end{aligned} \tag{3.6}$$

where $\tilde{w}$ is the solution of (3.5).

Remark 3.2. Note that the forcing terms in the above equation are zero near $t = 0$ because $\zeta'(t)$ and $(1 - \zeta(t))$ are zero near $t = 0$. Further, the initial condition in the above equation is also zero. Again, near $t = T$, $\zeta'(t)$ vanishes

and $(1 - \zeta(t))G_1$ is in the weighted L^2 space with weight $(T - t)^{3/2}\widehat{W}_\lambda$, where $\widehat{W}_\lambda$ is introduced in (3.3), if G_1 satisfies

$$\int_0^T \int_\Omega (T - t)^3 e^{\frac{2s\phi_\lambda(x)}{T-t}} |G_1(x, t)|^2 dx dt < \infty, \quad \forall \lambda > 1, \quad \forall s > 0. \quad (3.7)$$

We shall obtain the null controllability of (3.6) using the weight function $\widehat{W}_\lambda$ and Proposition 3.1. Unlike [11], our equation includes an inhomogeneous term. Our argument for null controllability is analogous to [9], using a duality argument and observability inequality.

To get an estimate of the solution of the control problem in the regular spaces with weight, for $\lambda > 2$, θ , ψ introduced in (3.3), we set

$$\Theta_\lambda(t) = \frac{e^{\lambda^2 \psi_{max}}}{\theta(t)}, \quad \forall t \in (0, T). \quad (3.8)$$

We note that for $\lambda > 2$, θ , ψ introduced in (3.3), we have

$$\begin{aligned} \frac{3}{4} e^{\lambda^2 \psi_{max}} &< \phi_\lambda(x) < e^{\lambda^2 \psi_{max}}, \quad \forall x \in \Omega, t \in (0, T), \\ e^{\frac{3}{4} \Theta_\lambda(t)} &< \widehat{W}_\lambda(x, t) < e^{\Theta_\lambda(t)}, \quad \forall t \in (0, T). \end{aligned} \quad (3.9)$$

Here and in the sequel, we denote by $D^j u$, all j th derivatives of the scalar function u with respect to the space variables. Let us define the weighted spaces for the function with periodic boundary conditions

$$\begin{aligned} \widehat{X}_2 &= \{u \mid \widehat{W}_\lambda^s u, \theta(t) \widehat{W}_\lambda^s D u, \theta^2(t) \widehat{W}_\lambda^s D^2 u, \theta^2(t) \widehat{W}_\lambda^s u_t \in L^2(0, T; L^2(\Omega)), \\ &\quad e^{\frac{s\Theta_\lambda(t)}{2}} u \in L^2(0, T; H_{per}^4(\Omega)) \cap H^1(0, T; H_{per}^2(\Omega))\}, \\ \widehat{Y}_2 &= \{g \mid \theta^{3/2}(t) \widehat{W}_\lambda^s g \in L^2(0, T; L^2(\mathcal{O})), e^{\frac{s\Theta_\lambda(t)}{2}} g \in L^2(0, T; H_0^2(\mathcal{O}))\}, \\ \widehat{Z}_2 &= \{\widehat{G} \mid \theta^{3/2}(t) \widehat{W}_\lambda^s \widehat{G} \in L^2(0, T; L^2(\Omega)), e^{\frac{s\Theta_\lambda(t)}{2}} \widehat{G} \in L^2(0, T; H_{per}^2(\Omega))\} \end{aligned} \quad (3.10)$$

for some $s > 0$ and $\lambda > 2$, to be determined later.

The following lemma helps us to study the null controllability of equation (3.6).

Lemma 3.3. *Let $T > 0$ and $\mathcal{O} \subset \Omega$. Equation (3.6) with forcing term $\widehat{G} \in L^2(0, T; L^2(\Omega))$ is null controllable at T with an admissible control g_1 in $L^2(0, T; L^2(\mathcal{O}))$ if and only if for all smooth functions z satisfying periodic boundary conditions, the solution $\widehat{w}$ of (3.6), in the sense of transposition with control g_1 , satisfies*

$$\begin{aligned} \int_0^T \int_\Omega \left(\partial_t z(x, t) + \nu \Delta z(x, t) + u_s \cdot \nabla z(x, t) - \nu_1 z(x, t) \right) \widehat{w}(x, t) dx dt \\ + \int_0^T \int_{\mathcal{O}} g_1(x, t) z(x, t) dx dt + \int_0^T \int_\Omega \widehat{G}(x, t) z(x, t) dx dt = 0. \end{aligned} \quad (3.11)$$

The proof is analogous to the argument on p. 199 of [9].

In view of Lemma 3.3, to solve the control problem (3.6), equivalently we seek a function $\widehat{w}$ satisfying (3.11). For that, we introduce the functional

$$\begin{aligned} J(z) = & \frac{1}{2} \int_0^T \int_{\Omega} |\widehat{W}_{\lambda}(x, t)|^{-2s} |\partial_t z + \nu \Delta z + u_s \cdot \nabla z - \nu_1 z|^2 dx dt \\ & + \frac{s^3}{2} \int_0^T \int_{\mathcal{O}} \chi_1^2 \widetilde{\phi}^3 |\widehat{W}_{\lambda}(x, t)|^{-2s} |z|^2 dx dt + \int_0^T \int_{\Omega} \widehat{G}(x, t) z(x, t) dx dt, \end{aligned} \quad (3.12)$$

for $z \in L^2(0, T; H_{per}^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and the norm defined by

$$\begin{aligned} \|z\|_{obs}^2 = & \int_0^T \int_{\Omega} |\widehat{W}_{\lambda}(x, t)|^{-2s} |\partial_t z + \nu \Delta z + u_s \cdot \nabla z - \nu_1 z|^2 dx dt \\ & + s^3 \int_0^T \int_{\mathcal{O}} \chi_1^2 \widetilde{\phi}^3 |\widehat{W}_{\lambda}(x, t)|^{-2s} |z|^2 dx dt. \end{aligned} \quad (3.13)$$

Let us also introduce the space

$$X_{obs} = \overline{C_{per}^2(\Omega \times [0, T])}^{\|\cdot\|_{obs}}. \quad (3.14)$$

Theorem 3.4. *Let $T > 0$. There exists a $\widehat{\lambda} > 2$, and an $s_0(\lambda)$, defined for $\lambda > \widehat{\lambda}$, such that for every $\lambda > \widehat{\lambda}$ and $s > s_0(\lambda)$ the following holds. For any $\widehat{G} \in \widehat{Z}_2$, there exists a control $g_1 \in \widehat{Y}_2$ such that (3.6) admits a unique solution in $\widehat{X}_2$ satisfying*

$$\widehat{w}(\cdot, T) = 0, \quad \Omega.$$

Further, the solution satisfies

$$\|\widehat{w}\|_{\widehat{X}_2} + \|g_1\|_{\widehat{Y}_2} \leq C_1 \|\widehat{G}\|_{\widehat{Z}_2}, \quad (3.15)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$, λ , s , and the coefficients in the equation.

Proof. Using (3.4), we can show that J is continuous and coercive on X_{obs} . Moreover, J is strictly convex on X_{obs} . Thus there exists a unique minimizer $\widehat{z}$ of J in X_{obs} . Hence, for any smooth function z satisfying periodic boundary conditions, we have

$$\begin{aligned} 0 = J'(\widehat{z})z = & \int_0^T \int_{\Omega} |\widehat{W}_{\lambda}(x, t)|^{-2s} (\partial_t z + \nu \Delta z + u_s \cdot \nabla z - \nu_1 z) \\ & (\partial_t \widehat{z} + \nu \Delta \widehat{z} + u_s \cdot \nabla \widehat{z} - \nu_1 \widehat{z}) dx dt + s^3 \int_0^T \int_{\mathcal{O}} \chi_1^2 \widetilde{\phi}^3 |\widehat{W}_{\lambda}(x, t)|^{-2s} z \widehat{z} dx dt \\ & + \int_0^T \int_{\Omega} \widehat{G}(x, t) z(x, t) dx dt. \end{aligned}$$

In view of (3.11), we set

$$\begin{aligned} \widehat{w}(x, t) = & |\widehat{W}_{\lambda}(x, t)|^{-2s} (\partial_t \widehat{z}(x, t) + \nu \Delta \widehat{z}(x, t) + u_s \cdot \nabla \widehat{z}(x, t) - \nu_1 \widehat{z}(x, t)), \\ g_1(x, t) = & s^3 \widetilde{\phi}^3(x, t) |\widehat{W}_{\lambda}(x, t)|^{-2s} \widehat{z}(x, t) \chi_1^2(x), \quad (x, t) \in \mathcal{O} \times (0, T). \end{aligned} \quad (3.16)$$

Then $(\widehat{w}, g_1)$ satisfies the identity (3.11) of the control problem (3.6). Hence, by Lemma 3.3, $\widehat{w}$ is the solution of (3.6) in the sense of transposition with control g_1 and it satisfies

$$\widehat{w}(\cdot, T) = 0 \quad \text{in } \Omega.$$

Since the source term $\widehat{G}$ and control g_1 , defined in (3.16), belong to $L^2(0, T; L^2(\Omega))$, $\widehat{w}$ is in fact a strong solution of (3.6) in $L^2(0, T; H_{per}^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$.

Now for a given $\widehat{G} \in \widehat{Z}_2$, we show that $(\widehat{w}, g_1)$ belongs to $\widehat{X}_2 \times \widehat{Y}_2$ along with the estimate (3.15). Since $\widehat{z}$ is the unique minimizer of J , we have

$$J(\widehat{z}) \leq J(0) = 0,$$

and hence from (3.12), it follows that

$$\frac{1}{2} \|\widehat{z}\|_{obs}^2 \leq - \int_0^T \int_{\Omega} \widehat{G}(x, t) \widehat{z}(x, t) \, dx \, dt. \quad (3.17)$$

Then using the Cauchy-Schwarz inequality and Carleman inequality (3.4), we obtain that there exists a positive constant C_1 such that

$$\|\widehat{z}\|_{obs}^2 \leq C_1 \left(\int_0^T \int_{\Omega} \theta^3(t) |\widehat{W}_{\lambda}(x, t)|^{2s} |\widehat{G}(x, t)|^2 \, dx \, dt \right).$$

Using the definition of $\widehat{w}$ and g_1 in (3.16), from the above inequality we derive

$$\begin{aligned} & \int_0^T \int_{\Omega} |\widehat{W}_{\lambda}(x, t)|^{2s} |\widehat{w}(x, t)|^2 \, dx \, dt + \int_0^T \int_{\mathcal{O}} \theta^3(t) |\widehat{W}_{\lambda}(x, t)|^{2s} |g_1(x, t)|^2 \, dx \, dt \\ & \leq C_1 \left(\int_0^T \int_{\Omega} \theta^3(t) |\widehat{W}_{\lambda}(x, t)|^{2s} |\widehat{G}(x, t)|^2 \, dx \, dt \right). \end{aligned} \quad (3.18)$$

To show $\widehat{w} \in \widehat{X}_2$, we need estimates on the higher derivatives of $\widehat{w}$. Following the arguments of Section 3.3 in [9], we multiply (3.6) first by $\widehat{w}(x, t) \theta^2(t) |\widehat{W}_{\lambda}(x, t)|^{2s}$ and then $\widehat{w}_t(x, t) \theta^4(t) |\widehat{W}_{\lambda}(x, t)|^{2s}$. Thus we get the following estimates of $\widehat{w}$:

$$\begin{aligned} & \int_0^T \int_{\Omega} |\widehat{W}_{\lambda}(x, t)|^{2s} |\widehat{w}(x, t)|^2 \, dx \, dt + \int_0^T \int_{\Omega} \theta^2(t) |\widehat{W}_{\lambda}(x, t)|^{2s} |\nabla \widehat{w}(x, t)|^2 \, dx \, dt \\ & + \int_0^T \int_{\Omega} \theta^4(t) |\widehat{W}_{\lambda}(x, t)|^{2s} \left(\sum_{i,j=1}^d |\partial_{x_i x_j}^2 \widehat{w}(x, t)|^2 + |\partial_t \widehat{w}(x, t)|^2 \right) \, dx \, dt \\ & + \int_0^T \int_{\mathcal{O}} \theta^3(t) |\widehat{W}_{\lambda}(x, t)|^{2s} |g_1(x, t)|^2 \, dx \, dt \\ & \leq C_1 \left(\int_0^T \int_{\Omega} \theta^3(t) |\widehat{W}_{\lambda}(x, t)|^{2s} |\widehat{G}(x, t)|^2 \, dx \, dt \right). \end{aligned} \quad (3.19)$$

Note that because of (3.9), from (3.17), we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} e^{-2s\Theta_\lambda(t)} |\partial_t \widehat{z} + \nu \Delta \widehat{z} + u_s \cdot \nabla \widehat{z} - \nu_1 \widehat{z}|^2 dx dt \\
& + \int_0^T \int_{\Omega} \chi_1^2 \theta^{-3}(t) e^{-2s\Theta_\lambda(t)} |\widehat{z}|^2 dx dt \\
& \leq C \left(\int_0^T \int_{\Omega} |\widehat{W}_\lambda(x, t)|^{-2s} |\partial_t \widehat{z} + \nu \Delta \widehat{z} + u_s \cdot \nabla \widehat{z} - \nu_1 \widehat{z}|^2 dx dt \right. \\
& \left. + \int_0^T \int_{\mathcal{O}} \chi_1^2 \widetilde{\phi}^3 |\widehat{W}_\lambda(x, t)|^{-2s} |\widehat{z}|^2 dx dt \right) < \infty.
\end{aligned} \tag{3.20}$$

For our further use, we want to give the following estimate which follows from the above estimate and the Carleman inequality (3.4). Since $\widehat{z} \in X_{obs}$, we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} \theta^{-3}(t) e^{-2s\Theta_\lambda(t)} |\widehat{z}|^2 dx dt \leq \int_0^T \int_{\Omega} \widetilde{\phi}^3 |\widehat{W}_\lambda(x, t)|^{-2s} |\widehat{z}|^2 dx dt \\
& \leq C \left(\int_0^T \int_{\Omega} |\widehat{W}_\lambda(x, t)|^{-2s} |\partial_t \widehat{z} + \nu \Delta \widehat{z} + u_s \cdot \nabla \widehat{z} - \nu_1 \widehat{z}|^2 dx dt \right. \\
& \left. + \int_0^T \int_{\mathcal{O}} \chi_1^2 \widetilde{\phi}^3 |\widehat{W}_\lambda(x, t)|^{-2s} |\widehat{z}|^2 dx dt \right) < \infty.
\end{aligned} \tag{3.21}$$

Now, to get $e^{\frac{s\Theta_\lambda(t)}{2}} g_1 \in L^2(0, T; H_0^2(\mathcal{O}))$, we first note from (3.16) that

$$e^{\frac{s\Theta_\lambda(t)}{2}} g_1(x, t) = s^3 \theta^{-3}(t) e^{3\lambda\psi(x)} e^{s \left(\frac{\Theta_\lambda(t)}{2} - \frac{2\phi_\lambda(x, t)}{\theta(t)} \right)} \widehat{z}(x, t) \chi_1^2(x). \tag{3.22}$$

Further, using (3.9), we can choose a small enough $\epsilon > 0$ such that

$$\left(\frac{\Theta_\lambda(t)}{2} - \frac{2\phi_\lambda(x, t)}{\theta(t)} \right) \leq -\Theta_\lambda(t) - \frac{\epsilon}{\theta(t)}.$$

In view of this, from (3.22) we write

$$e^{\frac{s\Theta_\lambda(t)}{2}} |g_1(x, t)| \leq s^3 |e^{3\lambda\psi(x)} \chi_1(x)| \theta^{-4}(t) e^{-s \frac{\epsilon}{\theta(t)}} \theta(t) e^{-s\Theta_\lambda(t)} |\widehat{z}(x, t)|, \tag{3.23}$$

and we consider the parabolic equation satisfied by $g_* = \theta(t) e^{-s\Theta_\lambda(t)} \widehat{z}$, on $\Omega \times (0, T)$,

$$\begin{aligned}
& \partial_t g_* + \nu \Delta g_* + u_s \cdot \nabla g_* - \nu_1 g_* = \theta(t) e^{-s\Theta_\lambda(t)} (\partial_t \widehat{z} + \nu \Delta \widehat{z} + u_s \cdot \nabla \widehat{z} - \nu_1 \widehat{z}) \\
& \quad + \partial_t \left(\theta(t) e^{-s\Theta_\lambda(t)} \right) \widehat{z}, \\
& g_*(\cdot, T) = 0.
\end{aligned} \tag{3.24}$$

We can see that by (3.20) the first term of the right hand side of the above equation is in $L^2(0, T; L^2(\Omega))$. For the second term of the right hand side of the above equation, by calculating we get

$$\partial_t \left(\theta(t) e^{-s\Theta_\lambda(t)} \right) \widehat{z} = \left(\theta'(t) e^{-s\Theta_\lambda(t)} \right) \widehat{z} + \left(s e^{\lambda^2 \psi_{max}} \theta'(t) \theta^{-1}(t) e^{-s\Theta_\lambda(t)} \right) \widehat{z}.$$

From the definition of θ , we have that $\theta'(t)$ is bounded over $(0, T)$ and hence using (3.21), we get that $\partial_t \left(\theta(t) e^{-s\Theta_\lambda(t)} \right) \widehat{z} \in L^2(0, T; L^2(\Omega))$. Thus, since the right hand side of (3.24) is in $L^2(0, T; L^2(\Omega))$, using the regularity of the parabolic equation we get $g_* \in L^2(0, T; H_{per}^2(\Omega))$. Now, using (3.23) and the fact that $|e^{3\lambda\psi(x)} \chi_1(x)| \theta^{-4}(t) e^{-s\frac{\epsilon}{\theta(t)}}$ is bounded over $\Omega \times (0, T)$, we can show that $g_1 \in \widehat{Y}_2$.

