A note on: “A Gaussian-product stochastic Gent–McWilliams parameterization”.

Malte Jansen

December 6, 2016

Abstract

This note builds on a recent article by Grooms (2016), which introduces a new stochastic parameterization for eddy buoyancy fluxes. The closure proposed by Grooms accounts for the fact that eddy fluxes arise as the product of two approximately Gaussian variables, which in turn leads to a distinctly non-Gaussian distribution. The directionality of the stochastic eddy fluxes, however, remains somewhat ad-hoc and depends on the reference frame of the chosen coordinate system. This note presents a modification of the approach proposed by Grooms, which eliminates this shortcoming. Eddy fluxes are computed based on a stochastic mixing length model, which leads to a frame invariant formulation. As in the original closure proposed by Grooms, eddy fluxes are proportional to the product of two Gaussian variables, and the parameterization reduces to the Gent and McWilliams parameterization for the mean buoyancy fluxes.

Introduction

Grooms (2016) (hereinafter: G16) proposes a Gaussian-product stochastic GM parameterization (Gent and McWilliams, 1990), to account for the chaotic nature of eddy fluxes.

Previous stochastic GM parameterizations made use of the stochastically perturbed parameterization tendency (SPPT) scheme (Buizza et al., 1999; Andrejczuk et al., 2016) where the eddy fluxes (as calculated by a deterministic parameterization) are simply multiplied by Gaussian noise. This approach has at least two major shortcomings: 1) it does not take into account the non-Gaussian nature of eddy fluxes, and 2) it does not adequately account for physical constraints on the direction of quasi-random eddy fluxes. G16 notes both of these shortcomings, and suggests an improved stochastic GM parameterization. The proposed closure accounts for the fact that eddy fluxes are the result of a product of two approximately Gaussian variables, which in turn leads to a distinctly non-Gaussian distribution. The parameterization proposed by G16 also implies a relationship between zonal and meridional eddy buoyancy fluxes, noting that both are linked to an approximately non-divergent flow field. I will argue, however, that the directionality of stochastic eddy fluxes in the parameterization of G16 remains somewhat unphysical, and that a relatively simple modification of the approach leads to a more physically consistent expression.

G16 computes horizontal buoyancy fluxes¹ as

$$F_b^x = \tilde{u}'\tilde{b}'M^{-1}\partial_x\bar{b}, \quad F_b^y = \tilde{v}'\tilde{b}'M^{-1}\partial_y\bar{b}, \quad (1)$$

where $\tilde{b}'$, $\tilde{u}'$, and $\tilde{v}'$ are stochastically generated pseudo-buoyancy and velocity perturbation fields, and M is a globally constant normalization factor with the same units as a buoyancy gradient. As acknowledged in G16, $\tilde{b}'$, $\tilde{u}'$, and $\tilde{v}'$ cannot be interpreted as the actual buoyancy and velocity perturbations, with which $F_b^x = u'b'$ and $F_b^y = v'b'$. As a result, the physical interpretation of Eq. (1) remains somewhat unclear.

Notably, the approach in Eq. (1) is not frame invariant, as can readily be seen by expressing the flux in (1) in terms of a “diffusivity” tensor, which relates the parameterized

¹G16 formulates the closure for density rather than buoyancy. Using the Boussinesq approximation, the two formulations are exactly analogous.

40 eddy fluxes to the large-scale mean field:

$$\mathbf{F}_b = M^{-1} \begin{pmatrix} \tilde{u}'\tilde{b}' & 0 \\ 0 & \tilde{v}'\tilde{b}' \end{pmatrix} \nabla \bar{b}, \quad (2)$$

41 Eq. (2) shows that the major and minor axes of the stochastically generated diffusivity
 42 ellipses are tied to the axes of the coordinate system, which breaks frame-invariance.
 43 Frame invariance is desirable on physical grounds, as well as for the grid-independent
 44 formulation of numerical models (e.g. Griffies, 2004). Below I argue that a frame-invariant
 45 closure that is otherwise very similar to that of G16 can readily be derived based on a
 46 stochastic mixing length argument.

47 **A modified Gaussian-product stochastic eddy parameterization**

48 Rather than considering the buoyancy flux specifically, we start by assuming a general
 49 conservative tracer, c , advected by the horizontal flow. The goal is to find a closure for
 50 the horizontal eddy flux of c :

$$F_c^x = u'c', \quad F_c^y = v'c' \quad (3)$$

51 Motivated by mixing length theory (Taylor, 1921; Prandtl, 1925; Vallis, 2006), we can
 52 relate tracer perturbations to the background gradient as

$$c' \approx -\boldsymbol{\eta}' \cdot \nabla \bar{c}, \quad (4)$$

53 where $\boldsymbol{\eta}'$ denotes a (two-dimensional) fluid parcel displacement, and $\nabla = (\partial_x, \partial_y)$ is
 54 the horizontal Nabla operator. An accurate calculation of the displacement requires an
 55 integration of the Lagrangian particle velocity. Instead, we here want to relate $\boldsymbol{\eta}'$ to the
 56 local Eulerian velocity using

$$\boldsymbol{\eta}' = \mathbf{v}'\tau + \tilde{\boldsymbol{\eta}}, \quad (5)$$

where $\mathbf{v}' = (u', v')$ is the Eulerian eddy velocity, τ denotes the Lagrangian decorrelation time-scale and $\tilde{\boldsymbol{\eta}}$ denotes a residual displacement, which is not correlated with the local Eulerian velocity.

