Impacts of wave spreading and multidirectional waves on estimating Stokes drift

A. Webb^{a,*}, B. Fox-Kemper^b

^a*Dept. of Ocean Technology, Policy, and Environment, The University of Tokyo, Japan*

^b*Dept. of Earth, Environmental, and Planetary Sciences, Brown University, Providence,
Rhode Island*

Abstract

The Stokes drift, and its leading-order approximation, for a random sea depend upon the interaction of different wave groups and the process of wave spreading. Here Stokes drift direction and magnitude from prescribed spectra, local observational buoy data, and global model WAVEWATCH III output are used to analyze approximations of Stokes drift for directional random seas in deep water. To facilitate analysis, a new approximation is defined to incorporate the systematic effects of wave spreading. Stokes drift is typically overestimated by ignoring these effects or by ignoring directional differences in swell and wind seas. These two errors are differentiated and found to be largely uncorrelated. These errors depend strongly on depth, with deeper Stokes drift favoring narrow-banded swell and shallower Stokes drift favoring wind seas. Results are consistent among the data examined. Mean Stokes drift magnitude reductions from wave spreading and multidirectional wave

*Corresponding author

Email addresses: adrian.webb@gmail.com (A. Webb), baylor@brown.edu (B. Fox-Kemper)

URL: <http://www.adrianwebb.com> (A. Webb), <http://www.fox-kemper.com> (B. Fox-Kemper)

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effects alone are 14–20% and 7–23% respectively, giving a combined reduction of 20–40% versus unidirectional waves, depending on wave age and depth. Approximations that do not include these reductions however, will on average overestimate Stokes drift by 16–26%, 26–43%, and 45–71% respectively. In addition to magnitude, the direction of Stokes drift is also affected and multi-directional waves generate a directional veer with depth: the 30/60/90% confidence intervals are bounded (approximately) by $\pm 0.12/0.28/0.84$ radians ($\pm 7/16/48$ deg) at the surface, with smaller intervals at depth. Complementary depth-integrated approximations are also investigated and directional effects are similar with depth-dependent subsurface results. Furthermore, an optimized directional spread correction for the surface is nearly identical for global simulations and a buoy located at Ocean Weather Station P (50°N 145°W), and does not require directional wave spectrum data.

Keywords: Stokes drift, unidirectional waves, wave spreading

1. Introduction

Stokes drift is an important quantity in categorizing the effects of waves generally, but ocean surface gravity waves in particular. Stokes drift quantifies the forcing of wave-driven upper ocean mixing (Craik and Leibovich, 1976; Thorpe, 1985; D’Asaro, 2014; D’Asaro et al., 2014), is crucial to understanding the open ocean interaction between waves and mean flows (McWilliams et al., 2012; McWilliams and Fox-Kemper, 2013), and is significant in understanding open ocean sea surface transport (McWilliams and Restrepo, 1999; Ardhuin et al., 2009; Sullivan and McWilliams, 2010; Röhrs et al., 2014). Interest in parameterizing the effects of wave-driven mixing in

11 global climate models has spurred an interest in parameterizations relating
12 Stokes drift and other wave parameters to mixing (Kantha and Clayson, 2004;
13 Harcourt and D’Asaro, 2008; Babanin and Haus, 2009; Grant and Belcher,
14 2009; Huang et al., 2011; Van Roekel et al., 2012; Belcher et al., 2012; Har-
15 court, 2013; Fan and Griffies, 2014) and climate impacts generally (Cavaleri
16 et al., 2012).

17 It is often overlooked that the depth-dependent and depth-integrated
18 Stokes drifts are not easily measured or generated from wind and wave data.
19 To simplify, an unidirectional seas assumption, in which all the waves are
20 moving in the same direction, is commonly used in practice. Kenyon (1969)
21 was the first to infer this unidirectional Stokes drift from the wave spec-
22 trum,¹ and Dalrymple and Dean (1984) provide a formulation of the depth-
23 integrated Stokes drift. McWilliams and Restrepo (1999) chose to infer the
24 unidirectional Stokes drift from wind data, not trusting the wave observa-
25 tion or modeling of the time. While some wave parameters are easily mea-
26 sured even by satellite (Gommenginger et al., 2003), Webb and Fox-Kemper
27 (2011) show that it is difficult to accurately measure the higher moments of
28 the wave spectrum needed to estimate surface or subsurface unidirectional
29 Stokes drift, with about half of the error stemming from amplitude errors and
30 half from frequency errors. Nonetheless, Webb and Fox-Kemper (2011) find
31 that the model to model discrepancy in depth-varying unidirectional Stokes
32 drift is of a similar magnitude to the model to satellite discrepancy. Since
33 the depth-integrated unidirectional Stokes drift depends primarily on lower

¹See Section 1.1 for a full definition.

spectral moments (Myrhaug, 2013a), it does not amplify frequency errors so profoundly. Care is required however in evaluating errors in the nonlinear functions that relate wave fields to the surface and depth-integrated unidirectional Stokes drift, and using conditional probabilities is a recent advance (Myrhaug, 2013b,a; Myrhaug et al., 2014). Recently several papers have appeared that are closely related to this one; Liu et al. (2014), Carrasco et al. (2014), and Breivik et al. (2014) discuss similar issues to those addressed here in passing.

1.1. Stokes drift for directional and unidirectional random seas

Before proceeding further, it is prudent to briefly review the definitions of Stokes drift for directional and unidirectional random seas. Here we will follow and build upon notation from Webb and Fox-Kemper (2011) and Webb (2013). In addition, a list of common abbreviations are given in Table 1 for quick reference.

The directional random sea approximation is a power spectral density estimate derived from the leading-order Stokes drift velocity using small-amplitude linear surface gravity waves. Using directional-frequency spectra, $\mathcal{S}_{f\theta}$, and the deep-water dispersion relation,² a cell-averaged estimate can be written as

$$\begin{aligned} \mathbf{u}_{2D_h}^S &= \mathbf{u}_{2D_h}^S(\mathbf{x}, t; \mathbf{L}_h, T) \\ &\approx \frac{16\pi^3}{g} \int_0^\infty \int_{-\pi}^\pi \left(\cos \theta, \sin \theta, 0 \right) f^3 \mathcal{S}_{f\theta}(f, \theta) \exp \left[\frac{8\pi^2 f^2 z}{g} \right] d\theta df, \end{aligned} \quad (1)$$

²The deep-water dispersion relation is used for consistency with previous work.

53 where $\mathbf{L}_h$ and T represent the implicit horizontal length and time scales
 54 necessary to resolve the discrete spectra. To be precise, $\mathbf{x} = (\mathbf{x}_h, z)$ is a generic
 55 horizontal cell center with an arbitrary depth. In addition, (1) is typically
 56 valid for $L_{\min} > 1$ km and $T \gg \sqrt{2L_{\max}/\pi g}$; these dimensions ensure scale
 57 separation from the wavelength and period of typical surface gravity waves.³
 58 This horizontally-two-dimensional Stokes drift vector uses 2D directional-
 59 frequency spectra and is termed $2D_h$ -SD to distinguish it from later
 60 horizontally-one-dimensional approximations that use 1D frequency spectra.

61 Due to a limitation in directional-frequency spectral data from both ob-
 62 servations and models, the following unidirectional wave assumption is often
 63 used to simplify calculations of Stokes drift:

$$\mathcal{S}_{f\theta}(f, \theta) = \delta(\theta - \theta') \mathcal{S}_f(f). \quad (2)$$

64 Here, δ is the Dirac delta function and θ' is used to define the assumed Stokes
 65 drift direction as $\hat{\mathbf{e}}^w = (\cos \theta', \sin \theta', 0)$. This simplifies the interior integral
 66 in (1) as

$$\int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f, \theta) d\theta = \hat{\mathbf{e}}^w \mathcal{S}_f(f), \quad (3)$$

67 and results in a simpler horizontally-one-dimensional form of Stokes drift
 68 (henceforth $1D_h$ -SD) and its surface value, given as:

$$\mathbf{u}_{1D_h}^S = \hat{\mathbf{e}}^w \frac{16\pi^3}{g} \int_0^\infty f^3 \mathcal{S}_f(f) \exp\left[\frac{8\pi^2 f^2 z}{g}\right] df \quad (4)$$

$$\mathbf{u}_{1D_h}^S|_{z=0} = \hat{\mathbf{e}}^w \frac{16\pi^3}{g} \int_0^\infty f^3 \mathcal{S}_f(f) df = \hat{\mathbf{e}}^w \frac{16\pi^3 m_3}{g}. \quad (5)$$

³ $L_{\min}$ and $L_{\max}$ simply refer to the minimum and maximum horizontal dimensions for a rectangular grid cell.