Similarly, to prove that $e^{\frac{s\Theta_\lambda(t)}{2}} \widehat{w} \in L^2(0, T; H_{per}^4(\Omega)) \cap H^1(0, T; H_{per}^2(\Omega))$, we use the regularity of the parabolic equation satisfied by $\widehat{w}_* = e^{\frac{s\Theta_\lambda(t)}{2}} \widehat{w}$ on $\Omega \times (0, T)$,

$$\begin{aligned} \partial_t \widehat{w}_* + u_s \cdot \nabla \widehat{w}_* + \nu_1 \widehat{w}_* - \nu \Delta \widehat{w}_* &= e^{\frac{s\Theta_\lambda(t)}{2}} (\widehat{G} + g_1) + \partial_t \left(e^{\frac{s\Theta_\lambda(t)}{2}} \right) \widehat{w}, \\ \widehat{w}_*(x, 0) &= 0. \end{aligned} \quad (3.25)$$

Now, we note that in the above equation

$$|\partial_t \left(e^{\frac{s\Theta_\lambda(t)}{2}} \right) \widehat{w}(x, t)| \leq C \left(\theta^{-4}(t) e^{\frac{-s\Theta_\lambda(t)}{4}} \right) \left(\theta^2(t) e^{\frac{3s\Theta_\lambda(t)}{4}} |\widehat{w}(x, t)| \right),$$

for some positive constant C and $\left(\theta^{-4}(t) e^{\frac{-s\Theta_\lambda(t)}{4}} \right)$ is bounded over $(0, T)$. Using (3.9) and (3.19), we get that $\left(\theta^2(t) e^{\frac{3s\Theta_\lambda(t)}{4}} |\widehat{w}(x, t)| \right) \in L^2(0, T; H_{per}^2(\Omega))$ and hence $\partial_t \left(e^{\frac{s\Theta_\lambda(t)}{2}} \right) \widehat{w} \in L^2(0, T; H_{per}^2(\Omega))$. Further, we have that $\widehat{G} \in \widehat{Z}_2$ and $g_1 \in \widehat{Y}_2$. Thus, $\widehat{w}_*$ belongs to $L^2(0, T; H_{per}^4(\Omega))$, since the right hand side of (3.25) is in $L^2(0, T; H_{per}^2(\Omega))$. Finally, we get that $\widehat{w} \in \widehat{X}_2$. From the above estimates, (3.15) follows and hence Theorem 3.6 is proved. $\square$

Let us recall the existence and uniqueness of the solution of (3.5) in a proper Sobolev space.

Proposition 3.5. *For $G_1 \in L^2(0, T; H_{per}^2(\Omega))$, $w_0 \in H_{per}^3(\Omega)$, equation (3.5) admits a unique solution $\widetilde{w}$ in $L^2(0, T; H_{per}^4(\Omega)) \cap H^1(0, T; H_{per}^2(\Omega))$ satisfying*

$$\|\widetilde{w}\|_{L^2(0, T; H_{per}^4(\Omega)) \cap H^1(0, T; H_{per}^2(\Omega))} \leq C \left(\|G_1\|_{L^2(0, T; H_{per}^2(\Omega))} + \|w_0\|_{H_{per}^3(\Omega)} \right), \quad (3.26)$$

for some positive constant C .

The proof of this proposition is standard and hence we skip it here.

Now we fix a $\lambda > \widehat{\lambda}$ and $s > s_0(\lambda)$, where $\widehat{\lambda}$ and s_0 are obtained from Theorem 3.4. Let us set for all $x \in \Omega$ and $t \in (0, T)$,

$$\phi(x) = e^{\lambda^2 \psi_{max}} - e^{\lambda \psi(x)}, \quad W(x, t) = e^{\frac{s\phi(x)}{T-t}}, \quad \Theta(t) = \frac{e^{s\lambda^2 \psi_{max}}}{2\theta(t)}. \quad (3.27)$$

Note that for a given interval Ω , a proper subset $\mathcal{O}_1$ and a smaller subset $\mathcal{O}'$ contained in $\mathcal{O}_1$, $\phi \in C^\infty(\Omega; \mathbb{R}^+)$, is a periodic function such that

$$|\nabla \phi| > 0 \quad \text{on } \Omega \setminus \mathcal{O}'. \quad (3.28)$$

Further, ϕ satisfies

$$\phi(x) = \min_{x \in \Omega} \phi(x) = \phi_{min}, \quad \forall x \in \mathcal{O}_2, \quad (3.29)$$

where $\mathcal{O}_2$ is also a given proper subset of $\mathcal{O}'$.

Here and in the sequel, we consider the following weighted spaces for functions with periodic boundary conditions. We define

$$\begin{aligned} X_2 &= \{u \mid Wu, (T-t)WDu, (T-t)^2WD^2u, (T-t)^2Wu_t \in L^2(0, T; L^2(\Omega)), \\ &\quad e^{\Theta(t)}u \in L^2(0, T; H_{per}^4(\Omega)) \cap H^1(0, T; H_{per}^2(\Omega))\}, \\ Z_2 &= \{G \mid (T-t)^{3/2}WG \in L^2(0, T; L^2(\Omega)), e^{\Theta(t)}G \in L^2(0, T; H_{per}^2(\Omega))\}, \\ Y_2 &= \{g \mid (T-t)^{3/2}Wg \in L^2(0, T; L^2(\mathcal{O})), e^{\Theta(t)}g \in L^2(0, T; H_0^2(\mathcal{O}))\}, \end{aligned} \quad (3.30)$$

where W is introduced in (3.27). We have the following theorem combining Proposition 3.5 and Theorem 3.4.

Theorem 3.6. *For weight functions Θ and W defined in (3.27), where ϕ satisfies (3.28)-(3.29), the following holds: Let $w_0 \in H_{per}^3(\Omega)$ and $G_1 \in Z_2$. Then for any $T > 0$, there exists a control $g_1 \in Y_2$ such that (3.1) has a unique solution w in X_2 with*

$$w(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|w\|_{X_2} + \|g_1\|_{Y_2} \leq C_1(\|G_1\|_{Z_2} + \|w_0\|_{H_{per}^3(\Omega)}), \quad (3.31)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

Proof. We decompose w as in the line following (3.5) and separately estimate $\hat{w}$ and $\tilde{w}$. $\square$

To study the equation (3.1) in weaker space for initial condition in $H_{per}^2(\Omega)$ and forcing term in $L^2(0, T; H_{per}^1)$, we define

$$\begin{aligned} \tilde{X}_2 &= \{u \mid Wu, (T-t)WDu, (T-t)^2WD^2u, (T-t)^2Wu_t \in L^2(0, T; L^2(\Omega)), \\ &\quad e^{\Theta(t)}u \in L^2(0, T; H_{per}^3(\Omega)) \cap H^1(0, T; H_{per}^1(\Omega))\}, \\ \tilde{Z}_2 &= \{G \mid (T-t)^{3/2}WG \in L^2(0, T; L^2(\Omega)), e^{\Theta(t)}G \in L^2(0, T; H_{per}^1(\Omega))\}, \\ \tilde{Y}_2 &= \{g \mid (T-t)^{3/2}Wg \in L^2(0, T; L^2(\mathcal{O})), e^{\Theta(t)}g \in L^2(0, T; H_0^1(\mathcal{O}))\}, \end{aligned} \quad (3.32)$$

where W is introduced in (3.27). We have the following null controllability result for (3.1) in the weaker spaces.

Corollary 3.7. *Consider equation (3.1) with forcing term $G_1 \in \tilde{Z}_2$ and initial condition $w_0 \in H_{per}^2(\Omega)$. Then for any $T > 0$, there exists a control $g_1 \in \tilde{Y}_2$ such that (3.1) has a unique solution w in $\tilde{X}_2$ with*

$$w(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|w\|_{\tilde{X}_2} + \|g_1\|_{\tilde{Y}_2} \leq C_1(\|G_1\|_{\tilde{Z}_2} + \|w_0\|_{H_{per}^2(\Omega)}), \quad (3.33)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

The proof of the above corollary follows similarly to that of Theorem 3.6.

Now let us define the vector valued spaces associated to the above weighted spaces:

$$\begin{aligned} \mathbb{X}_2 &= \{u \mid (u)_i \in X_2, \quad \forall i = 1, \dots, d\}, \quad \mathbb{Z}_2 = \{G \mid (G)_i \in Z_2, \quad \forall i = 1, \dots, d\}, \\ \mathbb{Y}_2 &= \{g \mid (g)_i \in Y_2, \quad \forall i = 1, \dots, d\}, \\ \tilde{\mathbb{X}}_2 &= \{u \mid (u)_i \in \tilde{X}_2, \quad \forall i = 1, \dots, d\}, \quad \tilde{\mathbb{Z}}_2 = \{G \mid (G)_i \in \tilde{Z}_2, \quad \forall i = 1, \dots, d\}, \\ \tilde{\mathbb{Y}}_2 &= \{g \mid (g)_i \in \tilde{Y}_2, \quad \forall i = 1, \dots, d\}, \end{aligned} \quad (3.34)$$

Now we consider the parabolic equation for vector a ,

$$a_t + (u_s \cdot \nabla)a - \mu \Delta a = g_2 + G_2 \quad \text{in } \Omega \times (0, T), \quad a(\cdot, 0) = a_0, \quad (3.35)$$

where G_2 is the forcing term and g_2 is the control to drive the solution a to rest at time $T > 0$. We can easily see that each component of a satisfies the estimates (3.31) and (3.33) provided the forcing terms and initial conditions satisfy the required conditions. Thus combining Theorem 3.6 and Corollary 3.7, we get the following theorem.

Theorem 3.8. *For weight functions Θ and W defined in (3.27), where ϕ satisfies (3.28)-(3.29), the following holds:*

1. *Let $a_0 \in (H_{per}^3(\Omega))^d$ and $G_2 \in \mathbb{Z}_2$. Then for any $T > 0$, there exists a control $g_2 \in \mathbb{Y}_2$ such that (3.35) has a unique solution a in $\mathbb{X}_2$ with*

$$a(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|a\|_{\mathbb{X}_2} + \|g_2\|_{\mathbb{Y}_2} \leq C_1(\|G_2\|_{\mathbb{Z}_2} + \|a_0\|_{H_{per}^3(\Omega)}), \quad (3.36)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

2. Let $a_0 \in (H_{per}^2(\Omega))^d$ and $G_2 \in \tilde{\mathbb{Z}}_2$. For any $T > 0$, there exists a control $g_2 \in \tilde{\mathbb{Y}}_2$ such that (3.35) has a unique solution a in $\tilde{\mathbb{X}}_2$ with

$$a(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|a\|_{\tilde{\mathbb{X}}_2} + \|g_2\|_{\tilde{\mathbb{Y}}_2} \leq C_1(\|G_2\|_{\tilde{\mathbb{Z}}_2} + \|a_0\|_{H_{per}^2(\Omega)}), \quad (3.37)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

We introduce a function $\chi_0 \in C^\infty(\Omega; \mathbb{R})$ with a compact support in $\mathcal{O}_2$, mentioned in (3.29), such that $\int_{\mathcal{O}_2} \chi_0(x) dx = 1$. Note that $\mathcal{O}_2$ is a proper subset of $\mathcal{O}$ and the weight function $W(x, t)$ takes its minimum value for all $(x, t) \in \mathcal{O}_2 \times (0, T)$.

Finally, let us consider the equation

$$v_t + u_s \cdot \nabla v - \mu \Delta v - (\mu_0 + \mu) \nabla \operatorname{div} v = g + G \quad \text{in } \Omega \times (0, T), v(\cdot, 0) = u_0 \quad \text{in } \Omega, \quad (3.38)$$

where G is a forcing term. We seek a control g localized in $\mathcal{O}$ such that $v(\cdot, T) = 0$. To obtain it, let us recall the decomposition of u_0 , G and v as mentioned in (2.7). Then w satisfies (3.1) for $\nu_1 = 0$ and a satisfies (3.35), whereas v_m satisfies

$$\begin{aligned} \partial_t v_m &= g_m + G_m \\ -\langle v \rangle(t) (u_s \cdot \nabla \chi_0 - \mu \Delta \chi_0) + \nu_0 \nabla \operatorname{div} (\langle v \rangle(t) \chi_0) &\quad \text{in } \Omega \times (0, T), \\ \partial_t \langle v \rangle(t) &= g_3(t) + \langle G \rangle(t) \quad \text{in } (0, T), \\ v_m(\cdot, 0) &= \langle u_0 \rangle \chi_0, \quad \langle v \rangle(0) = \langle u_0 \rangle \quad \text{in } \Omega. \end{aligned} \quad (3.39)$$

Now we define the weighted spaces for periodic vector valued functions:

$$\begin{aligned} \mathbb{X}_4 &= \{u \mid (T-t)W(u)_i, (T-t)^2WD(u)_i \in L^2(0, T; L^2(\Omega)), \\ &\quad e^{\Theta(t)}(u)_i \in L^2(0, T; H_{per}^3(\Omega)) \cap H^1(0, T; H_{per}^1(\Omega)), \quad \forall i = 1, \dots, d\}, \\ \mathbb{Z}_4 &= \{G \mid e^{2\Theta(t)}(G)_i \in L^2(0, T; H_{per}^1(\Omega)), \quad \forall i = 1, \dots, d\}, \\ \mathbb{Y}_4 &= \{g \mid e^{\Theta(t)}(g)_i \in L^2(0, T; H_0^1(\mathcal{O})), \quad \forall i = 1, \dots, d\}, \\ \tilde{\mathbb{X}}_4 &= \{u \mid (T-t)W(u)_i, (T-t)^2WD(u)_i \in L^2(0, T; L^2(\Omega)), \\ &\quad e^{\Theta(t)}(u)_i \in L^2(0, T; H_{per}^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad \forall i = 1, \dots, d\}, \\ \tilde{\mathbb{Z}}_4 &= \{G \mid e^{2\Theta(t)}(G)_i \in L^2(0, T; L^2(\Omega)), \quad \forall i = 1, \dots, d\}, \\ \tilde{\mathbb{Y}}_4 &= \{g \mid e^{\Theta(t)}(g)_i \in L^2(0, T; L^2(\mathcal{O})), \quad \forall i = 1, \dots, d\}. \end{aligned} \quad (3.40)$$

Proposition 3.9. *For weight functions Θ and W defined in (3.27), where ϕ satisfies (3.28)-(3.29), the following holds:*

1. *Let $u_0 \in (H_{per}^2(\Omega))^d$ and $G \in \mathbb{Z}_4$. For any $T > 0$, there exists a control $g \in \mathbb{Y}_4$ such that (3.39) has a unique solution v_m in $\mathbb{X}_4$ with*

$$v_m(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|v_m\|_{\mathbb{X}_4} + \|g_m\|_{\mathbb{Y}_4} \leq C_1(\|G\|_{\mathbb{Z}_4} + \|u_0\|_{H_{per}^2(\Omega)}), \quad (3.41)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

2. *Let $u_0 \in (H_{per}^1(\Omega))^d$ and $G \in \tilde{\mathbb{Z}}_4$. For any $T > 0$, there exists a control $g \in \tilde{\mathbb{Y}}_4$ such that (3.39) has a unique solution v_m in $\tilde{\mathbb{X}}_4$ with*

$$v_m(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|v_m\|_{\tilde{\mathbb{X}}_4} + \|g_m\|_{\tilde{\mathbb{Y}}_4} \leq C_1(\|G\|_{\tilde{\mathbb{Z}}_4} + \|u_0\|_{H_{per}^1(\Omega)}), \quad (3.42)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

Proof. First, we control (3.39)₂ such that $\langle v \rangle(T) = 0$ and the solution $\langle v \rangle$ and the control g_3 belong to the required weighted spaces provided the forcing term $\langle G \rangle$ is so.

Let us consider a smooth cut-off function $\zeta(t)$ having support inside $(0, T_2)$, for some $T_2 < T$, i.e., ζ vanishes near T , and $\int_0^T \zeta(t) dt = 1$. Now we set

$$g_3(t) = -\langle u_0 \rangle \zeta(t) - \langle G \rangle(t), \quad \forall t \in (0, T). \quad (3.43)$$

Since $\langle v \rangle$ solution of (3.39)₂ can be written in the form

$$\langle v \rangle(t) - \langle u_0 \rangle = \int_0^t g_3(\tau) d\tau + \int_0^t \langle G \rangle(\tau) d\tau,$$

we see that $\langle v \rangle(T) = 0$, where g_3 is defined as in (3.43).

Next, we set for all $x \in \Omega$ and $t \in (0, T)$,

$$\begin{aligned} v_m(x, t) = \langle v \rangle(t) \chi_0(x), \quad g_m(x, t) &= g_3(t) \chi_0(x) + \langle v \rangle(t) \left(u_s \cdot \nabla \chi_0 - \mu \Delta \chi_0 \right) \\ &\quad - \nu_0 \nabla \operatorname{div} \left(\langle v \rangle(t) \chi_0 \right), \end{aligned} \quad (3.44)$$

where $(\langle v \rangle, g_3)$ is the solution of (3.39)₂ as constructed above. We note that (v_m, g_m) defined above is the solution of the control problem (3.39)₁. Further, the estimates in the proposition follow directly. $\square$

Thus using Theorem 3.6, Corollary 3.7, Theorem 3.8 and Proposition 3.9, we have the following result for (3.38).

