Pressure Decimation and Interpolation (PDI) method for a Baroclinic Non-hydrostatic Model

Jian Shi^{1,2}, Fengyan Shi², James T. Kirby², Gangfeng Ma³, Guoxiang Wu^{2,4},
Chaofeng Tong¹, Jinhai Zheng¹

¹*College of Harbor, Coastal and Offshore Engineering, Hohai University, Nanjing, 210098, China*

²*Center for Applied Coastal Research, Department of Civil and Environmental Engineering, University of Delaware, Newark, DE 19716, USA*

³*Department of Civil and Environmental Engineering, Old Dominion University, Norfolk, VA 23529, USA*

⁴*College of Engineering, Ocean University of China, Qingdao, 266100, China*

Corresponding author: Tel: 86-13770984093; Email:address: Jianshi@udel.edu

Abstract

Non-hydrostatic models are computationally expensive in simulating density flows and mass transport problems due to the requirement of sufficient grid resolution to resolve density and flow structures. Numerical tests based on the Non-Hydrostatic Wave Model, NHWAVE [Ma et al., 2012], indicated that up to 70% of the total computational cost may be born by the pressure Poisson solver in cases with high grid resolution in both vertical and horizontal directions. However, recent studies using Poisson solver-based non-hydrostatic models have shown that an accurate prediction of wave dispersion does not require a large number of vertical layers if the dynamic pressure is properly discretized. In this study, we explore the possibility that the solution for the dynamic pressure field may, in general, be decimated to

a resolution far coarser than that used in representing velocities and other transported quantities, without sacrificing accuracy of solutions. Following van Reeuwijk [2002], we determine the dynamic pressure field by solving the Poisson equation on a coarser grid and then interpolate the pressure field onto a finer grid used for solving for the remaining dynamic variables. With the Pressure Decimation and Interpolation (PDI) method, computational efficiency is greatly improved. We use three test cases to demonstrate the model's accuracy and efficiency in modeling density flows.

Keywords:

Non-hydrostatic wave model, Pressure Decimation and Interpolation,
Baroclinic model

1. Introduction

Since the late 1990s, with the rapid growth of computing power, many efforts have been made to develop non-hydrostatic models for applications to modeling surface waves, internal waves and tides, and nearshore density flows. In contrast to models based on the hydrostatic pressure assumption, non-hydrostatic models are also valid for simulating flows in which the effects of vertical acceleration are not negligible, such as dispersive waves, strongly stratified fluids and flows over rapid bed slope changes. In the nearshore research community, several non-hydrostatic constant-density models based on vertical σ -coordinates have been developed for modeling surface wave transformation from deep water to the swash zone [Yuan and Wu, 2004; Zijlema

12 et al., 2011, Bradford 2011, Ma et al., 2012]. These models are attractive due
 13 to their computational efficiency in modeling waves in a large computational
 14 domain, compared to traditional Navier-Stokes solvers using surface track-
 15 ing methods such as the volume-of-fluid (VOF) [Hirt and Nichols, 1981] or
 16 the marker-and-cell (MAC) [Harlow and Welch, 1965] methods. In addition,
 17 the models are conceptually simple in comparison to the Boussinesq class of
 18 models for dispersive waves [Shi et al., 2012] and their further extensions to
 19 incorporate turbulence and stratification effects [Kim et al., 2009, Kim and
 20 Lynett, 2011].

21 As in traditional Navier-Stokes solvers, the two-step projection method
 22 (or the pressure-correcting projection method, Casulli, 1999) is usually used
 23 in solving the σ -coordinate based equations. The total pressure is divided
 24 into two parts: dynamic pressure and hydrostatic pressure. In the first step,
 25 intermediate values of velocities are computed from the momentum equations
 26 with the dynamic pressure term ignored. In the second step, the dynamic
 27 pressure is determined by solving the pressure Poisson equation, which is ob-
 28 tained by coupling the continuity equation and momentum equations forced
 29 by the dynamic pressure only. An alternative approach to implement the
 30 two-step projection method is that the intermediate values of velocities are
 31 estimated using the dynamic pressure at the previous time level in the first
 32 step. In the second step, the Poisson equation obtained in the same way as in
 33 the first approach becomes the equation with respect to the dynamic pressure
 34 increment (in time) instead of the dynamic pressure itself (Goda, 1979, van

35 Kan, 1986 and Guermond et al., 2006). In both approaches, the velocity
36 fields are corrected using the updated dynamic pressure. The solution of the
37 pressure Poisson equation is the most time consuming part of the numerical
38 scheme; our own tests based on the Non-Hydrostatic Wave Model NHWAVE
39 [Ma et al., 2012] also indicated that up to 70% of the computational cost
40 resides in the pressure Poisson solver in cases with high grid resolution.

41 Recent studies also show that the numerical scheme for solving the dy-
42 namic pressure is crucial for model accuracy and efficiency. The Navier-
43 Stokes solvers based on a staggered grid framework in which the pressure
44 is set in the grid cell center, e.g., Mahadevan et al. [1996], Namin et al.
45 [2001] and Chen [2003], have difficulties in accurately implementing the
46 boundary condition for the pressure at the free surface, and thus require a
47 considerable number of vertical layers (10-20) to get accurate wave disper-
48 sion characteristics. Stelling and Zijlema [2003] proposed an approximation
49 of the vertical gradient of the non-hydrostatic pressure based on the Keller-
50 box scheme, which locates the pressure at the upper cell faces so that the
51 dynamic pressure at the water surface can be exactly set to zero without any
52 approximation. Ma et al., [2012] followed the idea of the Keller-box scheme
53 to locate the dynamic pressure at the cell faces at which the velocity values
54 are estimated by interpolating the cell-centered values to the cell faces when
55 constructing the pressure Poisson equation. Young and Wu [2009, 2010] used
56 a non-hydrostatic pressure distribution at the top layer provided by Boussi-
57 nesq equations to optimize wave dispersion property. The models using either

58 Keller-box scheme or Boussinesq type scheme can efficiently and accurately
59 simulate highly dispersive waves with a very small number (3-5) of vertical
60 layers, indicating a weak dependence of the wave dispersion prediction on
61 the vertical resolution of the model.

62 For modeling stratified flows and mass transport problems, a high vertical
63 resolution for velocity and density fields is needed. To improve computing
64 efficiency, the traditional shallow water hydrostatic models, such as ROMS
65 [Shchepetkin and McWilliams, 2005], Delft3D [Stelling, 1984, 1986], FVCOM
66 [Chen et al., 2003], GETM [Burchard and Bolding, 2002] and HHU-ECR3D
67 [Zheng et al., 2002] use mode splitting schemes. The models are separated in-
68 to two components: external mode and internal mode. In the external mode,
69 the governing equations are the depth-integrated continuity and momentum
70 equations, which are less expensive to be solved. However, as a requirement
71 for simulating gravity waves, the external mode time step is limited by the
72 Courant-Friedrichs-Lewy condition. The baroclinic equations govern the in-
73 ternal mode, which are more expensive to be solved but can be solved at
74 much larger time steps due to the smaller celerity of internal waves.

75 An ocean model with the hydrostatic pressure assumption is not able to
76 predict non-hydrostatic effects, as significant vertical velocity on small scales
77 is present. Vertical velocity and acceleration may be large in dispersive waves,
78 strongly stratified fluids, turbulent fluctuations, and flows over a rapidly
79 varying bottom topography. Therefore, many efforts have been made to
80 extend the hydrostatic ocean models into non-hydrostatic solvers [Jankowski,

1999, Kanarska and Maderich, 2003, Heggelund et al., 2004, Berntsen and
 Furnes, 2005, Kanarska et al., 2007, Lai et al., 2010a, b, and Auclair et al.,
 2011] for applications of non-hydrostatic processes. Those extensions were
 usually implemented using the pressure-correcting projection method, which
 involves the time consuming pressure Poisson solver.

An alternative approach to include the non-hydrostatic pressure contri-
 bution into an existing hydrostatic model can be made based on the direct
 implementation of the non-hydrostatic pressure which is integrated from the
 vertical momentum equation into the horizontal pressure gradient terms in
 the horizontal momentum equations. This procedure is explicit and decou-
 pled between the mass conservation equation and the momentum equations.
 The approach is suitable for weakly non-hydrostatic flows, assuming that the
 dominant part of the pressure field is hydrostatic and the non-hydrostatic ef-
 fects represent a small correction. This straight-forward inclusion of the non-
 hydrostatic pressure contribution into a hydrostatic model does not require
 the inversion of a Poisson equation and have been applied to several model-
 ing systems such as in Dietrich and Lin [2002], Johns [1991], Johns and Xing
 [1993], Li and Johns [2001], and Gaarthuis [1995]. Recently, Klingbeil and
 Burchard [2013] carried out a validation of the non-hydrostatic capability of
 the straight-forward method used in the full 3D mode-splitting model GET-
 M. They pointed out that the treatment of the incompressibility constraint
 and the balance of momentum in a decoupled manner strongly affected the
 model stability. A cumulative temporal filter was suggested to be used in the

104 solver to suppress instabilities. In addition, the coupling convergence could
 105 be improved by a high number of iterations which could strongly hamper the
 106 model efficiency. It was concluded in Klingbeil and Burchard [2013] that the
 107 straight-forward inclusion of non-hydrostatic dynamics cannot replace the
 108 pressure correcting projection method but can be used in the limit of weakly
 109 non-hydrostatic processes.

110 Many efforts have been conducted to improve the efficiency of non-hydrostatic
 111 models by simplifying the Poisson equation. Berntsen and Furnes [2005] ob-
 112 tained a simpler set of Poisson equation by modelling the non-hydrostatic
 113 pressure in the σ -coordinates. This method is an approximation to the
 114 Poisson equation and the errors increase with the steepness of the topogra-
 115 phy (Keilegavlen and Berntsen, 2009). Scotti and Mitran [2008] approximated
 116 the solution of the pressure Poisson equation by at most three terms of an
 117 expansion in powers of the ratio $\Delta x/H$. This method is valid for simulating
 118 nonlinear internal waves for grids with the ratio less than 2. Cui [2013] and
 119 Cui et al. [2014] reduced the size of the Poisson equation by using a parame-
 120 terized non-hydrostatic pressure distribution in the bottom cell. The benefit
 121 of this method is significant in two layer models but the model is less efficien-
 122 t as a larger number of vertical layers is used. Vitousek and Fringer [2014]
 123 developed a non-hydrostatic, isopycnal-coordinate ocean model, which is effi-
 124 cient for reducing the number of vertical layers, but the isopycnal-coordinate
 125 system can be invalid for modeling over-turning processes such as internal
 126 wave breaking and Kelvin-Helmholtz instabilities.