69 Notice now that only the third moment of the 1D wave spectrum, m_3 , is
70 required to estimate the surface $1D_h$ -SD.

71 *1.2. Motivation for current study*

72 While the $1D_h$ -SD approximation is commonly used in literature, the uni-
73 directional seas assumption is a strong one that affects both the magnitude
74 and direction of Stokes drift. Without access to the full 2D wave spectra
75 however, estimating errors in any calculation poses a challenge since the de-
76 gree of wave spreading is unknown and the presence of multiple wave groups
77 can be hidden. These features affect both the magnitude and direction in
78 a nonlinear fashion that is sensitive to depth. By assumption, the $1D_h$ -SD
79 approximation ignores these features and as a result overestimates the mag-
80 nitude. In addition, the assumed direction (usually the wind seas direction)
81 is often misaligned with the actual direction. These differences can be sub-
82 stantial and are not readily quantifiable; such a quantification is the primary
83 purpose here.

84 Despite this, the $1D_h$ -SD approximation has often been preferable since
85 it simplifies calculations and access to 2D spectral data has been historically
86 limited. Recently however, there has been a trend toward using the $2D_h$ -SD
87 approximation for calculations that are sensitive to Stokes drift despite the
88 exponential increase in computation. These depth-dependent values can be
89 calculated for wave parameterizations online as needed without increasing
90 storage or using an assumed profile.⁴ While this is appropriate and rec-

⁴An exponentially decaying profile can sometimes be inaccurate. See Fig. 4 for an example.

ommended, the missing physics are unclear and comparisons of calculations using $1D_h$ -SD and $2D_h$ -SD approximations, as well as other intermediate approximations, are needed.

1.3. Overview of current study

In this paper, a first attempt has been made to better understand and quantify the influence of *wave spreading* and *multidirectional waves* on Stokes drift. While the former is essentially a stochastic process that is not sensitive to local or remote forcing conditions, the latter is strongly dependent on the interaction of different wave groups, particularly wind seas (local) and swell waves (remotely-generated). To achieve this, the Stokes drift estimation methods of Webb and Fox-Kemper (2011) are expanded to include directional information about the wave field. Again, prescribed wave spectra and global WAVEWATCH III simulations (version 2.22; Tolman, 2002) are used as a reference and model basis (respectively), while a deep-ocean moored buoy located at Ocean Weather Station P (50°N 145°W) is used as a directional-wave-information observational dataset (Thomson et al., 2013; Belka et al., 2014).⁵ In addition, a new $1D_h$ -SD approximation is introduced to quantify and differentiate these directional effects on Stokes drift.

In Section 2, several examples are selected to illustrate different directional effects in the random seas Stokes drift. In Section 3, differences be-

⁵It should be noted that Rascle and Ardhuin (2013) find sensitivity in Stokes drift with parameterizations used, so this work ought to be repeated with different versions of the model for robustness. However in this paper, our results are less sensitive since we only examine Stokes drift ratios. In addition, we find good agreement among comparisons so WAVEWATCH III simulations analyzed here cannot be grossly incorrect.

111 tween 1D_h-SD and 2D_h-SD are formally explored and a new 1D_h-SD approxi-
 112 mation is developed to include a form of wave spreading advanced by Donelan
 113 et al. (1985). In Sections 4–6, the directional effects of wave spreading and
 114 multidirectional waves on the depth-dependent and depth-integrated Stokes
 115 drift are analyzed using prescribed wave spectra, observations, and model
 116 simulations. Finally, a treatment of error is given in Section 7 and results
 117 are discussed and summarized in Sections 8 and 9 respectively.

118 2. Pitfalls of the unidirectional seas assumption

119 Several simple examples are given below to illustrate some of the chal-
 120 lenges of using 1D wave spectra to calculate Stokes drift. The last example
 121 will provide some insight on how the 1D_h-SD approximation can be improved.

122 2.1. Multidirectional waves and Stokes drift magnitude

123 When multiple wave groups are present, the magnitude of Stokes drift
 124 will depend on the angle of incidence between them. To illustrate, consider
 125 two monochromatic waves passing through the same point from different
 126 directions (see Fig. 1). For simplicity, let the peak frequency and amplitude
 127 of the waves be the same, and suppose that their phase varies slowly and
 128 sufficiently so that interference patterns are not persistent. First recall that
 129 a monochromatic spectrum for a peak frequency f_p , wave amplitude a , and
 130 direction $\tilde{\theta}$, can be defined as

$$\mathcal{S}_{\text{mono},f\theta}(f, \theta) = \frac{a^2}{2} \delta(f - f_p) \delta(\theta - \tilde{\theta}). \quad (6)$$

131 Using the 2D_h-SD approximation, the resulting Stokes drift for this wave is

$$\mathbf{u}_{2D_h, \text{mono}}^S|_{\mathbf{x}_0} = \left(\cos \tilde{\theta}, \sin \tilde{\theta}, 0 \right) \frac{8\pi^3 a^2 f_p^3}{g} \exp \left[\frac{8\pi^2 f_p^2 z_0}{g} \right]. \quad (7)$$

132 Similarly, a bichromatic spectra can be defined for some mean direction $\bar{\theta}$
 133 and angle of incidence θ' as

$$\mathcal{S}_{\text{bi},f\theta}(f, \theta) = \frac{a^2}{2} \delta(f - f_p) [\delta(\theta - \bar{\theta} - \theta') + \delta(\theta - \bar{\theta} + \theta')] , \quad (8)$$

134 and the resulting Stokes drift is then

$$\begin{aligned} \mathbf{u}_{2D_h, \text{bi}}^S \big|_{\mathbf{x}_0} &= (\cos(\bar{\theta} + \theta') + \cos(\bar{\theta} - \theta'), \sin(\bar{\theta} + \theta') + \sin(\bar{\theta} - \theta'), 0) \\ &\quad \times \frac{8\pi^3 a^2 f_p^3}{g} \exp \left[\frac{8\pi^2 f_p^2 z_0}{g} \right] \\ &= 2 \cos \theta' \mathbf{u}_{2D_h, \text{mono}}^S \big|_{\mathbf{x}_0} . \end{aligned} \quad (9)$$

135 This is analogous to comparing 1D_h-SD and 2D_h-SD approximations and it
 136 can be shown that $\mathbf{u}_{1D_h, \text{bi}}^S \big|_{\mathbf{x}_0} = 2 \mathbf{u}_{2D_h, \text{mono}}^S \big|_{\mathbf{x}_0}$ and hence

$$\mathbf{u}_{2D_h, \text{bi}}^S \big|_{\mathbf{x}_0} = \cos \theta' \mathbf{u}_{1D_h, \text{bi}}^S \big|_{\mathbf{x}_0} . \quad (10)$$

137 This implies that in general, the 1D_h-SD approximation will overestimate
 138 Stokes drift whenever both wind seas and swell (or other multidirectional
 139 waves) are present. Note that to the order of approximation considered here,
 140 linear superposition of the waves is assumed to continue to hold for linear
 141 superposition of Stokes drift (see Webb and Fox-Kemper, 2011).