Theorem 3.10. *For weight functions Θ and W defined in (3.27), where ϕ satisfies (3.28)-(3.29), the following holds:*

1. *Let $u_0 \in (H_{per}^2(\Omega))^d$ and $G \in \mathbb{Z}_4$. For any $T > 0$, there exists a control $g \in \mathbb{Y}_4$ such that (3.38) has a unique solution v in $\mathbb{X}_4$ with*

$$v(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|v\|_{\mathbb{X}_4} + \|g\|_{\mathbb{Y}_4} \leq C_1(\|G\|_{\mathbb{Z}_4} + \|u_0\|_{H_{per}^2(\Omega)}), \quad (3.45)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

2. *Let $u_0 \in (H_{per}^1(\Omega))^d$ and $G \in \tilde{\mathbb{Z}}_4$. For any $T > 0$, there exists a control $g \in \tilde{\mathbb{Y}}_4$ such that (3.38) has a unique solution v in $\tilde{\mathbb{X}}_4$ with*

$$v(\cdot, T) = 0 \quad \text{in } \Omega.$$

Moreover, this solution satisfies

$$\|v\|_{\tilde{\mathbb{X}}_4} + \|g\|_{\tilde{\mathbb{Y}}_4} \leq C_1(\|G\|_{\tilde{\mathbb{Z}}_4} + \|u_0\|_{H_{per}^1(\Omega)}), \quad (3.46)$$

where C_1 is a positive constant depending on T , $\mathcal{O}$ and the coefficients in the equation.

3.2. Controllability of the transport equation

We consider any $T > T_0$, where T_0 is defined in (1.4). Let us define the following weighted spaces:

$$\begin{aligned} X_3 &= \{ \rho \mid (T-t)W\rho \in C([0, T]; L^2(\Omega)), e^{\Theta(t)}\rho \in C([0, T]; H_{per}^2(\Omega)) \}, \\ Z_3 &= \left\{ F \mid (T-t)^2WF \in L^2(0, T; L^2(\Omega)), (T-t)e^{\Theta(t)}F \in L^2(0, T; H_{per}^2(\Omega)) \right\}, \\ Y_3 &= \{ f \mid f \in L^2(0, T_0; H_0^2(\mathcal{O})) \}, \end{aligned} \quad (3.47)$$

and the corresponding weaker spaces

$$\begin{aligned} \tilde{X}_3 &= \{ \rho \mid (T-t)W\rho \in C([0, T]; L^2(\Omega)), e^{\Theta(t)}\rho \in C([0, T]; H_{per}^1(\Omega)) \}, \\ \tilde{Z}_3 &= \left\{ F \mid (T-t)^2WF \in L^2(0, T; L^2(\Omega)), (T-t)e^{\Theta(t)}F \in L^2(0, T; H_{per}^1(\Omega)) \right\}, \\ \tilde{Y}_3 &= \{ f \mid f \in L^2(0, T_0; H_0^1(\mathcal{O})) \}, \end{aligned} \quad (3.48)$$

where T_0 is as defined in (1.4).

The transport equation we need to solve in each iteration is of the form:

$$\begin{aligned}\gamma_t + P(x, t) \cdot \nabla \gamma &= h + H \quad \text{in } \Omega \times (0, T), \\ \gamma(x, 0) &= \gamma_0(x) \quad \text{in } \Omega.\end{aligned}\tag{3.49}$$

In (3.49), the control h belongs to Y_3 and the forcing term H belongs to Z_3 , where Y_3 and Z_3 are defined in (3.47). In the iteration defined later, the function $P(x, t)$ is the velocity $u^n + u_s$ from the previous step of the iteration; throughout the iteration process P will therefore remain close to u_s .

We note that the weight W is singular only at $t = T$, and we ensure that the control lies in a weighted space by making it vanish near $t = T$. However, the inhomogeneous term H does not vanish near T . We handle this situation by splitting the time interval $(0, T)$ into two subintervals $(0, T_0)$ and (T_0, T) . We use the wellposedness of the backward transport equation corresponding to (3.49) with terminal condition zero, without h , on the time interval $(0, T)$. Then we use null controllability of the transport equation (3.49) in $(0, T_0)$ with control h . More precisely, we consider γ^b satisfying

$$\begin{aligned}\gamma_t^b + P(x, t) \cdot \nabla \gamma^b &= H \quad \text{in } \Omega \times (0, T), \\ \gamma^b(x, T) &= 0 \quad \text{in } \Omega.\end{aligned}\tag{3.50}$$

Let $\gamma^* = \gamma^b(\cdot, 0)$. We then choose γ^f , the solution of

$$\begin{aligned}\gamma_t^f + P(x, t) \cdot \nabla \gamma^f &= h \quad \text{in } \Omega \times (0, T), \\ \gamma^f(x, 0) &= \gamma_0^f(x) = \gamma_0(x) - \gamma^*(x) \quad \text{in } \Omega,\end{aligned}\tag{3.51}$$

obeying $\gamma^f(\cdot, T_0) = 0$ for some control h with support inside $\mathcal{O} \times (0, T_0)$. Moreover, it yields that $\gamma^f(x, t) = 0$ on $\Omega \times [T_0, T)$. Finally, setting

$$\gamma(x, t) = \gamma^b(x, t) + \gamma^f(x, t), \quad \forall (x, t) \in \Omega \times (0, T),$$

we obtain γ , the solution of (3.49) on $\Omega \times (0, T)$, satisfying $\gamma(\cdot, T) = 0$ with control h vanishing near T .

In the following subsections, we study the equations (3.50) and (3.51) and the estimates of their solutions γ^b and γ^f in terms of H and the initial condition.

3.2.1. Transport equation without forcing term

Let us consider the transport equation (3.51), where P is a periodic function on Ω belonging to a small neighborhood of the constant u_s in $\mathbb{X}_4$. We show the null controllability of system (3.51) at $T \geq T_0$. Let $\Gamma(x_0, t)$ be the characteristic passing through $(x_0, 0)$ satisfying

$$\frac{d\Gamma(x_0, \tau)}{d\tau} = P(\Gamma(x_0, \tau), \tau), \quad \Gamma(x_0, 0) = x_0.\tag{3.52}$$

Equivalently, Γ satisfies

$$\Gamma(x_0, t) = x_0 + \int_0^t P(\Gamma(x_0, \tau), \tau) d\tau.$$

Then we have the following lemma.

Lemma 3.11. *Let $P(x, t) = u_s + \Psi(x, t)$, where Ψ belongs to $H^1(0, T; H_{per}^1(\Omega)) \cap L^2(0, T; H_{per}^3(\Omega))$. There exists a positive constant κ_1 depending on T such that if*

$$\|\Psi\|_{\mathbb{X}_4} \leq \kappa_1, \quad (3.53)$$

then

1. $\Gamma - x$, where Γ is the solution of (3.52), belongs to $C_b^1([0, T]; H_{per}^2(\Omega)) \cap C_b([0, T]; H_{per}^3(\Omega))$ satisfying

$$\|\nabla \Gamma - I_d\|_{L^\infty(\Omega \times (0, T))} \leq |e^{C\kappa_1} - 1| < \frac{1}{2}, \quad (3.54)$$

where C is a positive constant depending only on T , Ψ and its derivatives and I_d is the identity $d \times d$ matrix.

2. For each $t > 0$, the mapping $x_0 \rightarrow \Gamma(x_0, t)$ is one-one from Ω onto itself. Therefore, the inverse of $\Gamma(\cdot, t)$ exists. Denoting it by $\Upsilon(\cdot, t)$ for each $t > 0$, we have

$$\Upsilon(\Gamma(x_0, t), t) = x_0, \quad \Gamma(\Upsilon(y_0, t), t) = y_0, \quad \forall x_0, y_0 \in \Omega, \quad \forall t > 0.$$

3. For each $t > 0$, the mapping $x_0 \rightarrow \Gamma(x_0, t)$ is a C^1 diffeomorphism from Ω onto itself.
4. $\Upsilon(x, t)$ satisfies the equation $\Upsilon_t + P\Upsilon_x = 0$ with initial condition $\Upsilon(x, 0) = x$ and standard energy estimates can be used to show that $\Upsilon - x$ belongs to $C_b^1([0, T]; H_{per}^2(\Omega)) \cap C_b([0, T]; H_{per}^3(\Omega))$ and Υ remains close to $x - u_s t$ provided $\|\Psi\|$ is small enough. In particular $\nabla \Upsilon - I_d$ remains small in the L^∞ norm.

We have the following controllability theorem.

Proposition 3.12. *Assume that $T > T_0$, where T_0 is as defined in (1.4), and $\|P - u_s\|_{\mathbb{X}_4}$ is sufficiently small. For any $\gamma_0^f \in H_{per}^2(\Omega)$, there exists a control h in $L^2(0, T_0; H_0^2(\mathcal{O}))$ with support inside $\mathcal{O} \times (0, T_0)$, such that γ^f , the solution of (3.51), reaches zero at $t = T_0$ and remains zero on $\Omega \times [T_0, T]$. Moreover, we have the estimate*

$$\|\gamma^f\|_{C([0, T]; H_{per}^2(\Omega))} + \|h\|_{L^2(0, T_0; H_0^2(\mathcal{O}))} \leq C \|\gamma_0^f\|_{H_{per}^2(\Omega)}. \quad (3.55)$$

The control h and the solution γ^f depend linearly on the initial data. The constant C depends on $\mathcal{O}$ and the bound on $\|P - u_s\|_{\mathbb{X}_4}$.

Proof. To obtain $\gamma^f(\cdot, T_0) = 0$ on Ω with a control h localized in $\mathcal{O} \times (0, T_0)$, we first recall that the solution of the transport equation (3.51) is

$$\gamma^f(\Gamma(x, t), t) = \gamma_0^f(x) + \int_0^t h(\Gamma(x, s), s) ds, \quad \forall (x, t) \in \Omega \times (0, T_0).$$

Now we choose h as follows. Let χ be any nonnegative C^∞ function with compact support in $\mathcal{O} \times (0, T_0)$ and the property that every line $x = u_s t + c$

passes through the (periodically continued) set where $\chi > 0$ for some $t \in (0, T_0)$. Now pick

$$h(x, t) = -\chi(\Upsilon(x, t) + u_s t, t) \gamma_0^f(\Upsilon(x, t)) / \int_0^{T_0} \chi(\Upsilon(x, t) + u_s \tau, \tau) d\tau. \quad (3.56)$$

Now from the representation of γ^f , solution of (3.51), it can be shown that for control (3.56) $\gamma^f(\cdot, T_0) = 0$ on Ω . Moreover, h and hence γ^f become 0 on $\Omega \times [T_0, T)$ and the estimate follows. $\square$

We need to track how the solution γ changes with P and γ_0 . For this, we shall prove the following lemma.

Lemma 3.13. *Let γ be the solution, as constructed above, of the control problem*

$$\gamma_t + P \cdot \nabla \gamma = h_1, \quad \gamma(x, 0) = \gamma_0(x), \quad \gamma(x, T) = 0, \quad (3.57)$$

and let ξ be the solution of the control problem

$$\xi_t + Q \cdot \nabla \xi = h_2, \quad \xi(x, 0) = \xi_0(x), \quad \xi(x, T) = 0. \quad (3.58)$$

Assume that $\|P - u_s\|_{\mathbb{X}_4}$ and $\|Q - u_s\|_{\mathbb{X}_4}$ are sufficiently small, as required by Proposition 3.12 and γ_0 and ξ_0 are within a fixed bounded set of $H_{per}^2(\Omega)$. Then we have a bound of the form

$$\begin{aligned} \|\gamma - \xi\|_{L^\infty(0, T; H_{per}^1(\Omega))} + \|h_1 - h_2\|_{L^2(0, T_0; H_0^1(\mathcal{O}))} &\leq C \left(\|\gamma_0 - \xi_0\|_{H_{per}^1(\Omega)} \right. \\ &\quad \left. + \|P - Q\|_{\mathbb{X}_4} \|\xi_0\|_{H_{per}^2(\Omega)} \right). \end{aligned} \quad (3.59)$$

Proof. We shall first estimate the difference $h_1 - h_2$. To facilitate notation, we shall write $h_1 = h(P, \gamma_0)$, $h_2 = h(Q, \xi_0)$. Moreover, (3.56) involves $\Upsilon(x, t)$, which depends on the velocity P ; we shall use the notations Υ_P and Υ_Q . Now we note that

$$\frac{\partial}{\partial t}(\Upsilon_P - \Upsilon_Q) + (P \cdot \nabla)(\Upsilon_P - \Upsilon_Q) = -((P - Q) \cdot \nabla) \Upsilon_Q. \quad (3.60)$$

By using energy estimates on this equation and its derivative along with the bounds on Υ_Q from Lemma 3.11, we find a bound of the form

$$\|\Upsilon_P - \Upsilon_Q\|_{L^\infty(0, T; H_{per}^2(\Omega))} \leq C \|P - Q\|_{L^2(0, T; H_{per}^2(\Omega))}. \quad (3.61)$$

We have $h(P, \gamma_0) - h(Q, \xi_0) = h(P, \gamma_0 - \xi_0) + h(P, \xi_0) - h(Q, \xi_0)$. We begin by estimating the first contribution, since only P is involved, we shall simply write Υ for Υ_P and Γ for Γ_P in the following. To estimate the norm of $h(P, \gamma_0 - \xi_0)$ in $L^2(0, T_0; H_0^1(\mathcal{O}))$, substituting $x = \Gamma(y, t)$, we find that

$$h(P, \gamma_0 - \xi_0)(\Gamma(y, t), t) = -\chi(y + t, t)(\gamma_0 - \xi_0)(y) / \int_0^{T_0} \chi(y + u_s \tau, \tau) d\tau. \quad (3.62)$$

Now, for $\gamma_0 - \xi_0 \in H_{per}^1(\Omega)$, using the smoothness and non-negativity of χ with support in $\mathcal{O} \times (0, T_0)$ and the regularity of Γ from Lemma 3.11, we get that there exists a positive constant C depending on T , Ω , $\mathcal{O}$ and $\|P - u_s\|_{\tilde{\mathbb{X}}_4}$ such that

$$\|h(P, \gamma_0 - \xi_0)\|_{L^2(0, T_0; H_0^1(\mathcal{O}))} \leq C \|\gamma_0 - \xi_0\|_{H_{per}^1(\Omega)}. \quad (3.63)$$

Next we consider $h(P, \xi_0) - h(Q, \xi_0)$ and we define

$$\psi(y, t) = -\chi(y + u_s t, t) \xi_0(y) / \int_0^{T_0} \chi(y + u_s \tau, \tau) d\tau, \quad \forall (y, t) \in \Omega \times (0, T).$$

From the definition of h in (1.3), we can see that for all $(x, t) \in \Omega \times (0, T)$, there exists a $s \in (0, 1)$ such that

$$\begin{aligned} (h(P, \xi_0) - h(Q, \xi_0))(x, t) &= \nabla \psi \left(s \Upsilon_Q(x, t) \right. \\ &\quad \left. + (1 - s) \Upsilon_P(x, t), t \right) \cdot (\Upsilon_P(x, t) - \Upsilon_Q(x, t)). \end{aligned}$$

From the above representation, using the property of χ and (3.61), we obtain that

$$\|h(P, \xi_0) - h(Q, \xi_0)\|_{L^2(0, T_0; H_0^1(\mathcal{O}))} \leq C \|P - Q\|_{\tilde{\mathbb{X}}_4} \|\xi_0\|_{H_{per}^2}, \quad (3.64)$$

for some constant $C > 0$ depending on T , Ω , $\mathcal{O}$ and $\|P - u_s\|_{\tilde{\mathbb{X}}_4}$, $\|Q - u_s\|_{\tilde{\mathbb{X}}_4}$.

Having bounded the difference in the controls, we can now treat them as given inhomogeneous terms. The inhomogeneous transport equation without control is considered in the next subsection, and we can complete the proof of the lemma by referring to Lemma 3.14, which we state next. $\square$

Lemma 3.14. *Consider the two equations*

$$\begin{aligned} \gamma_t + P \cdot \nabla \gamma &= H_1, \\ \xi_t + Q \cdot \nabla \xi &= H_2, \end{aligned} \quad (3.65)$$

with periodic conditions and final condition $\gamma = \xi = 0$ at $t = T$. Assume that γ and ξ lie in a given ball of X_3 , and P and Q lie in a given ball of $\mathbb{X}_4$. For the difference, we have the bound

$$\|\gamma - \xi\|_{\tilde{X}_3} \leq C (\|H_1 - H_2\|_{\tilde{Z}_3} + \|P - Q\|_{\tilde{\mathbb{X}}_4} \|\xi\|_{X_3}). \quad (3.66)$$

3.2.2. Transport equation with forcing term

The next lemma assures that the solution of (3.50) belongs to X_3 provided $H \in Z_3$.

Lemma 3.15. *Let q be any real number. We assume that $\|P - u_s\|_{\mathbb{X}_4} \leq \kappa_1$, where κ_1 is sufficiently small. Then the followings hold:*

1. For any $(T-t)^{q+1}WH \in L^2(0, T; L^2(\Omega))$, (3.50) admits a unique solution γ^b such that $(T-t)^q W \gamma^b \in C([0, T]; L^2(\Omega))$ and $(T-t)^{q-1} W \gamma^b \in L^2(0, T; L^2(\Omega))$, with a bound on the respective norms,

$$\begin{aligned} & \| (T-t)^q W \gamma^b \|_{C([0, T]; L^2(\Omega))} + \| (T-t)^{q-1} W \gamma^b \|_{L^2(0, T; L^2(\Omega))} \\ & \leq C \| (T-t)^{q+1} W H \|_{L^2(0, T; L^2(\Omega))}, \end{aligned} \quad (3.67)$$

for some positive constant C , depending on q , T and κ_1 .

2. For $(T-t)^{q+1} e^{\Theta(t)} H \in L^2(0, T; H_{per}^1(\Omega))$, γ^b , the solution of (3.50) satisfies

$$\begin{aligned} & \| (T-t)^q e^{\Theta(t)} \gamma^b \|_{C([0, T]; H_{per}^1(\Omega))} + \| (T-t)^{q-1} e^{\Theta(t)} \gamma^b \|_{L^2(0, T; H_{per}^1(\Omega))} \\ & \leq C \| (T-t)^{q+1} e^{\Theta(t)} H \|_{L^2(0, T; H_{per}^1(\Omega))}, \end{aligned} \quad (3.68)$$

for some positive constant C , depending on q , T and κ_1 .

