A terrain-following model of wave boundary layers

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Abstract

Over a variable seabed, conventional boundary layer approximations are rendered to be inadequate because of the large variations in bed elevation in the direction of wave propagation. Applying the method of conformal transformation to map the flow domain with a corrugated boundary onto a uniform strip, we put forward a terrain-following modeling approach for Stokes boundary layer flows, accompanying the recent development of the exact Floquet theory of water waves over a generally periodic seabed. For a non-steep seabed profile, but not necessarily small undulation height compared with the water depth, we solve the vorticity equation to obtain the analytical solutions for the boundary layer velocities, bed shear stress and rate of viscous dissipation, explicitly showing the variations both across the boundary layer and along the bed. For a relatively steep bed profile, a remedy is proposed that allows the velocity profiles to be locally determined across the boundary layer avoiding solving the 2-D differential equation for the vorticity. The modeling methodology is presented here for a constant viscosity, including the case of constant eddy viscosity, but can be extended to the case of variable eddy viscosity to improve turbulence modeling.

Keywords: Wave boundary layer, terrain-following, conformal map, periodic seabed

1. Introduction

Wave boundary layers at the seabed, also known as the Stokes boundary layers for oscillatory flows, are typically very thin compared with the water depth,

4 ranging from a few millimeters to centimeters depending on the bed and flow
 5 conditions (Mei 1989, Fredsøe & Deigaard 1992, Nielsen 1992). The flow struc-
 6 ture in this thin layer, however, is of great importance since it determines the
 7 bottom shear stress and near-bed turbulent intensity, hence having significant
 8 influence on sediment transport, physical, chemical and biological processes near
 9 the seabed. The wave boundary layer can also affect the vertical structure of a
 10 co-existing current in shallow coastal waters (Fredsøe & Deigaard 1992). The
 11 fluctuating (in space and/or in time) bed stresses and pressure due to wave mo-
 12 tion can induce, and interact with, groundwater flows (Mei 1989, Belibassakis
 13 2012).

14 It is generally accepted in the literature that outside the wave boundary
 15 layer the potential flow field of wave propagation applies, which by itself is a
 16 mathematically challenging problem for generally varying topographies, even
 17 assuming the linear dynamics (Rhine & Bretherton 1973, Athanassoulis & Be-
 18 libassakis 1999). Various approximate theories have been developed, manip-
 19 ulating the geometric constraints of seabed profiles, e.g. assuming a gentle
 20 slope, small bottom amplitude, or localized abrupt variation such as a step (see
 21 Athanassoulis & Belibassakis 1999, and the references within). When the water
 22 wavelength, depth and the scale of bed variations are comparable, numerical
 23 or semi-numerical methods are generally sought. The computational costs can
 24 be significant even with the restriction to linear waves, depending on the com-
 25 plexity of seabed geometry. This has motivated studies to explore the methods
 26 for improvements. For instance, Athanassoulis & Belibassakis (1999) developed
 27 the coupled-mode theory to amend the shortcoming in satisfying the bottom
 28 boundary conditions in numerical representations, thereby accelerating the nu-
 29 merical convergence. These authors also extended the coupled-mode method
 30 to treat oscillatory boundary layer effects over a generally variable bathymetry
 31 (Belibassakis & Athanassoulis 2008). Porter & Porter (2003) formulated a trans-
 32 fer matrix method for wave scattering by a patch of small-amplitude periodic
 33 bed, significantly reducing the computation. Most recently, Fokas & Nach-
 34 bin (2012) proposed a set of non-local equations in the propagation space for

35 weakly nonlinear shallow water waves over a variable bottom using a numerical
36 Schwarz-Christoffel mapping.

37 For a periodic seabed of arbitrary amplitude and shape, the recent advance
38 in the Floquet theory of linear water waves (Yu & Howard 2012) has provided
39 a new and general approach that can be regarded as the definitive solution to
40 this type of problems. What distinguishes this theory the most is not that it
41 gives exact solutions but that for any frequency it gives a complete set of linear
42 modes. For 2-D time harmonics motions over a horizontal flat seabed, it is well
43 known that the general mathematical treatment provides a complete basis of
44 solutions: Given a frequency, it consists of two oppositely directed propagating
45 waves and two infinite families of evanescent waves (with rapidly exponentially
46 growing wave amplitudes in opposite directions). This set of flat-bottom linear
47 modes has, for many years, been the only known complete basis for water waves
48 and played a significant role in various engineering and scientific applications.
49 For a periodic seabed, the set of Floquet modes provided in Yu & Howard (2012)
50 is analogous to the above mentioned set of flat-bottom modes and in fact ap-
51 proaches the latter for the same given frequency when the bed undulation height
52 is reduced to zero. Therefore, it becomes the second complete basis for water
53 waves. It has been demonstrated that this set of Floquet modes can be used as
54 a complete basis to construct solutions to boundary-value problems involving
55 a finite extent of periodic bed, similar to the way that the flat-bottom basis is
56 used (Howard & Yu 2007, Yu & Howard 2010, Yu & Zheng 2012, Weidman *et*
57 *al.* 2015).

58 This study is to provide the accompanying boundary layer formulation for
59 linear waves over a periodic bed. Although the nonlinear effects become strong
60 as the wave approaches the shoreline, complete and explicit linear solutions are
61 still very valuable in engineering applications and studies of coastal environ-
62 ments and sustainability. In most practical situations, the wave boundary layer
63 is thin for non-breaking waves and without a resulting rectified current. Even
64 in the case of a turbulent condition, the turbulence is confined to the thin layer
65 just above the bed (Jonsson 1966, Johns 1967, Fredsøe & Deigaard 1992). The

66 flow closely follows the seabed terrain, meaning that the Stokes boundary layer
 67 should be measured everywhere in the direction normal to the bed, not in the
 68 vertical z direction. However, the conventional boundary layer approximation
 69 in the vertical (x, z) plane assumes $\partial/\partial z \gg \partial/\partial x$. This becomes inappropriate
 70 when the seabed elevation varies at the scale comparable to the water depth.
 71 The conformal transformation developed in Yu & Howard (2012) maps the flow
 72 domain over a periodic seabed onto a uniform strip with a flat bottom; see figure
 73 1. Since the transformation preserves angles, a vertical distance just above the
 74 flat bottom in the mapped plane (ξ, η) corresponds to a normal distance mea-
 75 sured from the bed surface in the physical (x, z) plane. Thus, a boundary layer
 76 approach formulated in the mapped plane is terrain-following, and the approx-
 77 imations would be made by comparing the derivatives normal and tangent to
 78 the bed surface, which respectively measure the variations across the boundary
 79 layer and along the bed.

80 In this study, we assume a constant mean water depth over the general peri-
 81 odic seabed. For many practical applications at scales of engineering interests,
 82 this assumption is appropriate since typical beach slopes are gentle (of a few
 83 degrees) and the mean water depths do not change significantly. In real coastal
 84 regions, a long stretch of periodic (or nearly periodic) seabed is uncommon, but
 85 patches of bed undulations of different shapes can occur over a long distance on
 86 a shoaling beach. The terrain-following boundary layer approach presented here
 87 can be applied locally, considering a finite stretch of periodic seabed. The cor-
 88 responding potential wave field in such a finite domain can readily be obtained,
 89 following the procedure in Yu & Zheng (2012). Here, for the simplicity and clear
 90 illustration, we shall only consider a free wave (a Floquet propagating mode)
 91 over an indefinitely long periodic bed. For long range propagation of waves,
 92 the attenuation of wave energy occurs due to the boundary layer effects. The
 93 mathematical tools exist in the literature to deal with long range wave propa-
 94 gation, in particular over a flat or mildly varying seabed (see Mei 1989 and the
 95 references within), and in principle can be adapted to analyze the coupling ef-
 96 fects of wave boundary layer and wave amplitude attenuation in the conformally

mapped plane. These applications are of scientific and engineering interest, but beyond the scope of the present work and will be explored elsewhere.