142 2.2. Multidirectional waves and Stokes drift direction

143 In the previous example, the resulting Stokes drift magnitude follows sim-
 144 ple vector addition. This generalizes for any n -chromatic wave, $\mathcal{S}_{n\text{-chro},f\theta} =$
 145 $\sum_{i=1}^n \frac{a_i^2}{2} \delta(f - f_i) \delta(\theta - \theta_i)$, as

$$\mathbf{u}_{n\text{-chro}}^S \big|_{\mathbf{x}_0} = \sum_{i=1}^n (\cos \theta_i, \sin \theta_i, 0) A_i(z_0) \quad (11)$$

146 where $A_i(z) = \frac{8\pi^3 a_i^2 f_i^3}{g} \exp\left[\frac{8\pi^2 f_i^2 z}{g}\right]$ for some amplitude and peak frequency a_i
 147 and f_i . Again for simplicity, consider a bichromatic wave but with different
 148 peak frequencies such that $a_1 = a_2$, $\theta_1 \neq \theta_2$ and $f_2 = e^1 f_1$. Then for the
 149 depth

$$z = \frac{-3g}{8\pi^2 (e^2 - 1) f_1^2} \approx \frac{-0.058}{f_1^2}, \quad (12)$$

150 the Stokes drift direction will be the directional average of both waves,
 151 $(\theta_1 + \theta_2)/2$. However, for any depth above this value, the dominant Stokes
 152 drift direction will be directed more toward the higher-frequency monochro-
 153 matic wave. Likewise, for any depth below this value, the direction will be
 154 oriented more towards the lower frequency wave. We will refer to this effect
 155 as *veer* (to distinguish it from the *directional magnitude loss* in the previous
 156 section) and this will be discussed in more detail later. However in gen-
 157 eral, the direction of Stokes drift should be expected to align with higher
 158 frequency, shorter wavelength wind seas at the surface and lower frequency,
 159 longer wavelength swell at greater depths, whenever both are present.

160 2.3. Wave spreading and Stokes drift magnitude

161 In addition to multidirectional waves, the degree of wave spreading also
 162 contributes to directional magnitude loss in Stokes drift. To illustrate, con-
 163 sider the following modification to the unidirectional spectra in (2) (shown
 164 here again):

$$\mathcal{S}_{1,f\theta}(f, \theta) = \sqrt{\frac{2}{\pi}} \exp[-2\theta^2] \mathcal{S}_f(f),$$

$$\mathcal{S}_{2,f\theta}(f, \theta) = \delta(\theta) \mathcal{S}_f(f).$$

165 Notice that both directional-frequency spectra are normalized to have the
 166 same frequency spectra after integrating in θ and are directed on average in
 167 the $\hat{e}_1$ direction. However due to wave spreading, the magnitudes are not
 168 equivalent and $\|\mathbf{u}_1^S\| \approx 0.882 \|\mathbf{u}_2^S\|$. The $\|\cdot\|$ notation is defined here (and
 169 throughout) as the Euclidian norm.

170 In all three examples, the magnitude and/or direction of Stokes drift is
 171 affected by the use of directionally-averaged 1D wave spectra. In the first
 172 two examples, the differences between 1D_{*h*}-SD and 2D_{*h*}-SD are dependent
 173 upon properties of coincidences of a random sea (e.g., how often are the waves
 174 multidirectional, what are typical frequency and directional differences, etc.).
 175 In this example however, the differences are systematic (i.e., determined by
 176 the wave spreading function) and not a coincidental process. It will be later
 177 shown that the error due to wave spreading in 1D_{*h*}-SD approximations can
 178 be removed if the frequency-dependent directional spread is known.

179 **3. An improved 1D_{*h*}-SD estimate**

180 To aid understanding, formal relations between the 1D_{*h*}-SD and 2D_{*h*}-
 181 SD approximations are defined and utilized in this section to develop an
 182 improved 1D_{*h*}-SD approximation. Incorporating previous work (Webb and
 183 Fox-Kemper, 2011, Appendices A.3 and A.4), this new approximation uses
 184 an empirical spreading function to include the effects of wave spreading and
 185 is used in later sections to differentiate between directional magnitude losses
 186 in Stokes drift due to wave spreading and multidirectional waves.

187 *3.1. Defining a generic directional distribution*

188 The 1D frequency spectrum is defined such that

$$\mathcal{S}_f(f) = \int_{-\pi}^{\pi} \mathcal{S}_{f\theta}(f, \theta) d\theta. \quad (13)$$

189 This is equivalent to splitting the directional-frequency spectrum into a fre-
 190 quency component ($\mathcal{S}_f$) and a directional distribution (D) and integrating in
 191 direction (θ). For a fixed frequency f_α , a fixed directional distribution can
 192 be defined as

$$D_{f_\alpha}(\theta) = \begin{cases} \frac{\mathcal{S}_{f\theta}(f_\alpha, \theta)}{\int_{-\pi}^{\pi} \mathcal{S}_{f\theta}(f_\alpha, \theta) d\theta}, & \mathcal{S}_{f\theta}(f_\alpha, \theta) \neq 0, \\ \frac{1}{2\pi}, & \mathcal{S}_{f\theta}(f_\alpha, \theta) = 0. \end{cases} \quad (14)$$

193 Assuming D can be defined continuously in f (as $D(f, \theta) \equiv D_f(\theta)$), the
 194 directional distribution has the property

$$\int_{-\pi}^{\pi} D(f, \theta) d\theta = 1, \quad (15)$$

195 for every f , and the directional-frequency and frequency spectrum can be
 196 related by

$$\int_0^\infty \int_{-\pi}^{\pi} \mathcal{S}_{f\theta}(f, \theta) d\theta df = \int_0^\infty \int_{-\pi}^{\pi} D(f, \theta) \mathcal{S}_f(f) d\theta df = \int_0^\infty \mathcal{S}_f(f) df. \quad (16)$$

197 *3.2. Defining a generic directional component*

198 Due to the vector component $(\cos \theta, \sin \theta, 0)$ in the integrand, the behavior
 199 of Stokes drift is not always intuitive. To identify the role it plays more
 200 concretely, let the combined integral of any generic directional distribution
 201 and vector component, or

$$\mathbf{H}(f) = \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) D(f, \theta) d\theta, \quad (17)$$

202 be termed the *directional-Stokes-drift-component*.⁶ Then the 2D_h-SD can be
 203 rewritten as

$$\mathbf{u}_{2D_h}^S = \frac{16\pi^3}{g} \int_0^\infty \mathbf{H}(f) f^3 S_f(f) \exp\left[\frac{8\pi^2 f^2}{g} z\right] df. \quad (18)$$

204 Notice that for all unidirectional waves (i.e., $\mathcal{S}_{f,\theta}(f, \theta) = \delta(\theta - \bar{\theta})\mathcal{S}_f(f)$),
 205 $\|\mathbf{H}(f)\| = 1$ for all $f \in \mathbb{R}_+$. In this context, the 1D_h-SD can be thought
 206 of as a 2D_h-SD approximation with $\|\mathbf{H}_{1D_h}\| = 1$ and $\mathbf{H}_{1D_h}/\|\mathbf{H}_{1D_h}\| =$
 207 $(\cos \bar{\theta}, \sin \bar{\theta}, 0)$ for some assumed direction $\bar{\theta}$. The directional-Stokes-drift-
 208 component can now be used to quantify the directional magnitude loss due
 209 to wave spreading alone.

210 3.3. The DHH wave spreading function

211 There have been numerous studies on the topic of directional wave spread-
 212 ing and several well-known spreading functions proposed by Mitsuyasu et al.
 213 (1975), Hasselmann et al. (1980), and Donelan et al. (1985) were consid-
 214 ered for the directional-Stokes-drift-component. In each study, an assumed
 215 parametric form was used with observational wave spectra to derive an em-
 216 pirical function for fetch-limited conditions in deep water.⁷ Mitsuyasu et al.
 217 (1975) and Hasselmann et al. (1980) used pitch-roll-heave buoys to mea-
 218 sure the wave field whereas Donelan et al. (1985) used a spatial array of 14

⁶This was previously referred to as *directional spread loss* in Webb and Fox-Kemper (2011).

⁷Since these functions have been parametrized for fetch-limited conditions, they are likely to be inaccurate for fully-developed or even mixed sea conditions (Alves and Melo, 1999). However, the authors believe they are good candidates for the purposes of this initial analysis.

219 wave staffs (Young, 2010). Of these three studies, the Donelan-Hamilton-Hui
 220 (henceforth abbreviated *DHH*) field campaign had the highest spectral res-
 221 olution and subsequent (observational and numerical) analyses have found
 222 better agreement with the DHH wave spreading function (Young, 1994; For-
 223 ristall and Ewans, 1998; Ewans, 1998).