127 The motivation of this study is to find an efficient method for modeling
128 dispersive waves in a stratified environment using a non-hydrostatic model.
129 As mentioned above, a higher vertical resolution may be needed for simu-
130 lations involving baroclinic forcing and mass transport, while an accurate
131 prediction of dynamic pressure field may not require a large number of ver-
132 tical layers. It is expected that a reduction of the number of vertical layers
133 in solving the dynamic pressure can significantly improve the computation-
134 al efficiency of a non-hydrostatic model. van Reeuwijk [2002] has used the
135 strategy to develop a non-hydrostatic 2DV free surface wave model. In van
136 Reeuwijk, independent numbers of pressure layers and horizontal velocity
137 layers were implemented with a smaller number of pressure layers than that
138 of velocity layers. An extensive investigation was carried out to examine the
139 interpolation method which was used to re-distribute the dynamic pressure
140 field from the pressure points into the velocity points. The model with fewer
141 pressure layers than velocity layers showed a greater efficiency in modeling
142 short wave propagation without losing much accuracy.

143 In this study, we focus on stratified flow problems with the wave dispersion
144 characteristics. Our hypothesis is that an accurate prediction of the dynamic
145 pressure field may not require a high resolution grid, especially in the vertical
146 direction. The solution for the dynamic pressure can be decimated by solving
147 the Poisson equation with a coarser grid compared to the regular grid used for
148 solving the rest of the equations for velocity, turbulence and passive tracers.
149 Then, an interpolation can be performed to specify the pressure field on the

150 finer grid used for solving the remaining equations. A coarser grid is used in
 151 solving the pressure Poisson equation and a finer grid is used for the velocity,
 152 turbulence and equations for passive tracer components such as salinity and
 153 temperature. Numerical tests with different grid configurations are carried
 154 out against theoretical solutions or measured data. The PDI method is also
 155 used in the horizontal direction to examine if the model efficiency can be
 156 further improved.

157 The paper is organized as follows. The model formulation is summarized
 158 in section 2, followed by the introduction to the PDI algorithm. In section
 159 3, three numerical experiments, the lock exchange problem, internal wave
 160 breaking and a two-layer flow over a fixed sill, were carried out to demonstrate
 161 the capability of PDI in simulating density flows. Section 4 is devoted to a
 162 discussion of the accuracy and computational efficiency of the model. Finally,
 163 conclusions are drawn in section 5.

164 **2. Model description**

165 *2.1. Governing equations*

166 The complete NHWAVE equations, describing non-hydrostatic motion,
 167 turbulent mixing, baroclinic forcing and mass transport, can be found in Ma
 168 et al. [2013]. Here, we only list the equations related to the PDI method in
 169 the present study.

170 The generalized conservative form of mass and momentum equations can

171 be written as

$$\frac{\partial \Psi}{\partial t} + \nabla \cdot \Theta(\Psi) = \mathbf{S} \quad (1)$$

172 where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \sigma} \right)$. Ψ and $\Theta(\Psi)$ are the vector of conserved variables
 173 and the flux vector respectively.

$$\Psi = \begin{pmatrix} D \\ Du \\ Dv \\ Dw \end{pmatrix} \quad (2)$$

$$\Theta = \begin{pmatrix} Dui + Dvj + \omega \mathbf{k} \\ (Duu + (\frac{1}{2}g\eta^2 + gh\eta))\mathbf{i} + Duv\mathbf{j} + u\omega \mathbf{k} \\ Duv\mathbf{i} + (Dvv + (\frac{1}{2}g\eta^2 + gh\eta))\mathbf{j} + v\omega \mathbf{k} \\ Duw\mathbf{i} + Dvw\mathbf{j} + w\omega \mathbf{k} \end{pmatrix}. \quad (3)$$

174 where $D = h + \eta$, h is water depth and η is surface elevation, (u, v, w)
 175 represent velocity components in the Cartesian coordinate system (x^*, y^*, z^*)
 176 and ω is the velocity normal to the σ levels. The source term on the right
 177 hand side includes several source components:

$$\mathbf{S} = \mathbf{S}_h + \mathbf{S}_p + \mathbf{S}_\rho + \mathbf{S}_\tau \quad (4)$$

178 where $\mathbf{S}_h$, $\mathbf{S}_p$, $\mathbf{S}_\rho$ and $\mathbf{S}_\tau$ represent the bottom slope term, dynamic pressure
 179 gradient, baroclinic forcing and turbulent mixing respectively. These terms
 180 are expressed as below.

$$\begin{aligned} \mathbf{S}_h &= \begin{pmatrix} g\eta \frac{\partial h}{\partial x} \\ g\eta \frac{\partial h}{\partial y} \\ 0 \end{pmatrix} \quad \mathbf{S}_p = \begin{pmatrix} -\frac{D}{\rho} \left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} \right) \\ -\frac{D}{\rho} \left(\frac{\partial p}{\partial y} + \frac{\partial p}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} \right) \\ -\frac{1}{\rho} \frac{\partial p}{\partial \sigma} \end{pmatrix} \\ \mathbf{S}_\rho &= \begin{pmatrix} -gD \left(\frac{\partial r}{\partial x} + \frac{\partial r}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} \right) \\ -gD \left(\frac{\partial r}{\partial y} + \frac{\partial r}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} \right) \\ -g \frac{\partial r}{\partial \sigma} \end{pmatrix} \quad \mathbf{S}_\tau = \begin{pmatrix} DS_{\tau x} \\ DS_{\tau y} \\ DS_{\tau z} \end{pmatrix} \end{aligned} \quad (5)$$

181 where p represents the dynamic pressure and r is the baroclinic pressure head
 182 calculated by

$$r = \frac{D}{\rho_0} \int_{\sigma}^1 \rho d\sigma \quad (6)$$

183 where ρ_0 is the reference density and ρ is the dynamic density defined in Ma
 184 et al. [2013]. The turbulent diffusion term $S_{\tau x}$, $S_{\tau y}$ and $S_{\tau z}$ are given by

$$\begin{aligned} S_{\tau x} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xx}}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} + \frac{\partial \tau_{xz}}{\partial \sigma} \frac{\partial \sigma}{\partial z^*} \\ S_{\tau y} &= \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yy}}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} + \frac{\partial \tau_{yz}}{\partial \sigma} \frac{\partial \sigma}{\partial z^*} \\ S_{\tau z} &= \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} + \frac{\partial \tau_{zz}}{\partial \sigma} \frac{\partial \sigma}{\partial z^*} \end{aligned} \quad (7)$$

185 where the (x^*, y^*, z^*) are the Cartesian coordinate system. The stresses are

186 calculated as

$$\begin{aligned}
\tau_{xx} &= 2\nu_t \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} \right) & \tau_{yy} &= 2\nu_t \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} \right) & \tau_{zz} &= 2\nu_t \left(\frac{\partial w}{\partial \sigma} \frac{\partial \sigma}{\partial z^*} \right) \\
\tau_{xy} &= \tau_{yx} = \nu_t \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} \right) \\
\tau_{xz} &= \tau_{zx} = \nu_t \left(\frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial z^*} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} \right) \\
\tau_{yz} &= \tau_{zy} = \nu_t \left(\frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial z^*} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} \right)
\end{aligned} \tag{8}$$

187 where the ν_t is the turbulent kinematic viscosity. In the first two cases in
188 Section 3, the values of viscosity ν_t are set to be constant as described in the
189 specific cases. In the last case, the model is run using a LES approach (Pope,
190 2000) and the standard Smagorinsky (1963) model for turbulent viscosity
191 given by

$$\nu_t = (C_s \delta)^2 \sqrt{2S_{ij}S_{ij}} \tag{9}$$

192 where C_s is the Smagorinsky coefficient, $\delta = (\Delta x \Delta y \Delta z)^{1/3}$ and $S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j^*} + \right.$
193 $\left. \frac{\partial u_j}{\partial x_i^*} \right)$ is the stress tensor. The value of the C_s is taken as 0.15 in the last test
194 case.

195 The governing equation for free surface can be written as

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial x} \left(D \int_0^1 u d\sigma \right) + \frac{\partial}{\partial y} \left(D \int_0^1 v d\sigma \right) = 0 \tag{10}$$

196 The advection and diffusion equation governing the motion of passive
197 tracers can be expressed in the similar conservative form as in (1)

$$\frac{\partial \Psi_c}{\partial t} + \nabla \cdot \Theta_c(\Psi_c) = S_c \tag{11}$$

198 where Ψ_c is a passive tracer scalar such as water temperature, salinity, turbu-
 199 lent kinetic energy and dissipation, etc. $\mathbf{S}_c$ represents the source/sink term.
 200 Here, Ψ_c is a concentration variable C of a passive tracer; then

$$\Theta_c(C) = (DuC\mathbf{i} + DvC\mathbf{j} + \omega C\mathbf{k}) \quad (12)$$

201 Numerically, NHWAVE uses a combined finite-volume and finite-difference
 202 scheme with a Godunov-type method for the spatial discretization and a two-
 203 stage (second-order) SSP Runge-Kutta (R-K) scheme [Gottlieb et al., 2001]
 204 for time stepping. The two-step projection method, which splits the time
 205 integration into a hydrostatic predictor phase and the non-hydrostatic cor-
 206 rector phase, is used within each R-K stage. Here, we only illustrate the first
 207 stage of R-K for brevity.

208 In the hydrostatic phase, an intermediate quantity of $\mathbf{U}$, where $\mathbf{U} =$
 209 $(Du, Dv, Dw)^T$, is evaluated using the momentum equations with the dy-
 210 namic pressure term neglected:

$$\frac{\mathbf{U}^* - \mathbf{U}^n}{\Delta t} = -\nabla \cdot \Theta(\mathbf{U})^n + \mathbf{S}_h^n + \mathbf{S}_\rho^n + \mathbf{S}_\tau^n \quad (13)$$

211 where $\mathbf{U}^n$ represents $\mathbf{U}$ value at time level n and $\mathbf{U}^*$ is the intermediate value
 212 from the hydrostatic phase. The fluxes in Eq. (13) are discretized using a
 213 second-order Godunov-type finite volume method. In the present version of
 214 NHWAVE, the Superbee (Roe, 1986) and van Leer (van Leer, 1974) limiters
 215 are implemented. Our tests show that the van Leer limiter is more dissipative

216 than the Superbee limiter and thus the Superbee limiter is used for all test
 217 cases in the present study.