3. For $(T-t)^{q+1} e^{\Theta(t)} H \in L^2(0, T; H_{per}^2(\Omega))$, γ^b , the solution of (3.50) satisfies

$$\begin{aligned} & \| (T-t)^q e^{\Theta(t)} \gamma^b \|_{C([0, T]; H_{per}^2(\Omega))} + \| (T-t)^{q-1} e^{\Theta(t)} \gamma^b \|_{L^2(0, T; H_{per}^2(\Omega))} \\ & \leq C \| (T-t)^{q+1} e^{\Theta(t)} H \|_{L^2(0, T; H_{per}^2(\Omega))}, \end{aligned} \quad (3.69)$$

for some positive constant C , depending on q , T and κ_1 .

Proof. By using the characteristic Γ and its inverse Υ , we get that γ^b , the solution of (3.50), is

$$\gamma^b(x, t) = - \int_t^T H(\Gamma(\Upsilon(x, t), s), s) ds. \quad (3.70)$$

To obtain the estimate of γ^b in X_3 , here we just give the estimate of $(T-t)W\gamma^b$ in $C([0, T]; L^2(\Omega))$ and the other terms can be estimated similarly. We calculate, for any fixed y ,

$$\begin{aligned} \frac{\partial}{\partial t} \left[(T-t)^q W(\Gamma(y, t), t) \gamma^b(\Gamma(y, t), t) \right] &= (T-t)^q W(\Gamma(y, t), t) H(\Gamma(y, t), t) \\ &+ \frac{\partial}{\partial t} \left[(T-t)^q W(\Gamma(y, t), t) \right] \gamma^b(\Gamma(y, t), t) \end{aligned} \quad (3.71)$$

Next, we deduce that

$$\begin{aligned} \frac{\partial}{\partial t} \left[(T-t)^q W(\Gamma(y, t), t) \right] &= -q(T-t)^{q-1} W(\Gamma(y, t), t) \\ &+ (T-t)^q \nabla W(\Gamma(y, t), t) \cdot \Gamma_t(y, t) + (T-t)^q W_t(\Gamma(y, t), t). \end{aligned} \quad (3.72)$$

Now we note that Γ_t is bounded, ∇W is proportional to $W/(T-t)$, and W_t is positive and proportional to $W/(T-t)^2$. For t close to T , we therefore

conclude that the right hand side of (3.72) is positive and bounded below by $CW(T-t)^{q-2}$.

Now multiplying (3.71) by $(T-t)^q W(\Gamma(y, t), t) \gamma^b(\Gamma(y, t), t)$ and using the lower bound for $\frac{\partial}{\partial t} \left[(T-t)^q W(\Gamma(y, t), t) \right]$ obtained above, we get for each fixed $y \in \Omega$ and t close to T

$$\begin{aligned} & -\frac{\partial}{\partial t} \left[(T-t)^{2q} W^2(\Gamma(y, t), t) (\gamma^b(\Gamma(y, t), t))^2 \right] \\ & + C(T-t)^{2q-2} W^2(\Gamma(y, t), t) (\gamma^b(\Gamma(y, t), t))^2 \\ & \leq -(T-t)^{2q} W^2(\Gamma(y, t), t) H(\Gamma(y, t), t) \gamma^b(\Gamma(y, t), t). \end{aligned}$$

Integrating it over $\Omega \times (t, T)$ and using a Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \int_{\Omega} (T-t)^{2q} W^2(\Gamma(y, t), t) (\gamma^b(\Gamma(y, t), t))^2 dy \\ & + \int_t^T \int_{\Omega} (T-s)^{2q-2} W^2(\Gamma(y, s), s) (\gamma^b(\Gamma(y, s), s))^2 dy ds \\ & \leq C\epsilon \left(\int_t^T \int_{\Omega} (T-s)^{2q-2} W^2(\Gamma(y, s), s) (\gamma^b(\Gamma(y, s), s))^2 dy ds \right) \\ & + \frac{C}{\epsilon} \left(\int_t^T \int_{\Omega} (T-s)^{2q+2} W^2(\Gamma(y, s), s) (H(\Gamma(y, s), s))^2 dy ds \right). \quad (3.73) \end{aligned}$$

Finally, choosing ϵ small enough and using a change of variables $y = \Upsilon(x, s)$ for each fixed s , we derive (3.67). $\square$

In particular, this implies the following theorem for (3.49).

Theorem 3.16. *Let $T > T_0$, where T_0 is defined in (1.4). Let the forcing term H be in Z_3 and assume that $\|P - u_s\|_{\mathbb{X}_4}$ is sufficiently small as required by Proposition 3.12. Then for any $\gamma_0 \in H_{per}^2(\Omega)$, there exists a control h in Y_3 such that γ , the solution of system (3.49), belongs to X_3 satisfying*

$$\gamma(\cdot, T) = 0 \quad \text{in } \Omega,$$

and the control h vanishes over $\Omega \times [T_0, T)$. Moreover, there exists a positive constant C such that

$$\|\gamma\|_{X_3} + \|h\|_{Y_3} < C(\|\gamma_0\|_{H_{per}^2(\Omega)} + \|H\|_{Z_3}). \quad (3.74)$$

Proof. We first solve for γ^b and using (3.67), we obtain

$$\|\gamma^b\|_{X_3} \leq C\|H\|_{Z_3}, \quad \|\gamma^b(\cdot, 0)\|_{H_{per}^2(\Omega)} \leq C\|H\|_{Z_3}.$$

Then we solve γ^f for modified initial condition $\gamma_0^f = \gamma_0 - \gamma^b(\cdot, 0)$, as indicated above, and obtain the estimate (3.55). Finally setting

$$\gamma(x, t) = \gamma^b(x, t) + \gamma^f(x, t), \quad \forall (x, t) \in \Omega \times (0, T),$$

the theorem follows. $\square$

We have already stated Lemma 3.14 concerning differences between solutions. We next indicate the proof of this lemma.

The proof is obtained by using an argument similar to the proof of Lemma 3.15 for the equation

$$(\gamma - \xi)_t + P \cdot \nabla(\gamma - \xi) = H_1 - H_2 + (P - Q) \cdot \nabla \xi. \quad (3.75)$$

In this fashion, we get the estimate in (3.14).

4. Inner iteration

In this section, we give the detailed analysis of Section 2.2. Here for velocity, we use the weighted Sobolev spaces defined in (3.30), (3.34) and (3.40). For density γ , in the intermediate stage, we use the weighted spaces as defined in (3.47). In addition, we define the following space for the control in density obtained after modification due to the coupling with heat equation.

$$\hat{Y}_3 = \{f \mid f \in L^2(0, T_1; H_0^2(\mathcal{O}))\},$$

where $T_0 < T_1 < T$. Further, for density ϱ at the final stage, we need to define the weighted spaces which are similar to the spaces defined in (3.47) satisfying, in addition, the zero average.

$$\begin{aligned} X &= X_1 \times \mathbb{X}_4, \quad Z = Z_1 \times \mathbb{Z}_4, \quad Y = Y_1 \times \mathbb{Y}_4, \\ X_1 &= \left\{ \rho \mid (T - t)W\rho \in C([0, T]; L^2(\Omega)), e^{\Theta(t)}\rho \in C([0, T]; H_{per}^2(\Omega)), \right. \\ &\quad \left. \int_{\Omega} \rho(x, t) dx = 0, \quad \forall t \in (0, T) \right\}, \\ Z_1 &= \left\{ F \mid (T - t)^2 WF \in L^2(0, T; L^2(\Omega)), (T - t)e^{\Theta(t)}F \in L^2(0, T; H_{per}^2(\Omega)), \right. \\ &\quad \left. \int_{\Omega} F(x, t) dx = 0, \quad \forall t \in (0, T) \right\}, \\ Y_1 &= \left\{ f \mid f \in L^2(0, T_1; H_0^2(\mathcal{O})), \int_{\Omega} f(x, t) dx = 0, \quad \forall t \in (0, T_1) \right\}, \end{aligned} \quad (4.1)$$

where $\mathbb{X}_4$, $\mathbb{Z}_4$ and $\mathbb{Y}_4$ are defined in (3.40).

Our goal of this section is to solve the control problem (2.4).

Theorem 4.1. *Let us assume that $\Psi \in \mathbb{X}_4$ satisfying*

$$\|\Psi\|_{\mathbb{X}_4} \leq \kappa_2,$$

where κ_2 is a positive constant. We also assume that $G \in \mathbb{Z}_4$ is given. Then there exists a sufficiently small $\kappa_2 > 0$ such that the null control problem (2.4) admits a solution and the solution satisfies

$$\|\varrho\|_{X_1} + \|v\|_{\mathbb{X}_4} + \|f\|_{Y_1} + \|g\|_{\mathbb{Y}_4} \leq C(\|G\|_{\mathbb{Z}_4} + \|\rho_0\|_{\dot{H}_{per}^2(\Omega)} + \|u_0\|_{H_{per}^2(\Omega)}). \quad (4.2)$$

4.1. Steps 1 and 2

In Step 1, we study the null controllability of (2.14), (2.15) and hence (2.16) with proper estimates.

In (2.14), v_m is a known term coming by solving (2.10)₄ and hence $\operatorname{div} v_m$ will be treated as a known forcing term in (2.14). To study (2.14), we shall treat the terms $(I - \Pi_*)w^1$, $\Psi \cdot \nabla(w^1/\nu)$ and $(\operatorname{div} \Psi)(\gamma^1 - w^1/\nu)$ as perturbations. Let us recall that

$$(I - \Pi_*)w^1 = w^1\chi_{[T_*, T]}(t) + (I - \Pi)w^1\chi_{[0, T_*]}(t). \quad (4.3)$$

For the first term, we have

$$\|w^1\chi_{[T_*, T]}(t)\|_{Z_3} \leq (T - T_*)\|w^1\|_{X_2}. \quad (4.4)$$

For the second term, we have a constant C , depending on T_* , such that

$$\|(I - \Pi)w^1\chi_{[0, T_*]}(t)\|_{Z_3} \leq \frac{C(T_*)}{N^2}\|w^1\|_{X_2}. \quad (4.5)$$

Finally, we have

$$\|\Psi \cdot \nabla(w^1/\nu)\|_{Z_3} \leq C\|w^1\|_{X_2}\|\Psi\|_{\mathbb{X}_4}, \quad (4.6)$$

$$\|(\operatorname{div} \Psi)(\gamma^1 - w^1/\nu)\|_{Z_3} \leq C\|\Psi\|_{\mathbb{X}_4}(\|\gamma^1\|_{X_3} + \|w^1\|_{X_2}). \quad (4.7)$$

If we choose T_* sufficiently close to T , N sufficiently large and $\|\Psi\|_{\mathbb{X}_4}$ sufficiently small, we can therefore take these terms to the right hand side of (2.14) and solve using a fixed point argument. For given $\zeta \in X_3$ and $\eta \in X_2$, we are solving the simpler system

$$\begin{aligned} \gamma_t^{\zeta, \eta} + u_s \cdot \nabla \gamma^{\zeta, \eta} + \Psi \cdot \nabla \gamma_x^{\zeta, \eta} + \frac{p'(1)\gamma^{\zeta, \eta}}{\nu} &= f^{\zeta, \eta} + \frac{g_1^{\zeta, \eta}}{\nu} + \frac{G_1}{\nu} - \operatorname{div} v_m \\ &+ \frac{p'(1)}{\nu^2}(I - \Pi_*)\eta + \psi \cdot \nabla(\eta/\nu) + (\operatorname{div} \Psi)(\zeta - \eta/\nu) \quad \text{in } \Omega \times (0, T), \\ w_t^{\zeta, \eta} + u_s \cdot \nabla w^{\zeta, \eta} - \nu \Delta w^{\zeta, \eta} + p'(1) \left(\gamma^{\zeta, \eta} - \frac{w^{\zeta, \eta}}{\nu} \right) &= g_1^{\zeta, \eta} + G_1 \quad \text{in } \Omega \times (0, T), \\ \gamma^{\zeta, \eta}(x, 0) &= \gamma_0, \quad w^{\zeta, \eta}(x, 0) = w_0, \quad \forall x \in \Omega, \end{aligned} \quad (4.8)$$

Proposition 4.2. *Let $T > T_0$ and $\|\Psi\|_{\mathbb{X}_4} \leq \kappa_2$, where κ_2 is sufficiently small. Let us assume that $(\zeta, \eta) \in X_3 \times X_2$ and the inhomogeneous term G_1 is given in Z_2 along with $v_m \in \mathbb{X}_4$, the solution of (2.10)₄. For any $(\gamma_0, w_0) \in H_{\text{per}}^2(\Omega) \times H_{\text{per}}^3(\Omega)$, there exist controls $f^{\zeta, \eta}$ in $\hat{Y}_3$ and $g_1^{\zeta, \eta}$ in Y_2 such that $(\gamma^{\zeta, \eta}, w^{\zeta, \eta})$, the solution of (4.8) belongs to $X_3 \times X_2$ and obeys*

$$(\gamma^{\zeta, \eta}, w^{\zeta, \eta})(\cdot, T) = (0, 0) \quad \text{on } \Omega.$$

Moreover, this solution satisfies

$$\begin{aligned} & \|\gamma^{\zeta,\eta}\|_{X_3} + \|w^{\zeta,\eta}\|_{X_2} + \|f^{\zeta,\eta}\|_{\widehat{Y}_3} + \|g_1^{\zeta,\eta}\|_{Y_2} \leq C_0 \left(\|\gamma_0\|_{H_{per}^2(\Omega)} + \|w_0\|_{H_{per}^3(\Omega)} \right. \\ & + \|v_m\|_{\mathbb{X}_4} + \|G_1\|_{Z_2} + \|\zeta\|_{X_3} \|\Psi\|_{\mathbb{X}_4} + \|\eta\|_{X_2} \|\Psi\|_{\mathbb{X}_4} + \frac{C(T_*)}{N^2} \|\eta\|_{X_2} \\ & \left. + (T - T_*) \|\eta\|_{X_2} \right). \end{aligned} \quad (4.9)$$

Proof. There is still a coupling, since $g_1^{\zeta,\eta}$ appears in the first equation, and we want $f^{\zeta,\eta}$ to be zero near $t = T$. We note, however, that, according to Lemma 3.15, we can solve the transport equation, with no control, backward with zero terminal condition at $t = T$, and according to the estimate (3.67), we only need to have g_1 in Z_3 to bound the norm of γ . We pick $T_1 \in (T_0, T)$ and set

$$f^{\zeta,\eta} + \frac{g_1^{\zeta,\eta}}{\nu} = \tilde{f}^{\zeta,\eta} + \frac{1}{\nu} g_1^{\zeta,\eta} \chi_{[T_1, T]}(t). \quad (4.10)$$

We can make $\tilde{f}^{\zeta,\eta}$ independent of $g_1^{\zeta,\eta}|_{[0, T_1]}$, and such that $\tilde{f}^{\zeta,\eta}$ is supported on the time interval $(0, T_0)$. We note that

$$\|g_1^{\zeta,\eta} \chi_{[T_1, T]}\|_{Z_3} \leq (T - T_1)^{1/2} \|g_1^{\zeta,\eta}\|_{Y_2}. \quad (4.11)$$

We can thus solve the control problem for $\gamma^{\zeta,\eta}$ by treating this term as a perturbation. Note that a lower order term $\frac{p'(1)\gamma^{\zeta,\eta}}{\nu}$ is present in $(4.8)_1$. After a slight modification due to the addition of this term in (3.56) and (3.70), we obtain a controllability result similar to Theorem 3.16 for $(4.8)_1$ and the estimate

$$\begin{aligned} & \|\gamma^{\zeta,\eta}\|_{X_3} + \|\tilde{f}^{\zeta,\eta}\|_{Y_3} \leq C \left((T - T_1)^{1/2} \|g_1^{\zeta,\eta}\|_{Y_2} + \|\gamma_0\|_{H_{per}^2} + \|H\|_{Z_3} \right. \\ & \left. + \|\zeta\|_{X_3} \|\Psi\|_{\mathbb{X}_4} + \|\eta\|_{X_2} \|\Psi\|_{\mathbb{X}_4} + \frac{C(T_*)}{N^2} \|\eta\|_{X_2} + (T - T_*) \|\eta\|_{X_2} \right), \end{aligned} \quad (4.12)$$

where $H = \frac{G_1}{\nu} - \text{div} v_m$ and we also have $\|H\|_{Z_3} \leq C(\|G_1\|_{Z_2} + \|v_m\|_{\mathbb{X}_4})$. Next, using Theorem 3.6, we solve the control problem for the heat equation to get

$$\|w^{\zeta,\eta}\|_{X_2} + \|g_1^{\zeta,\eta}\|_{Y_2} \leq C(\|\gamma^{\zeta,\eta}\|_{X_3} + \|G_1\|_{Z_2} + \|w_0\|_{H_{per}^3(\Omega)}). \quad (4.13)$$

Combining the estimates thus far discussed and choosing T_1 close enough to T , we finally obtain (4.9). $\square$

Now we can use a fixed point argument and get the null controllability of (2.14) with the required estimates. We choose T_* and T_1 sufficiently close to T and N sufficiently large (relative to T_*). We also have to pick $\|\Psi\|_{\mathbb{X}_4}$ smaller than some constant to obtain a contraction. In view of (4.9), for the same positive constant C_0 appearing there, let us set

$$\|\Psi\|_{\mathbb{X}_4} \leq \frac{1}{2C_0}, \quad (T - T_*) \leq \frac{1}{2C_0}. \quad (4.14)$$

After that, we fix N_0 as

$$N_0 = \sqrt{2C_0C(T_*)}, \quad (4.15)$$

and take $N > N_0$.