224 For these reasons and others to be discussed later, the DHH wave spread-
 225 ing function has been chosen for the directional-Stokes-drift-component. How-
 226 ever it has been slightly modified here; the amplitude and domain have been
 227 adjusted to be normal and periodic in θ (Webb and Fox-Kemper, 2011) as

$$D_{\text{DHH}}(f, \theta; f_p, \bar{\theta}) = \frac{\beta(f/f_p)}{2 \tanh[\beta(f/f_p)\pi]} \times \begin{cases} \text{sech}^2[\beta(f/f_p) \theta], & \bar{\theta} = 0, \\ \text{sech}^2[\beta(f/f_p) (\theta - \bar{\theta})], & -\pi + |\bar{\theta}| \leq \bar{s} \theta \leq \pi, \\ \text{sech}^2[\beta(f/f_p) (\theta - \bar{\theta} + 2\pi \bar{s})], & -\pi \leq \bar{s} \theta \leq -\pi + |\bar{\theta}|. \end{cases} \quad (19)$$

228 Here, f_p is the peak frequency, $\bar{\theta}(f) \in [-\pi, \pi]$ is the mean direction for a
 229 particular f , $\bar{s}$ is the sign of $\bar{\theta}$, and β is given by

$$\beta(f/f_p) = \begin{cases} 2.61 (f/f_p)^{1.3}, & 0.56 < f/f_p \leq 0.95, \\ 2.28 (f/f_p)^{-1.3}, & 0.95 < f/f_p < 1.6, \\ 1.24, & \text{otherwise.} \end{cases} \quad (20)$$

230 In addition, it should be noted that β is only piecewise continuous.

231 3.4. The *DHH* directional-Stokes-drift-component

232 Unfortunately, calculating the directional-Stokes-drift-component for the
 233 DHH wave spreading function is fairly complicated and requires the use of
 234 hypergeometric functions with complex arguments (Luke, 1969). To simplify,

235 a Padé approximate of order $[2/2]$ (Bender and Orszag, 1978) has been used
 236 to approximate the exact solution within a relative error of 3×10^{-3} , given
 237 by

$$\begin{aligned} \mathbf{H}_{\text{DHH}}(f; f_p, \bar{\theta}) &= (\cos \bar{\theta}, \sin \bar{\theta}, 0) \\ &\times \begin{cases} \frac{0.52f_p^2 - 3.3f_p f + 8.9f^2}{f_p^2 - 3.4f_p f + 8.9f^2}, & 0.56 < f/f_p \leq 0.95, \\ \frac{0.98f_p^2 - 0.19f_p f + 0.0058f^2}{f_p^2 - 0.26f_p f + 0.12f^2}, & 0.95 < f/f_p < 1.6, \\ 0.777, & \text{otherwise.} \end{cases} \end{aligned} \quad (21)$$

238 Notice that the DHH directional-Stokes-drift-component requires knowledge
 239 of the peak frequency and the mean direction and is still a function in terms of
 240 frequency (i.e., the mean direction is not necessarily constant for all frequen-
 241 cies). While not the focus of this paper, this approximation is significantly
 242 faster computationally than using (19) to solve (17), requiring about the
 243 same number of multiplications and additions but *no special function calls*
 244 *or numerical integration*.

245 The magnitude of the DHH directional-Stokes-drift-component is depicted
 246 in Fig. 2. Notice that it is bounded as $0.777 \leq \|\mathbf{H}_{\text{DHH}}\| \leq 0.934$. This im-
 247 plies $1D_h$ -SD approximations may be overestimating $2D_h$ -SD by as much as
 248 30% due to wave spreading alone (since $\max \{\|\mathbf{H}_{1D_h}\| / \|\mathbf{H}_{DHH}\|\} \approx 1.287$).

249 3.5. The new $1D_h$ -DHH-SD approximation

250 As mentioned in the example in Section 2.3, the $1D_h$ -SD approximation
 251 can be improved by using a wave spreading function to approximate the full
 252 2D spectra for Stokes drift calculations. Here, the Padé-approximated, DHH
 253 directional-Stokes-drift-component, (21), will be used to modify (4) as

$$\mathbf{u}_{\text{DHH}}^S = \hat{\mathbf{e}}^w \frac{16\pi^3}{g} \int_0^\infty H_{\text{DHH}}(f/f_p) f^3 S_f(f) \exp\left[\frac{8\pi^2 f^2 z}{g}\right] df, \quad (22)$$

254 where $H_{\text{DHH}}(f/f_p) = \|\mathbf{H}_{\text{DHH}}(f; f_p, \bar{\theta} = \hat{\mathbf{e}}^w)\|$ and $\hat{\mathbf{e}}^w$ is the chosen mean
 255 direction. The approximation requires knowledge of the peak frequency but
 256 this is fairly straight-forward to calculate in a model implementation. As
 257 with any $1D_h$ -SD approximation, the choice of $\hat{\mathbf{e}}^w$ is not always clear and
 258 will be discussed in Section 5. The improved $1D_h$ estimate will be termed
 259 *1D_h-DHH-SD* to distinguish it from the unidirectional approximation ($1D_h$ -
 260 SD). As previously pointed out, there are multiple spreading functions and
 261 hence, the improved estimate is not limited to the use of the DHH directional-
 262 Stokes-drift-component. While rarely used here, an alternative directional-
 263 Stokes-drift-component (also displayed in Fig. 2) is derived in Appendix A
 264 and is based on a modification to the DHH wave spreading function proposed
 265 by Banner (1990) (henceforth abbreviated DHH-B).

266 4. Depth-dependent Stokes drift magnitude

267 Here, the $1D_h$ -DHH-SD will be used with prescribed wave spectra (i.e.,
 268 empirically formulated), observations, and model output to differentiate the
 269 directional magnitude losses in Stokes drift due to wave spreading and multi-
 270 directional waves. For *surface* Stokes drift magnitudes, it will be shown that
 271 estimates from all three types are in close agreement and that it is possible
 272 to correct $1D_h$ -SD for wave spreading by multiplying by a constant.

273 4.1. Analysis using *1D_h-DHH-SD* with prescribed wave spectra

274 Here, spectra from three different empirically-derived formulas will be
 275 used to examine the role of wave spreading. Since prescribed spectra require
 276 knowledge of the peak frequency, *e*-folding depths will be used instead of

277 depth to remove the peak frequency dependence and analyze the results. Let
 278 $n \in \mathbb{R}_+$ and $z_n = -ng/(8\pi^2 f_p^2)$. Then for a monochromatic wave,

$$\|\mathbf{u}_{\text{mono}}^S(z_n)\| = e^{-n} \|\mathbf{u}_{\text{mono}}^S(z=0)\|. \quad (23)$$

279 In Table 2, three different peak frequencies are chosen (from high to low) to
 280 illustrate a typical range of e -folding depths. Notice that at each e -folding
 281 depth, the depth for the lowest frequency is approximately 50 times larger
 282 than the highest. This will aid an intuitive understanding shortly.

283 In addition to JONSWAP (Hasselmann et al., 1976) and Pierson and
 284 Moskowitz (1964, *PM* hereafter) prescribed spectra, empirical spectral shapes
 285 from Donelan et al. (1985) (abbreviated *DHH* as well) are also used to an-
 286 alyze the effects of wave spreading. Fetch-limited (wave age of 0.25) and
 287 fully-developed (wave age of 1.2) cases are chosen for the DHH spectra to
 288 mirror JONSWAP and PM shapes respectively. See Webb and Fox-Kemper
 289 (2011) for details and a visual comparison of the spectra.

290 To examine the reduction in Stokes drift magnitude due to wave spread-
 291 ing, ratios of $1D_h$ -DHH-SD to $1D_h$ -SD for e -folding depths are presented in
 292 Table 3 (for select values) and in Fig. 3. For the prescribed spectra selected,
 293 there is approximately a 8–20% directional magnitude loss when using $1D_h$ -
 294 DHH-SD instead of $1D_h$ -SD. This loss is greatest for all spectra at the surface
 295 and flattens out at an e -folding depth near 3 before slowly increasing again.
 296 This is expected since initially the exponential component of Stokes drift will
 297 filter mostly higher frequencies as depth is increased (see Table 2). Eventu-
 298 ally however, the majority of Stokes drift will be comprised of spectra with
 299 frequencies less than the peak and the magnitude loss will asymptotically
 300 increase towards 22%.