218 In the non-hydrostatic phase, the velocity field is corrected using the
 219 dynamic pressure term:

$$\frac{\mathbf{U}^{(1)} - U^*}{\Delta t} = \mathbf{S}_p^{(1)} \quad (14)$$

220 where the superscription $()^{(1)}$ represents the first stage for the R-K scheme.

221 Substituting (14) into the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} + \frac{1}{D} \frac{\partial w}{\partial \sigma} = 0 \quad (15)$$

222 yields the Poisson equation of the dynamic pressure

$$\begin{aligned} & \frac{\partial}{\partial x} \left[\frac{\partial p}{\partial x} + \frac{\partial p}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} \right] + \frac{\partial}{\partial y} \left[\frac{\partial p}{\partial y} + \frac{\partial p}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} \right] + \frac{\partial}{\partial \sigma} \left(\frac{\partial p}{\partial x} \right) \frac{\partial \sigma}{\partial x^*} + \\ & \frac{\partial}{\partial \sigma} \left(\frac{\partial p}{\partial y} \right) \frac{\partial \sigma}{\partial y^*} + \left[\left(\frac{\partial \sigma}{\partial x^*} \right)^2 + \left(\frac{\partial \sigma}{\partial y^*} \right)^2 + \frac{1}{D^2} \right] \frac{\partial}{\partial \sigma} \left(\frac{\partial p}{\partial \sigma} \right) = \\ & \frac{\rho}{\Delta t} \left(\frac{\partial u^*}{\partial x} + \frac{\partial u^*}{\partial \sigma} \frac{\partial \sigma}{\partial x^*} + \frac{\partial v^*}{\partial y} + \frac{\partial v^*}{\partial \sigma} \frac{\partial \sigma}{\partial y^*} + \frac{1}{D} \frac{\partial w^*}{\partial \sigma} \right) \end{aligned} \quad (16)$$

223 The same procedure is used for the second stage of the R-K scheme. The
 224 transport of a passive tracer is calculated using (11) with the velocity field
 225 corrected in the non-hydrostatic phase.

226 For the computational grid arrangement, the points for velocity and trac-
 227 er variables are defined at the cell centers. The dynamic pressure for the
 228 Keller-box scheme is defined at the cell faces. The velocity values at the

cell faces, (u^*, v^*, w^*) , are obtained by linear interpolation using the adjacent cell-centered values for the discretization of the Poisson equation (16).

2.2. Pressure decimation and interpolation (PDI)

The PDI method is implemented by choosing a subset of vertical grid points from the full velocity grid and implements the Poisson solver for dynamic pressure only on the subset. Fig. 1 demonstrates the grid arrangement in the vertical direction with a coarse grid with less vertical layers shown in solid lines and a fine grid with more vertical layers in dashed lines. The same grid arrangement is done in the horizontal directions. In the coarse grid, the dynamic pressure (p') points are located at the cell faces and velocity points (u', w') at the cell centers. The same configuration is made for the fine grid and the corresponding variables are denoted by (p, u, w) in the figure. (u, w) and (u', w') are overlapped at the cell center of the coarse grid, which is unnecessary. It should be mentioned that, the pressure points in the coarse grid are uniformly distributed in the σ coordinate, in contrast to van Reeuwijk who applied a non-uniform distribution with more pressure points at surface and less pressure points at the bottom for unstratified surface wave simulations.

Fig. 2 illustrates the flow chart of PDI. The variables in a coarse grid and in a fine grid are solved alternatively in the time integration loops. Detailed procedures are listed as follows, and again, only the first stage of R-K scheme is shown for demonstration.

251 Step 1: the intermediate values $\mathbf{U}^*$ is evaluated on the fine grid by solving
 252 (13).

253 Step 2: $\mathbf{U}^{*'} on the coarse grid is obtained by piecewise-averaging $\mathbf{U}^*$ from
 254 the fine grid. The piecewise-averaging method can be given by:$

$$\mathbf{U}_i^{*'} = \sum_{j=J_i}^{J_i+N_i-1} \frac{\mathbf{U}_j^*}{N_i} \quad (17)$$

255 where i is the index on the coarse grid and N_i is the number of the fine
 256 grid points inside the coarse grid cell i , J_i is the first fine grid index
 257 inside the coarse grid cell i .

258 Step 3: the dynamic pressure p' is calculated on the coarse grid by solving
 259 (16).

260 Step 4: the dynamic pressure p on the fine grid is obtained using an inter-
 261 polation method.

262 Step 5: the velocity field is corrected on the fine grid using the equation
 263 (14).

264 Step 6: free surface positions are obtained using equation (10). Equation
 265 (11) for the passive tracers are solved on the fine grid. The turbulent
 266 eddy viscosity and baroclinic forcing are calculated and used for the
 267 next time step on the fine grid.

268 Note that either $\mathbf{U}^*$ on the fine grid or $\mathbf{U}^{*'} on the coarse grid does not$

269 guarantee mass or volume conservation. However, the incompressible con-
 270 dition in the coarse grid is satisfied by using (16) in Step 3, while the mass
 271 conservation in the fine grid is guaranteed by using the vertically integrated
 272 continuity equation (10).

273 Different interpolation methods, including the linear, quadratic, cubic
 274 spine interpolation and the fourth-order spline interpolation [Van Reeuwijk,
 275 2002], were tested. The cubic spline interpolation gives the most accurate
 276 estimation. So we only described the detail of the cubic spline interpolation.

277 A cubic spline is a spline constructed of piecewise third-order polynomials
 278 which can be expressed as

$$p_i(\sigma) = a_i(\sigma - \sigma_i)^3 + b_i(\sigma - \sigma_i)^2 + c_i(\sigma - \sigma_i) + d_i \quad (18)$$

279 where $p_i(\sigma)$ is the cubic polynomial of the dynamic pressure in each interval.
 280 a_i , b_i , c_i and d_i are the coefficients. i is the index on the coarse grid.

281 We define n_u as the number of vertical grid cells for velocity calculation
 282 and n_p as the number of vertical grid cells for pressure calculation. There are
 283 $n_p + 1$ control points and n_p intervals, which means there are $4n_p$ independent
 284 conditions needed to fix the spline. We assume that the cubic polynomials
 285 match the values at both ends of the intervals and the first and second
 286 derivatives are continuous, which can provides $4n_p - 2$ conditions. Two extra
 287 conditions are needed. The following boundary conditions of the dynamic

288 pressure at the bottom and surface are applied:

$$\begin{aligned}\frac{\partial p}{\partial \sigma} &= 0; & \sigma &= 0 \\ \frac{\partial^2 p}{\partial \sigma^2} &= 0; & \sigma &= 1\end{aligned}\tag{19}$$

289 3. Numerical experiments

290 We conducted three numerical experiments to test the PDI method in
 291 simulating density flows. The first case is the lock exchange problem, which
 292 demonstrates the efficiency and accuracy of PDI in simulating the extreme
 293 baroclinic dynamics. The second case is a simulation of internal wave break-
 294 ing, which is compared to the laboratory experiment data. The third case is
 295 a simulation of two-layer flow over a fixed sill. The first two cases are carried
 296 out in two dimensions, with the domain in a vertical plane. The third case is
 297 three dimensional and utilized to explore the benefits of the PDI method in
 298 a 3D application. In this section, we focused on tests of PDI in the vertical
 299 direction only. The applications of the PDI in the horizontal direction ap-
 300 peared to be less promising than the vertical PDI in terms of model efficiency
 301 improvement. The PDI method in the horizontal direction will be discussed
 302 in Section 4.

303 3.1. The lock-exchange problem

304 The lock exchange problem, which is the gravitational adjustment of two
 305 fluids of different density initially separated by a vertical gate, has become

306 a standard case for testing the baroclinic dynamics component of numerical
 307 models [Bourgault and Kelley, 2004; Fringer et al., 2006; Lai et al., 2010a;
 308 Ma et al., 2013]. The density is assumed to be expressed as follows:

$$\rho = 999.972 \times (1 + 0.75 \times 10^{-3}S) \quad (20)$$

309 The light fluid is set to be freshwater ($\rho_1 = 999.972\text{kg/m}^3$) with zero salinity
 310 and the heavy fluid($\rho_2 = 1000.991\text{kg/m}^3$) is set to be salt water with a
 311 salinity of 1.3592 psu. The two kinds of fluids are separated by a gate in the
 312 middle of the domain. The salinity is defined as:

$$S(x, z) = \begin{cases} 1.3592 & x \geq 0.4\text{m} \\ 0 & x < 0.4\text{m} \end{cases} \quad (21)$$

313 The numerical domain is a 2D flume with a length ($L = 0.8$ m) and a
 314 static water depth ($H = 0.1$ m). The horizontal resolution is $\Delta x = 0.001$
 315 m. Five cases with, respectively, (200,200)((n_u, n_p), thereafter), (200,20),
 316 (100,20), (50,20) and (20,20) vertical layers were conducted to investigate
 317 the influence of PDI on the model accuracy and efficiency. The molecular
 318 viscosity is $10^{-6}\text{m}^2/\text{s}$. The eddy viscosities are set to be zero. The slope
 319 limiter used in the shock-capturing scheme is the Superbee limiter.

320 After the gate between the fresh water and the heavy fluid is removed
 321 at $t = 0$ s, a gravitational current develops between the interface of the two
 322 fluids due to the baroclinic forcing. With an increase of velocity shear, the

323 Kelvin-Helmholtz billows arise across the interface. Fig. 3(a)-(c) compare
 324 the snapshots of density distribution at $t = 20$ s between the cases with
 325 (20,20), (200,20) and (200,200) vertical layers. The results of (20,20) shows
 326 that the vorticity cores are not well resolved at the low grid resolution, while
 327 the comparison between the cases of (200,20) and (200,200) does not show an
 328 obvious difference. It indicates that the PDI model with less pressure layers
 329 can successfully predict salinity and velocity fields. Fig. 3(d) shows the
 330 snapshot of the hydrostatic simulation. The billows cannot be predicted by
 331 the model without dynamic pressure, indicating that the dynamic pressure
 332 component plays an important role in the generation and evolution of the
 333 Kelvin-Helmholtz billows. The comparisons of the horizontal velocities at the
 334 midpoint of the computational domain between the model simulations and
 335 the theoretical value (1.59 cm/s) derived by Jankowski [1999] are presented
 336 in Fig. 4. The maximum velocities at the surface and bottom are very close
 337 to the theoretical values. The model slightly underestimates the terminal
 338 velocity from $t = 17.0$ s, which may be induced by the influence of the
 339 left and right boundaries. As the case is simulated in a finite domain, the
 340 velocities cannot reach an asymptotic value.