Real seabed topographies are in general two-dimensional, but approximately one-dimensional periodic bed features are common (Mei 1989, Komar 1998, Pietrzak, Kranenburg & Abraham 1990, Elgar, Raubenheimer & Herbers 2003). The theoretical approach presented here can be applied to obtain an accurate first order approximation in those situations. On the other hand, periodic seabeds are often used in laboratory to study wave-topography interactions. The theoretical predictions of the potential wave field and wave boundary layer over a periodic bed are desirable in planning and designing experiments, as well as analyzing data.

Whereas the fully nonlinear and terrain-following wave boundary layer dynamics is described by the vorticity equation in the mapped plane, cf. (3.10), the linearized equation for the Stokes boundary layer (3.13) is based on the assumption of small amplitude waves of the outer potential flows. Neither of these equations makes assumption on the steepness of seabed undulations, nor on the relative importance of the normal and tangential derivatives. They can be solved numerically if needed, matching with the outer potential wave field that can be provided by the Floquet theory (Yu & Howard 2012) if linear waves are considered or by other wave theories deemed to be appropriate. However, for non-steep topographies (but not necessarily small undulation height compared with the water depth), we can further simplify (3.13), obtaining explicit analytical solutions of the boundary layer velocity distribution and bed shear stress. This is the main focus of this study.

The rest of the paper is structured, as follows. In §2, the Floquet theory of waves over a periodic seabed (Yu & Howard 2012) is summarized, where the conformal transformation is described. The terrain-following modeling of boundary layer flows is formulated in §3. For a non-steep but large bed undulation compared with the water depth, the analytical solutions are given in §3.1. Degradation of the solutions as the seabed becomes steep is discussed and, a simple remedy is proposed in §3.2. Summarizing remarks follow in §4.

128 2. The exact Floquet theory of waves over a periodic seabed

129 Deferring the technical details to Yu & Howard (2012), we here summarize
 130 the Floquet theory of water waves over a generally periodic seabed for the
 131 purpose of providing the potential flow field outside the boundary layer and
 132 introducing the conformal map that is needed in §3 to formulate the terrain-
 133 following boundary layer approach.

134 2.1. The conformal map

135 Consider a layer of homogeneous fluid over a periodic seabed. Denoting the
 136 dimensional variables by the prime, in the vertical (x', z') plane $z' = 0$ is the
 137 undisturbed free surface and the seabed is at $z' = -h'_0 + h'_b(x')$, where $h'_b(x')$
 138 has a spatial period λ_{bed} and zero average in x' so that h'_0 represents the mean
 139 water depth. The bed elevation varies over a height of $\Delta h'_b = h'_{b,max} - h'_{b,min}$.
 140 A wavenumber

$$k_B = \pi / \lambda_{bed} \quad (2.1)$$

141 is defined which corresponds to a wavelength $2\lambda_{bed}$ and is used for normalization,
 142 as follows.

$$x = k_B x', \quad z = k_B z', \quad h_b(x) = k_B h'_b(x'), \quad h_0 = k_B h'_0. \quad (2.2)$$

143 It is clear that $\Delta h_b \equiv h_{b,max} - h_{b,min} = k_B \Delta h'_b$ is a measure of the steepness
 144 of a seabed profile. The normalized seabed wavelength is π , which sets the
 145 intrinsic periodicity of the medium in which waves propagate. This is for the
 146 mathematical convenience to employ the Floquet theory, following the common
 147 practice. The conformal map between the (x, z) and (ξ, η) plane is given by (Yu
 148 & Howard 2012)

$$x = \xi - h \sum_{j=1}^{\infty} (b_j \sin 2j\xi - c_j \cos 2j\xi) \cosh 2j\eta / \sinh 2jh \quad (2.3a)$$

$$z = \eta - h \sum_{j=1}^{\infty} (b_j \cos 2j\xi + c_j \sin 2j\xi) \sinh 2j\eta / \sinh 2jh. \quad (2.3b)$$

149 It transforms the flow domain $-h_0 + h_b(x) \leq z \leq 0$ into a uniform strip $-h \leq$
 150 $\eta \leq 0$; see figure 1. The fluid depth h in the mapped plane (ξ, η) and the Fourier
 151 coefficients b_j, c_j are implicitly determined according to

$$-h_0 + h_b(x) = -h + h \sum_{j=1}^{\infty} (b_j \cos 2j\xi + c_j \sin 2j\xi), \quad (2.4)$$

where

$$x = \xi - h \sum_{j=1}^{\infty} (b_j \sin 2j\xi - c_j \cos 2j\xi) \coth(2jh) \quad (2.5)$$

and

$$h_0 = h + h^2 \sum_{j=1}^{\infty} j (b_j^2 + c_j^2) \coth(2jh). \quad (2.6)$$

152 The constraint (2.6) results from $h_b(x)$ having zero average in x . The map
 153 (2.3) is π -periodic in both x and ξ , therefore preserves the intrinsic periodicity
 154 π of the problem. The Cauchy-Riemann conditions, $x_\xi = z_\eta$ and $x_\eta = -z_\xi$,
 155 are readily verified from (2.3). Under the transformation, the Cartesian grids
 156 $\xi = \text{constant}$ and $\eta = \text{constant}$ are mapped onto the orthogonal curvilinear
 157 grids in the (x, z) plane, which are terrain-following near the seabed; see figure
 158 1(c). An iteration method using FFT to compute the map parameters b_j, c_j
 159 and h (collectively denoted as $\mathcal{B}$) from (2.4)–(2.6) is detailed in Yu & Howard
 160 (2012), in which the convergence of the series is also discussed. From this point
 161 on, for a given profile $h_b(x)$ we consider $\mathcal{B}$ to be known.

162 2.2. The Floquet solutions for time-harmonic flows

163 In the mapped plane, the Laplace equation for the velocity potential ϕ is
 164 solved, exactly satisfying the conformally transformed boundary conditions at
 165 the free surface and the seabed. Using the normalization

$$t = t' \sqrt{gk_B}, \quad \phi = \phi' (a \sqrt{gk_B} / k_B), \quad (2.7)$$

where a characterises the surface wave amplitude and g is the acceleration gravity, we write the solution

$$\phi = e^{\mu\xi} \sum_{n=-\infty}^{\infty} D_n e^{in\xi} \frac{\cosh[(n - i\mu)(\eta + h)]}{\cosh[(n - i\mu)h]} e^{-i\sigma t} + \text{c.c.}, \quad (2.8)$$

where the Floquet exponent μ is the generalized wavenumber for the given (normalized) frequency $\sigma = \omega/\sqrt{gk_B}$. Given a seabed, μ and σ must satisfy the dispersion relationship $\Delta(\sigma, \mu; \mathcal{B}) = 0$, where the function Δ is expressed as the determinant of a matrix which arises from satisfying the free surface boundary conditions. The Fourier coefficients D_n are then given by the vector in the null space of the matrix. The entries of the matrix are explicit analytical functions of μ , σ and $\mathcal{B}$. Thus, the determinant Δ is an analytical function (similar to the Hill's infinite determinants) despite its complex appearance. The behavior of Δ and the convergence of the Fourier series in (2.8) are discussed in Yu & Howard (2012).