301 4.2. Analysis using $1D_h$ -DHH-SD with observational data

302 Since interesting comparisons with $2D_h$ -SD are not possible with pre-
303 scribed spectra (as it is the superposition of different wave packets from dif-
304 ferent sources that leads to an interesting two-dimensional wave spectrum),
305 it is necessary to use real or realistic data to study 2D spectrum effects on
306 Stokes drift. Wave buoys have commonly been used to measure vertical ve-
307 locities to infer the frequency spectra at a point (Holthuijsen, 2007). It is
308 also possible for some buoys to estimate wave direction and thus infer the
309 directional-frequency spectra. Care is needed however in interpreting this
310 inferred spectra since an array of buoys would be needed to measure the full
311 2D spectra.

312 Here, observational data from Ocean Weather Station P (henceforth *OWS-*
313 *P*), a CDIP⁸ Datawell directional buoy (O'Reilly et al., 1996) stationed in
314 deep water in the northeastern Pacific Ocean (50°N, 145°W), is used to
315 compare $1D_h$ -SD, $1D_h$ -DHH-SD, and $2D_h$ -SD approximations. A contin-
316 uous 27 month period, 30 minute intervals from 2010/6/21 to 2012/9/20,
317 has been selected for comparison. Processed data (obtained from [http:](http://cdip.ucsd.edu)
318 [//cdip.ucsd.edu](http://cdip.ucsd.edu)) is comprised of 1D frequency spectra with 64 frequency
319 bands (with an initial and cutoff frequency of 0.025 Hz and 0.58 Hz respec-
320 tively) and the first four non-constant Fourier coefficients of the directional
321 distribution (for each frequency band).⁹ The directional-frequency spectra
322 is generated using the Maximum Entropy Method (henceforth *MEM*; Lygre

⁸Coastal Data Information Program, Scripps Institution of Oceanography.

⁹By definition, the first complex coefficient of the directional distribution is a constant ($c_0 = 1/2\pi$).

323 and Krogstad, 1986) with a 9 deg directional discretization, chosen both to
 324 minimize abrupt transitions between the frequency bins and to use a sim-
 325 ilar discretization ratio ($\Delta f/\Delta\theta$) as WAVEWATCH III (see Fig. 4b for an
 326 example of the MEM-generated directional distribution). In addition, all
 327 approximations assume a spectral f^{-5} tail (also for later comparison with
 328 WAVEWATCH III output).

329 Casual inspection of the 2D spectral data shows that both wave spreading
 330 and multidirectional waves are naturally present. As a result, occasional dif-
 331 ferences between the approximations are strikingly noticeable. In Fig. 4a, a
 332 snapshot from 2012/04/12 at 05:32 of Stokes drift magnitudes are displayed
 333 for the first 9 m of the three different approximations. In the absence of
 334 multidirectional waves, Stokes drift has an exponentially decaying profile.
 335 In this figure interestingly, the surface magnitude of the $2D_h$ -SD is **half its**
 336 **subsurface value at a depth of 9 m**. This increase of Stokes drift with
 337 depth can be attributed to the presence of multiple wave groups since the
 338 $1D_h$ approximations do not exhibit this behavior. Examining the directional
 339 distribution for the same time period shown in Fig. 4b, we find that there is
 340 sharply peaked swell and diffuse wind seas traveling in approximately oppo-
 341 site directions (roughly east and west respectively). The observations at this
 342 time are an extreme example of real-world multidirectional waves.

343 From examples presented in Section 2, it is clear that both wave spread-
 344 ing and multidirectional waves can affect the magnitude of Stokes drift. To
 345 determine if one is more influential, median ratios of the different Stokes
 346 drift approximations have been calculated for the two year period and are
 347 displayed in Fig. 5 (closed circles). Medians are used for analysis here due to

348 large skews and comparisons with mode-centered distributions in Section 4.3;
 349 means (open circles) are included for later reference (with Fig. 11). In ad-
 350 dition, the shaded interval indicates two-thirds of the distribution centered
 351 about the median and is generally not equivalent with the $33.\bar{3}$ percentile
 352 (for a non-normal distribution). The ratios of $1D_h$ -DHH-SD to $1D_h$ -SD have
 353 also been displayed for comparison with the prescribed spectra. While it
 354 is unclear if the results from prescribed spectra fall within the shaded blue
 355 region (due to the use of an e -folding depth), the trend is similar with the
 356 largest loss at the surface. Since the calculations were limited to depths of
 357 9 m however, it is also unclear if the observational curve will flatten out as
 358 well.

359 Observational ratios between the $1D_h$ and $2D_h$ approximations are illus-
 360 trated in the other two curves, $2D_h$ -SD to $1D_h$ -SD and $2D_h$ -SD to $1D_h$ -DHH-
 361 SD. The (two-third centered) distributions about the median are large in both
 362 and are likely caused by multidirectional waves rather than spreading since
 363 the distribution for the $1D_h$ -DHH-SD to $1D_h$ -SD ratio is narrowly banded.
 364 In Webb and Fox-Kemper (2011), it was identified that $1D_h$ -SD tended to
 365 overestimate the $2D_h$ -SD approximation in the model data analyzed by about
 366 33% (median value) at the surface. This reduction is of a similar magnitude
 367 to that found by Ardhuin et al. (2009) and Breivik et al. (2014). While
 368 calculated but not shown, **the unidirectional overestimation median is**
 369 **nearly double in these observations at 62%**. In comparison, the $1D_h$ -
 370 DHH-SD overestimates the $2D_h$ -SD by 28% at the surface and levels out to
 371 about 14% for depths 3 m and below. This is still a sizable observational
 372 difference but it is a definite improvement. Further discussion on the role of

373 wave spreading and multidirectional waves using the OWS-P buoy data will
374 be delayed until Section 7.

375 *4.3. Analysis using $1D_h$ -DHH-SD with WAVEWATCH III*

376 While buoys can provide good spectral data, the data is sparse spatially
377 and it is useful to also compare the results with global model-generated data.
378 A 1-year 2D spectral data set (three-hourly output for the year 2000) has
379 been generated here by WAVEWATCH III (ver. 2.22) and is used to compare
380 differences in Stokes drift approximations.

381 With the exception of output frequency, model setup is identical with
382 Webb and Fox-Kemper (2011) and implementation details include 25 fre-
383 quency and 24 directional bins (with an initial and cutoff frequency of
384 0.0418 Hz and 0.411 Hz respectively), f^{-5} tail, and Tolman and Chalikov
385 source terms. In addition, all simulations are forced with CORE2 (Large
386 and Yeager, 2008) winds¹⁰ with appropriate sea surface temperatures (Hadley
387 SST: Rayner et al., 2006) and sea ice concentrations (Bootstrap Sea Ice Con-
388 centrations from Nimbus-7 SMMR and DMSP SSM/I, v2: Comiso, 1999) on
389 a $1^\circ \times 1.25^\circ$ latitude-longitude grid of $(-78:78) \times (0:358.75)$ respectively.

390 In the OWS-P buoy data analyzed in Fig. 5, the majority of uncertainty
391 (in the $1D_h$ approximations) is likely due to the presence of multidirectional
392 waves. This exercise has been repeated with WAVEWATCH III output and
393 an even larger amount of uncertainty is found on a global scale.¹¹ To ex-
394 plore why, scatter plots have been generated in Figs. 6 and 7 to help identify

¹⁰Available from <http://data1.gfdl.noaa.gov/nomads/forms/mom4/COREv2.html>.

¹¹See Fig. 11 for a comparison of mean Stokes drift magnitude ratios (the two-third centered distributions are not shown).

395 correlations. The dependent values, Stokes drift magnitudes (m/s) at vari-
396 ous depths, are normalized by the $2D_h$ -SD sample maximum (listed in each
397 figure) and shaded by their density distribution. The regions are colored
398 red, green, and blue by the highest 0–30%, 31–60%, and 61–90% centered
399 distributions respectively.