341 The vertical distributions of velocity u and w , salinity S and dynamic
 342 pressure p predicted from models with the grid configurations of (200,200),
 343 (100,20), (50,20) and (20,20), at $t = 20$ s, are compared at two sections
 344 as shown in Figs. 5 and 6. One of the sections is located between two
 345 vortex cells at $x = 0.31$ m, and the other is located across a vortex cell

346 at $x = 0.19$ m. The comparisons show that the results of (200,20) layers
 347 agree well with the results of (200,200) layers for all compared variables
 348 (black lines for (200,200) and red lines for (200,20) are indistinguishable).
 349 There are not much differences in u between the cases of (200,200), (100,20),
 350 (50,20), except for the case of (20,20) as shown in the left panels of Fig. 5.
 351 In the comparisons of the vertical distribution of salinity, large discrepancies
 352 can be found at the selected sections (right panels of Fig. 5) between the
 353 case of (20,20) and other cases. The large discrepancies are caused by a big
 354 difference in the vortex core locations predicted by the (20,20) model and
 355 other models. More significant differences between the results from the full
 356 grid model ((200,200), thereafter) and the PDI models can be found in the
 357 comparisons of w and p as shown in Fig. 6. Except the case of (200,20), w
 358 and p predicted by the PDI models apparently deviate from those predicted
 359 by the full grid model. The deviation increases with a decrease in the velocity
 360 grid resolution. The case of the (20,20) grid configuration shows the largest
 361 pressure discrepancy from the full grid model, especially at $x = 0.31$ m,
 362 where the opposite pressure values were predicted compared to the full grid
 363 model.

364 Tests on further reducing the number of pressure layers to less than 20 did
 365 not show a successful generation of the Kelvin-Helmholtz billows, probably
 366 because the low vertical resolution could not resolve the vertical structure of
 367 dynamic pressure in the lock exchange case.

368 To investigate why the non-hydrostatic pressure field can be decimat-

ed, but not the velocity/tracer field, in modeling non-hydrostatic, baroclinic flows, we examined the vorticity generation mechanism in the lock exchange problem. The vorticity conservation equation for the lock exchange problem can be written as

$$\begin{aligned} \frac{d\boldsymbol{\Omega}}{dt} &= (\boldsymbol{\Omega} \cdot \nabla) \mathbf{U} \\ &+ \frac{1}{\rho^2} (\nabla \rho \times \nabla p_s + \nabla \rho \times \nabla p_{nh}) \\ &+ \nabla \times \left(\frac{\nabla \cdot \boldsymbol{\tau}}{\rho} \right) \end{aligned} \quad (22)$$

where $\boldsymbol{\Omega}$ represents the vorticity vector. p_s and p_{nh} represent the static pressure and dynamic pressure, respectively. $\boldsymbol{\tau}$ is the shear stress. The first term is the vorticity tilting or stretching term, which vanishes for the 2D vertical problem. If diffusion is neglected, which is true in the simulation with a small viscosity coefficient, vorticity is generated by the cross-product of the density gradient and the gradient of the static pressure (static pressure production, the second term) and dynamic pressure (dynamic pressure production, the third term), respectively. Fig. 7 shows snapshots of the ratio of the dynamic pressure production to the static pressure production at $t = 6, 13$ and 20 s. The static pressure production is generally $O(10) - O(10^5)$ order of magnitude larger than the dynamic pressure production. The maximum value of the dynamic pressure production is at least an order of magnitude smaller than the static pressure production during the entire simulation, suggesting that the static pressure production plays a dominant role in vorticity

387 generation. The accuracy in modeling the baroclinic force depends on the
 388 accuracy in predicting density distributions and thus is closely related to the
 389 mass transport advected by the current velocity field. It may be the rea-
 390 son why the calculation of velocity and baroclinic pressure requires a higher
 391 resolution than the dynamic pressure calculation. However, the comparison
 392 between the non-hydrostatic model and the hydrostatic model (Fig. 3(d))
 393 indicated that the dynamic pressure production plays an important role in
 394 triggering the Kelvin-Helmholtz instability, and thus is also an important
 395 component in modeling the physical process.

396 *3.2. An internal solitary wave shoaling and breaking over a sloping bottom*

397 Mixing induced by the breaking of internal waves on sloping boundaries
 398 plays an important role in controlling the mass transport and the dissipa-
 399 tion of internal wave energy in stratified fluids. The case of an internal wave
 400 breaking on a slope is to test whether the PDI method can preserve the mod-
 401 el's capability for simulating the process of internal solitary wave breaking on
 402 a uniform slope. The corresponding laboratory experiment was conducted
 403 by Michallet and Ivey [1999].

404 The computational domain (Fig. 8) is a 2D rectangular flume with a
 405 length of 165 cm, which is discretized by 660 grids in the horizontal direc-
 406 tion. The water depth remains constant at $H = 15$ cm from the left boundary
 407 (-102 cm) to the toe of the slope and then linearly decreases from this posi-
 408 tion to the right boundary with a slope $s = 0.214$. Five numerical scenarios

409 with (200,200), (200,20), (100,20), (50,20) and (20,20) vertical layers were
 410 calculated. The initial density profile is adopted from Bourgault and Kelley
 411 [2004] and is given by

$$\rho = \rho_1 + \frac{\Delta\rho}{2} \left\{ 1 + \tanh\left(\frac{z - z_i - \zeta}{\Delta h}\right) \right\} \quad (23)$$

412 where $\rho_1 = 1000 \text{ kg m}^{-3}$ is the density in the upper layer, $\Delta\rho = 12 \text{ kg m}^{-3}$,
 413 $\rho_2 = \rho_1 + \Delta\rho$ is the density of the lower layer, $z_i = 0.16H$ is the depth of the
 414 interface, $\Delta h = 1.4 \text{ cm}$ is the thickness of the pycnocline, ζ is the vertical
 415 displacement of the initial isopycnals and is given by

$$\zeta = 2a_0 \text{sech}^2 \left[\frac{(x - x_0)}{2W} \right] \quad (24)$$

416 where $a_0 = 3.1 \text{ cm}$, x_0 is the origin which is -102 cm in this case, and W is
 417 the half width defined as

$$W = \frac{2h_1h_2}{\sqrt{3a_0(h_2 - h_1)}} \quad (25)$$

418 where h_1 and h_2 are the upper and lower layer thickness, respectively, and
 419 $h_1 = 0.024 \text{ m}$ and $h_2 = 0.126 \text{ m}$. The model was run with a constant viscosity
 420 and diffusivity of $10^{-6} \text{ m}^2/\text{s}$. A constant bottom roughness height (0.001 m)
 421 was used in the bottom boundary condition.

422 Fig. 9 illustrates snapshots of the density distribution and velocity fields
 423 using the (200,200) grid configuration. As the internal wave approaches the

424 slope, the volume of the lower layer fluid reduces and the front half of the
 425 wave becomes parallel to the slope as shown in Fig. 9 (b), which increases
 426 the down-slope velocities near the bottom. In the back half of the wave,
 427 velocities cross the interface perpendicularly, causing the wave breaking and
 428 the generation of a bolus as shown in Fig. 9 (c).

429 Fig. 10 compares the velocity and density distributions obtained by the
 430 physical experiment and the model with the (200,200) grid configuration.
 431 The model can capture the formation and evolution of the bolus, with the
 432 position of the bolus and the velocity field comparable to the measurement.

433 Fig. 11 shows the comparison of the velocity and density distributions
 434 between cases of (20,20) and (200,20) layers at $t = 33.5, 34.6$ and 36 s,
 435 respectively. Compared with the full grid model shown in Fig. 10, the size
 436 and position of the bolus cannot be well resolved by the (20,20) layers model,
 437 indicating that more vertical layers are needed to retain consistency with the
 438 full grid model. It can also be seen that the results of (200,20) layers are
 439 much closer to those of (200,200) layers. There is no apparent difference in
 440 the density and velocity distributions between the results of (200,200) and
 441 (200,20) layers case.

442 Figs. 12 and 13 show comparisons of the horizontal velocity u , salinity
 443 S , vertical velocity w and dynamic pressure p between the full grid model
 444 with (200,200) and (20,20) layers and the PDI models with (200,20), (100,20)
 445 and (50,20) layers at two sections at $x = 33$ and $x = 40$ cm, at time $t =$
 446 33.5 s. The results from the model with (200,20) layers fit well with that

447 from the full grid model with (200, 200) layers for all compared variables.
448 The comparisons also show that, as the resolution of velocity/tracer grid
449 decreases, the variables predicted by the PDI model deviate from the full
450 grid model, similar to the trend found in the lock exchange case. Again, the
451 deviation increases with decrease in the velocity/tracer grid resolution.

452 *3.3. A two-layer flow over a fixed sill*

453 The problem of two layer flow over topography has been studied in geo-
454 physical fluid mechanics over several decades since the work of Queney [1948].
455 In estuaries, stratified flows can develop in area with relative high river dis-
456 charges and small tidal oscillations. The strong stratified flows can be simpli-
457 fied as two layer flows in the laboratory experiments and numerical modeling.
458 Lawrence [1993] conducted a laboratory experiment on steady two-layer flow
459 over a fixed obstacle. The experiment emphasized a flow regime of a two-layer
460 flow, called approach-controlled flow, where an internal hydraulic control is
461 located near the foot of the obstacle (supercritical leap, Baines, 1995). In the
462 approach-controlled flow, the interface level around the downstream of the
463 obstacle crest drops rapidly, with the flow changing from a supercritical flow
464 where the upper layer is thinner to one where the lower layer is thinner. The
465 transition involves significant non-hydrostatic processes and is not predicted
466 by the traditional hydraulic theory (Lawrence, 1993).