Given a frequency σ , a set of Floquet exponents (occurring in $\pm\mu$ pairs) can be found to satisfy $\Delta(\sigma, \mu; \mathcal{B}) = 0$. Correspondingly, a set of solutions can be identified from (2.8): Two are referred to as the Floquet wave modes, which approach the flat-bottom propagating waves (in $\pm x$ directions) of the same frequency when the bed undulation height $\Delta h_b \rightarrow 0$ (or approach the flat-bottom standing waves if the frequency is in a ‘forbidden band’, i.e. Bragg resonant). The two infinite families of Floquet evanescent modes are given by a sequence of large real $\pm\mu$, having the wave amplitudes grow rapidly exponentially in the $+x$ or $-x$ direction. Each member of the families approaches a flat-bottom evanescent wave as $\Delta h_b \rightarrow 0$. For a particular problem involving a finite extent of periodic bed, the solution can be expressed as the superposition of this set of Floquet modes, with the proportionality coefficients being determined by satisfying the lateral boundary conditions in the x direction (Yu & Zheng 2012, Weidman *et al.* 2015).

Over a flat bed, waves with an arbitrary wavelength (i.e. the horizontal distance between two adjacent wave crests or troughs) are possible, since the

length scale of the medium is indefinite. This is manifested by the one-to-one relationship between the frequency ω and wavenumber k (or $-k$). For a periodic seabed, however, the spatial periodicity of the bed sets an intrinsic length scale for the wave motions. Within the realm of linear dynamics, exactly spatially periodic waves with identical adjacent wave crests and troughs can only exist when the wavelength is an integer multiple of λ_{bed} . These occur for the frequencies that correspond to $\pm\mu = i(1 - 2/m)$, giving the water wavelengths $\lambda = m\lambda_{bed}$, where the integer $m \geq 2$; see figure 2(a). For other frequencies, $\lambda \neq m\lambda_{bed}$, hence the two adjacent wave crests (or troughs) cannot be identical as they see different parts of the bed profile at any instance. As a result, the wave must modulate, slowly varying in space; see figure 2(b). These are the quasi-spatially periodic waves (similar to the Bloch waves in physics) that can freely propagate in an open domain of continuous periodic bed. Bragg resonant waves occur when $\lambda \simeq (2/m')\lambda_{bed}$, $m' = 1, 2, \dots$, where the wave amplitudes vary in space exponentially but slowly since μ is real and small. (It is the slow variation that distinguishes the resonant waves from the evanescent waves.) Obviously, these resonant waves can only exist over a finite extent of periodic bed, where the lateral boundary conditions limit the spatial growth (or decay) of the wave amplitude. This was discussed in Yu & Maas (2016) for internal waves in a two-layer fluid system, and is stressed here for free surface waves.

For time-harmonic wave motions over a large periodic seabed, since the spatial variation is not sinusoidal (even in the case of exactly spatially periodic waves), the propagation of wave is no longer seen as the simple translation of waveform in the space-time plane. The waveform changes as the wave propagates; see figure 2. Such modulations, which would have arisen from nonlinear effects in the case of wave propagation over a flat bottom, are here solely due to linear waves interacting with the seabed geometry.

221 **3. A terrain-following boundary layer approach**

222 For 2-D incompressible fluid motions, the stream function $\psi(x, z, t)$ is so
 223 defined that the velocity vector

$$(u, w) = (\psi_z, -\psi_x), \quad (3.1)$$

224 where the x, z subscripts (as well as the ξ, η subscripts below) are the stan-
 225 dard notations for partial derivatives. The stream function is normalized by
 226 $a\sqrt{gk_B}/k_B$, same as for ϕ . The vorticity $\nabla \times \mathbf{u} = \nabla^2 \psi \mathbf{j}$ is a pseudo vector per-
 227 pendicular to the (x, z) plane. In the (ξ, η) plane, the exact vorticity equation
 228 is written as

$$\frac{\partial}{\partial t} \nabla^2 \psi - \alpha \frac{\partial (\psi, J^{-1} \nabla^2 \psi)}{\partial (\xi, \eta)} = \frac{1}{2} \delta^2 \sigma \nabla^2 (J^{-1} \nabla^2 \psi), \quad (3.2)$$

229 where the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \quad (3.3)$$

230 and the Jacobian of transformation

$$J(\xi, \eta) = x_\xi^2 + x_\eta^2 \neq 0. \quad (3.4)$$

231 The parameters are

$$\alpha = ak_B, \quad \delta = k_B \delta', \quad (3.5)$$

232 where $\delta' = \sqrt{2\nu/\omega}$ is the usual definition of the Stokes layer thickness and ν is
 233 the viscosity. Under the transformation, the no-slip boundary conditions at the
 234 seabed become

$$\psi = \psi_\eta = 0 \quad \text{at} \quad \eta = -h. \quad (3.6)$$

235 Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit vectors along $\eta = \text{constant}$ and $\xi = \text{constant}$,
 236 pointing in the directions of increasing ξ and η , respectively. Since

$$\mathbf{e}_1 = J^{-1/2} (x_\xi \mathbf{i} + z_\xi \mathbf{k}), \quad \mathbf{e}_2 = J^{-1/2} (x_\eta \mathbf{i} + z_\eta \mathbf{k}), \quad (3.7)$$

237 the velocity vector (3.1) becomes

$$\begin{aligned} \mathbf{u} &= J^{-1} (x_\xi \psi_\eta - x_\eta \psi_\xi) \mathbf{i} - J^{-1} (z_\eta \psi_\xi - z_\xi \psi_\eta) \mathbf{k} \\ &= J^{-1/2} \psi_\eta \mathbf{e}_1 - J^{-1/2} \psi_\xi \mathbf{e}_2. \end{aligned} \quad (3.8)$$

238 Thus, we define the velocity components in the (ξ, η) plane

$$U \equiv J^{-1/2} \psi_\eta, \quad W \equiv -J^{-1/2} \psi_\xi, \quad (3.9)$$

239 so that the velocity vector is the same in both planes, i.e. $\mathbf{u} = u\mathbf{i} + w\mathbf{k} =$
 240 $U\mathbf{e}_1 + W\mathbf{e}_2$.

241 The oscillatory boundary layer above a seabed is characterized by the Stokes
 242 layer thickness δ' . The surface waves are affected by the bed when the wave-
 243 lengths are comparable to, or longer than, the bed wavelength, i.e. k/k_B is finite
 244 or small. For short waves, the bed becomes insignificant since the wave motions
 245 exponentially diminish downward away from the free surface. Thus, the viscos-
 246 ity parameter $\delta \ll 1$ effectively compares the Stokes layer thickness with the
 247 water wavelength. Let us introduce the boundary layer scaling $\tilde{\eta} = \delta^{-1}(\eta + h)$.
 248 Upon rescaling the stream function using δ' by setting $\tilde{\psi} = \delta^{-1}\psi$, we rewrite
 249 (3.2) as

$$\frac{\partial}{\partial t} (\tilde{\nabla}^2 \tilde{\psi}) - \alpha \frac{\partial (\tilde{\psi}, J^{-1} \tilde{\nabla}^2 \tilde{\psi})}{\partial (\xi, \tilde{\eta})} = \frac{1}{2} \sigma \tilde{\nabla}^2 (J^{-1} \tilde{\nabla}^2 \tilde{\psi}) \quad \text{for } \tilde{\eta} > 0, \quad (3.10)$$

where

$$\tilde{\nabla}^2 = \delta^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \tilde{\eta}^2} \quad (3.11)$$

250 is the distorted Laplacian operator. The boundary conditions in (3.6) become

$$\tilde{\psi} = \tilde{\psi}_{\tilde{\eta}} = 0 \quad \text{at} \quad \tilde{\eta} = 0. \quad (3.12)$$

251 For linear waves $\alpha \ll 1$, we obtain

$$\frac{\partial}{\partial t} \left(\tilde{\nabla}^2 \tilde{\psi} \right) = \frac{1}{2} \sigma \tilde{\nabla}^2 \left(J^{-1} \tilde{\nabla}^2 \tilde{\psi} \right) \quad \text{for} \quad \tilde{\eta} > 0. \quad (3.13)$$

252 The effects of seabed are brought in via the Jacobian of transformation $J(\xi, \eta)$.
 253 This equation, together with the boundary conditions in (3.12), describes the
 254 wave boundary layer flow following the bed surface. Given a periodic bed, it
 255 does not necessarily reduce to a 1-D differential equation (as in the conventional
 256 boundary layer approximations) since the bottom variation can be so rapid that
 257 $\delta \frac{\partial}{\partial \xi} \sim \frac{\partial}{\partial \eta}$.