Theorem 4.3. *Let $T > T_0$, $G_1 \in Z_2$ and v_m , the solution of $(2.10)_4$, be in $\mathbb{X}_4$. We assume that $\|\Psi\|_{\mathbb{X}_4}$ is sufficiently small as required in Proposition 4.2 and (4.15). Then for any $(\gamma_0, w_0) \in H_{per}^2(\Omega) \times H_{per}^3(\Omega)$, there exist controls f^1 in $\widehat{Y}_3$ and g_1^1 in Y_2 such that (γ^1, w^1) , the solution of (2.14) belongs to $X_3 \times X_2$ and obeys*

$$(\gamma^1, w^1)(\cdot, T) = (0, 0) \quad \text{on } \Omega.$$

Moreover, this solution satisfies

$$\begin{aligned} \|\gamma^1\|_{X_3} + \|w^1\|_{X_2} + \|f^1\|_{\widehat{Y}_3} + \|g_1^1\|_{Y_2} &\leq C \left(\|\gamma_0\|_{H_{per}^2(\Omega)} + \|w_0\|_{H_{per}^3(\Omega)} \right. \\ &\quad \left. + \|G_1\|_{Z_2} + \|v_m\|_{\mathbb{X}_4} \right). \end{aligned} \quad (4.16)$$

Proof. Using estimate (4.9) for sufficiently large N , we show that the map

$$(\zeta, \eta) \mapsto (\gamma^{\zeta, \eta}, w^{\zeta, \eta}, f^{\zeta, \eta}, g_1^{\zeta, \eta}) \mapsto (\gamma^1, w^1)$$

is a contraction. Then the theorem follows from Proposition 4.2. $\square$

Now we study the system (2.15). This is a backward coupled system involving the nonhomogeneous term $\frac{p'(1)}{\nu^2} \Pi_* w^1$, which is already determined by solving (2.14). Note that $\Pi_* w^1$ is a finite dimensional term and vanishes near T . Moreover, the parameters N and T_* involved in the definition of Π_* are as in (4.15) and (4.14). Thus we have the following theorem.

Theorem 4.4. *Let $T > T_0$ and $\Pi_* w^1$ be determined from (2.14). Then (γ^2, w^2) is the solution of (2.15) vanishing near T and satisfies*

$$\|\gamma^2\|_{X_3} + \|w^2\|_{X_2} \leq C \|w^1\|_{X_2}. \quad (4.17)$$

Proof. The theorem follows from solving the finite dimensional backward system. $\square$

Combining Theorems 4.3 and 4.4, we have the null controllability result for (2.16).

Theorem 4.5. *In (2.16), let the forcing term $G_1 \in Z_2$ and v_m , the solution of $(2.10)_4$ be given in $\mathbb{X}_4$. Let the control $(f^1, g_1^1) \in \widehat{Y}_3 \times Y_2$ be as determined in Theorem 4.3. Let γ^2 be as determined in Theorem 4.4. Let us also assume that $\|\Psi\|_{\mathbb{X}_4}$ is sufficiently small. For $T > T_0$ and any $(\gamma_0, w_0) \in H_{per}^2(\Omega) \times H_{per}^3(\Omega)$, (γ^3, w^3) , the solution of (2.16), belongs to $X_3 \times X_2$ obeying*

$$(\gamma^3, w^3)(\cdot, T) = (0, 0) \quad \text{on } \Omega.$$

Moreover, the solution satisfies

$$\begin{aligned} \|\gamma^3\|_{X_3} + \|w^3\|_{X_2} + \|f^1\|_{\widehat{Y}_3} + \|g_1^1\|_{Y_2} &\leq C(\|G_1\|_{Z_2} + \|v_m\|_{\mathbb{X}_4} \\ &\quad + \|\gamma_0\|_{H_{per}^2(\Omega)} + \|w_0\|_{H_{per}^3(\Omega)}). \end{aligned} \quad (4.18)$$

The estimates of the terms arising in Step 2 are fairly straightforward. We find that

$$\|\tilde{\gamma}^3\|_{X_3} + \|\tilde{f}^1\|_{Y_1} \leq C(\|\gamma^3\|_{X_3} + \|w^3\|_{X_2} + \|f^1\|_{\widehat{Y}_3}). \quad (4.19)$$

Finally, we obtain the bound for ϱ^3 in X_1 using a change of variables and estimates (4.18) and (4.19). We thus end up with a bound of the form

$$\begin{aligned} \|\varrho^3\|_{X_1} + \|w^3\|_{X_2} + \|\tilde{f}^1\|_{Y_1} + \|\tilde{g}_1^1\|_{Y_2} &\leq C(\|G_1\|_{Z_2} + \|v_m\|_{\mathbb{X}_4} \\ &\quad + \|\rho_0\|_{\dot{H}_{per}^2(\Omega)} + \|w_0\|_{H_{per}^3(\Omega)}). \end{aligned} \quad (4.20)$$

In particular, this gives a bound for ϱ_0^4 and w_0^4 .

4.2. Step 3

Here we correct the initial condition considering the system (2.23) and for that, first we consider the approximate controllability problem (2.20). For (2.20), we have the following approximate controllability result.

Theorem 4.6. *Let $T > T_0$, where T_0 is introduced in (1.4). The system (2.20) is approximately controllable in $\dot{L}^2(\Omega) \times L^2(\Omega)$ at T using a control in $L^2(0, T; L^2(\mathcal{O}))$ with support inside $\mathcal{O}$.*

Proof. First for any $T > T_0$, we prove the approximate controllability of (2.20) in $L^2(0, T; \dot{L}^2(\Omega) \times L^2(\Omega))$ using a L^2 -control with support inside $\mathcal{O}$ and average zero, by showing that for any

$$v|_{\mathcal{O} \times (0, T)} = 0 \quad \text{implies that} \quad (\sigma, v) \equiv (0, 0) \quad \text{on} \quad \Omega \times (0, T), \quad (4.21)$$

where $(\sigma, v) \in L^2(0, T; \dot{L}^2(\Omega) \times L^2(\Omega))$, is the solution of the adjoint problem of (2.20)

$$\begin{aligned} -\sigma_t - u_s \cdot \nabla \sigma + p'(1)(v - \langle v \rangle) &= 0 \quad \text{in} \quad \Omega \times (0, T), \\ -v_t - u_s \cdot \nabla v - \nu \Delta v + \Delta \sigma &= 0 \quad \text{in} \quad \Omega \times (0, T), \\ \sigma(\cdot, T) = \sigma_T, \quad v(\cdot, T) &= v_T \quad \text{in} \quad \Omega, \end{aligned} \quad (4.22)$$

for any terminal condition $(\sigma_T, v_T) \in \dot{L}^2(\Omega) \times L^2(\Omega)$.

Now, if $v|_{\mathcal{O} \times (0, T)} = 0$, it gives that $\Delta v|_{\mathcal{O} \times (0, T)} = 0$ and from (4.22)₂, it can be derived that $\Delta \sigma|_{\mathcal{O} \times (0, T)} = 0$.

Let us set $(\psi^1, \psi^2) = (\Delta \sigma, \Delta v)$ on $\Omega \times (0, T)$. Then (ψ^1, ψ^2) satisfies

$$\begin{aligned} -\psi_t^1 - u_s \cdot \nabla \psi^1 + p'(1)\psi^2 &= 0 \quad \text{in} \quad \Omega \times (0, T), \\ -\psi_t^2 - u_s \cdot \nabla \psi^2 - \nu \Delta \psi^2 + \Delta \psi^1 &= 0 \quad \text{in} \quad \Omega \times (0, T), \end{aligned} \quad (4.23)$$

and we have that

$$(\psi^1, \psi^2) \equiv 0 \quad \text{on} \quad \mathcal{O} \times (0, T). \quad (4.24)$$

To apply Holmgren's uniqueness theorem to prove that $(\psi^1, \psi^2) \equiv 0$ on $\mathcal{O} \times (0, T)$, we see that the characteristic hyperplanes of (4.23) in variable $(x, t) \in \mathbb{R}^{d+1}$ are

$$t = C_1, \quad \text{and} \quad \alpha \cdot x - (u_s \cdot \alpha)t = C_2$$

where C_1 and C_2 are arbitrary constants and $\alpha = (1, \dots, 1) \in \mathbb{R}^d$.

Now, following the proof of Theorem 2.9 in [4], using Holmgren's uniqueness theorem along with the backward uniqueness of the solution of the system, we derive that

$$(\psi^1, \psi^2) = (\Delta\sigma, \Delta v) \equiv 0 \quad \text{on} \quad \Omega \times (0, T).$$

Thus, σ satisfies the following equation with periodic boundary conditions and average zero

$$\Delta\sigma = 0 \quad \text{in} \quad \Omega \times (0, T), \quad \int_{\Omega} \sigma(x, t) dx = 0, \quad \forall t \in (0, T),$$

and hence, we obtain that $\sigma(x, t) = 0$ for all $(x, t) \in \Omega \times (0, T)$. Further, we have that

$$\Delta v(x, t) = 0 \quad \text{in} \quad \Omega \times (0, T), \quad v(x, t) = 0 \quad \text{in} \quad \mathcal{O} \times (0, T).$$

From this equation, we get that $v(x, t) = 0$ for all $(x, t) \in \Omega \times (0, T)$. Hence, the theorem is proved. $\square$

Using the above theorem, we obtain the approximate controllability of (2.20) in higher Sobolev spaces.

Corollary 4.7. *Let $\epsilon > 0$ any small number such that $T_0 + \frac{\epsilon}{2} < T$, where T_0 is introduced in (1.4). The system (2.20) is approximately controllable in $\dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega)$ at $T_0 + \frac{\epsilon}{2}$ using a control in $L^2(0, T_0 + \frac{\epsilon}{2}; H_0^3(\mathcal{O}))$ with support inside $\mathcal{O}$.*

Proof. We have the approximate controllability of (2.20) in $\dot{L}^2(\Omega) \times L^2(\Omega)$ using a localized L^2 -control with average zero. To get it in higher Sobolev spaces: Let ϕ_0 be a mollifier. If g controls zero initial data to (ϱ_f, v_f) , then the smooth control $\phi_0 * g$ controls zero initial data to $(\phi_0 * \varrho_f, \phi_0 * v_f)$. Moreover, functions of the form $\phi_0 * u$ are dense in any Sobolev space, in particular they include all finite Fourier series. We find a control, which at time $T_0 + \epsilon/2$ controls the given initial data to final values which are as small as we wish and in any Sobolev norm we choose. Let us denote these final values by $\varrho_f^4(x) = \varrho^4(x, T_0 + \frac{\epsilon}{2})$ and $v_f^4(x) = v^4(x, T_0 + \frac{\epsilon}{2})$ for all $x \in \Omega$. Let us choose the space $\dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega)$ where the approximate controllability of (2.20) is considered. In $\dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega)$, the wellposedness of (2.20), without control, follows in an usual manner. Hence, for any small $\tilde{\epsilon} > 0$, we get a control $g_4 \in L^2(0, T_0 + \epsilon/2; H_0^3(\mathcal{O}))$ such that the solution (ϱ^4, v^4) of (2.20) belongs to $C([0, T_0 + \epsilon/2]; \dot{H}_{per}^3(\Omega) \times H_{per}^4(\Omega))$ and satisfies (2.21). $\square$

Using the above corollary, we obtain that

$$\begin{aligned} & \|\varrho^4(\cdot, T_0 + \epsilon/2)\|_{H_{per}^3(\Omega)} + \|w^4(\cdot, T_0 + \epsilon/2)\|_{H_{per}^4(\Omega)} \\ & \leq \tilde{\epsilon}(\|\varrho^4(\cdot, 0)\|_{H_{per}^3(\Omega)} + \|w^4(\cdot, 0)\|_{H^2}), \\ & \|g_1^4\|_{Y_2} \leq C_{\tilde{\epsilon}}(\|\varrho^4(\cdot, 0)\|_{H_{per}^3(\Omega)} + \|w^4(\cdot, 0)\|_{H_{per}^4(\Omega)}). \end{aligned} \quad (4.25)$$

Here we can make $\tilde{\epsilon}$ as small as we wish, at the expense of making $C_{\tilde{\epsilon}}$ large. Finally, we have a bound of the form

$$\|F^4\|_{Z_1} + \|G_1^4\|_{Z_2} \leq C(\|\varrho^4(\cdot, T_0 + \epsilon/2)\|_{H_{per}^3(\Omega)} + \|w^4(\cdot, T_0 + \epsilon/2)\|_{H_{per}^4(\Omega)}). \quad (4.26)$$

For the solution of (2.26) we thus have a bound of the form

$$\begin{aligned} \|\tilde{\varrho}\|_{X_1} + \|\tilde{w}\|_{X_2} + \|\tilde{f}^1\|_{Y_1} + \|\tilde{g}^1 + g_1^4\|_{Y_2} & \leq C(\|G_1\|_{Z_2} + \|v_m\|_{\mathbb{X}_4} \\ & + \|\rho_0\|_{\dot{H}_{per}^2(\Omega)} + \|w_0\|_{H_{per}^3(\Omega)}). \end{aligned} \quad (4.27)$$

Moreover, for the inhomogeneous error terms, we have a bound of the form

$$\begin{aligned} & \|F^3 + F^4 + \Psi \cdot \nabla \varrho^4 + (\operatorname{div} \Psi) \varrho^4\|_{Z_1} + \|G^4\|_{Z_2} \\ & \leq \epsilon_1(\|G_1\|_{Z_2} + \|v_m\|_{\mathbb{X}_4} + \|\rho_0\|_{\dot{H}_{per}^2(\Omega)} + \|w_0\|_{H_{per}^3(\Omega)}). \end{aligned} \quad (4.28)$$

Here we can choose ϵ_1 as small as we wish by restricting the size of Ψ . Since $\tilde{\varrho}$ and $\tilde{w}$ satisfy the initial conditions exactly, we can now go through Steps 1-3 iteratively to obtain a solution of (2.4).

That is, we iteratively define the error terms

$$\begin{aligned} P^j & = F^3(\varrho^{j,3}, w^{j,3}) + F^4(\varrho^{j,4}, w^{j,4}) + \Psi \cdot \nabla \varrho^{j,4} + (\operatorname{div} \Psi) \varrho^{j,4}, \\ \check{G}_1^j & = G_1^4(\varrho^{j,4}, w^{j,4}), \end{aligned} \quad (4.29)$$

where F^3 , F^4 , G^4 are defined in (2.18) and (2.24). Let us denote the control for velocity equation in (2.26) by

$$\check{g}_1^j = \tilde{g}_1^{j,1} + g^{j,4},$$

where s_3 is defined in Step 2.

For $j = 1$, let us denote $(\tilde{\varrho}^j, \tilde{w}^j)$ the solution of (2.26) and $(\varrho^{j,\ell}, w^{j,\ell})$, for $\ell = 1, 2, 3, 4$ be the corresponding solutions in the intermediate three steps. For $j \geq 2$, let $(\tilde{\varrho}^j, \tilde{w}^j)$ be the solution of

$$\begin{aligned} & \tilde{\varrho}_t^j + u_s \cdot \nabla \tilde{\varrho}^j + \Psi \cdot \nabla \tilde{\varrho}^j + (\operatorname{div} \Psi) \tilde{\varrho}^j + \Delta \tilde{w}^j + \operatorname{div} v_m \\ & = \tilde{f}^j - P^{j-1} + P^j \quad \text{in } \Omega \times (0, T), \\ & \tilde{w}_t + u_s \cdot \nabla \tilde{w} - \nu \Delta \tilde{w} + p'(1) \tilde{\varrho} = \check{g}_1^j + G_1 - \check{G}_1^{j-1} + \check{G}^j \quad \text{in } \Omega \times (0, T), \\ & \tilde{\varrho}^j(\cdot, 0) = \rho_0, \quad \tilde{w}^j(\cdot, 0) = w_0 \quad \text{in } \Omega. \end{aligned} \quad (4.30)$$

For each fixed j , $\{(\varrho^{j,\ell}, w^{j,\ell})\}_{\ell=1,2,3,4}$ are the solutions in the intermediate steps. Using the estimates obtained above and standard arguments, we obtain convergence of the iterates to a solution of $(2.10)_1 - (2.10)_2$ and the required estimates for the solution. Finally, combining this result along with Proposition 3.9 and Theorem 3.8, we obtain the solution of the null control problem for (2.10) with the required estimates for the solution in the weighted Sobolev spaces.

Proof of Theorem 4.1:

Proof. By our analysis done so far in this section, we obtain that $(\varrho, w, a, v_m, \langle v \rangle)$ is the solution of (2.10) with controls (f, g_1, g_2, g_m, g_3) such that the solution reaches zero at time T . Now setting

$$\begin{aligned} v(x, t) &= \nabla w(x, t) + \nabla \times a(x, t) + v_m(x, t), \\ g(x, t) &= \nabla g_1(x, t) + \nabla \times g_2(x, t) + g_m(x, t), \end{aligned}$$

we get that (ϱ, v) satisfies (2.4) with controls f and g in both equations for density and velocity such that

$$(\varrho, v)(\cdot, T) = 0 \quad \text{in } \Omega.$$

Further, the estimates done in this section yield (4.2). $\square$

Now, to accomplish our analysis done in Section 5 to absorb the density control f in (2.4) by a suitable control in velocity equation, we need to have f with support in $\mathcal{O} \times (\delta, T_1)$ and the velocity control g in (2.4) with support in $\mathcal{O} \times (\delta, T)$. Here $\delta > 0$ is chosen in a way such that $T_0 + \delta < T_1$.