467 The approach-controlled flow is supercritical along the sill and a hydraulic
468 jump occurs, which can involve 3D structures to the flow at the lee side of

the sill. A 2D model would underestimate the turbulence induced by the jump and cannot predict 3D turbulence coherence structures. Here, we use models with and without the PDI method to demonstrate the efficiency and accuracy of the PDI method in a 3D simulation.

Fig. 14 shows a schematic diagram of the experiment. The center of the sill is located at $x = 0$. The bathymetry is uniform in the y direction (cross-section). The two-layer flow with the density ρ_1 in the upper layer and ρ_2 in the lower layer move from left to right with the flux rates q_1 and q_2 , respectively. Following the experimental setup, the length and width of the computational domain are specified as 20 m and 0.38 m, respectively. The shape of the sill satisfies the following equation:

$$h(x) = h_m \cos^2(x/L) \quad (26)$$

where $h_m = 0.15$ m, the length scale $L = 4h_m$ and the range $|x/L| \leq \frac{1}{2}\pi$.

The computational grid has a uniform resolution of $\Delta x = 0.05$ m and $\Delta y = 0.0475$ m in the horizontal directions. In the vertical direction, three grid configurations are set up with (40, 40) for the full grid model, (40,20) and (40, 10) for the PDI models.

Based on the experimental setup for the regime of the approach-controlled flow (Regime III, case 21 in Lawrence, 1993), the initial density profile in the

487 model is described as

$$\rho = \rho_0 + \frac{\Delta\rho}{2}[1 + \tanh(\frac{z - h_1}{\Delta h})] \quad (27)$$

488 where $\rho_0 = 1000\text{kg/m}^3$ is the density of freshwater in the upper layer, $\Delta\rho =$
489 8.98kg/m^3 is density difference between the two layers; $h_1 = h_2 = 25.55\text{cm}$
490 are the thicknesses of the upper layer and the lower layer, respectively. The
491 density interface thickness (Δh) is 1 cm for a smooth transition at the in-
492 terface. The flow rates of the upper layer and the lower layer are $q_1 = 204$
493 cm^2/s and $q_2 = 205 \text{ cm}^2/\text{s}$, respectively. The simulation is carried out using
494 a LES approach, which is the Smagorinsky model(Smagorinsky, 1963). The
495 total simulation time is 300 s and the time step Δt is adaptive based on the
496 CFL criterion.

497 Fig. 15 shows the interfacial positions along the x direction predicted
498 by the full grid non-hydrostatic model, the hydrostatic model, and the PDI
499 models with the two grid configurations in comparison to the data. The
500 interfacial positions are determined by finding the maximum vertical density
501 gradient in the computational domain as the flow reached the steady state
502 at $t = 200$ s, and are averaged over the y direction. All non-hydrostatic
503 models successfully reproduced the internal hydraulic jump (supercritical
504 leap) occurring as the flow passes through the lee side of the sill, matching
505 the laboratory data very well. The interfaces predicted by the full grid model
506 and the PDI models are almost identical, indicating that the PDI model with

fewer pressure layers can successfully capture the supercritical leap. The result from the hydrostatic model deviates significantly from the data as well as the non-hydrostatic results at the jump location, indicating the important role of the dynamic pressure in the regime of the approach-controlled flow. The non-hydrostatic model revealed significant 3D turbulence distributions and coherent structures not shown here due to scope of the paper.

Fig. 16 shows the comparisons of velocity profile between the experimental data and the numerical results upstream of the obstacle crest at $x = -1\text{m}$, and downstream at $x = 1\text{m}$ and $x = 5\text{m}$, respectively. The quantities were obtained by averaging the values along the y direction at $t = 200$ s. Generally good agreements with the data are achieved at all three locations for all models. Some discrepancy between the model with the $(40, 10)$ grid configuration and other models is noticeable at $x = 1\text{m}$ but is small. At $x = 5\text{m}$, the models agree well with each other but some under-prediction can be found in the lower layer in all models compared with the data. The reason for the under-prediction is unknown.

Fig. 17 shows the y -averaged density profiles at the same locations and times as the velocity profiles in comparison to the measurements. All models reproduced the transition from the supercritical flow with a thinner upper layer to subcritical flow with the thicker upper layer as the two-layer flow passes over the sill, with good agreement with the data. The results of hydrostatic simulations are also shown in Figs. 16 and 17. Compared with the results of non-hydrostatic model, hydrostatic model cannot predict the

530 evolution of the hydraulic jump well, which is consistent with the comparison
531 in Fig. 15.

532 Overall, the 3D models with the PDI method are able to capture the
533 internal hydraulic jump and flow features in the approach-controlled flow
534 regime with the similar accuracy with the full grid model. The tests also in-
535 dicated that the predictions from the PDI models deviate from the full grid
536 model as the resolution of the pressure grid decreases. However, decreasing
537 the pressure layers from 40 to 10 does not affect the model predictions of ve-
538 locity and density fields. Detailed model results with more extensive analysis
539 and physical explanations will be reported later in a new paper.

540 4. Discussion

541 4.1. Interpolation method

542 van Reeuwijk [2002] carried out an extensive investigation of the inter-
543 polation method used in his version of a PDI method. The interpolation
544 method was constrained to fulfill four conditions: 1) zero dynamic pressure
545 at free surface, 2) zero dynamic pressure gradient at the bottom, 3) con-
546 sistency condition and 4) piecewise monotonic pressure distribution in each
547 pressure layer. It was found that first- and fourth-order spline interpolations
548 satisfy the four conditions and that the fourth-order spline interpolation pro-
549 vided more accurate results in surface wave tests. In the present study, we
550 focus on applications to stratified flows and implemented cubic spline and
551 fourth-order spline interpolation methods, respectively, where the fourth-

552 order follows van Reeuwijk. The cubic spline interpolation used here match-
 553 es the values at both ends of the coarse grid points, ensuring zero-pressure
 554 at the surface. The cubic spline interpolation does not satisfy the piecewise
 555 monotonic condition. However, our numerical experiments in the test cases
 556 indicated the cubic spline interpolation used in PDI provided better fits to
 557 the original full grid model compared to the fourth-order spline interpolation
 558 (not shown in the paper). The fourth-order spline interpolation encountered
 559 a stability problem as the velocity-pressure grid ratio became large. The
 560 reason causing the numerical stability problem is unknown.

561 *4.2. Accuracy*

562 The idea of PDI is based on the fact that an accurate prediction of wave
 563 dispersion associated with the dynamic pressure field does not require a large
 564 number of vertical layers. Our hypothesis is that the velocity field corrected
 565 by the dynamic pressure field is weakly dependent on the vertical resolution
 566 of the dynamic pressure. Therefore, it is necessary to check the accuracy
 567 of PDI in predicting dynamic pressure and its influence on predicting ve-
 568 locity field and distribution of tracer variables. Here, we set the case with
 569 the (200,200) grid configuration as the baseline case for the lock exchange
 570 problem and the application of internal wave breaking. The baseline of the
 571 internal hydraulic jump case is the scenario with (40,40) layers. The model
 572 accuracy in predicting dynamic pressure, velocity (u, w) and salinity is eval-
 573 uated using the normalized root-mean-square error (NRMSE) with respect

574 to the results of the baseline case. The NRMSE formula is defined as

$$\text{NRMSE}(t) = \sqrt{\frac{\sum_{i=1}^N (X_i - X'_i)^2}{\sum_{i=1}^N X_i'^2}} \quad (28)$$

575 where N is the number of data, X'_i represents the vertical mean result at
576 point i in the baseline case, X_i is the vertical mean value from the cases with
577 different grid configurations. The data are collected from all points of the
578 grid configurations at $t = 20$ s for the lock exchange problem, $t = 30$ s for
579 the internal wave breaking case and $t = 100$ s for the internal hydraulic jump
580 case. Table 1, 2 and 3 list the NRMSE of the three cases. All cases show
581 that NRMSE decreases with increase of the number of vertical pressure layers.
582 More-over, NRMSE also decreases with the increase of vertical velocity layers
583 (tests for different velocity layers are not conducted in the internal hydraulic
584 jump case).

585 Different interpolation methods in PDI, including the linear, quadratic
586 and cubic spine interpolation, were examined in all test cases. Results show
587 that the cubic spline interpolation gives the most accurate estimation in
588 terms of NRMSE.

589 4.3. Efficiency

590 In order to eliminate the differences caused by different parallel com-
591 puting configurations, we calculated all cases using a single processor. The
592 improvement in model efficiency was evaluated by comparing the computa-
593 tional time used in solving the Poisson equation in the full grid model versus

the time used in solving the Poisson equation plus the overhead associated with the interpolations in the PDI model. Fig. 18 shows the computational ratio (r_p) which is defined as

$$r_p = \frac{t}{t_{20}} \quad (29)$$

where t_{20} is the computational time for the (20,20) layers case.

Notice that the computational time in the cases without PDI increases linearly with the increase of velocity and pressure layer numbers. Increasing the number of vertical layers from (20,20) to (200,200) results in 9.4 and 8.5 times rise in computational time for the lock exchange problem and internal wave breaking case, respectively. However, the computational ratio in the cases with PDI method remains within 1.5 for the lock exchange case and 1.3 for the internal wave case when the vertical grid configuration changes from (20,20) to (200,20). The increased computational time is associated with the interpolation, thus the overhead of the PDI method is two times less than that of solving Poisson equation in the cases with vertical grid configuration of (200,20).

Tests on the 3D case of the internal hydraulic jump also show that the PDI method can greatly improve the model's efficiency. The computational time in solving the Poisson equation reduces by 80% and 56% in the cases with the pressure layer number decreased by 75% and 50%, respectively. Because the 3D case uses different grid configurations from the 2D cases, the

614 computational time ratio r_p is not included in Fig. 18.