258 3.1. The analytical solutions for a non-steep periodic seabed

259 For a non-steep seabed profile, the boundary layer flows can be sufficiently
 260 smooth so that $\delta \frac{\partial}{\partial \xi} \ll \frac{\partial}{\partial \eta}$. It then follows from (3.13) that

$$\frac{\partial}{\partial t} \tilde{\psi}_{\tilde{\eta}\tilde{\eta}} = \frac{1}{2} \sigma \frac{\partial^2}{\partial \tilde{\eta}^2} \left(J^{-1} \tilde{\psi}_{\tilde{\eta}\tilde{\eta}} \right) \quad \text{for} \quad \tilde{\eta} > 0. \quad (3.14)$$

261 Close to the bed for $\tilde{\eta} = O(1)$, $J = J_b(\xi) + O(\delta \tilde{\eta})$, where $J_b(\xi) \equiv J|_{\eta=-h}$. To
 262 the lowest order approximation, we can further simplify (3.14) into

$$\frac{\partial}{\partial t} \tilde{\psi}_{\tilde{\eta}\tilde{\eta}} = \frac{1}{2} \sigma J_b^{-1} \tilde{\psi}_{\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}} \quad \text{for} \quad \tilde{\eta} > 0. \quad (3.15)$$

263 Over a non-steep seabed, the Stokes boundary layer flow is essentially 1-D fol-
 264 lowing the bed terrain with only a parametrical dependence on the along-bed
 265 variable ξ . This is similarly found in the study of oscillatory boundary layer
 266 flows over sand ripples (Hara & Mei 1990). For time periodic motions, the so-
 267 lution that is bounded in $\tilde{\eta}$ and satisfies the boundary conditions in (3.12) is
 268 found to be

$$\tilde{\psi} = C(\xi) \left[\tilde{\eta} + \frac{1}{2} (1 + i) J_b^{-1/2} \left(e^{-(1-i) J_b^{1/2} \tilde{\eta}} - 1 \right) \right] e^{-i\sigma t} + \text{c.c.}, \quad (3.16)$$

269 where $C(\xi)$ is to be determined by matching with the outer (potential) flow.

270 As is discussed in §2, the linear potential flow above the seabed can be ob-
 271 tained using the appropriate superposition of the set of Floquet modes given by
 272 (2.8). For the illustration of completing the boundary layer solution (3.16), we
 273 suppose here that the potential flow in the inviscid core is given by a propagat-
 274 ing Floquet wave mode. (For such a freely propagating wave to exist over an
 275 indefinite extent of periodic bed, the frequency is necessarily away from the for-
 276 bidden bands.) For potential flows, ϕ and ψ are conjugate harmonic functions,
 277 satisfying the Cauchy-Riemann conditions $\phi_\xi = \psi_\eta$ and $\phi_\eta = -\psi_\xi$. Thus, the
 278 velocities outside the boundary layer are given by

$$U^{(o)} = J^{-1/2}\phi_\xi, \quad W^{(o)} = J^{-1/2}\phi_\eta. \quad (3.17)$$

279 Comparing (3.17) and (3.9) at the outer edge of the boundary layer, we get

$$\phi_\xi|_{\eta \rightarrow -h} = \tilde{\psi}_\eta|_{\tilde{\eta} \rightarrow \infty}, \quad \phi_\eta|_{\eta \rightarrow -h} = -\delta\tilde{\psi}_\xi|_{\tilde{\eta} \rightarrow \infty}. \quad (3.18)$$

280 It follows from (2.8) that at $\eta = -h + \delta\tilde{\eta}$

$$\phi_\xi = e^{\mu\xi} \sum_{n=-\infty}^{\infty} D_n e^{in\xi} \frac{i(n-i\mu)}{\cosh[(n-i\mu)h]} e^{-i\sigma t} + \text{c.c.} + O(\delta\tilde{\eta}), \quad (3.19a)$$

$$\phi_\eta = e^{\mu\xi} \sum_{n=-\infty}^{\infty} D_n e^{in\xi} \frac{(n-i\mu)^2 \delta\tilde{\eta}}{\cosh[(n-i\mu)h]} e^{-i\sigma t} + \text{c.c.} + O(\delta\tilde{\eta})^3. \quad (3.19b)$$

281 At the outer boundary, (3.16) gives

$$\tilde{\psi}_\eta \rightarrow C e^{-i\sigma t} + \text{c.c.} \quad (3.20a)$$

$$\tilde{\psi}_\xi \rightarrow \left[C'\tilde{\eta} - \frac{1}{2}(1+i)J_b^{-1/2} \left(C' - \frac{1}{2}CJ'_b/J_b \right) \right] e^{-i\sigma t} + \text{c.c.}, \quad (3.20b)$$

282 where C' and J'_b are, respectively, the derivatives of $C(\xi)$ and $J_b(\xi)$. Within
 283 the thin boundary layer, U and W are essentially tangent and normal to the
 284 seabed, respectively. Matching U at the outer boundary, cf. (3.19a) and (3.20b),

285 it yields

$$C(\xi) = e^{\mu\xi} \sum_{n=-\infty}^{\infty} D_n e^{in\xi} \frac{i(n - i\mu)}{\cosh[(n - i\mu)h]}. \quad (3.21)$$

286 By the scaling of ϕ , $C(\xi) \sim O(1)$. At the outer boundary,

$$W^{(o)} - W^{(i)} = \delta \frac{1}{4} (1 + i) (J'_b J_b^{-2} C - 2 J_b^{-1} C') e^{-i\sigma t} + \text{c.c.} \sim O(\delta). \quad (3.22)$$

287 Both tangential and normal velocities are matched up to the accuracy of $O(\delta)$ for
 288 non-steep seabeds, being consistent with the overall accuracy of the boundary
 289 layer approximation.

290 Writing $(U, W) = \frac{1}{2}(U_m, W_m)e^{-i\sigma t} + \text{c.c.}$, the amplitudes $|U_m|$ and $|W_m|$
 291 vary considerably over a bed period, whereas the instantaneous velocities vary
 292 at the scale of water wavelength (or wave modulation length). In figures 3(a)
 293 and 3(b), $|U_m|$ and $|W_m|$ are plotted varying in z at different locations $\xi = \xi_0$
 294 over a doubly sinusoidal bed, showing the structure of boundary layer velocities
 295 and matching with the outer velocities. Locally, the U velocity profile across
 296 the boundary layer is qualitatively similar to that over a flat bed, but there is
 297 a significant variability in the velocity shear over a bed period π , obviously due
 298 to the large variation of bed elevation. Since the velocity gradients determine
 299 the bed shear stresses, these spatial variabilities in the boundary layer flow
 300 can have significant influence on the near-bed processes of sediments, chemical
 301 species and nutrients, as well as the submarine groundwater flows.

302 It must be mentioned that a vertical line $\xi = \xi_0$ in the mapped plane in
 303 general corresponds to a curve in the (x, z) plane, which is perpendicular to the
 304 seabed and free surface. If the line $\xi = \xi_0$ is situated at a bed crest or trough
 305 (or other local elevation maximums or minimums), it corresponds to a vertical
 306 line $x = x_0$ in the (x, z) plane; see figure 1(c). This is due to the fact that a
 307 conformal map preserves angles while stretching lengths. Therefore, U and W
 308 along $\xi = \xi_0$ are, respectively, the velocity components normal and tangent to
 309 the corresponding vertical curve in the physical plane: Close to $z = -h_0 + h_b(x)$,

310 U is tangent and W is normal to the seabed, whereas close to the free surface,
 311 they become the u and w velocities, respectively.