Combining Eqs. (3), (4) and (5) yields

$$\mathbf{F}_c = -\tau(\mathbf{v}'^\top \mathbf{v}') \nabla c - (\mathbf{v}'^\top \tilde{\boldsymbol{\eta}}) \nabla c, \quad (6)$$

where $\top$ denotes the transpose. The matrix in the first term on the R.H.S. is symmetric and positive semi-definite, giving a down-gradient, “diffusive”, flux in the direction of the eddy velocity. The second term represents a random flux component with zero mean. Assuming that u' and v' are not correlated and have identical variance, V^2 , (as expected if both are computed from a random isotropic streamfunction field), we arrive at the well known diffusive closure for the mean flux:

$$\overline{\mathbf{F}}_c = -\tau V^2 \nabla \bar{c}. \quad (7)$$

However, moving beyond the mean, Eq. (6), can be used to derive a more general stochastic eddy flux closure. With $u' = V\hat{u}$, $v' = V\hat{v}$, $\tilde{\eta}_x = L_{rand}\hat{\eta}_x$, $\tilde{\eta}_y = L_{rand}\hat{\eta}_y$, $K = V^2\tau$, and $\alpha = L_{rand}/(V\tau)$, Eq. (6) can be written as

$$\mathbf{F}_c = -K\hat{\mathbf{K}}\nabla c \quad \text{with} \quad \hat{\mathbf{K}} = [\hat{\mathbf{v}}^\top \hat{\mathbf{v}} + \alpha \hat{\mathbf{v}}^\top \hat{\boldsymbol{\eta}}]. \quad (8)$$

$\hat{\mathbf{v}}$ can be represented as a normal process consistent with a non-divergent flow field, as discussed in G16, whereas $\hat{\eta}_x$ and $\hat{\eta}_y$ may be approximated as independent normal processes. Spatial and temporal auto-correlations of $\hat{\mathbf{v}}$ and $\hat{\boldsymbol{\eta}}$ can be implemented similarly as in G16. It shall be noted that the anti-symmetric component of $\hat{\mathbf{v}}^\top \hat{\boldsymbol{\eta}}$ does not contribute to the flux convergence and may thus be eliminated for implementation in numerical

models.

Analogous to G16 we can apply Eq. (8) to the horizontal (or more strictly: isopycnal) buoyancy flux to obtain a stochastic GM parameterization. The eddy bolus streamfunction can then be calculated from the horizontal eddy buoyancy flux using Eq. (7) (or Eqs. 11 and 12) in G16. A 2-dimensional mixing length argument for the horizontal buoyancy flux component is not strictly justifiable as it neglects vertical buoyancy advection, but it remains a useful description of the horizontal stirring effect of the largest and most barotropic eddies (e.g. Larichev and Held, 1995). Moreover, the approach reduces to a simple stochastic modification of the successful GM parameterization, with the modified eddy transport

$$\mathcal{T}_{stoch} = \hat{\mathbf{K}} \mathcal{T}_{GM}, \quad (9)$$

where $\mathcal{T}_{GM} = -K(\partial_z b)^{-1}(\partial_x b, \partial_y b)$ is the deterministic GM eddy transport streamfunction and $\mathcal{T}_{stoch} = (\partial_z b)^{-1} \mathbf{F}_b$ its stochastic generalization.²

Discussion

Eq. (9) resembles the closure of G16 in that the GM transport is multiplied by a Gaussian-product noise. However, the transport is here multiplied by a stochastic 2x2 tensor, while the result in G16 amounts to multiplication by a diagonal matrix. The multiplication by a (generally non-diagonal) 2x2 tensor arguably makes the revised approach slightly more complicated than the one proposed in G16, but it is exactly this property that makes the revised parameterization frame-invariant³. Computation of $\hat{\mathbf{K}}$ also requires

²We are here using the two dimensional eddy transport streamfunction, $\mathcal{T}_{GM}$, defined such that the eddy transport velocity is given as $(u^\dagger, v^\dagger, w^\dagger) = (-\partial_z \mathcal{T}_x, -\partial_z \mathcal{T}_y, \nabla \cdot \mathcal{T})$, which is consistent with the formulation typically used in ocean circulation models. $\mathcal{T}$ is related to the three dimensional vector streamfunction in G16 as $\Psi = (-\mathcal{T}_y, \mathcal{T}_x, 0)$. Notice also that Eq. (9) holds similarly if the eddy transport streamfunction is defined using the boundary-value problem approach of Ferrari et al. (2010) (Eqs. 8/9 and 11/12 in G16), as long as $\hat{\mathbf{K}}$ is chosen to be vertically constant (see also Discussion below).