615 4.4. Convergence

616 Numerical convergence rates usually depend on the order of accuracy of
617 a numerical scheme and can be examined by grid refinement and time refine-
618 ment. The model convergence with respect to the pressure layer refinement
619 in the PDI method is different from general numerical convergence (usually
620 evaluated by convergence rates with respect to mesh refinement and time
621 step refinement) because it not only depends on the numerical scheme but
622 also is related to physics modeled in a specific case. A model convergence
623 test for the PDI method can examine what degree of pressure decimation can
624 affect non-hydrostatic processes. The NRMSE in Table 1-3 decreases with
625 increase of the vertical pressure layer number, which has indicated that the
626 PDI method is convergent to the full grid model. To further test the model
627 convergence with respect to the pressure layer refinement, we carried out
628 extra tests with different n_p in the test cases. We did not calculate the con-
629 vergence rate which is usually obtained by integrating computational errors
630 over the whole computational domain and used to evaluate orders of numer-
631 ical accuracy. Instead, we use a single point-comparison to demonstrate the
632 model convergence with the pressure layer refinement. Fig. 19 shows the
633 vertical pressure distributions at $x = 0.4$ m and $t = 5$ s from the PDI model
634 with $n_p = 10, 15$, and 20 , in comparison to the full grid model ($n_p = 200$)
635 for the case of the lock exchange problem. The figure shows that the vertical

636 distribution of dynamic pressure from the PDI models are approaching that
 637 of the full grid model with increasing n_p , indicating the convergence of the
 638 PDI model. It is also noted that the model results from $n_p = 15$ and 10
 639 deviate largely from the PDI model with $n_p = 20$ and the full grid model
 640 even at the beginning of the simulation ($t = 5$ s), and the differences become
 641 more significant for later time. Our tests with $n_p = 50$ and 100 provide
 642 similar results as the test with $n_p = 20$ (Table 1 and 2), suggesting that
 643 the PDI model converges at $n_p = 20$ for the case. Fig. 20 shows the same
 644 comparisons for the case of the internal wave breaking at $x = -65$ cm and t
 645 $= 5$ s. The same conclusion can be drawn. The model convergence for the
 646 case of the internal hydraulic jump is demonstrated in the comparisons of the
 647 dynamic pressure distributions at $x = 1$ m and at the steady state as shown
 648 in Fig. 21. Different from the other two cases, the differences in the pressure
 649 distribution between the PDI models and the full grid model are generally
 650 small, but the tendency of convergence can still be found as n_p increasing
 651 from 10 to 40.

652 As indicated in the three test cases, the model convergence with respect to
 653 the pressure layer refinement is highly related to specific physical processes.
 654 The convergence rates can change case by case and depend on a specific non-
 655 hydrostatic process. We carried out the pressure decimation on the basis of
 656 trial-and-error. The question how to select the number of the pressure layers
 657 in the PDI model is not answered in the present study. Further work is needed
 658 to explain why the dynamic pressure can be decimated in a non-hydrostatic

659 model for simulating baroclinic, non-hydrostatic flows.

660 *4.5. PDI in horizontal direction*

661 The extension of the PDI method in the horizontal direction was exam-
662 ined using the lock exchange case as an example. The test was done in the
663 similar procedures as the PDI in the vertical direction but reducing the hor-
664 izontal resolution of the pressure grid in contrast to the vertical resolution in
665 the vertical PDI.

666 The case was run by reducing 50% of the horizontal resolution of pressure
667 grid and keeping the same number of vertical pressure layers as the full grid
668 model (200 layers) in order to examine the influence from the horizontal PDI
669 method only. The flow patterns predicted by the model with the horizontal
670 PDI are similar to the full grid model (not shown here). However, further
671 reducing the horizontal pressure grid resolution did not yield flow patterns
672 comparable to the full grid model. The test may indicate that the effects
673 of dynamic pressure gradient are more significant in the horizontal direction
674 than in the vertical direction in the non-hydrostatic, baroclinic processes.

675 **5. Conclusions**

676 A pressure decimation and interpolation method was applied in the non-
677 hydrostatic wave model NHWAVE in order to improve the computational
678 efficiency. The idea was to use a coarser grid in solving the pressure Poisson
679 equation and a finer grid in the rest of equations for velocity and passive

680 tracers. With a lower resolution grid in solving the Poisson equation, the
681 computational efficiency was greatly improved.

682 Test cases in the study include simulations of the lock exchange problem,
683 the internal waves breaking and a two-layer flow over a fixed sill, all of which
684 used several grid configurations. The cases showed that a considerable large
685 number of vertical layers is necessary for resolving the velocity structure and
686 density field for modeling non-hydrostatic, stratified flows. The results in-
687 dicated that the reduction of vertical layer number in solving the Poisson
688 equation did not obviously alter the numerical results but improved compu-
689 tational efficiency significantly. The applications of the horizontal PDI to
690 the test cases did not show much improvement in model efficiency compared
691 to PDI in the vertical direction.

692 **Acknowledgements** The authors wish to thank Dr. Stephan Kramer and
693 two anonymous reviewers for their constructive comments and suggestion-
694 s. This work was supported by the National Science Foundation, Physical
695 Oceanography Program (OCE-1334325 and OCE-1435147) and the Office
696 of Naval Research, Littoral Geosciences and Optics Program (N00014-10-1-
697 0188) and Arctic and Global Prediction Program(N00014-14-1-0569) (Kirby
698 and F. Shi). J. Shi and G. Wu were supported by the National Natural
699 Science Foundation of China (51339005), the National Science Foundation
700 for Distinguished Young Scholars (51425901) and 111 Project through Grant
701 No. B14028. Development and testing of the parallel code was carried out

on UD's Community Cluster System operated by UD IT group, supported
by NSF award CNS-0958512.

References

- Auclair, F., Estournel, C., Floor, J.W., Herrmann, M., Nguyen, C. and
Marsaleix, P., 2011. A non-hydrostatic algorithm for free-surface ocean
modelling, *Ocean Modelling*, 36, 49-70, doi: 10.1016/j.ocemod.2010.09.006.
- Baines, P.G., 1995. *Topographic Effects in Stratified Flows*, Cambridge Univ.
Press, 482 pp.
- Berntsen, J., Furnes, G., 2005. Internal pressure errors in sigma-coordinate
ocean models—sensitivity of the growth of the flow to the time step-
ping method and possible non-hydrostatic effects, *Continental Shelf
Research*, 25, 829-848, doi: 10.1016/j.csr.2004.09.025.
- Bourgault, D., and Kelley, D.E., 2004. A laterally averaged nonhydro-
static ocean model, *J. Atmos. Oceanic Technol.*, 21 (12), 1910-1924,
doi:10.1175/JTECH-1674.1.
- Bradford, S., 2011. Nonhydrostatic model for surf zone simulation, *J. Wa-
terway, Port, Coastal, Ocean Eng*, 137, 163-174, doi: 10.1061/(ASCE)
WW.1943-5460.0000079.
- Burchard, H. and Bolding, K., 2002. GETM C A general estuarine trans-
port model. *Scientific Documentation*. Technical Report EUR 20253

722 EN. European Commission.

723 Casulli, V., and Stelling, G., 1998. Numerical smulation of 3D quasi-
 724 hydrostatic, free-surface flows, *J. Hydraul. Eng.*, 124, 678-686, doi:10.10
 725 61(ASCE)0733-9429(1998)124:7(678)

726 Casulli, V., 1999. A semi-implicit finite difference method for non-hydrostatic,
 727 free-surface flows, *Int. J. Numer. Meth. Fluids*, 30(4), 425-440, doi:
 728 10.1002/(SICI)1097-0363(19990630)30:4<425::AID-FLD847>3.0.CO;2-
 729 D.

730 Cui, H., 2013, A new numerical model for simulating the propagation
 731 of and inundation by tsunami waves, Ph.D. Thesis, Delft University
 732 of Technology, Netherland, doi:10.4233/uuid:1440c86c-2e53-4a3c-a43c-
 733 77ef9b75fa82.

734 Cui, H., Pietrzak, J. and Stelling, G., 2014, Optimal dispersion with mini-
 735 mized Poisson equations for non-hydrostatic free surface flows, *Ocean*
 736 *Modelling*, 81, 1-12, doi:10.1016/j.ocemod.2014.06.004.

737 Chen, C., Liu, H. and Beardsley, R.C., 2003. An unstructured grid, finite-
 738 volume, three-dimensional, primitive equations ocean model: Applica-
 739 tion to coastal ocean and estuaries, *J. Atmos. Oceanic Technol.*, 20 (1),
 740 159-186, doi:10.1175/1520-0426(2003)020<0159:AUGFVT>2.0.CO;2.

741 Chen, X., 2003. A fully hydrodynamic model for three-dimensional, free-
 742 surface flows, *Int. J. Numer. Meth. Fluids*, 42 (9), 929-952.

- 743 Dietrich D. and Lin, C.A., 2002. Effects of hydrostatic approximation
744 and resolution on the simulation of convective adjustment, *Tellus A*,
745 *Munksgaard International Publishers*, 54, 34-43, doi: 10.1034/j.1600-
746 0870.2002.00162.x.
- 747 Fringer, O., Gerritsen, M. and Street, R., 2006. An unstructured-grid,
748 finite-volume, nonhydrostatic, parallel coastal ocean simulator, *Ocean*
749 *Modelling*, 14, 139-173, doi: 10.1016/j.ocemod.2006.03.006.
- 750 Gaarthuis, J., 1995. A non-hydrostatic pressure model for shallow water
751 flow: Analysis and application of three methods to compute a non-
752 hydrostatic correction term, *Ph.D. Thesis*, Technische University, Ein-
753 shoven.
- 754 Goda, K., 1979. A multistep technique with implicit difference schemes for
755 calculating two- or three-dimensional cavity flows, *J. Comput. Phys*,
756 30, 76-95, doi: 10.1016/0021-9991(79)90088-3.
- 757 Gottlieb, S., Shu, C.W. and Tadmor, E., 2001. Strong stability-preserving
758 high-order time discretization methods, *SIAM Rev*, 43, 89-112.
- 759 Guermond, J., Mineev, P. and Shen, J., 2006. An overview of projection
760 methods for incompressible flows, *Computer Methods in Applied Me-*
761 *chanics and Engineering*, 195, 6011-6045, doi: 10.1016/j.cma.2005.10.010.
- 762 Harlow, F.H. and Welch, J.E., 1965. Numerical calculation of time-dependent
763 viscous incompressible flow of fluid with free surface, *Physics of Fluids*,

764 8, 2182-2189, doi: 10.1063/1.1761178.

765 Heggelund, Y., Vikebø, F., Berntsen, J. and Furnes, G., 2004. Hydrostat-
766 ic and non-hydrostatic studies of gravitational adjustment over a slope,
767 *Continental Shelf Research*, 24, 2133-2148, doi: 10.1016/j.csr.2004.07.005.

768 Hirt, C. and Nichols, B., 1981. Volume of fluid (VOF) method for the
769 dynamics of free boundaries, *Journal of Computational Physics*, 39,
770 201-225, doi: 10.1016/0021-9991(81)90145-5.