312 *3.2. Approximation for a steep periodic seabed*

313 As the seabed becomes steep, the derivative J'_b can be large at some locations,
 314 resulting in an appreciable discrepancy $W^{(o)} - W^{(i)}$, cf. (3.22). To show an
 315 example, we take the doubly sinusoidal seabed in figure 3 and increase h_0 from
 316 $\pi/4$ to $2\pi/5$, keeping the ratio $\Delta h_b/h_0 = \frac{1}{2}$. This corresponds to shorten λ_{bed} by
 317 a factor of $5/8$, while fixing the corrugation height and mean water depth. The
 318 steepness of the bed profile is therefore increased from $\Delta h_b = \pi/8$ to $\pi/5$. For
 319 this steep bed profile, it is seen in figure 4 that J_b^{-1} , and $J'_b J_b^{-2}$ in particular,
 320 vary rapidly and in large magnitude over a spatial period π . Consequently, the
 321 normal velocity W cannot be matched properly at the outer boundary at some
 322 locations along the bed, even though the tangential velocity U calculated using
 323 the solution (3.16) remains satisfactory; see in figure 5 the velocity profiles at
 324 $\xi_0 = 9.08, 8.58$ and 7.88 .

325 This is of course due to the neglect of along-bed variations $\partial/\partial\xi$ in the
 326 vorticity equation (3.15), that are no longer small compared with the normal
 327 derivatives $\partial/\partial\eta$ across the Stokes layer due to the rapid changes in bed eleva-
 328 tion. While the ultimate fix is to solve the fully 2-D linear equation (3.13), a
 329 relatively simple remedy can be provided by solving (3.14). Although both are
 330 1-D differential equations, (3.14) retains the full evaluation of $J(\xi, \eta)$, hence ac-
 331 counts for the 2-D effects of bed topography to a greater degree than (3.15) does.
 332 The equation (3.14), the seabed boundary conditions (3.12) and the matching
 333 conditions at the outer boundary form a ‘two-point’ boundary value problem
 334 which allows $U^{(i)}$ and $W^{(i)}$ to be locally determined along $\xi = \xi_0$. This improves
 335 the boundary layer velocity $W^{(i)}$, allowing a transition to match with the outer
 336 potential flow, but does not significantly affect the velocity $U^{(i)}$; see figure 5. A
 337 shooting method for solving a boundary layer equation (Cebeci & Keller 1971)
 338 can be adapted to numerically integrate (3.14) along $\xi = \xi_0$. This is detailed in
 339 Appendix A.

340 The steepness of a seabed obviously also depends on its shape. For instance,
 341 for the same parameters $h_0 = 2\pi/5$ and $\Delta h_b = \frac{1}{2}h_0$, a simple sinusoidal profile
 342 is much smoother than the doubly sinusoidal profile in figure 5. It is seen in
 343 figure 6 that the analytical solution (3.16) can still satisfactorily provide the
 344 boundary layer velocities for the sinusoidal profile despite the large value of the
 345 steepness parameter Δh_b .

346 3.3. The bed shear stress and rate of viscous dissipation

347 Let $\hat{n} = (n_1, n_2)$ be the unit vector normal to a surface. Thus, the unit
 348 vector tangent to the surface is $\hat{s} = (-n_2, n_1)$. The stress vector on this surface
 349 is

$$\boldsymbol{\tau} = (n_1, n_2) \begin{pmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{pmatrix} = (\tau_{xx}n_1 + \tau_{zx}n_2, \tau_{xz}n_1 + \tau_{zz}n_2). \quad (3.23)$$

350 For stress tensors, $\tau_{xz} = \tau_{zx}$. The shear stress is the component tangent to the
 351 surface, i.e.

$$\tau_s = \boldsymbol{\tau} \cdot \hat{s} = (\tau_{zz} - \tau_{xx})n_1n_2 + \tau_{xz}(n_1^2 - n_2^2). \quad (3.24)$$

352 In view of the time scale chosen for the normalization in (2.7), the appropriate
 353 normalization for the stress is ρga (the same as for the dynamic pressure), where
 354 ρ is the fluid density. In dimensionless form,

$$\tau_{xx} = \delta^2 \psi_{xz}, \quad \tau_{xz} = \frac{1}{2} \delta^2 (\psi_{zz} - \psi_{xx}), \quad \tau_{zz} = -\delta^2 \psi_{zx}. \quad (3.25)$$

355 For the seabed surface $z + h_0 - h_b(x) = 0$, the unit normal vector pointing
 356 outwards is

$$n_1 = h_{b,x} / \sqrt{1 + h_{b,x}^2}, \quad n_2 = -1 / \sqrt{1 + h_{b,x}^2}. \quad (3.26)$$

357 Upon using the map (2.3),

$$n_1 = J_b^{-1/2} x_\eta, \quad n_2 = J_b^{-1/2} x_\xi \quad \text{at} \quad \eta = -h. \quad (3.27)$$

358 Under the transformation,

$$\psi_{zz} - \psi_{xx} = J_b^{-2} (x_\xi^2 - x_\eta^2) \psi_{\eta\eta}, \quad \psi_{xz} = J_b^{-2} x_\xi x_\eta \psi_{\eta\eta} \quad \text{at} \quad \eta = -h. \quad (3.28)$$

359 Substituting (3.27) and (3.28) into (3.24), we obtain the bed shear stress $\tau_b \equiv$
 360 $\tau_s|_{\eta=-h}$, i.e.

$$\tau_b = \frac{1}{2} \delta^2 J_b^{-1} \psi_{\eta\eta}|_{\eta=-h} = \frac{1}{2} \delta J_b^{-1} \tilde{\psi}_{\tilde{\eta}\tilde{\eta}}|_{\tilde{\eta}=0}. \quad (3.29)$$

361 For a non-steep bed, using (3.16) we get

$$\tau_b = \frac{1}{2} (1 - i) \delta J_b^{-1/2} C e^{-i\sigma t} + \text{c.c.}, \quad (3.30)$$

362 where the function C is given in (3.21). Writing $\tau_b = \frac{1}{2} \tau_{bm} e^{-i\sigma t} + \text{c.c.}$, the am-
 363 plitude $|\tau_{bm}|$ fluctuates at the short-scale of topography, while the instantaneous
 364 τ_b varies over the scale of water wavelength (or wave modulation length, which
 365 is determined by the Floquet exponent μ for a given frequency σ). The short-
 366 wavelength fluctuation reflects the bed profile, but is not a copy or rescaled
 367 version of it; see figure 7. Recall that $U^{(o)} = J_b^{-1/2} C e^{-i\sigma t} + \text{c.c.}$ at $\eta = -h$.
 368 For a constant viscosity and simple time-harmonic motions, the phase shift of
 369 $\pi/4$ (in time) remains between the bed shear stress and the outer velocity, same
 370 as in the case of a flat bed. Some mathematical details in deriving (3.28) and
 371 (3.29) are given in Appendix B.