400 *4.3.1. Surface comparisons*

401 A comparison of Figs. 6a and 7a, surface magnitudes of $2D_h$ -SD (y -
402 axis) versus $1D_h$ -SD and $1D_h$ -DHH-SD (x -axes) respectively, reveals that
403 the sources of error can be cleanly separated. The shaded regions in both
404 figures are approximately equivalent but oriented about different slopes. In
405 Fig. 7a, the upper part of the shaded regions aligns well with the line $y = x$
406 and indicates that the too shallow slope in Fig. 6a is due to ignoring the
407 effects of wave spreading. The improvement found in slope in Fig. 7a is the
408 direct result from including these effects. It should be pointed out that the
409 alignment in Fig. 7a is not guaranteed. As an example, the slope could have
410 easily been higher than unity if the directional distribution had been too
411 diffuse for higher frequencies. Verifying this agreement is a good test for any
412 potential directional-Stokes-drift-component.

413 Errors due to multidirectional waves are also present. In both figures, the
414 density distribution normal to the slope of orientation is sharply peaked near
415 the upper part and then gradually descends (in the normal direction with the
416 negative y component). This sharp peak is expected since the ocean surface
417 is rarely a mix of wind seas and swell. In addition, the asymmetry is expected
418 as well. Unlike with wave spreading, the presence of multidirectional waves
419 will only cause an overestimation by the $1D_h$ approximations and never an

underestimation.

The correlation holds in Fig. 8 as well, where the surface magnitudes of $1D_h$ -DHH-SD (y -axis) versus $1D_h$ -SD (x -axis) have been plotted for both WAVEWATCH III model and OWS-P buoy data (respectively). In both subfigures, the density distribution normal to the slope of orientation is singularly peaked, which is expected since the magnitudes are invariant to the angle of incidence between the multidirectional waves. In addition, the slopes of the lines the scatter plots fall on are identical (plotted in red with $m = 0.795$) and are in close agreement with the values derived from empirical shapes (0.799, 0.812). This implies that reducing the surface $1D_h$ -SD magnitudes by 20% is a good rule of thumb to approximate directional magnitude losses and correct for wave spreading. This could be very useful since surface magnitudes can be calculated without full knowledge of the 1D wave spectra (as shown in Webb and Fox-Kemper, 2011). This approach could be extended to include the effects of multidirectional waves globally but it would probably be more effective in a specific region where the statistics of the wave field are known.

4.3.2. Subsurface comparisons

Scatter plots of subsurface magnitude ratios for two depths, 3 m and 9 m, have been generated and are displayed in Figs. 6b,c and 7b,c. As previously, the $1D_h$ -DHH-SD approximation corrects for systematic wave spreading errors while the slope of orientation for the $1D_h$ -SD approximation is less than unity. Interestingly, the slope of the latter increases with the depths displayed and implies that the role of wave spreading diminishes with depth (since the minimum overestimation error decreases). This is consistent with

445 prescribed wave spectra results and also suggests that the trend will reverse
446 for deeper depths (albeit for nominal Stokes drift magnitudes).

447 In both approximations, the density distributions parallel to the slope of
448 orientation become more sharply peaked with depth and can be explained
449 by the higher filtering of wind sea spectra by the exponential component. At
450 the surface, Stokes drift is largely determined by the higher frequencies of the
451 broad-banded wind seas (when multidirectional waves are not significantly
452 present) and this contributes to the broader distribution of magnitudes. How-
453 ever at subsurface depths, the wind sea contribution is quickly attenuated
454 in comparison with the narrow-banded swell. This can be seen in Table 2,
455 where e -folding depths of 1 and 3 corresponded to approximately 1 m and
456 3 m for a monochromatic wave with a higher wind sea frequency of 0.34 Hz.
457 In general, the directional magnitude losses due to the interaction of wind
458 seas and swell will decrease with depth (due to the declining role wind seas
459 play) while losses due to the interaction of other multidirectional waves will
460 persist.

461 5. Depth-dependent Stokes drift direction

462 Up until this point, the discussion has largely focused on Stokes drift mag-
463 nitude and not direction. Without full knowledge of the 2D wave spectra, it
464 is not immediately clear what the actual direction should be, particularly for
465 subsurface calculations. At the surface, approximations typically assume the
466 direction of Stokes drift is the same as the wind sea and thus the wind di-
467 rection. But what about subsurface Stokes drift orientation? Should it align

468 with the direction of the wind, the mean wave direction,¹² or a combination
 469 of both? Here, the topic of veer is addressed using the same approach as the
 470 last section.

471 In the example in Section 2.2, the direction of the $2D_h$ -SD (for a bichro-
 472 matic wave with different peak frequencies) varied with depth, with a pref-
 473 erence for the directions of the higher and lower frequencies for relatively
 474 shallower and deeper depths respectively. This insight is explored further in
 475 Figs. 9 and 10 (using scatter plots again) to determine how well the depth-
 476 dependent and depth-integrated $2D_h$ -SD aligns with the 10 m surface wind
 477 and mean wave directions.

478 In Figs. 9a and 10a, the $2D_h$ -SD at the surface aligns best with the 10 m
 479 wind. This result is expected since as previously mentioned, the higher fre-
 480 quencies of the wind seas contribute the most to the surface Stokes drift
 481 magnitude. However, this correlation no longer holds as depth is increased.
 482 As the higher frequencies are filtered by the exponential depth component,
 483 the direction of $2D_h$ -SD quickly begins to reorientate toward the mean wave
 484 direction. At a depth of 1 m, the correlations for both wind and mean wave
 485 directions are already comparable (not shown). By 3 m and 9 m, the $2D_h$ -SD
 486 directions are in good and near complete agreement (respectively) with the
 487 mean wave direction (Figs. 9b,c and 10b,c).

488 This veer is consistent with the notion that wind seas dominate surface
 489 Stokes drift and swell dominates subsurface Stokes drift. While it is possible
 490 to parameterize this transition, the large degree of uncertainty suggests that

¹²Following Tolman (2002), the mean wave direction is functionally computed as
 $\theta_w = \text{atan2}[b, a]$, with $\theta_w \in (-\pi, \pi]$ and $(a, b) = \int_0^\infty \int_{-\pi}^\pi (\cos \theta, \sin \theta) \mathcal{S}_{f,\theta}(f, \theta) d\theta df$.

491 it might be unwise to do so. If the intermediate subsurface Stokes drift
 492 direction is important (as in Sullivan et al., 2012), it is recommended that
 493 a spectral wave model is used to calculate this nonlocal, nonlinear vector
 494 component.

495 6. Depth-integrated Stokes drift magnitude and direction

496 Observations of surface slicks (Qazi et al., 2013), computer cards (Weller
 497 et al., 1985), high-frequency (HF) radar systems (Chapman et al., 1997), and
 498 small drifters (Carrier et al., 2013) tend to emphasize Lagrangian transport
 499 of the upper meter of the ocean or less, but the surface Stokes drift is not
 500 always the most useful measure of dynamical implications. For one, if the
 501 wave field has an f^{-4} spectrum, then the surface $2D_h$ -SD integral does not
 502 converge, even though the integral at just 1% of the peak e-folding depth
 503 does (Webb and Fox-Kemper, 2011). As a driving mechanism for Langmuir
 504 turbulence, the surface unidirectional Stokes drift may not be as predictive as
 505 the depth-integrated form (Harcourt and D’Asaro, 2008; Van Roekel et al.,
 506 2012; Sullivan et al., 2012). Furthermore, the depth-integrated Stokes drift
 507 depends on a different moment of the wave spectrum than the surface Stokes
 508 drift, which lends it very distinct statistics (Myrhaug, 2013b). Thus, in
 509 addition to the directional effects of Stokes drift at various depths including
 510 the surface, we consider the depth-integrated Stokes drift.¹³

511 Formally, the Stokes drift approximations ($1D_h$ -SD, $1D_h$ -DHH-SD, and

¹³Sometimes also called the Stokes volume transport.