In view of this, we redefine the spaces for controls f with support in the reduced time interval (δ, T_1) and g with support in the reduced time interval (δ, T) as follows

$$\begin{aligned} Y_1 &= \left\{ f \mid f \in L^2(\delta, T_1; H_0^2(\mathcal{O})), \int_{\Omega} f(x, t) dx = 0, \quad \forall t \in (\delta, T_1) \right\} \\ \mathbb{Y}_4 &= \{g \mid e^{\Theta(t)}(g)_i \in L^2(\delta, T; H_0^1(\mathcal{O})), \quad \forall i = 1, \dots, d\}. \end{aligned} \quad (4.31)$$

The null controllability of (2.4) using controls (f, g) in $Y_1 \times \mathbb{Y}_4$, defined above, can be obtained by studying the existence of the solution of (2.4) without control in the time interval $(0, \delta)$ and then applying Theorem 4.1 to the control problem (2.4) on (δ, T) for the initial conditions at $t = \delta$ instead of $t = 0$. Thus, we get the following corollary.

Corollary 4.8. *Let us assume that $\Psi \in \mathbb{X}_4$ satisfying*

$$\|\Psi\|_{\mathbb{X}_4} \leq \kappa_2,$$

where κ_2 is a positive constant. We also assume that $G \in \mathbb{Z}_4$ is given. Then there exists a sufficiently small $\kappa_2 > 0$ such that the null control problem (2.4) admits a solution and the solution satisfies

$$\|\varrho\|_{X_1} + \|v\|_{\mathbb{X}_4} + \|f\|_{Y_1} + \|g\|_{\mathbb{Y}_4} \leq C(\|G\|_{\mathbb{Z}_4} + \|\rho_0\|_{\dot{H}_{per}^2(\Omega)} + \|u_0\|_{H_{per}^2(\Omega)}). \quad (4.32)$$

5. Two controls to one control

5.1. A dual observability problem

So far, we have discussed controls in both the mass and momentum equation. This of course is not physically feasible. It turns out that it is sufficient to control just the momentum equation.

To absorb f by a control acting in the velocity equation, we consider the following linearized system with the f coming from the controllability of (2.4) as a forcing term in the density equation and a control in the velocity equation.

$$\begin{aligned} \rho_t^f + (u_s \cdot \nabla) \rho^f + \operatorname{div}(\Psi \rho^f) + \operatorname{div} u^f &= f \quad \text{in } \Omega \times (0, T), \\ u_t^f + (u_s \cdot \nabla) u^f - \mu \Delta u^f - (\mu_0 + \mu) \nabla \operatorname{div} u^f + p'(1) \nabla \rho^f &= \hat{g} \quad \Omega \times (0, T), \\ \rho^f(\cdot, 0) &= 0, \quad u^f(\cdot, 0) = 0 \quad \text{in } \Omega, \end{aligned} \tag{5.1}$$

where f is supported in $\mathcal{O} \times (\delta, T_1)$ and $\hat{g}$ is supported in $\hat{\mathcal{O}} \times (0, T_1 + \delta_1)$, for some $\delta_1 > 0$ chosen in a way such that $T_1 + \delta_1 < T$. Moreover, we have $f \in L^2(\delta, T_1; H_0^2(\mathcal{O}))$ with $\int_{\Omega} f(x, t) dx = 0$ and we require the regularity $\hat{g} \in (L^2(0, T_1 + \delta_1; H_0^1(\hat{\mathcal{O}})))^d$. Clearly, it now suffices to show that there exists a null control for any given f .

Let us define the space for $\hat{g}$ which is same as $\mathbb{Y}_4$, defined in (3.40), except the support of the control functions:

$$\hat{\mathbb{Y}}_4 = \{g \mid e^{\Theta(t)}(g)_i \in L^2(0, T; H_0^1(\hat{\mathcal{O}})), \quad \forall i = 1, \dots, d\}. \tag{5.2}$$

In the usual fashion, this can be transformed into a dual observability problem. The dual equation is

$$\begin{aligned} -\sigma_t - (u_s \cdot \nabla) \sigma - (\Psi \cdot \nabla) \sigma - p'(1) \operatorname{div} v &= 0 \quad \text{in } \Omega \times (0, T), \\ -v_t - (u_s \cdot \nabla) v - \mu \Delta v - (\mu_0 + \mu) \nabla \operatorname{div} v - \nabla \sigma &= 0 \quad \text{in } \Omega \times (0, T), \\ \sigma(\cdot, T) &= \sigma_T, \quad v(\cdot, T) = v_T \quad \text{in } \Omega \times (0, T). \end{aligned} \tag{5.3}$$

Remark 5.1. Since, by our previous analysis, (5.1) is well-posed in $\dot{H}_{per}^2(\Omega) \times (H_{per}^1(\Omega))^d$, the adjoint system (5.3) with $(\sigma_T, v_T) \in \dot{H}_{per}^{-2}(\Omega) \times (H_{per}^{-1}(\Omega))^d$ is well-posed, with a solution in $L^2(0, T; \dot{H}_{per}^{-2}(\Omega)) \times (L^2(0, T; H_{per}^{-1}(\Omega)))^d$.

Theorem 5.2. Assume that $\|\Psi\|_{\mathbb{X}_4}$ is sufficiently small. Then, for a given $f \in L^2(\delta, T_1; \dot{H}_0^2(\mathcal{O}))$, there exists $\hat{g} \in (L^2(0, T_1 + \delta_1; H_0^1(\hat{\mathcal{O}})))^d$ with support inside $\hat{\mathcal{O}} \times (0, T_1 + \delta_1)$ such that (ρ^f, u^f) , the solution of (5.1), satisfies

$$(\rho^f, u^f)(\cdot, t) = (0, 0), \quad \forall t \in (T_1 + \delta_1, T).$$

Moreover, (ρ^f, u^f) belongs to $X_1 \times \mathbb{X}_4$ and the control $\hat{g} \in \hat{\mathbb{Y}}_4$.

Proof. Consider any smooth solution (σ, v) of the dual problem (5.3). We compute

$$\begin{aligned} \int_0^{T_1+\delta_1} \int_{\Omega} [f\sigma + \widehat{g}.v] dx dt &= \int_{\Omega} \left[\rho^f(x, T_1 + \delta_1) \sigma(x, T_1 + \delta_1) \right. \\ &\quad \left. + u^f(x, T_1 + \delta_1).v(x, T_1 + \delta_1) - \rho^f(x, 0) \sigma(x, 0) - u^f(x, 0).v(x, 0) \right] dx. \end{aligned} \quad (5.4)$$

By using $\rho^f(\cdot, 0) = u^f(\cdot, 0) = 0$ on Ω and the supports of f and $\widehat{g}$, we get

$$\begin{aligned} \int_{\delta}^{T_1} \int_{\mathcal{O}} f\sigma dx dt + \int_0^{T_1+\delta_1} \int_{\widehat{\mathcal{O}}} \widehat{g}.v dx dt \\ = \int_{\Omega} [\rho^f(x, T_1 + \delta_1) \sigma(x, T_1 + \delta_1) + u^f(x, T_1 + \delta_1).v(x, T_1 + \delta_1)] dx. \end{aligned} \quad (5.5)$$

By a density argument we get

$$\begin{aligned} \int_{\delta}^{T_1} \langle \sigma(\cdot, t), f(\cdot, t) \rangle_{H^{-2}, H_0^2(\mathcal{O})} dt + \int_0^{T_1+\delta_1} \langle v(\cdot, t), \widehat{g}(\cdot, t) \rangle_{H^{-1}, H_0^1(\widehat{\mathcal{O}})} dt \\ = \langle \sigma(\cdot, T_1 + \delta_1), \rho^f(\cdot, T_1 + \delta_1) \rangle_{H_{per}^{-2}, H_{per}^2} + \langle v(\cdot, T_1 + \delta_1), \widehat{g}(\cdot, T_1 + \delta_1) \rangle_{H_{per}^{-1}, H_{per}^1} \end{aligned} \quad (5.6)$$

for all $(\sigma(\cdot, T_1 + \delta_1), v(\cdot, T_1 + \delta_1)) \in \dot{H}_{per}^{-2}(\Omega) \times (H_{per}^{-1}(\Omega))^d$. The idea is to deduce, from the equations of (5.3), that the norm of σ modulo constants in $L^2(\delta, T_1; H^{-2}(\mathcal{O}))$ is bounded by the norm of v in $(L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^d$, i.e.,

$$\|\sigma\|_{L^2(\delta, T_1; H^{-2}(\mathcal{O}))} \leq C \|v\|_{L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))}. \quad (5.7)$$

More precisely, a bounded operator K from $(L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^d$ to $L^2(\delta, T_1; H^{-2}(\mathcal{O}))$ is to be constructed in the next subsection such that $\sigma = Kv$. Using this, we set $\widehat{g} = -K^*f$. Then we must have

$$\begin{aligned} \int_{\delta}^{T_1} \langle \sigma(\cdot, t), f(\cdot, t) \rangle_{H^{-2}, H_0^2(\mathcal{O})} dt + \int_0^{T_1+\delta_1} \langle v(\cdot, t), \widehat{g}(\cdot, t) \rangle_{H^{-1}, H_0^1(\widehat{\mathcal{O}})} dt \\ = \int_0^{T_1+\delta_1} \langle v(\cdot, t), K^*f + \widehat{g}(\cdot, t) \rangle_{H^{-1}, H_0^1(\widehat{\mathcal{O}})} dt = 0. \end{aligned} \quad (5.8)$$

But since on the other hand $\sigma(\cdot, T_1)$ and $v(\cdot, T_1)$ can be chosen arbitrarily, it follows that $\rho^f(\cdot, T_1)$ and $u^f(\cdot, T_1)$ must be zero. Moreover, since f and $\widehat{g}$ are zero over $\Omega \times (T_1 + \delta_1, T)$ together with $(\rho^f, u^f)(\cdot, T_1 + \delta_1) = (0, 0)$, (ρ^f, u^f) becomes zero over $\Omega \times [T_1 + \delta_1, T]$. $\square$

5.2. Construction of K

Let us recall $\widehat{\mathcal{O}}$ and $\mathcal{O}$ from (1.3) and (2.5). We also note that $\mathcal{O} \subset \widehat{\mathcal{O}}$.

To construct the operator K , we therefore first consider the ‘‘hyperviscous’’ Stokes system on $\widehat{\mathcal{O}} \times (0, T_1 + \delta_1)$. Here n is the normal direction on $\partial\widehat{\mathcal{O}}$, and Δ_{d+1}

is the $(d+1)$ -dimensional Laplacian $\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial t^2}$. Then the “hyperviscous” Stokes system is

$$\begin{aligned}
& \nabla_x \sigma - \Delta_{d+1}^3 \eta_1 = \alpha_1 \quad \text{on } \widehat{\mathcal{O}} \times (0, T_1 + \delta_1), \\
& \sigma_t - \Delta_{d+1}^3 \eta_2 = \alpha_2 \quad \text{on } \widehat{\mathcal{O}} \times (0, T_1 + \delta_1), \\
& \operatorname{div}_x(\eta_1) + (\eta_2)_t = 0 \quad \text{on } \widehat{\mathcal{O}} \times (0, T_1 + \delta_1), \\
& \eta_j(\cdot, 0) = \eta_j(\cdot, T_1 + \delta_1) = \frac{\partial \eta_j}{\partial t}(\cdot, 0) = \frac{\partial \eta_j}{\partial t}(\cdot, T_1 + \delta_1) = 0, \\
& \frac{\partial^2 \eta_j}{\partial t^2}(\cdot, 0) = \frac{\partial^2 \eta_j}{\partial t^2}(\cdot, T_1 + \delta_1) = 0, \quad \text{on } \widehat{\mathcal{O}}, \quad \forall j = 1, 2, \\
& \eta_j|_{\partial \widehat{\mathcal{O}}} = 0, \quad \frac{\partial \eta_j}{\partial n} \Big|_{\partial \widehat{\mathcal{O}}} = \frac{\partial^2 \eta_j}{\partial n^2} \Big|_{\partial \widehat{\mathcal{O}}} = 0, \quad \text{on } (0, T_1 + \delta_1), \quad j = 1, 2,
\end{aligned} \tag{5.9}$$

where, on the right hand side, $\alpha_1 \in (H^{-3}(\widehat{\mathcal{O}} \times (0, T_1 + \delta_1)))^d$ and $\alpha_2 \in H^{-3}(\widehat{\mathcal{O}} \times (0, T_1 + \delta_1))$. We note that the hyperviscous Stokes system is elliptic; indeed, in $(d+1)$ dimensions it is related to the tetraharmonic equation in the same way as the Stokes system is related to the biharmonic equation.

In an analogous fashion as for the Stokes problem (see e.g. [2], it can be shown that (5.9) has a unique solution except for an undetermined constant in the “pressure” σ . For given $\alpha_1 \in (H^{-3}(\widehat{\mathcal{O}} \times (0, T_1 + \delta_1)))^d$ and $\alpha_2 \in H^{-3}(\widehat{\mathcal{O}} \times (0, T_1 + \delta_1))$, the solution satisfies $\eta_1, \eta_2 \in H^3(\mathcal{O} \times (\delta, T_1))$, $\sigma \in H^{-2}(\mathcal{O} \times (\delta, T_1))$, where $\mathcal{O} \times (\delta, T_1)$ is properly contained in $\widehat{\mathcal{O}} \times (0, T_1 + \delta_1)$. However, if the given forcing term is $(\alpha_1, \alpha_2) \in (L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})))^{d+1}$, then we can show that $\sigma \in L^2(\delta, T_1; H^{-2}(\mathcal{O}))$ satisfying

$$\|\sigma\|_{L^2(\delta, T_1; H^{-2}(\mathcal{O}))} \leq C \|(\alpha_1, \alpha_2)\|_{L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}}))}, \tag{5.10}$$

for some constant $C > 0$. Finally, if the right hand side in (5.9) is a gradient to begin with, then $\eta_1 = \eta_2 = 0$.

Remark 5.3. *Note that $\widehat{\mathcal{O}} \times (0, T_1 + \delta_1)$ is domain with corners in the boundary. So, it imposes difficulty to get the regularity of the solution of (5.9) in $\widehat{\mathcal{O}} \times (0, T_1 + \delta_1)$. To overcome this problem, we estimate the solution in a smaller domain $\mathcal{O} \times (\delta, T_1)$ and using interior regularity of an elliptic equation, we get the required estimates. Here this is enough for us how reduce two controls to one control, whereas we use a slightly different method to accomplish this in [15].*

Proposition 5.4. *Let us assume that $(\alpha_1, \alpha_2) \in (L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})))^{d+1}$. In addition, we assume that there exists a scalar function $\phi \in L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}}))$ such that*

$$\nabla_{d+1} \phi = (\alpha_1, \alpha_2).$$

Then $(\phi, 0, 0)$ is a solution of (5.9) and by the uniqueness of the solution of (5.9), we have

$$(\sigma, \eta_1, \eta_2) = (\phi, 0, 0).$$

Using Proposition 5.4, we define projections of any vector field to a gradient and it will help us to construct K . Let us define the space

$$\mathcal{H} = \{(\alpha_1, \alpha_2) \in (L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})))^{d+1} \mid$$

$$(\alpha_1, \alpha_2) = (\nabla_x \phi, \phi_t) \text{ for some } \phi \in L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}}))\}.$$

Now we return to (5.3). In order to write (5.3) in operator form, we introduce the following notations:

$$Qv = \begin{pmatrix} -p'(1)\text{div}_x v \\ -v_t - (u_s \cdot \nabla)v - \mu \Delta_x v \end{pmatrix},$$

$$M = \begin{pmatrix} (u_s + \Psi)^T & 1 \\ I_{d \times d} & 0_d \end{pmatrix}, \quad (5.11)$$

where $I_{d \times d}$ is the $d \times d$ identity matrix and 0_d is the d -dimensional column vector having 0 at each entity. With the notations defined in (5.11), we can write (5.3) in the form

$$\begin{pmatrix} \nabla_x \sigma \\ \sigma_t \end{pmatrix} = M^{-1}(Qv). \quad (5.12)$$

Let L denote the operator that maps (α_1, α_2) to $\sigma|_{\mathcal{O} \times (\delta, T_1)}$ by solution of (5.9). From (5.10), it follows that L is a bounded linear operator from $(L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})))^{d+1}$ to $L^2(\delta, T_1; H^{-2}(\mathcal{O}))$. For further use below, we also note that if (α_1, α_2) is in $(L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^{d+1}$, then σ is in $L^2(\delta, T_1; H^{-1}(\mathcal{O}))$. We note also that if α_1, α_2 is actually in $\mathcal{H}$, then $L(\alpha_1, \alpha_2)$ is the function ϕ in the definition of $\mathcal{H}$.

From the definition of the operator L , we obtain

$$\sigma = L(M^{-1}(Qv)). \quad (5.13)$$

We can now set

$$K(\Psi) = LM^{-1}Q, \quad (5.14)$$

and give the required bounds.

Lemma 5.5. *Let us assume that $v \in (L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^d$ and $\Psi \in \mathbb{X}_4$. Let the operators Q and M be as in (5.11). Then the following holds.*

1. Q is a bounded operator from $(L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^d$ to

$$\widetilde{\mathcal{H}} = \left(L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) \right)^{d+1}.$$

2. From the representation of M , we have that M^{-1} is defined by

$$M^{-1} = \begin{pmatrix} 0_d & I_{d \times d} \\ 1 & -(u_s + \Psi) \end{pmatrix}.$$

Moreover, $M^{-1}Q$ is a bounded operator from $(L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^d$ to $\widetilde{\mathcal{H}}$, and the operator norm of $M^{-1}Q$ satisfies a bound depending on $\|\Psi\|_{\mathbb{X}_4}$.