372 Normalized by $\rho a^2 (g k_B)^{3/2}$, the rate of viscous dissipation per volume in the
 373 fluid is

$$\begin{aligned} \dot{d} &= \frac{1}{2} \delta^2 \left[2 (u_x)^2 + 2 (w_z)^2 + (u_z + w_x)^2 \right] \\ &= \frac{1}{2} \delta^2 \left[4 (\psi_{xz})^2 + (\psi_{zz} - \psi_{xx})^2 \right]. \end{aligned} \quad (3.31)$$

374 Close to the seabed at $\eta = -h + \sigma \tilde{\eta}$,

$$\psi_{xz} \simeq \delta^{-1} J_b^{-2} x_\eta x_\xi \tilde{\psi}_{\tilde{\eta}\tilde{\eta}} + O(1), \quad (3.32)$$

$$\psi_{zz} - \psi_{xx} \simeq \delta^{-1} J_b^{-2} (x_\xi^2 - x_\eta^2) \tilde{\psi}_{\tilde{\eta}\tilde{\eta}} + O(1). \quad (3.33)$$

375 To the lowest order approximation, the rate of viscous dissipation inside the
376 boundary layer and over a bed period $0 < \xi < \pi$ is

$$\dot{D} = \int_0^\pi \int_{-h}^{-h+\delta\tilde{\eta}_\infty} dJ \, d\xi d\eta \simeq \frac{1}{2}\delta \int_0^\pi \int_0^{\tilde{\eta}_\infty} J_b^{-1} \left(\tilde{\psi}_{\tilde{\eta}\tilde{\eta}} \right)^2 d\xi d\tilde{\eta}, \quad (3.34)$$

377 where $\tilde{\eta}_\infty$ is the outer edge of the Stokes boundary layer that can easily be
378 determined by examining the matching of velocities. For a non-steep seabed,
379 using (3.16) we obtain the rate of dissipation averaged over a wave period

$$\overline{\dot{D}} \simeq \delta \int_0^\pi CC^* J_b^{-1/2} \left(1 - e^{-2J_b^{1/2}\tilde{\eta}_\infty} \right) d\xi. \quad (3.35)$$

380 The integration above can be evaluated fairly easily, e.g. using Matlab.

381 4. Concluding remarks

382 For 2-D boundary layer flows under the wave motions over a generally pe-
383 riodic seabed, we have formulated a terrain-following modeling approach. Ap-
384 plying the conformal transformation developed in Yu & Howard (2012), the
385 Cartesian grids in the mapped plane correspond to the orthogonal curvilinear
386 grids in the physical plane, which are terrain-following near the seabed. This
387 allows the vorticity equation to be solved to advantage in the Cartesian coordi-
388 nates in the mapped plane, obtaining the description of flow dynamics following
389 the boundary terrain. For linear dynamics, the potential flow field above the pe-
390 riodic seabed is provided by the Floquet theory of water waves in Yu & Howard
391 (2012).

392 For a non-steep periodic seabed (but not necessarily small undulation height
393 compared with the water depth), the linearized vorticity equation in the mapped
394 plane is formally similar to that of the Stokes boundary layer over a flat bed,
395 with only a parametric dependence on the along-bed variable. The analytical
396 solutions are attainable, including the boundary layer velocity profiles, the bed

397 shear stress and the rate of viscous dissipation. For a relatively steep seabed,
 398 the 1-D differential equation (3.14) can account for the 2-D effects of bed to a
 399 greater degree since it retains the full evaluation of the Jacobian of transfor-
 400 mation. Thus, the velocity profiles across the Stokes layer can still be locally
 401 determined. For still steeper bed profiles, the fully 2-D differential equation
 402 (3.13) is necessary to obtain the Stokes boundary layer flow.

403 Although we have used the molecular viscosity when presenting the exam-
 404 ples, a constant eddy viscosity for turbulent flows equally applies. To improve
 405 turbulence modeling, the terrain-following modeling methodology developed in
 406 this study can be extended to the case of spatially varying eddy viscosity. The
 407 conformal transformation of the vorticity equation may involve some tedious
 408 algebra, but is straightforward. The coefficients of the vorticity equation will be
 409 more complicated, combining the Jacobian of the transformation (the effects of
 410 bed geometry) and the functions describing the spatial variations of turbulence
 411 intensity.

412 The nonlinear dynamics of wave boundary layer flows can be studied in the
 413 mapped plane, in principle following the standard approaches in the literature
 414 (e.g. Hara & Mei 1990, Yu & Mei 2000). For a periodic bed, the Floquet
 415 theory of water waves (Yu & Howard 2012) is yet to be extended to include
 416 the surface wave nonlinearity, though other existing theories may be applied to
 417 provide the potential flow above the seabed (e.g. Fokas & Nachbin 2012). Of
 418 particular interest is the spatial variability of the induced steady streaming and
 419 mass transport in the wave boundary layer, due to the non-uniformity of viscous
 420 resistance along the variable seabed, and the implications for near-bed processes.
 421 In a recent study of viscous flows driven by vibrating sawtooth channel walls
 422 (Yu 2014), the method of conformal mapping presented here has been shown to
 423 have a superior capability in modeling nonlinear boundary layer flows.

424 For non-periodic seabeds, the terrain-following modeling methodology pre-
 425 sented here should in principle be applicable, provided that a conformal map is
 426 attainable, either analytically or numerically, and most importantly the poten-
 427 tial flow field of waves over such a seabed is available. Of course, the mathe-

428 matical explicitness is likely to be lost in such general cases.

429 The attention in this study is on large scale seabed topographies of wave-
430 length comparable to, or greater than, the water wavelength, which cause wave
431 scattering and modulation. For small scale bed ripples of wavelength compa-
432 rable to the water wave amplitude under the essentially oscillatory flows (see
433 the reviews in Blondeaux 2001, Charru, Andreotti & Claudin 2013, Seminara
434 1998), the method of conformal mapping has been applied to study the boundary
435 layer flows. In such studies, the known transformation functions are employed,
436 limiting the ripple profiles to certain shapes, e.g. Hara & Mei (1990). The
437 method of constructing conformation transformation functions for arbitrary pe-
438 riodic shapes, as presented here, can be modified so that the mapping is more
439 confined to the bed and less effective in the inviscid core. Using such a map, we
440 can study the Stokes boundary layer and wave induced steady streaming over
441 a rippled bed of large amplitude and complex form, e.g. following the analyses
442 in Hara & Mei (1990).

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450 **Appendix A. A shooting method for solving (3.14)**

451 Integrating (3.14) twice, we obtain

$$\frac{\partial^2 \tilde{\psi}}{\partial \tilde{\eta}^2} = -2iJ(\xi, \eta) \left[\tilde{\psi} - C_2(\xi)\tilde{\eta} - C_1(\xi) \right]. \quad (\text{A.1})$$

452 In view of the matching conditions in (3.18), as well as (3.19), we set

$$\tilde{\psi} = C(\xi)\tilde{\eta} + G(\xi, \tilde{\eta}). \quad (\text{A.2})$$

453 From (A.1), (3.12) and (3.18), the equations for G are obtained, i.e.

$$\frac{\partial^2 G}{\partial \tilde{\eta}^2} = -2iJ [G + (C - C_2)\tilde{\eta} - C_1], \quad (\text{A.3})$$

$$G = 0, \quad \frac{\partial G}{\partial \tilde{\eta}} = -C \quad \text{at} \quad \tilde{\eta} = 0, \quad (\text{A.4a})$$

$$G = \frac{\partial G}{\partial \tilde{\eta}} = 0 \quad \text{at} \quad \tilde{\eta} = \tilde{\eta}_\infty, \quad (\text{A.4b})$$

454 where $\tilde{\eta}_\infty$ is some “sufficiently large” value that can be easily determined in the
 455 calculations. To get the normal velocity W , we need an equation for $\tilde{\psi}_\xi$, hence
 456 G_ξ in view of (A.2). Taking the derivative with respect to ξ , we obtain from
 457 (A.3) and (A.4),

$$\frac{\partial^2 G_\xi}{\partial \tilde{\eta}^2} = -2iJ_\xi [G + (C - C_2)\tilde{\eta} - C_1] - 2iJ [G_\xi + (C' - C'_2)\tilde{\eta} - C'_1], \quad (\text{A.5})$$

$$G_\xi = 0, \quad \frac{\partial G_\xi}{\partial \tilde{\eta}} = -C' \quad \text{at} \quad \tilde{\eta} = 0, \quad (\text{A.6a})$$

$$G_\xi = \frac{\partial G_\xi}{\partial \tilde{\eta}} = 0 \quad \text{at} \quad \tilde{\eta} = \tilde{\eta}_\infty, \quad (\text{A.6b})$$

458 where the prime denotes the derivative in ξ .