771 Jankowski J.A., 1999. A non-hydrostatic model for free surface flows. *Ph.D.*
772 *Thesis*, Hannover University, Germany.

773 Johns, B., 1991. The modelling of the free surface flow of water over topog-
774 raphy, *Coastal Engineering*, 15, 257-278, doi: 10.1016/0378-3839(91)900
775 05-2.

776 Johns, B. and Xing , J., 1993. Three-dimensional modelling of the free sur-
777 face turbulent flow of water over a bedform, *Continental Shelf Research*,
778 13, 705-721, doi: 10.1016/0278-4343(93)90024-R.

779 Kanarska, Y. and Maderich, V., 2003. A non-hydrostatic numerical model
780 for calculating free-surface stratified flows, *Ocean Dynamics*, 53, 176-
781 185, doi: 10.1007/s10236-003-0039-6.

782 Kanarska, Y., Shchepetkin, A. and McWilliams, J., 2007. Algorithm for
783 non-hydrostatic dynamics in the Regional Oceanic Modeling System,
784 *Ocean Modelling*, 18, 143-174, doi: 10.1016/j.ocemod.2007.04.001.

785 Keilegavlen, E. and Berntsen, J., 2009. Non-hydrostatic pressure in σ -
786 coordinate ocean models, *Ocean Modelling*, 28, 240-249, doi:10.1016/j.
787 ocemod.2009.02.006.

788 Kim, D.H., and Lynett, P.J., 2011. Turbulent mixing and passive scalar
789 transport in shallow flows, *Physics of Fluids*, 23, 016603, doi: 10.1063/1.3
790 531716.

791 Kim, D.H., Lynett, P.J., and Socolofsky, S.A., 2009. A depth-integrated
792 model for weakly dispersive, turbulent, and rotational fluid flows, *Ocean*
793 *Model*, 27, 198-214, doi: 10.1016/j.ocemod.2009.01.005.

794 Klingbeil, K. and Burchard, H., 2013. Implementation of a direct nonhy-
795 drostatic pressure gradient discretisation into a layered ocean model,
796 *Ocean Modelling*, 65, 64-77, doi: 10.1016/j.ocemod.2013.02.002.

797 Lai, Z., Chen, C., Cowles, G.W. and Beardsley, R.C., 2010a. A nonhy-
798 drostatic version of FVCOM: 1. Validation experiments, *J. Geophys.*
799 *Res.*, 115, C11010, doi: 10.1029/2009JC005525.

800 Lai, Z., Chen, C., Cowles, G.W. and Beardsley, R.C., 2010b. A nonhydro-
801 static version of FVCOM: 2. Mechanistic study of tidally generated
802 nonlinear internal waves in Massachusetts Bay, *J. Geophys. Res.*, 115,
803 C12049, doi:10.1029/2010JC006331.

804 Lawrence, G. A., 1993. The hydraulics of steady two-layer flow over a fixed
805 obstacle. *J. Fluid Mech*, 254, pp 605-633.

- 806 Li, Z. and Johns, B., 2001. A numerical method for the determination of
807 weakly non-hydrostatic non-linear free surface wave propagation, *Int.*
808 *J. Numer. Meth. Fluids*, 35, 299-317, doi: 10.1002/1097-0363(20010215)
809 35:3<299::AID-FLD94>3.0.CO;2-B.
- 810 Ma, G., Shi, F. and Kirby, J.T., 2012. Shock-capturing non-hydrostatic
811 model for fully dispersive surface wave processes, *Ocean Modelling*, 43-
812 44 , 22 - 35, doi: 10.1016/j.ocemod.2011.12.002.
- 813 Ma, G., Kirby, J.T. and Shi, F., 2013. Numerical simulation of tsunami
814 waves generated by deformable submarine landslides, *Ocean Modelling*,
815 69, 146 - 165, doi: 10.1016/j.ocemod.2013.07.001.
- 816 Mahadevan, A., Oliger, J. and Street, R., 1996. A nonhydrostatic mesoscale
817 ocean model. Part i: Well-posedness and scaling, *J. Phys. Oceanogr.*,
818 26 (9), 1868-1880, doi:10.1175/1520-0485(1996)026<1868:ANMOMP>
819 2.0.CO;2.
- 820 Michallet, H. and Ivey, G.N., 1999. Experiments on mixing due to internal
821 solitary waves breaking on uniform slopes, *J. Geophys. Res.*, 104 (C6),
822 13,467-13,477, doi: 10.1029/1999JC900037.
- 823 Namin, M.M., Lin, B., and Falconer, R.A., 2001. An implicit numeri-
824 cal algorithm for solving non-hydrostatic free-surface flow problems,
825 *Int. J. Numer. Meth. Fluids*, 35 (3), 341-356, doi: 10.1002/1097-
826 0363(20010215)35:3<341::AID-FLD96>3.0.CO;2-R.

- 827 Pope, S., 2000. *Turbulent Flows*, Cambridge University Press, 522pp
- 828 Queney, P., 1948. The problem of air flow over mountains: A summary of
829 theoretical results, *Bull. AMS*, 29, 16-26.
- 830 Roe, P.L., 1986. Characteristic-based schemes for the Euler equations, *Ann.*
831 *Rev. Fluid Mech.* 18, 337-365.
- 832 Scotti, A. and Mitran, S., 2008. An approximated method for the solution
833 of elliptic problems in thin domains: Application to nonlinear internal
834 waves. *Ocean Modelling*, 25, 144-153, doi:10.1016/j.ocemod.2008.07.005.
- 835 Shchepetkin, A.F., and McWilliams, J.C., 2005. The regional oceanic
836 modeling system (ROMS): a split-explicit, free-surface, topography-
837 following-coordinate oceanic model, *Ocean Modelling*, 9 (4), 347-404,
838 doi:10.1016/j.ocemod.2004.08.002.
- 839 Shi, F., Kirby, J.T., Harris, J.C., Geiman, J.D., and Grilli, S.T., 2012. A
840 high-order adaptive timestepping TVD solver for Boussinesq modeling
841 of breaking waves and coastal inundation, *Ocean Modelling*, 43-44, 36-
842 51, doi: 10.1016/j.ocemod.2011.12.004.
- 843 Smagorinsky, J., 1963. General circulation experiments with the primitive
844 equations: I. the basic experiment*, *Monthly weather review*, 91(3),
845 99-164.
- 846 Stelling, G.W., 1984. On the construction of computational methods for

847 shallow water flow problems, *Technical Report 35*, Rijkwaterstaat, Nether-
848 lands.

849 Stelling, G.W., 1986. Practical aspects of accurate tidal computations, *J.*
850 *Hydr. Eng*, 112, 802-817, doi: 10.1061/(ASCE)0733-9429(1986)112:9(802).

851 Stelling, G. and Zijlema, M., 2003. An accurate and efficient finite-difference
852 algorithm for non-hydrostatic free-surface flow with application to wave
853 propagation, *Int. J. Numer. Meth. Fluids*, 43, 1-23, doi: 10.1002/fld.595

854 van Kan, J., 1986. A second-order accurate pressure correction scheme for
855 viscous incompressible flow, *SIAM J. Sci. Stat. Comput.*, 7, 870-891,
856 doi:10.1137/0907059.

857 van Leer, B., 1974. Towards the ultimate conservative difference scheme II.
858 Monotonicity and conservation combined in a second order scheme, *J.*
859 *Comp. Phys.* 14 (4): 361-370.

860 van Reeuwijk, M. 2002. Efficient simulation of non-hydrostatic free-surface
861 flow, *Master's thesis*, Delft University of Technology, Netherlands.

862 Vitousek, S. and Fringer, O. B., 2014. A nonhydrostatic, isopycnal-coordinate
863 ocean model for internal waves, *Ocean Modelling*, 83, 118-144, doi:10.1016
864 /j.ocemod.2014.08.008.

865 Young, C.-C. and Wu, C.H., 2009. An efficient and accurate non-hydrostatic
866 model with embedded Boussinesq-type like equations for surface wave

- 867 modeling, *Int. J. Numer. Meth. Fluids*, 60, 27-53, doi: 10.1002/fld.1876.
- 868 Young, C.-C. and Wu, C.H., 2010. A σ -coordinate non-hydrostatic model
869 with embedded Boussinesq-type-like equations for modeling deep-water
870 waves, *Int. J. Numer. Meth. Fluids*, 63, 1448-1470, doi: 10.1002/fld.2150.
- 871 Yuan, H. and Wu, C.H., 2004. An implicit three-dimensional fully non-
872 hydrostatic model for free-surface flows, *Int. J. Numer. Meth. Fluids*,
873 46, 709-733, doi: 10.1002/fld.778.
- 874 Zheng, J., Yan, Y. and Zhu, Y., 2002. Three dimensional baroclinic numeri-
875 cal model for simulating fresh and salt water mixing in the Yangtze Es-
876 tuary, *China ocean Engineering*, 16(2), 227-238, doi:10.3321/j.issn:0890-
877 5487.2002.02.008.
- 878 Zijlema, M., Stelling, G. and Smit, P., 2011. SWASH: An operational pub-
879 lic domain code for simulating wave fields and rapidly varied flows in
880 coastal waters, *Coastal Engineering*, 58, 992-1012, doi: 10.1016/j.coastal
881 eng.2011.05.015.