Proof. 1. For all $v \in (L^2(0, T_1 + \delta_1; H^{-1}(\mathcal{O})))^d$, from the definition of Q , it is clear that Q is a bounded operator from $(L^2(0, T_1 + \delta_1; H^{-1}(\mathcal{O})))^d$ to $\left(L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))\right)^{d+1}$. Since $\left(L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))\right)^{d+1}$ is a subspace of $\widetilde{\mathcal{H}}$, Q is a bounded operator from $(L^2(0, T_1 + \delta_1; H^{-1}(\mathcal{O})))^d$ to $\widetilde{\mathcal{H}}$.

2. Note that $\left(L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))\right)^{d+1}$ is the dual space of $\left(L^2(0, T_1 + \delta_1; H_0^3(\widehat{\mathcal{O}})) \cap H_0^1(0, T_1 + \delta_1; H_0^1(\widehat{\mathcal{O}}))\right)^{d+1}$. Since $\Psi \in \mathbb{X}_4$, it is a multiplier in the latter space and hence Ψ is also the multiplier of $\left(L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))\right)^{d+1}$. Thus, $M^{-1}Q$ is a bounded operator from $(L^2(0, T_1 + \delta_1; H^{-1}(\mathcal{O})))^d$ to $\left(L^2(0, T_1 + \delta_1; H^{-3}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))\right)^{d+1}$. Since the latter space is a subspace of $\widetilde{\mathcal{H}}$, the claim for $M^{-1}Q$ follows with a bound depending of $\|\Psi\|_{\mathbb{X}_4}$. $\square$

As a consequence, we obtain the following theorem.

Theorem 5.6. *Let M and Q be defined in (5.12) and L be the solution operator for (5.9) as defined above. Let us also assume that Ψ belongs to $\mathbb{X}_4$. Then $K = K(\Psi)$, defined in (5.14), is a bounded operator from $(L^2(0, T_1 + \delta_1; H^{-1}(\mathcal{O})))^d$ into $L^2(\delta, T_1; H^{-2}(\mathcal{O}))$. Moreover, there exists a positive C , independent of Ψ , such that*

$$\|K(\Psi)\|_{\mathcal{L}(L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})), L^2(\delta, T_1; H^{-2}(\mathcal{O})))} \leq C(1 + \|\Psi\|_{\mathbb{X}_4}). \quad (5.15)$$

Correspondingly, K^* maps $L^2(\delta, T_1; H_0^2(\mathcal{O}))$ to $(L^2(0, T_1 + \delta_1; H_0^1(\widehat{\mathcal{O}})))^d$.

Proof. By Lemma 5.5 and the definition of L , it follows that $LM^{-1}Q$ is bounded from $(L^2(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^d$ to $L^2(\delta, T_1; H^{-2}(\mathcal{O}))$, with a bound depending on the norm of Ψ . Hence K is well-defined and the theorem follows. $\square$

To keep track of the dependence on Ψ , we need to consider K in spaces with a stronger norm (corresponding to a weaker norm for K^*). We show that K maps $(L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}})))^d$ into $L^2(\delta, T_1; H^{-1}(\mathcal{O}))$.

Lemma 5.7. *Let us recall that K is defined in (5.14). Then K is a bounded operator from $(L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}})))^d$ into $L^2(\delta, T_1; H^{-1}(\mathcal{O}))$ with a bound depending on $\|\Psi\|_{\mathbb{X}_4}$. Consequently, K^* is a bounded operator from $L^2(\delta, T_1; H_0^1(\mathcal{O}))$ into $(L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}})))^d$.*

Proof. We first note that Q maps $(L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}})))^d$ into $\left(L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}}))\right)^{d+1}$. Since Ψ is a multiplier in this space,

$M^{-1}Q$ also maps to $\left(L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}}))\right)^{d+1}$, which is a subspace of $(L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}})))^{d+1}$. Hence $LM^{-1}Q$ maps $(L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}})))^d$ into $L^2(\delta, T_1; H^{-1}(\mathcal{O}))$, with a norm depending on the norm of Ψ in $\mathbb{X}_4$. This shows that K is well defined in these stronger norms. Correspondingly, K^* maps $L^2(\delta, T_1; H_0^1(\mathcal{O}))$ into $(L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}})))^d$. $\square$

We can now keep track of the dependence on Ψ . Note that M depends on Ψ . To indicate this dependence, let us write $K(\Psi)$, $M(\Psi)$ and give the following lemma.

Lemma 5.8. *The operator $K(\Psi) - K(\Phi)$ belongs to $\mathcal{L}((L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}})))^d; L^2(\delta, T_1; H^{-2}(\mathcal{O})))$ with a bound proportional to the norm of $\|\Psi - \Phi\|$ in $L^2(0, T_1 + \delta_1; H^2(\widehat{\mathcal{O}})) \cap H^1(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}}))^d$.*

Proof. Let us say, $\sigma = K(\Psi)v$ and $\tau = K(\Phi)v$. We have

$$\sigma - \tau = L\left((M^{-1}(\Psi) - M^{-1}(\Phi))Qv\right). \quad (5.16)$$

We estimate as follows: for all $v \in L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}}))^d$,

$$\begin{aligned} & \| (M^{-1}(\Psi) - M^{-1}(\Phi))Qv \|_{L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))} \\ & \leq C \|\Psi - \Phi\|_{L^2(H^2) \cap H^1(L^2)} \|Qv\|_{L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}}))} \\ & \leq C \|\Psi - \Phi\|_{L^2(H^2) \cap H^1(L^2)} \|v\|_{L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}}))}. \end{aligned} \quad (5.17)$$

Using the fact that L is a bounded operator from $L^2(0, T_1 + \delta_1; H^{-2}(\widehat{\mathcal{O}})) + H^{-1}(0, T_1 + \delta_1; H^{-1}(\widehat{\mathcal{O}}))$ to $L^2(\delta, T_1; H^{-1}(\mathcal{O}))$ and $L^2(\delta, T_1; H^{-1}(\mathcal{O})) \subset L^2(\delta, T_1; H^{-2}(\mathcal{O}))$, we get the required estimate. $\square$

Corollary 5.9. *Let Ψ^1, Ψ^2 be in $\mathbb{X}_4$ where $\|\Psi^1\|_{\mathbb{X}_4}$ and $\|\Psi^2\|_{\mathbb{X}_4}$ are sufficiently small to satisfy the requirements of Section 4 and of Theorem 5.6. Let f^1, f^2 be in $L^2(\delta, T_1; \dot{H}_0^2(\mathcal{O}))$ and $\widehat{g}^1, \widehat{g}^2$ be the controls obtained from Theorem 5.2. Let us consider that for each $j = 1, 2$, $(\rho^{f,j}, u^{f,j})$ is the solution of (5.1) with f^j as forcing term and control $\widehat{g}^j \in (L^2(0, T_1 + \delta_1; H_0^1(\widehat{\mathcal{O}})))^d$ with support inside $\widehat{\mathcal{O}} \times (0, T_1 + \delta_1)$ satisfying all the conditions mentioned in Theorem 5.2. Then there exists a positive constant C such that*

$$\begin{aligned} & \|\rho^{f,2} - \rho^{f,1}\|_{L^2(0, T_1 + \delta_1; H^1(\Omega))} + \|u^{f,2} - u^{f,1}\|_{L^2(0, T_1 + \delta_1; H^2(\Omega))} \\ & + \|\widehat{g}^2 - \widehat{g}^1\|_{L^2(0, T_1 + \delta_1; L^2(\widehat{\mathcal{O}}))} \leq C \left(\|f^2 - f^1\|_{L^2(\delta, T_1; H_0^1(\mathcal{O}))} \right. \\ & \quad \left. + \|\Psi^2 - \Psi^1\|_{\mathbb{X}_4} \|f^2\|_{L^2(\delta, T_1; H_0^2(\mathcal{O}))} \right). \end{aligned} \quad (5.18)$$

5.3. Absorption of the control in the density equation

Let us denote for all $n \in \mathbb{N}$,

$$G^n = G(\rho^n, \nabla \rho^n, u^n, \nabla u^n, u_t^n),$$

where G is defined in (2.2) and (ρ^n, u^n) is the solution of (2.3) at the n th iteration.

Theorem 5.10. *Let us assume that for each $n \in \mathbb{N}$, $G^n \in \mathbb{Z}_4$ and $(\rho_0, u_0) \in \dot{H}_{per}^2(\Omega) \times (H_{per}^2(\Omega))^d$. Let, for all $n \in \mathbb{N}$, $\|u^n\|_{\mathbb{X}_4}$ be small enough. Then for any $T > T_0$, there exists a control $g^{n+1} \in \widehat{\mathbb{Y}}_4$ such that (2.3) is null controllable at T and (ρ^{n+1}, u^{n+1}) belongs to $X_1 \times \mathbb{X}_4$ with the following estimate:*

$$\|\rho^{n+1}\|_{X_1} + \|u^{n+1}\|_{\mathbb{X}_4} + \|g^{n+1}\|_{\widehat{\mathbb{Y}}_4} \leq C \left(\|(\rho_0, u_0)\|_{\dot{H}_{per}^2(\Omega) \times H_{per}^2(\Omega)} + \|G^n\|_{\mathbb{Z}_4} \right), \quad (5.19)$$

where C is a positive constant depending on T and $\|u^n\|_{\mathbb{X}_4}$.

Proof. Now let us define

$$\rho^{n+1} = \varrho - \rho^f, \quad u^{n+1} = v - u^f, \quad g^{n+1} = g - \widehat{g} = g + K^*(u^n)f, \quad (5.20)$$

where (ϱ, v) is the solution of (2.4) and (ρ^f, u^f) is the solution of (5.1). Therefore, (ρ^{n+1}, u^{n+1}) belongs to X and it satisfies (2.3) with

$$(\rho^{n+1}, u^{n+1})(\cdot, T) = (0, 0).$$

Moreover, by using the estimate (4.2) for (ϱ, v) and the fact that (ρ^f, u^f) is $(0, 0)$ near T , we get (5.19). Hence Theorem 5.10 is proved. $\square$

6. Outer iteration

In this section, we study the convergence of the iterates (ρ^n, u^n, g^n) , the solution of the control problem (2.3) for each $n \in \mathbb{N}$ and show that the limit of $\{(\rho^n, u^n, g^n)\}_{n \in \mathbb{N}}$ exists in a proper Sobolev space and the limit is the solution of the control problem (2.1).

6.1. Useful estimates

Let us recall G defined in (2.2):

$$\begin{aligned} G(\zeta, \nabla \zeta, \eta, \nabla \eta, \eta_t) &:= -\zeta[\eta_t + ((\eta + u_s) \cdot \nabla) \eta] - (\eta \cdot \nabla) \eta \\ &\quad - [\nabla p(\zeta + 1) - p'(1) \nabla \zeta]. \end{aligned} \quad (6.1)$$

Now we have the following estimates. Recall the spaces defined in (3.40), (4.1) and (5.2).

Lemma 6.1. For any $\rho \in X_1$, let M be such that

$$\|\rho\|_{C([0,T],\dot{H}_{per}^2(\Omega))} \leq M\|\rho\|_{X_1}.$$

Let c_0 be the constant for the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^d$ for $d = 2, 3$. If

$$\|\rho\|_{X_1} \leq \frac{1}{2Mc_0}, \quad (6.2)$$

then

$$|\rho(x, t)| \leq \frac{1}{2}, \quad \forall (x, t) \in \Omega \times (0, T).$$

Proof. From

$$|\rho(x, t)| \leq \|\rho\|_{L^\infty(\Omega \times (0, T))} \leq c_0 \|\rho\|_{C([0, T], \dot{H}_{per}^2(\Omega))} \leq Mc_0 \|\rho\|_{X_1},$$

the lemma follows. $\square$

Lemma 6.2. Let c_0 and M be the constants introduced in Lemma 6.1. There exists a positive constant C_2 depending only on T , c_0 and M such that for all (ζ, η) in $X_1 \times \mathbb{X}_4$ satisfying

$$\|(\zeta, \eta)\|_{X_1 \times \mathbb{X}_4} \leq \frac{1}{2Mc_0},$$

we have

$$\|G\|_{\mathbb{Z}_4} \leq C_2 \|(\zeta, \eta)\|_{X_1 \times \mathbb{X}_4}^2. \quad (6.3)$$

Proof. To get the required estimates, we first recall the continuous Sobolev embeddings for a bounded domain Ω in $\mathbb{R}^d$, where $d = 2, 3$, $H_{per}^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H_{per}^1(\Omega) \hookrightarrow L^6(\Omega)$. Using these results, we have that for any $\zeta \in C([0, T]; H_{per}^2(\Omega))$ and $h \in L^2(0, T; H_{per}^1(\Omega))$

$$\|\zeta h\|_{L^2(0, T; H_{per}^1(\Omega))} \leq C \|\zeta\|_{C([0, T]; H_{per}^2(\Omega))} \|h\|_{L^2(0, T; H_{per}^1(\Omega))},$$

for some positive constant C depending on Ω , T and the constants for the Sobolev embeddings.

Now, we recall the definition of G from (6.1) and we assume that $\zeta \in X_1$ and $\eta \in \mathbb{X}_4$ and $\|(\zeta, \eta)\|_{X_1 \times \mathbb{X}_4} \leq \frac{1}{2Mc_0}$, where constants c_0 and M are introduced in Lemma 6.1. Thus, we have for a positive generic constant C_2

$$\|e^{2\Theta(t)} \zeta \eta_t\|_{L^2(0, T; H_{per}^1(\Omega))} \leq C_2 \|(\zeta, \eta)\|_{X_1 \times \mathbb{X}_4}^2,$$

and

$$\|e^{2\Theta(t)} \zeta ((\eta + u_s) \cdot \nabla) \eta\|_{L^2(0, T; H_{per}^1(\Omega))} \leq C_2 \|(\zeta, \eta)\|_{X_1 \times \mathbb{X}_4}^2,$$

$$\|e^{2\Theta(t)} (\eta \cdot \nabla) \eta\|_{L^2(0, T; H_{per}^1(\Omega))} \leq \|(\zeta, \eta)\|_{X_1 \times \mathbb{X}_4}^2.$$

Now, to estimate the last term in the expression of G in (6.1), we have

$$\nabla p(\zeta + 1) - p'(1) \nabla \zeta = (p'(1 + \zeta) - p'(1)) \nabla \zeta.$$

Using the Mean-value theorem and (6.2), we get

$$\|e^{2\Theta(t)}[\nabla p(\zeta + 1) - p'(1)\nabla \zeta]\|_{L^2(0,T;H_{per}^1(\Omega))} \leq C_2 \|\zeta\|_{X_1}^2.$$

Hence the lemma is proved. $\square$

By combining this with the estimates developed in the preceding section, we shall see that the map $\mathcal{F}$ defined by the iteration $(\rho^n, u^n) \mapsto (\rho^{n+1}, u^{n+1})$ takes a small ball in $X_1 \times \mathbb{X}_4$ into itself provided the norm of the initial data is sufficiently small. To complete the convergence argument, we need to show that $\mathcal{F}$ is a contraction with respect to a weaker norm. For this purpose, we define the following new weaker spaces.

$$\begin{aligned} \tilde{X} &= \tilde{X}_1 \times \tilde{\mathbb{X}}_4, \quad \tilde{Z} = \tilde{Z}_1 \times \tilde{\mathbb{Z}}_4, \\ \tilde{X}_1 &= \left\{ \rho \mid (T-t)W\rho \in C([0, T]; L^2(\Omega)), e^{\Theta(t)}\rho \in C([0, T]; H_{per}^1(\Omega)), \right. \\ &\quad \left. \int_{\Omega} \rho(x, t) dx = 0, \quad \forall t \in (0, T) \right\}, \\ \tilde{Z}_1 &= \left\{ F \mid (T-t)^2 WF \in L^2(0, T; L^2(\Omega)), (T-t)e^{\Theta(t)}F \in L^2(0, T; H_{per}^1(\Omega)), \right. \\ &\quad \left. \int_{\Omega} F(x, t) dx = 0, \quad \forall t \in (0, T) \right\}, \\ \tilde{Y}_1 &= \left\{ f \mid f \in L^2(\delta, T_1; H_0^1(\mathcal{O})), \int_{\Omega} f(x, t) dx = 0, \quad \forall t \in (\delta, T_1) \right\}, \\ \tilde{\mathbb{X}}_4 &= \{u \mid (T-t)W(u)_i, (T-t)^2 WD(u)_i \in L^2(0, T; L^2(\Omega)), \\ &\quad e^{\Theta(t)}(u)_i \in L^2(0, T; H_{per}^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad \forall i = 1, \dots, d\}, \\ \tilde{\mathbb{Z}}_4 &= \{G \mid e^{2\Theta(t)}(G)_i \in L^2(0, T; L^2(\Omega)), \quad \forall i = 1, \dots, d\}, \\ \tilde{\mathbb{Y}}_4 &= \{g \mid e^{\Theta(t)}(g)_i \in L^2(\delta, T; L^2(\mathcal{O})), \quad \forall i = 1, \dots, d\}, \\ \hat{\mathbb{Y}}_4^w &= \{g \mid e^{\Theta(t)}(g)_i \in L^2(0, T; L^2(\hat{\mathcal{O}})), \quad \forall i = 1, \dots, d\}. \end{aligned} \tag{6.4}$$

Now consider two pairs (ζ^1, η^1) and (ζ^2, η^2) in $X_1 \times \mathbb{X}_4$. Let G^1 and G^2 be the corresponding nonlinear terms. Taking advantage of the estimates in Sections 2.1.2 and 2.2.2, we have the following estimates.