459 For a given position $\xi = \xi_0$, the problem is to find the local values $C_1(\xi_0)$,
 460 $C_2(\xi_0)$, $C'_1(\xi_0)$ and $C'_2(\xi_0)$ such that the solutions of the (spatial) initial value
 461 problem (A.3), (A.4a), (A.5) and (A.6a) satisfy the outer boundary conditions
 462 (A.4b) and (A.6b). In this study, we employ the shooting technique developed
 463 in (Cebeci & Keller 1971), which combines the Newton’s method for the search.
 464 Defining

$$y_1 = G, \quad y_2 = \frac{\partial G}{\partial \tilde{\eta}}, \quad y_3 = G_\xi, \quad y_4 = \frac{\partial G_\xi}{\partial \tilde{\eta}}, \quad (\text{A.7})$$

we replace (A.3) and (A.5) by a system of first-order ordinary differential equations, i.e.

$$\frac{dy_1}{d\tilde{\eta}} = y_2, \quad (\text{A.8a})$$

$$\frac{dy_2}{d\tilde{\eta}} = -2iJ[y_1 + (C - C_2)\tilde{\eta} - C_1], \quad (\text{A.8b})$$

$$\frac{dy_3}{d\tilde{\eta}} = y_4, \quad (\text{A.8c})$$

$$\begin{aligned} \frac{dy_4}{d\tilde{\eta}} = & -2iJ_\xi[y_1 + (C - C_2)\tilde{\eta} - C_1] \\ & -2iJ[y_3 + (C' - C'_2)\tilde{\eta} - C'_1]. \end{aligned} \quad (\text{A.8d})$$

The boundary conditions in (A.4) and (A.6) are replaced by

$$y_1 = 0, \quad y_2 = -C, \quad y_3 = 0, \quad y_4 = -C' \quad \text{at} \quad \tilde{\eta} = 0 \quad (\text{A.9a})$$

$$y_1 = y_2 = y_3 = y_4 = 0 \quad \text{at} \quad \tilde{\eta} = \tilde{\eta}_\infty. \quad (\text{A.9b})$$

If we denote the solution to the initial value problem for a given location $\xi = \xi_0$ by $y_i(\tilde{\eta}; C_1, C_2, C'_1, C'_2)$, $i = 1, 2, 3, 4$, we seek for $C_1(\xi_0)$, $C_2(\xi_0)$, $C'_1(\xi_0)$ and $C'_2(\xi_0)$ such that

$$y_i(\tilde{\eta}_\infty; C_1, C_2, C'_1, C'_2) = 0, \quad i = 1, 2, 3, 4 \quad (\text{A.10})$$

Applying Newton's method to the above equation, we obtain the iterations for updating C_1 , C_2 , C'_1 and C'_2 , i.e.

$$\begin{bmatrix} C_1 \\ C_2 \\ C'_1 \\ C'_2 \end{bmatrix}^{(j+1)} = \begin{bmatrix} C_1 \\ C_2 \\ C'_1 \\ C'_2 \end{bmatrix}^{(j)} - \begin{bmatrix} \frac{\partial y_1}{\partial C_1} & \frac{\partial y_1}{\partial C_2} & \frac{\partial y_1}{\partial C'_1} & \frac{\partial y_1}{\partial C'_2} \\ \frac{\partial y_2}{\partial C_1} & \frac{\partial y_2}{\partial C_2} & \frac{\partial y_2}{\partial C'_1} & \frac{\partial y_2}{\partial C'_2} \\ \frac{\partial y_3}{\partial C_1} & \frac{\partial y_3}{\partial C_2} & \frac{\partial y_3}{\partial C'_1} & \frac{\partial y_3}{\partial C'_2} \\ \frac{\partial y_4}{\partial C_1} & \frac{\partial y_4}{\partial C_2} & \frac{\partial y_4}{\partial C'_1} & \frac{\partial y_4}{\partial C'_2} \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad (\text{A.11})$$

where y_i and their derivatives with respect to C_1 , C_2 , C'_1 and C'_2 are evaluated at $\tilde{\eta}_\infty$. To find these derivatives of y_i , we consider the variational equations for

475 (A.8). Defining

$$q_i = \frac{\partial y_i}{\partial C_1}, \quad q_{i+4} = \frac{\partial y_i}{\partial C_2}, \quad q_{i+8} = \frac{\partial y_i}{\partial C'_1}, \quad q_{i+12} = \frac{\partial y_i}{\partial C'_2}, \quad i = 1, 2, 3, 4, \quad (\text{A.12})$$

476 we write from (A.8),

$$\frac{dq_1}{d\tilde{\eta}} = q_2, \quad \frac{dq_2}{d\tilde{\eta}} = -2iJ(q_1 - 1), \quad \frac{dq_3}{d\tilde{\eta}} = q_4, \quad (\text{A.13a})$$

$$\frac{dq_4}{d\tilde{\eta}} = -2iJ_\xi(q_1 - 1) - 2iJq_3, \quad \frac{dq_5}{d\tilde{\eta}} = q_6, \quad \frac{dq_6}{d\tilde{\eta}} = -2iJ(q_5 - \tilde{\eta}), \quad (\text{A.13b})$$

$$\frac{dq_7}{d\tilde{\eta}} = q_8, \quad \frac{dq_8}{d\tilde{\eta}} = -2iJ_\xi[q_5 - \tilde{\eta}] - 2iJq_7, \quad \frac{dq_9}{d\tilde{\eta}} = q_{10}, \quad (\text{A.13c})$$

$$\frac{dq_{10}}{d\tilde{\eta}} = -2iJq_9, \quad \frac{dq_{11}}{d\tilde{\eta}} = q_{12}, \quad \frac{dq_{12}}{d\tilde{\eta}} = -2iJ_\xi q_9 - 2iJ(q_{11} - 1), \quad (\text{A.13d})$$

$$\frac{dq_{13}}{d\tilde{\eta}} = q_{14}, \quad \frac{dq_{14}}{d\tilde{\eta}} = -2iJq_{13}, \quad \frac{dq_{15}}{d\tilde{\eta}} = q_{16}, \quad (\text{A.13e})$$

$$\frac{dq_{16}}{d\tilde{\eta}} = -2iJ_\xi q_{13} - 2iJ(q_{15} - \tilde{\eta}). \quad (\text{A.13f})$$

477 From (A.9), the initial conditions are obtained, i.e.

$$q_1 = q_2 = \dots = q_{16} = 0 \quad \text{at} \quad \tilde{\eta} = 0. \quad (\text{A.14})$$

478 Once the initial value problem defined by (A.8), (A.9a), (A.13) and (A.14) is
 479 solved, the second term on the right-hand-side of (A.11) can be evaluated for
 480 $\tilde{\eta} = \tilde{\eta}_\infty$ and consequently the next approximation $[C_1, C_2, C'_1, C'_2]^{(j+1)}$ can be
 481 computed. The initial value problem can readily be solved, e.g. using the
 482 Matlab solver function *ode45*.

483 Upon finding the appropriate values C_1, C_2, C'_1, C'_2 at $\xi = \xi_0$, we integrate
 484 (A.8) from the bed $\tilde{\eta} = 0$ to the outer boundary $\tilde{\eta} = \tilde{\eta}_\infty$, with the initial
 485 conditions in (A.9). Recall that $y_2 \equiv G_{\tilde{\eta}}$ and $y_3 \equiv G_\xi$. From (A.2), $\tilde{\psi}_{\tilde{\eta}} =$
 486 $C(\xi) + y_2$ and $\tilde{\psi}_\xi = C'(\xi)\tilde{\eta} + y_3$ for $\xi = \xi_0$. The tangential and normal velocities,
 487 U and W , immediately follow. The numerical results are included in figures 5
 488 and 6, comparing with the analytical solutions given by (3.16) for a non-steep
 489 bed.