512 2D_h-SD) can be depth integrated as

$$\begin{aligned}
\mathbf{u}_*^{S\text{-int}} &= \int_{-\infty}^0 \mathbf{u}_*^S(z) dz \\
&= \int_{-\infty}^0 \left\{ \frac{16\pi^3}{g} \int_0^\infty \mathbf{H}_*(f) f^3 S_f(f) \exp\left[\frac{8\pi^2 f^2 z}{g}\right] df \right\} dz \\
&= 2\pi \int_0^\infty \mathbf{H}_*(f) f S_f(f) df.
\end{aligned} \tag{24}$$

513 The reordering of integration is valid here since the integrand is piecewise
514 continuous and non-negative (Roussos, 2013). As previously alluded, we can
515 see that the depth-integrated Stokes drift resembles the *first spectral moment*
516 (with the addition of a directional component that is dependent upon the
517 approximation used) and that (24) is well-defined for the aforementioned
518 f^{-4} spectrum (since $\|\mathbf{H}_*\| \leq 1$). The surface Stokes drift depends on the
519 *third moment* of the spectrum, and thus is significantly more sensitive to
520 high-frequency waves. Here the depth-integrated approximations will use
521 the prefix ‘int’ (int-1D_h-SD, int-1D_h-DHH-SD, int-2D_h-SD) to distinguish
522 from Stokes drift estimated at a particular depth.

523 Scatter plots using the depth-integrated Stokes drift approximations are
524 displayed in Figs. 6d, 7d, 9d, and 10d for comparison with previous results.
525 In both magnitude and direction, we can see that the depth-integrated Stokes
526 drift approximations resemble their non-depth-integrated counterparts at
527 subsurface depths, particularly for the case $z=-9$ m. Thus, *both the depth-*
528 *integrated and subsurface Stokes drift are aligned with the swell, not wind*
529 *seas*. While these results are intuitively expected, mathematically it is not
530 immediately clear why the resemblance favors a certain depth since there is
531 no z -component in (24) to filter out higher frequencies. However, a compar-
532 ison of the depth-dependent and depth-integrated integrands shows that the

533 normalized distributions will be similar as long as the expression,

$$g(f; z) = \alpha_z f^2 \exp\left[\frac{8\pi^2 f^2 z}{g}\right], \quad (25)$$

534 minimizes the filtering of $f\mathcal{S}_f$ about an important interval of frequencies for
 535 some normalizing constant α_z . To illustrate, consider a second order Taylor
 536 series approximation of (25) centered about f_a , the median in some frequency
 537 subset. Then the approximation will also be centered about f_a as long as
 538 $g'(f_a; z) = 0$, which is equivalent to

$$z = g/8\pi^2 f_a^2. \quad (26)$$

539 For simplicity, define the subset to contain all of the frequencies in the distri-
 540 butional median bin of $f\mathcal{S}_f$. Then solving for and substituting $f_a=0.12$ Hz
 541 into (26), gives a depth of 8.6 m, with the majority of $f\mathcal{S}_f$ attenuated by less
 542 than 33% for the interval covering the subset.

543 7. Directional magnitude loss and veer in Stokes drift

544 Using three different types of spectra, it has been shown that both wave
 545 spreading and multidirectional waves contribute to directional magnitude
 546 losses in Stokes drift. Analysis suggests that directional-magnitude-loss er-
 547 rors in the $1D_h$ -SD approximation due to wave spreading are both systematic
 548 and removable. Loss errors due to the interaction of locally and remotely-
 549 generated waves however appear to be random and add a large degree of
 550 uncertainty in both $1D_h$ approximations. A detailed quantification of both
 551 error types would be useful but only a brief treatment will be given here.

552 In Fig. 8, surface scatter plot comparisons of $1D_h$ -SD versus $1D_h$ -DHH-SD
 553 indicate that the addition of wave spreading does not increase uncertainty

554 in the improved $1D_h$ approximation using either OWS-P buoy or WAVE-
 555 WATCH III model data. This implies that the physical processes may be
 556 independent of each other. In fact, further inspection reveals that the di-
 557 rectional effects are largely uncorrelated since the following relation holds to
 558 within a percent ($< 0.2\%$) in both buoy and model data sets:

$$\begin{aligned} \left\langle \frac{\|\mathbf{u}_{2D_h}^S\|}{\|\mathbf{u}_{1D_h}^S\|} \right\rangle &= \left\langle \frac{\|\mathbf{u}_{2D_h}^S\|}{\|\mathbf{u}_{DHH}^S\|} \frac{\|\mathbf{u}_{DHH}^S\|}{\|\mathbf{u}_{1D_h}^S\|} \right\rangle \\ &\approx \left\langle \frac{\|\mathbf{u}_{2D_h}^S\|}{\|\mathbf{u}_{DHH}^S\|} \right\rangle \times \left\langle \frac{\|\mathbf{u}_{DHH}^S\|}{\|\mathbf{u}_{1D_h}^S\|} \right\rangle. \end{aligned} \quad (27)$$

559 Here, the notation $\langle \cdot \rangle$ is used to denote temporal and unweighted spatial
 560 (model only) means.¹⁴ Since the $2D_h$ -SD to $1D_h$ -DHH-SD and $1D_h$ -DHH-
 561 SD to $1D_h$ -SD ratios are largely uncorrelated, both can be compared side by
 562 side to measure the directional effects of wave spreading and multidirectional
 563 waves separately. In addition, the reciprocal form of (27) also holds ($< 0.4\%$)
 564 and the appropriate means can also be used to assess overestimation errors
 565 in the $1D_h$ -SD approximation. This additional calculation is necessary since
 566 $\langle 1/X \rangle$ is not necessarily equivalent to $1/\langle X \rangle$.

567 To facilitate discussion, mid-Pacific buoy (circles) and global model
 568 (squares) mean magnitude ratios have been plotted in Fig. 11 so that the **bot-**
 569 **tom** and **top** halves of the figure can be used to estimate depth-dependent
 570 **magnitude losses** and **overestimation errors** respectively. In addition,
 571 we will refer to the mean $1D_h$ -DHH-SD to $1D_h$ -SD, $2D_h$ -SD to $1D_h$ -DHH-SD,
 572 and $2D_h$ -SD to $1D_h$ -SD ratio curves (and their reciprocals) by the physical
 573 processes they represent, namely *wave spreading*, *multidirectional wave(s)*,

¹⁴No latitudinal adjustments have been made for changing grid cell sizes.

574 and *combined* respectively. Due to large skews in the two latter ratio dis-
575 tributions however (see Fig. 5), less emphasis should be placed on the mean
576 values and more on the trends they represent.

577 At first glance of Fig. 11, we find that the wave spreading curves are co-
578 incident between buoy and model data while in contrast the multidirectional
579 wave curves are dissimilar near the surface. The surface differences may be
580 due to the peculiarities of the OWS-P site or model deficiencies (such as
581 errors in long-distance swell propagation or an inability to resolve bimodal
582 wind seas). To illustrate, magnitude scatter plots of the $2D_h$ -SD (y -axis)
583 versus $1D_h$ -SD and $1D_h$ -DHH-SD (x -axes) have been generated for two dif-
584 ferent depths ($z = 0, -3$) using OWS-P buoy data in Fig. 12. Notice that
585 higher surface magnitudes slightly deviate from the general patterns observed
586 in WAVEWATCH III model data but do not for subsurface depths. These
587 differences could be caused by bimodal wind seas present in OWS-P buoy
588 data (see Fig. 4b for an example) but not in WAVEWATCH III output. This
589 interesting possibility will be elaborated on in Section 8.

590 Examining the bottom half of Fig. 11, we see that there is a 20% (0.795)
591 reduction in magnitude at the surface due to wave spreading alone and the
592 reduction diminishes somewhat with depth to 14% at 9 m. In comparison,
593 magnitude losses due to multidirectional wave effects range between 7–23% at
594 the surface before converging to 17% for depths 7 m and deeper. Comparing
595 side by side, wave spreading effects are similar with or dominate multidir-
596 rectional wave effects from the surface until about a depth of 5 m, where
597 both contribute about a 16% decrease. For deeper depths, there is better
598 agreement among buoy and model data and multidirectional wave effects

599 dominate the magnitude loss. The combined directional magnitude losses
600 range from 26–39% with a final and subsurface average of 29%.