Lemma 6.3. *There exists a positive constant C_3 depending only on T, c_0 such that for all (ζ^1, η^1) and (ζ^2, η^2) in $X_1 \times \mathbb{X}_4$ satisfying*

$$\|(\zeta^1, \eta^1)\|_{X_1 \times \mathbb{X}_4} \leq \frac{1}{2Mc_0}, \quad \|(\zeta^2, \eta^2)\|_{X_1 \times \mathbb{X}_4} \leq \frac{1}{2Mc_0}, \tag{6.5}$$

we have the Lipschitz estimates

$$\|(G^1 - G^2)\|_{\tilde{\mathbb{Z}}_4} \leq C_3 (\|(\zeta^1, \eta^1)\|_{X_1 \times \mathbb{X}_4} + \|(\zeta^2, \eta^2)\|_{X_1 \times \mathbb{X}_4}) \|(\zeta^1 - \zeta^2, \eta^1 - \eta^2)\|_{\tilde{X}_1 \times \tilde{\mathbb{X}}_4}. \tag{6.6}$$

Proof. Let (ζ^1, η^1) and (ζ^2, η^2) be in $X_1 \times \mathbb{X}_4$ satisfying (6.5). The first four terms in $(G^1 - G^2)$ can be estimated similarly to that of Lemma 6.2. Now, to estimate the last term in $(G^1 - G^2)$, we define

$$\psi(\zeta) = \nabla p(1 + \zeta) - p'(1)\nabla \zeta.$$

Now we have that

$$\psi(\zeta^1) - \psi(\zeta^2) = (\zeta^1 - \zeta^2) \int_0^1 \psi'(\tau\zeta^1 + (1-\tau)\zeta^2) d\tau,$$

and hence using Lemma 6.2, we get that for some positive constant C_3

$$\|\psi(\zeta^1) - \psi(\zeta^2)\|_{L^2(0,T;L^2(\Omega))} \leq C_3(\|(\zeta^1, \eta^1)\|_{X_1 \times \mathbb{X}_4} + \|(\zeta^2, \eta^2)\|_{X_1 \times \mathbb{X}_4}) \|\zeta^1 - \zeta^2\|_{\tilde{X}_1}.$$

Hence the lemma is proved. $\square$

Lemma 6.4. *Let us assume $T > T_0$ and for all $n \in \mathbb{N}$, $\|(\rho^n, u^n)\|_{X_1 \times \mathbb{X}_4}$ is sufficiently small as required by Sections 4 and 5 above and by Lemma 6.1. For each $n \in \mathbb{N}$, let $(\rho^{n+1}, u^{n+1}) \in X_1 \times \mathbb{X}_4$ be the solution of (2.3) with a control $g^{n+1} \in \hat{\mathbb{Y}}_4$ such that*

$$(\rho^{n+1}, u^{n+1})(\cdot, T) = (0, 0) \quad \text{on } \Omega.$$

We also assume that $f_^n \in Y_1$ is the control for density in (2.4) in the intermediate stage of each n th iteration, for all $n \in \mathbb{N}$. Then there exists a positive constant C_4 depending only on $T, \hat{O}$, such that*

$$\begin{aligned} & \|(\rho^{n+1}, u^{n+1}) - (\rho^n, u^n)\|_{\tilde{X}} + \|g^{n+1} - g^n\|_{\hat{\mathbb{Y}}_4^w} + \|f_*^{n+1} - f_*^n\|_{\tilde{Y}_1} \leq \\ & C_4 \left(\|(\rho^n, u^n)\|_X + \|(\rho^{n-1}, u^{n-1})\|_X + \|(\rho_0, u_0)\|_{\dot{H}_{per}^2 \times H_{per}^2} \right) \\ & \quad \left\| (\rho^n - \rho^{n-1}, u^n - u^{n-1}) \right\|_{\tilde{X}_1 \times \tilde{\mathbb{X}}_4}. \end{aligned} \quad (6.7)$$

Proof. We consider the equations satisfied by differences of the iterates with forcing terms in $\tilde{Z}_1 \times \tilde{\mathbb{Z}}_4$ and with zero initial conditions. We repeat the analysis formulated in Section 2 for this system. Using the results in Corollary 3.7 and Lemmas 3.13, 3.14, we have the estimates for the solutions and controls of heat and transport equations in weaker spaces. We handle the coupling in the weaker Sobolev spaces by repeating the analysis done in Section 4 in the weaker spaces and the estimates in Steps 1, 2 and 3 in Section 4 go through analogously for the difference terms in the weaker norms. In addition combining the result 2. in Proposition 3.9 and the result 2. in Theorem 3.10, we obtain the estimate similar to (4.32) for the difference terms.

Let ρ_*^n and u_*^n be the intermediate values obtained before reducing from two controls to one, and let f_*^n, g_*^n be the accompanying controls. Taking advantage of the estimate analogous to (4.32) for the difference terms in weaker norms and

Lemma 6.3, we end up with the estimate

$$\begin{aligned} & \|(\rho_*^{n+1} - \rho_*^n, u_*^{n+1} - u_*^n)\|_{\tilde{X}_1 \times \tilde{\mathbb{X}}_4} + \|f_*^{n+1} - f_*^n\|_{\tilde{Y}_1} \\ & + \|g_*^{n+1} - g_*^n\|_{\tilde{\mathbb{Y}}_4} \leq C(\|(\rho^n, u^n)\|_X + \|(\rho^{n-1}, u^{n-1})\|_X) \\ & \|(\rho^n - \rho^{n-1}, u^n - u^{n-1})\|_{\tilde{X}}. \end{aligned} \quad (6.8)$$

Finally, we take advantage of the estimates at the end of the previous section, to estimate $\|g^{n+1} - g^n\|_{\hat{\mathbb{Y}}_4^w}$, where using (5.20)

$$g^n = g_*^n + K^*(u^{n-1})f_*^n, \quad \forall n \in \mathbb{N}$$

and K is defined in (5.14). We have

$$\|g^{n+1} - g^n\|_{\hat{\mathbb{Y}}_4^w} \leq \|g_*^{n+1} - g_*^n\|_{\tilde{\mathbb{Y}}_4} + \|K^*(u^n)f_*^{n+1} - K^*(u^{n-1})f_*^n\|_{(L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d}.$$

Now we can write

$$\begin{aligned} & \|K^*(u^n)f_*^{n+1} - K^*(u^{n-1})f_*^n\|_{(L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d} \\ & \leq \|(K^*(u^n) - K^*(u^{n-1}))f_*^{n+1}\|_{(L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d} \\ & + \|K^*(u^{n-1})(f_*^{n+1} - f_*^n)\|_{(L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d}. \end{aligned}$$

Let us recall that from Lemmas 5.7 and 5.8, we have that for each $n \in \mathbb{N}$, $(K^*(u^n) - K^*(u^{n-1})) \in \mathcal{L}(L^2(\delta, T_1; H_0^2(\mathcal{O})); (L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d)$ and $K^*(u^{n-1}) \in \mathcal{L}(L^2(\delta, T_1; H_0^1(\mathcal{O})); (L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d)$ with the estimates

$$\begin{aligned} & \|(K^*(u^n) - K^*(u^{n-1}))f_*^{n+1}\|_{(L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d} \leq C_4 \|u^n - u^{n-1}\|_{\tilde{\mathbb{X}}_4} \|f_*^{n+1}\|_{Y_1}, \\ & \|K^*(u^{n-1})(f_*^{n+1} - f_*^n)\|_{(L^2(0, T_1 + \delta_1; L^2(\hat{\mathcal{O}})))^d} \leq C_4 (1 + \|u^{n-1}\|_{\mathbb{X}_4}) \|f_*^{n+1} - f_*^n\|_{\tilde{Y}_1}, \end{aligned}$$

where C_4 is a positive constant independent of n . Hence, we obtain

$$\begin{aligned} \|g^{n+1} - g^n\|_{\hat{\mathbb{Y}}_4^w} & \leq C_4 ((1 + \|u^{n-1}\|_{\mathbb{X}_4}) \|f_*^{n+1} - f_*^n\|_{\tilde{Y}_1} + \|g_*^{n+1} - g_*^n\|_{\tilde{\mathbb{Y}}_4} \\ & + \|f_*^{n+1}\|_{Y_1} \|u^n - u^{n-1}\|_{\tilde{\mathbb{X}}_4}). \end{aligned} \quad (6.9)$$

Now recall that for each $n \in \mathbb{N}$, $\rho^n = \rho_*^n - \rho^{f,n}$ and $u^n = u_*^n - u^{f,n}$, where $(\rho^{f,n}, u^{f,n})$ satisfies (5.1). Combining estimates (6.8) and (5.18), we finally end up with (6.7). $\square$

For any $\kappa > 0$, let us set D_κ and $\tilde{D}_\kappa$ by

$$\begin{aligned} D_\kappa & = \{(\zeta, \eta, f, g) \in X \times Y_1 \times \hat{\mathbb{Y}}_4 \mid \|(\zeta, \eta, f, g)\|_{X \times Y_1 \times \hat{\mathbb{Y}}_4} \leq \kappa\}, \\ \tilde{D}_\kappa & = \{(\zeta, \eta, f, g) \in \tilde{X} \times \tilde{Y}_1 \times \hat{\mathbb{Y}}_4^w \mid \|(\zeta, \eta, f, g)\|_{\tilde{X} \times \tilde{Y}_1 \times \hat{\mathbb{Y}}_4^w} \leq \kappa\}. \end{aligned} \quad (6.10)$$

Note that $D_\kappa \subset \tilde{D}_\kappa$, for all $\kappa > 0$.

Proposition 6.5. *There exist positive constants κ_0 and ι depending only on $T, \Omega, \widehat{\mathcal{O}}, c_0, \mu, \mu_0$ such that for $0 < \kappa \leq \kappa_0$ and any $(\rho_0, u_0) \in \dot{H}_{per}^2(\Omega) \times (H_{per}^2(\Omega))^d$ satisfying*

$$\|(\rho_0, u_0)\|_{\dot{H}_{per}^2(\Omega) \times H_{per}^2(\Omega)} \leq \iota \kappa, \quad (6.11)$$

the sequence of solutions of (2.3), $\{(\rho^n, u^n, g^n)\}_{n \in \mathbb{N}}$, for each $n \in \mathbb{N}$, is bounded in $X_1 \times \mathbb{X}_4 \times \widehat{\mathbb{Y}}_4$ obeying

$$\|(\rho^n, u^n, g^n)\|_{X_1 \times \mathbb{X}_4 \times \widehat{\mathbb{Y}}_4} \leq \kappa, \quad \forall n \in \mathbb{N},$$

and $\{(\rho^n, u^n)\}_{n \in \mathbb{N}}$ converges to (ρ, u) in $\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4$ satisfying

$$(\rho(\cdot, T), u(\cdot, T)) = (0, 0).$$

Moreover, there exists $g \in (L^2(0, T; H_0^1(\widehat{\mathcal{O}})))^d$ such that $\{g_n\}_{n \in \mathbb{N}}$ converges weakly to g in $(L^2(0, T; H_0^1(\widehat{\mathcal{O}})))^d$.

Proof. Our proof is based on successive iterative method. For that we first set $(\rho^0, u^0, g^0) = (\rho_0, u_0, 0)$ and for any $n \in \mathbb{N}$, $(\rho^{n+1}, u^{n+1}, g^{n+1})$ is the solution of (2.3) satisfying (5.19) and f^n is the control for density appearing in the intermediate state while solving (2.4).

Let, for κ_0 and ι which will be determined suitably later, for any $0 < \kappa \leq \kappa_0$, $(\rho_0, u_0) \in \dot{H}_{per}^2(\Omega) \times (H_{per}^2(\Omega))^d$ satisfy (6.11). For any $0 < \kappa \leq \kappa_0$, we assume $(\rho^n, u^n, f^n, g^n) \in D_\kappa$. Then applying (5.19), (6.3) and (4.2), we get that

$$\begin{aligned} & \|(\rho^{n+1}, u^{n+1}, f^{n+1}, g^{n+1})\|_{X \times Y_1 \times \widehat{\mathbb{Y}}_4} \\ & \leq C \left(\|(\rho_0, u_0)\|_{\dot{H}_{per}^2 \times H_{per}^2} + \|G(\rho^n, \nabla \rho^n, u^n, \nabla u^n, u_t^n)\|_{\mathbb{Z}_4} \right) \\ & \leq C \left(\|(\rho_0, u_0)\|_{\dot{H}_{per}^2 \times H_{per}^2} + C_2 \|(\rho^n, u^n)\|_{\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4}^2 \right) \leq C\kappa(\iota + \kappa). \end{aligned} \quad (6.12)$$

This is $\leq \kappa$, provided κ and ι are small enough. Hence $(\rho^{n+1}, u^{n+1}, f^{n+1}, g^{n+1}) \in D_\kappa$, if $(\rho^n, u^n, f^n, g^n) \in D_\kappa$ and by choosing κ_0 and ι sufficiently small.

Notice that $D_\kappa \subset \widetilde{D}_\kappa$, for any $\kappa > 0$. Now we show the convergence of (ρ^n, u^n) in $\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4$, provided $(\rho^n, u^n, f^n, g^n) \in D_\kappa$, for all $n \in \mathbb{N}$. From Lemma 6.4, it follows that

$$\begin{aligned} & \|(\rho^{n+1} - \rho^n, u^{n+1} - u^n)\|_{\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4} \leq (2 + \iota)\kappa C_4 \left\| (\rho^n - \rho^{n-1}, u^n - u^{n-1}) \right\|_{\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4} \\ & \leq \frac{1}{2} \left\| (\rho^n - \rho^{n-1}, u^n - u^{n-1}) \right\|_{\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4}, \quad \forall n \in \mathbb{N}, \end{aligned} \quad (6.13)$$

by choosing κ_0 and ι small enough. Finally, we get that

$$\|(\rho^{n+1} - \rho^n, u^{n+1} - u^n)\|_{\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4} \leq \frac{1}{2^n} \|(\rho^1 - \rho^0, u^1 - u^0)\|_{\widetilde{X}_1 \times \widetilde{\mathbb{X}}_4} \leq \frac{\iota \kappa_0}{2^n}, \quad \forall n \in \mathbb{N}. \quad (6.14)$$

It yields that

$$\sum_{j=1}^n \|(\rho^{n+1} - \rho^n, u^{n+1} - u^n)\|_{\tilde{X}_1 \times \tilde{\mathbb{X}}_4} \leq \iota \kappa_0 \sum_{j=1}^n \frac{1}{2^j}.$$

Since the geometric series $\sum \frac{1}{2^j}$ is convergent, $(\rho^{n+1} - \rho^n, u^{n+1} - u^n)$ is also convergent in $\tilde{X}_1 \times \tilde{\mathbb{X}}_4$. As we have that

$$(\rho^{n+1}, u^{n+1}) = \sum_{j=1}^n (\rho^{n+1} - \rho^n, u^{n+1} - u^n) + (\rho_0, u_0),$$

there exists (ρ, u) in $\tilde{X}_1 \times \tilde{\mathbb{X}}_4$ such that (ρ^n, u^n) converges to (ρ, u) in $\tilde{X}_1 \times \tilde{\mathbb{X}}_4$. The bound of $\{g_n\}_{n \in \mathbb{N}}$ in $\tilde{\mathbb{Y}}_4$ gives the existence of its weak limit g in $(L^2(0, T; H_0^1(\hat{\mathcal{O}})))^d$. $\square$

6.2. Proof of Theorem 1.1

Finally, we obtain the null control of (1.6).

Theorem 6.6. *Let us assume that $T > T_0$. There exists a positive constant $\hat{\kappa}$ depending only on $T, \Omega, \hat{\mathcal{O}}, c_0, \mu, \mu_0$ such that for any $(\rho_0, u_0) \in \dot{H}_{per}^2(\Omega) \times (H_{per}^2(\Omega))^d$ satisfying*

$$\|(\rho_0, u_0)\|_{\dot{H}_{per}^2(\Omega) \times H_{per}^2(\Omega)} \leq \hat{\kappa}, \quad (6.15)$$

system (1.6) admits a solution (ρ, u) in $C([0, T]; \dot{H}_{per}^2(\Omega)) \times (L^2(0, T; H_{per}^3(\Omega)) \cap H^1(0, T; H_{per}^1(\Omega)))^d$ with control $g \in (L^2(0, T; H_0^1(\hat{\mathcal{O}})))^d$, having support inside $\hat{\mathcal{O}} \subset \Omega$ and this solution satisfies

$$(\rho(\cdot, T), u(\cdot, T)) = (0, 0).$$

Proof. We choose κ_0, κ and ι as in the argument above and pick $\hat{\kappa} = \iota \kappa$. The boundedness of all the iterates in $X_1 \times \mathbb{X}_4$ implies that the weak limit is in $L^\infty(0, T; \dot{H}_{per}^2(\Omega)) \times (L^2(0, T; H_{per}^3(\Omega)) \cap H^1(0, T; H_{per}^1(\Omega)))^d$ and the corresponding control is in $(L^2(0, T; H_0^1(\hat{\mathcal{O}})))^d$. The limit (ρ, u) in $\tilde{X}_1 \times \tilde{\mathbb{X}}_4$ obtained in Proposition 6.5 is in fact in $L^\infty(0, T; \dot{H}_{per}^1(\Omega)) \times (L^2(0, T; H_{per}^2(\Omega)))^d$. Hence (ρ, u) has to coincide with the above weak limit and thus belongs to $L^\infty(0, T; \dot{H}_{per}^2(\Omega)) \times (L^2(0, T; H_{per}^3(\Omega)) \cap H^1(0, T; H_{per}^1(\Omega)))^d$. It only remains to show that (ρ, u) is a solution to the differential equations with control g . This requires passing to the limit in the nonlinear terms. But this follows by using the same estimate for $G(\rho, \nabla \rho, u, \nabla u, u_t) - G(\rho^n, \nabla \rho^n, u^n, \nabla u^n, u_t^n)$ as were used for $G^1 - G^2$ in Lemma 6.3. Finally, the continuity of the density follows from the differential equation in a standard fashion. $\square$

Theorem 1.1 follows from the preceding theorem by a change of variables.

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