Table 1: The results of *NRMSE* for the lock-exchange problem

Case	Dynamic pressure	Salinity	u	w
(20,20)	5.14×10^{-2}	3.73×10^{-2}	1.27×10^{-3}	4.29×10^{-3}
(50,20)	3.39×10^{-2}	1.32×10^{-2}	9.73×10^{-5}	6.03×10^{-4}
(100,20)	2.72×10^{-2}	7.71×10^{-3}	6.30×10^{-5}	2.71×10^{-4}
(200,20)	9.41×10^{-3}	3.79×10^{-3}	2.43×10^{-6}	6.70×10^{-5}
(200,50)	2.32×10^{-3}	2.51×10^{-4}	6.76×10^{-7}	5.50×10^{-6}
(200,100)	3.06×10^{-4}	7.13×10^{-5}	3.61×10^{-7}	2.01×10^{-6}

Table 2: The results of *NRMSE* for the internal wave breaking case

Case	Dynamic pressure	Salinity	u	w
(20,20)	2.06×10^{-1}	1.80×10^{-1}	1.72×10^{-3}	2.76×10^{-3}
(50,20)	9.71×10^{-3}	2.77×10^{-3}	2.31×10^{-5}	1.91×10^{-4}
(100,20)	7.02×10^{-3}	1.53×10^{-3}	1.42×10^{-5}	6.01×10^{-5}
(200,20)	5.46×10^{-3}	6.90×10^{-4}	8.05×10^{-6}	4.33×10^{-5}
(200,50)	2.34×10^{-3}	2.70×10^{-4}	4.30×10^{-6}	8.11×10^{-6}
(200,100)	6.77×10^{-4}	8.98×10^{-5}	1.01×10^{-6}	3.04×10^{-6}

Table 3: The results of *NRMSE* for the internal hydraulic jump case

Case	Dynamic pressure	Salinity	u	w
(40,10)	4.6×10^{-2}	1.3×10^{-4}	2.9×10^{-4}	4.4×10^{-3}
(40,20)	2.4×10^{-2}	3.0×10^{-5}	2.8×10^{-5}	2.4×10^{-3}

882 List of Figures

883	1	Grid arrangement in PDI. p' and (u', w') denote the dynamic	
884		pressure and velocity points, respectively in the coarse grid. p	
885		and (u, w) are the dynamic pressure and velocity points in the	
886		fine grid.	52
887	2	Flow chart of PDI.	53
888	3	Comparisons of density distributions for the lock exchange	
889		problem at 20.0 s. (a) (20,20) layers; (b) (200,20) layers; (c)	
890		(200,200) layers; (d) hydrostatic simulation	54
891	4	Time series of the surface (upper panel)and bottom (lower	
892		panel) horizontal velocity at $x = 0.4$ m. Dashed lines: the-	
893		oretical values (1.59 cm/s); Blue lines: (20,20) layers case;	
894		Black lines: (200,200) layers case; Red lines: (200,20) layers	
895		case.	54
896	5	Comparisons of u and salinity distributions at $t = 20$ s (Blue	
897		lines: (20,20) layers; Black lines: (200,200) layers; Red-solid	
898		lines: (200,20) layers; Dash-dotted lines (100,20) layers; Dashed	
899		lines: (50,20) layers).	55
900	6	Comparisons of w and dynamic pressure distributions at $t =$	
901		20 s (Blue lines: (20,20) layers; Black lines: (200,200) layers;	
902		Red-solid lines: (200,20) layers; Dash-dotted lines (100,20)	
903		layers; Dashed lines: (50,20) layers).	56

904	7	Snapshots of the ratio of the dynamic pressure production to	
905		the static pressure production at $t = 6, 13$ and 20 s.	57
906	8	Initial density distribution of the internal wave breaking case.	58
907	9	Snapshots of simulated density and velocity fields with (200,200)	
908		layers.	59
909	10	Comparisons of velocity field and density distributions be-	
910		tween the laboratory observation(left, Michallet and Ivey 1999)	
911		and the (200,200) layers case (right).	60
912	11	Velocity and density distributions for cases of (20,20) layers	
913		(A_1, A_2, A_3) ; (200,20) layers(B_1, B_2, B_3).	61
914	12	Comparisons of u and salinity distributions at $t = 33.5$ s(Blue	
915		lines: (20,20) layers; Black lines: (200,200) layers; Red-solid	
916		lines: (200,20) layers; Dash-dotted lines (100,20) layers; Dashed	
917		lines: (50,20) layers.	62
918	13	Comparisons of w and dynamic pressure distributions at $t =$	
919		33.5 s(Blue lines: (20,20) layers; Black lines: (200,200) layers;	
920		Red-solid lines: (200,20) layers; Dash-dotted lines (100,20)	
921		layers; Dashed lines: (50,20) layers.	63
922	14	A schematic for the initial conditions in the case of the internal	
923		hydraulic jump.	63

924	15	Comparisons of the interface positions (Black line: (40,40) lay-	
925		ers; Red line: (40,20) layers; Blue line: (40,10) layers; Dashed	
926		line: hydrostatic prediction; Circles: laboratory data conduct-	
927		ed by Lawrence [1993], $Y = h(0) + \eta(x)$ is the free surface	
928		position, $h(0)$ is the depth at the west boundary).	64
929	16	Comparisons of the u profiles at $x = -1.0\text{m}$, 1.0m , and 5.0m	
930		(from left to right) Black lines: (40,40) layers; Red lines:	
931		(40,20) layers; Blue lines: (40,10) layers; Dashed lines: Hy-	
932		drostatic predictions; Circles: laboratory data conducted by	
933		Lawrence [1993].	65
934	17	Comparisons of the density profile at $x = -1.0\text{m}$, 1.0m , and	
935		5.0m (from left to right) Black lines: (40,40) layers; Red lines:	
936		(40,20) layers; Blue lines: (40,10) layers; Dashed lines: Hy-	
937		drostatic predictions; Circles: laboratory data conducted by	
938		Lawrence [1993].	66
939	18	Computational time ratio for solving Poisson equation and in-	
940		terpolation. t_{20} is the computational time for solving Poisson	
941		equation for the (20,20) layers case. Black lines: the lock ex-	
942		change case; Red lines: the internal wave case; Solid lines: the	
943		cases using the same numbers of pressure layers and velocity	
944		layers; Dashed lines: the cases with the number of $n_p = 20$. . .	67

945	19	Comparisons of dynamic pressure distributions of the lock ex-	
946		change case at $t = 5$ s. Black line: (200,200) layers; Red line:	
947		(200,20) layers; Blue line: (200,15) layers; Green line: (200,10)	
948		layers	68
949	20	Comparisons of dynamic pressure distributions of the internal	
950		wave case at $t = 5$ s. Black line: (200,200) layers; Red line:	
951		(200,20) layers; Blue line: (200,15) layers; Green line: (200,10)	
952		layers	68
953	21	Comparisons of dynamic pressure distributions of the third	
954		case at $x = 1.0$ m. Black line: (40,40) layers; Red line: (40,20)	
955		layers; Blue line: (40,10) layers	69

Figure 1: Grid arrangement in PDI. p' and (u', w') denote the dynamic pressure and velocity points, respectively in the coarse grid. p and (u, w) are the dynamic pressure and velocity points in the fine grid.

Figure 2: Flow chart of PDI.

Figure 3: Comparisons of density distributions for the lock exchange problem at 20.0 s. (a) (20,20) layers; (b) (200,20) layers; (c) (200,200) layers; (d) hydrostatic simulation

Figure 4: Time series of the surface (upper panel) and bottom (lower panel) horizontal velocity at $x = 0.4 \text{ m}$. Dashed lines: theoretical values (1.59 cm/s); Blue lines: (20,20) layers case; Black lines: (200,200) layers case; Red lines: (200,20) layers case.

Figure 5: Comparisons of u and salinity distributions at $t = 20$ s (Blue lines: (20,20) layers; Black lines: (200,200) layers; Red-solid lines: (200,20) layers; Dash-dotted lines (100,20) layers; Dashed lines: (50,20) layers).

Figure 6: Comparisons of w and dynamic pressure distributions at $t = 20$ s (Blue lines: (20,20) layers; Black lines: (200,200) layers; Red-solid lines: (200,20) layers; Dash-dotted lines (100,20) layers; Dashed lines: (50,20) layers).

Figure 7: Snapshots of the ratio of the dynamic pressure production to the static pressure production at $t = 6, 13$ and 20 s.

Figure 8: Initial density distribution of the internal wave breaking case.

Figure 9: Snapshots of simulated density and velocity fields with (200,200) layers.

Figure 10: Comparisons of velocity field and density distributions between the laboratory observation(left, Michallet and Ivey 1999) and the (200,200) layers case (right).

Figure 11: Velocity and density distributions for cases of (20,20) layers (A_1, A_2, A_3); (200,20) layers (B_1, B_2, B_3).

Figure 12: Comparisons of u and salinity distributions at $t = 33.5$ s (Blue lines: (20,20) layers; Black lines: (200,200) layers; Red-solid lines: (200,20) layers; Dash-dotted lines: (100,20) layers; Dashed lines: (50,20) layers).

Figure 13: Comparisons of w and dynamic pressure distributions at $t = 33.5$ s (Blue lines: (20,20) layers; Black lines: (200,200) layers; Red-solid lines: (200,20) layers; Dash-dotted lines (100,20) layers; Dashed lines: (50,20) layers).

Figure 14: A schematic for the initial conditions in the case of the internal hydraulic jump.

Figure 15: Comparisons of the interface positions (Black line: (40,40) layers; Red line: (40,20) layers; Blue line: (40,10) layers; Dashed line: hydrostatic prediction; Circles: laboratory data conducted by Lawrence [1993], $Y = h(0) + \eta(x)$ is the free surface position, $h(0)$ is the depth at the west boundary).

Figure 16: Comparisons of the u profiles at $x = -1.0$ m, 1.0 m, and 5.0 m (from left to right) Black lines: (40,40) layers; Red lines: (40,20) layers; Blue lines: (40,10) layers; Dashed lines: Hydrostatic predictions; Circles: laboratory data conducted by Lawrence [1993].

Figure 17: Comparisons of the density profile at $x = -1.0\text{m}$, 1.0m , and 5.0m (from left to right) Black lines: (40,40) layers; Red lines: (40,20) layers; Blue lines: (40,10) layers; Dashed lines: Hydrostatic predictions; Circles: laboratory data conducted by Lawrence [1993].

Figure 18: Computational time ratio for solving Poisson equation and interpolation. t_{20} is the computational time for solving Poisson equation for the (20,20) layers case. Black lines: the lock exchange case; Red lines: the internal wave case; Solid lines: the cases using the same numbers of pressure layers and velocity layers; Dashed lines: the cases with the number of $n_p = 20$.

Figure 19: Comparisons of dynamic pressure distributions of the lock exchange case at $t = 5$ s. Black line: (200,200) layers; Red line: (200,20) layers; Blue line: (200,15) layers; Green line: (200,10) layers

Figure 20: Comparisons of dynamic pressure distributions of the internal wave case at $t = 5$ s. Black line: (200,200) layers; Red line: (200,20) layers; Blue line: (200,15) layers; Green line: (200,10) layers

Figure 21: Comparisons of dynamic pressure distributions of the third case at $x = 1.0\text{m}$.
 Black line: (40,40) layers; Red line: (40,20) layers; Blue line: (40,10) layers