³The only frame-invariant alternative would be multiplication by a random scalar, as in SPPT. However, this does not adequately account for the fact that zonal and meridional eddy fluxes are related via

one more random variable than the approach of G16, as the random component of the buoyancy anomaly is here the result of zonal and meridional displacements. However, the additional computational overhead is likely to be small and justified by the improved physical properties.

The approach discussed above can also be applied for along-isopycnal mixing of temperature, salinity and chemical tracers. Eq. (8) is then to be applied for the horizontal fluxes in an isopycnal coordinate system. In a z-coordinate model, this can be achieved by a suitable coordinate rotation, as discussed by Redi (1982).

In general the eddy velocity magnitude, V , and thus K may be chosen to have a vertical structure, while for computational simplicity we may want to keep $\hat{K}$ vertically constant (as in G16). Physically, this amounts to assuming an equivalent barotropic eddy velocity field, which is a reasonable approximation at least in the most energetic parts of the ocean, such as the Antarctic Circumpolar Current (Killworth, 1992). A more realistic three-dimensional closure for eddy buoyancy fluxes may take into account that the vertical structure of the meso-scale eddy velocity field is related to the eddy buoyancy field via the thermal wind relation. Incorporating a more sophisticated vertical structure for eddy fluxes by accounting for the thermal wind relationship or steering-level effects (e.g. Green, 1970; Ferrari and Nikurashin, 2010) represents a promising direction for further research, but is beyond the scope of this note. A fully three-dimensional stochastic model is also stated as a future goal in G16. The limitations of a quasi-two-dimensional closure notwithstanding, Eqs. (8) and (9), together with the recipes to generate the required random fields discussed in G16, provide a promising closure that can readily be implemented in ocean models as a stochastic modification of the existing GM and Redi (1982) schemes.

the eddy velocity field.

Acknowledgements

I would like to thank Ian Grooms for productive conversations and helpful comments. I also appreciate helpful comments from an anonymous reviewer. Funding for this work was provided by the National Science Foundation under award OCE-1536450.

References

- Andrejczuk, M., Cooper, F., Juricke, S., Palmer, T., Weisheimer, A., Zanna, L., 2016. Oceanic stochastic parameterizations in a seasonal forecast system. *Monthly Weather Review* 144, 1867–1875.
- Buizza, R., Milleer, M., Palmer, T., 1999. Stochastic representation of model uncertainties in the ecmwf ensemble prediction system. *Quarterly Journal of the Royal Meteorological Society* 125, 2887–2908.
- Ferrari, R., Griffies, S.M., Nurser, A.G., Vallis, G.K., 2010. A boundary-value problem for the parameterized mesoscale eddy transport. *Ocean Modelling* 32, 143–156.
- Ferrari, R., Nikurashin, M., 2010. Suppression of eddy mixing across jets in the southern ocean. *J. Phys. Oceanogr.* 40, 1501–1519.
- Gent, P.R., McWilliams, J.C., 1990. Isopycnal mixing in ocean circulation models. *J. Phys. Oceanogr.* 20, 150–155.
- Green, J.S.A., 1970. Transfer properties of the large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor. Soc.* 96, 157–185.
- Griffies, S.M., 2004. *Fundamentals of ocean climate models*. Princeton University Press.
- Grooms, I., 2016. A Gaussian-product stochastic Gent–McWilliams parameterization. *Ocean Modelling* 106, 27–43.

- 139 Killworth, P.D., 1992. An equivalent-barotropic mode in the fine resolution antarctic
140 model. *J. Phys. Oceanogr.* 22, 1379–1387.
- 141 Larichev, V.D., Held, I.M., 1995. Eddy amplitudes and fluxes in a homogeneous model
142 of fully developed baroclinic instability. *J. Phys. Oceanogr.* 25, 2285–2297.
- 143 Prandtl, L., 1925. Bericht über untersuchungen zur ausgebildeten turbulenz. *Z. Angew.*
144 *Math. Mech* 5, 136–139.
- 145 Redi, M., 1982. Oceanic isopycnal mixing by coordinate rotation. *J. Phys. Oceanogr.* 12,
146 1154–1158.
- 147 Taylor, G.I., 1921. Diffusion by continuous movements. *Proc. London Math. Soc* 20,
148 196–211.
- 149 Vallis, G.K., 2006. *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University
150 Press, Cambridge, U.K.