490 **Appendix B. Derivation of equations (3.28) and (3.29)**

491 Under the transformation

$$\frac{\partial}{\partial x} = J^{-1} \left(x_\xi \frac{\partial}{\partial \xi} + x_\eta \frac{\partial}{\partial \eta} \right), \quad \frac{\partial}{\partial z} = J^{-1} \left(x_\xi \frac{\partial}{\partial \eta} - x_\eta \frac{\partial}{\partial \xi} \right). \quad (\text{B.1})$$

492 Thus,

$$\psi_{xx} = J^{-1} \left(x_\xi \frac{\partial}{\partial \xi} + x_\eta \frac{\partial}{\partial \eta} \right) [J^{-1} (x_\xi \psi_\xi + x_\eta \psi_\eta)], \quad (\text{B.2a})$$

$$\psi_{zz} = J^{-1} \left(x_\xi \frac{\partial}{\partial \eta} - x_\eta \frac{\partial}{\partial \xi} \right) [J^{-1} (x_\xi \psi_\eta - x_\eta \psi_\xi)], \quad (\text{B.2b})$$

$$\psi_{xz} = J^{-1} \left(x_\xi \frac{\partial}{\partial \xi} + x_\eta \frac{\partial}{\partial \eta} \right) [J^{-1} (x_\xi \psi_\eta - x_\eta \psi_\xi)]. \quad (\text{B.2c})$$

493 At $\eta = -h$, $\psi = \psi_\eta = 0$, hence $\psi_\xi = \psi_{\xi\xi} = \psi_{\xi\eta} = 0$. After some algebra, we
494 obtain from (B.2) that at the seabed $\eta = -h$,

$$\psi_{xx} = J_b^{-2} x_\eta^2 \psi_{\eta\eta}, \quad \psi_{zz} = J_b^{-2} x_\xi^2 \psi_{\eta\eta}, \quad \psi_{xz} = J_b^{-2} x_\xi x_\eta \psi_{\eta\eta}. \quad (\text{B.3})$$

495 These lead to the expressions in (3.28). Substituting (3.25) and (3.27) into
496 (3.24), we have,

$$\tau_s|_{\eta=-h} = -2\delta^2 J_b^{-1} x_\xi x_\eta \psi_{xz} + \frac{1}{2} \delta^2 J_b^{-1} (x_\eta^2 - x_\xi^2) (\psi_{zz} - \psi_{xx}). \quad (\text{B.4})$$

497 Using (B.3), we obtain

$$\begin{aligned} \tau_s|_{\eta=-h} &= \delta^2 J_b^{-3} \psi_{\eta\eta} \left[-2x_\xi^2 x_\eta^2 - \frac{1}{2} (x_\xi^2 - x_\eta^2) \right]^2 \\ &= \frac{1}{2} \delta^2 J_b^{-3} \psi_{\eta\eta} (x_\xi^2 + x_\eta^2)^2. \end{aligned} \quad (\text{B.5})$$

498 Noting that $J_b = x_\xi^2 + x_\eta^2$ at $\eta = -h$, we obtain $\tau_s|_{\eta=-h} = \frac{1}{2} \delta^2 J_b^{-1} \psi_{\eta\eta}$. This is
499 the right-hand-side of the bed shear stress τ_b in (3.29).

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Figure 1: (a) A 2-D sketch of a homogeneous fluid layer over a periodic seabed, $-\infty < x < \infty$, $-h_0 + h_b(x) \leq z \leq 0$. (b) The uniform strip of the flow domain upon conformal transformation, $-\infty < \xi < \infty$, $-h \leq \eta \leq 0$. (c) The orthogonal curvilinear grids in the (x, z) plane that correspond to the Cartesian grids $\xi = \text{constant}$ and $\eta = \text{constant}$, showing the terrain-following contours near the seabed.

Figure 2: Surface elevations $\zeta(x, t) = -\phi_t|_{z=0}$ for a wave propagating in the $+x$ direction over a simple sinusoidal bed. (a) A perfectly spatially periodic wave of wavelength (crest-to-crest distance) $3\lambda_{bed}$: $\mu = -i/3$ and $\sigma = 0.5977$. (b) A quasi-spatially periodic wave of wavelength $3.3\lambda_{bed}$: $\mu = -i13/33$ and $\sigma = 0.5488$. The seabed profile $h_b(x) = \frac{1}{2}\Delta h_b \cos(2x)$, $\Delta h_b = \frac{3}{4}h_0$ and $h_0 = \pi/3$.

Figure 3: (a) The amplitude of U along $\xi = \xi_0$ for a propagating wave of wavelength $5\lambda_{bed}$ over a doubly sinusoidal bed. (b) The amplitude of W . —, the boundary layer velocity given by (3.16). Symbols for the outer potential flow given by (2.8): $\circ$, at $\xi_0 = 9.73$ (at a bed crest); $\bullet$, at $\xi_0 = 9.08$ (at a bed trough); $\triangle$, at $\xi_0 = 8.58$; $\square$, at $\xi_0 = 7.88$. (c) Contours of stream function ψ at time $t = 0$ over one wavelength in x : $\mu = -i3/5$, $\sigma = 0.3388$ and $\delta = 0.0064$. The bed profile $h_b(x) = \Delta h_b (\sin 4x + \sin 6x) / 3.8042$, $\Delta h_b = \frac{1}{2}h_0$ and $h_0 = \pi/4$.

Figure 4: Graphs of J_b^{-1} (dashed) and $J'_b J_b^{-2}$ (solid) for $0 < \xi < 2\pi$ for a steep doubly sinusoidal bed with $\Delta h_b = \frac{1}{2}h_0$ and $h_0 = 2\pi/5$.

Figure 5: Comparison of the boundary layer velocities given by the solution (3.16) and by numerical integration of (3.14), for a wave of wavelength $8\lambda_{bed}$ (keeping the same wavelength-depth-ratio as in figure 3) over a steep doubly sinusoidal bed, $\Delta h_b = \frac{1}{2}h_0$ and $h_0 = 2\pi/5$. In (a) and (b), the amplitude $|U_m|$ along $\xi = \xi_0$; in (c) and (d), the amplitude $|W_m|$. ---, solution by (3.16); —, solution to (3.14). For the outer potential flow: $\circ$, at $\xi_0 = 9.73$; $\bullet$, at $\xi_0 = 9.08$; $\triangle$, at $\xi_0 = 8.58$; $\square$, at $\xi_0 = 7.88$. The flow parameters, $\mu = -i3/4$, $\sigma = 0.2646$, $\delta = 0.01034$.

Figure 6: (a) Graphs of J_b^{-1} (dashed) and $J'_b J_b^{-2}$ (solid) for $0 < \xi < 2\pi$ for a steep sinusoidal bed $h_b = \frac{1}{2}\Delta h_b \cos 2x$, $\Delta h_b = \frac{1}{2}h_0$ and $h_0 = 2\pi/5$. (b) and (c) The amplitudes $|U_m|$ and $|W_m|$ along $\xi = \xi_0$ for a wave of wavelength $\lambda = 8\lambda_{bed}$. —, solution by (3.16); ---, solution to (3.14). The outer potential flow: $\circ$, at $\xi_0 = 0$ (bed crest); $\triangle$, at $\xi = \pi/2$ (bed trough); $\bullet$, at $\xi = \pi/4$. The flow parameters: $\mu = -i3/4$, $\sigma = 0.2661$ and $\delta = 0.0103$.

Figure 7: Spatial variations of bed shear stress over one water wavelength in x along a doubly sinusoidal bed. —, the amplitude $|\tau_{bm}|$; ---, τ_b at $\sigma t = 0$; $\cdots$, τ_b at $\sigma t = \pi/2$. The conditions of the wave field and the seabed profile are the same as in figure 3.