601 In the top half of Fig. 11, the multidirectional wave curves are even more
602 dissimilar due to large skew and many of the overestimation errors are larger
603 than what is predicted by their reciprocal counterparts. The overestimation
604 errors due to wave spreading alone are about 26% at the surface and decrease
605 to 16% at 9 m. For depths 1 m and greater, both buoy and model means
606 suggest overestimation errors due to multidirectional waves will dominate,
607 ranging from an additional 26–43%. The combined overestimation errors are
608 particularly substantial and range from approximately 45–71% (both are at
609 the surface).

610 As a companion comparison, mean ratios have been tabulated for the
611 depth-integrated approximations in Table 4 using WAVEWATCH III model
612 data. Magnitude losses from wave spreading and multidirectional wave effects
613 alone are 16% and 17% respectively, giving a combined loss of 30%. Interest-
614 ingly, overestimation errors from multidirectional wave effects are more than
615 double than those of wave spreading at 41% and 19% respectively, giving
616 a combined error of 68%. These results are similar with depth-dependent
617 Stokes drift subsurface ratios, which in turn is consistent with previous anal-
618 ysis in Section 6.

619 In addition to magnitude, the direction of Stokes drift is also affected and
620 multidirectional waves will generate a directional veer. Instead of examining
621 mean ratios however, a rough estimate of global directional errors are made
622 using confidence intervals from scatter plots. For surface spectral moment
623 approximations, a typical 10 m wind alignment assumption will lead to a

624 random directional error. Examining Fig. 9a, we find that the 30/60/90%
625 confidence intervals are bounded (approximately) by $\pm 0.12/0.28/0.84$ radi-
626 ans ($\pm 7/16/48$ deg). Contrast this with Fig. 9c where at a 9 m depth,
627 the 30/60/90% confidence interval bounds using a mean wave direction as-
628 sumption are only $\pm 0.05/0.09/0.21$ radians ($\pm 3/5/12$ deg). Likewise for the
629 depth-integrated form in Fig. 9d, the 30/60/90% confidence interval bounds
630 for the same assumption are $\pm 0.09/0.16/0.35$ radians ($\pm 5/9/20$ deg).

631 8. Discussion

632 Directional spreading is an active field of research and there have been
633 many advances since the classical measurements of Mitsuyasu et al. (1975),
634 Hasselmann et al. (1980), and Donelan et al. (1985). There are now a vari-
635 ety of techniques (Fourier Expansion Method, Maximum Entropy Method,
636 Maximum Likelihood Method, and Wavelet Directional Method) to extract
637 the directional spectra and accuracy will depend upon the method chosen
638 (Young, 1994; Alves and Melo, 1999; Young, 2010). In addition, there is
639 now strong evidence that fetch-limited waves in deep water are bimodal for
640 frequencies higher than the peak frequency (Young et al., 1995; Ewans, 1998;
641 Alves and Melo, 1999; Hwang et al., 2000). While not demonstrated here,
642 a bimodal directional distribution will generate multidirectional magnitude
643 losses (in addition to wave spreading ones) that are not present in a uni-
644 modal directional distribution. This could be a particularly important result
645 for any model simulation since the $2D_h$ -SD approximation will likely overes-
646 timate Stokes drift if the bimodal wind seas are not resolved properly.

647 As is evident from analysis, near surface Stokes drift magnitudes are

648 particularly sensitive to the higher-frequency wind seas. This implies that
649 any improved $1D_h$ -SD estimate will also be sensitive to wind seas near the
650 surface and any wave spreading function that is too broad (or narrow) will
651 overestimate (or underestimate) magnitude losses. At this early stage of in-
652 vestigation, no attempt has been made to measure the directional spreading
653 in the OWS-P buoy or WAVEWATCH III model data. This endeavor cer-
654 tainly would be useful in future studies for measuring the effectiveness of the
655 spreading function chosen (particularly for mixed seas conditions) and con-
656 ducting any sensitivity analyses of the improved estimate (particularly near
657 the surface). In addition, the measured directional spreading might also be
658 a useful metric for comparing $2D_h$ -SD calculations from different data sets.

659 One of the shortcomings of this study is that the spreading functions
660 considered here are designed for fetch-limited conditions and are likely to be
661 inaccurate at measuring swell (Alves and Melo, 1999). In addition, these
662 classical spreading functions are only tuned for one wave group. As a result,
663 the directional spread for wind seas is likely to be too broad when large
664 swell are also present (i.e., the peak frequency from swell is used instead to
665 determine the spread for wind seas) (Alves and Melo, 1999). This is the main
666 reason why the DHH-B directional-Stokes-drift-component was not used in
667 this analysis. Most other spreading functions, including DHH-B, broaden
668 sharply for large values of f/f_p . However, the DHH spreading function (and
669 more importantly the DHH directional-Stokes-drift-component) is constant
670 for $f/f_p > 1.6$ (see Fig. 2) and minimizes the risk of underestimating Stokes
671 drift when both swell and wind seas are present.

672 Regardless though of the wave spreading function chosen for the direc-

673 tional-Stokes-drift-component, the directional effects on magnitude (due to
 674 wave spreading and multidirectional waves) should remain largely uncorre-
 675 lated. This is due to the similarities of the wave spreading functions and
 676 on the mechanics involved. However, spreading losses associated with each
 677 improved $1D_h$ estimate should vary. Studies by Young (1994) have shown
 678 the Mitsuyasu et al. (1975) and Hasselmann et al. (1980) distributions are
 679 artificially broad; as a result they are likely to overestimate the magnitude
 680 losses. In addition, studies by Donelan et al. (1985), Young and Van Vledder
 681 (1993), and Young (1994) have questioned their dependence on wave age. As
 682 discussed in Sec. 4.3.1, agreement between any improved $1D_h$ -SD estimate
 683 and $2D_h$ -SD for wind seas is not guaranteed and fact that $1D_h$ -DHH-SD is
 684 in agreement (aside from OWS-P buoy data near the surface), should not be
 685 overlooked. Based on the reasons discussed here, the DHH wave spreading
 686 function appears to be an adequate choice for this exploratory study.

687 *8.1. Thoughts on improving 1D spectral calculations of Stokes drift*

688 Fully reconciling unidirectional and multidirectional approximations is a
 689 difficult task and this work has been a first step at identifying and quanti-
 690 fying differences in magnitude and direction. While the $1D_h$ -DHH-SD and
 691 int- $1D_h$ -DHH-SD approximations have been used here as a proxy to examine
 692 these differences, they certainly can be used to calculate depth-dependent
 693 and depth-integrated magnitudes when 2D spectra are not available. How-
 694 ever as shown, both approximations will not include the effects of multiple
 695 wave groups and these additional magnitude losses will need to be estimated
 696 (approximately 7–23% and 17% respectively in the data examined here).

697 Unfortunately, predicting the depth-dependent Stokes drift direction is

less straightforward and as shown, there can even be large differences at the surface with the commonly assumed (wind) direction. However, it may be possible to parametrize the Stokes drift veer from (approximate) alignment with wind at the surface to the mean wave direction at deeper depths using a method similar to Section 6. As also noted by Breivik et al. (2014), direction is less of an issue with the depth-integrated Stokes drift since it will mostly align with the mean wave direction.

The best approach however for improving 1D spectral calculations would likely involve spectral partitioning. In theory, this could remove errors both in magnitude and direction and would not require any prior knowledge or assumptions about the sea state. If the 1D frequency spectra were partitioned into swell and wind seas as $\mathcal{S}_f = \mathcal{S}_{\text{swell},f} + \mathcal{S}_{\text{wind},f}$ (such as using the method outlined in Portilla et al. (2009)), then (22) could be rewritten as

$$\mathbf{u}_{\text{Par}}^S = \frac{16\pi^3}{g} \left\{ \hat{\mathbf{e}}^{\text{swell}} \int_0^\infty H_{\text{DHH}}(f/f_p) f^3 S_{\text{swell},f}(f) \exp\left[\frac{8\pi^2 f^2 z}{g}\right] df + \hat{\mathbf{e}}^{\text{wind}} \int_0^\infty H_{\text{DHH}}(f/f_p) f^3 S_{\text{wind},f}(f) \exp\left[\frac{8\pi^2 f^2 z}{g}\right] df \right\}, \quad (28)$$

where $\hat{\mathbf{e}}^{\text{swell}}$ and $\hat{\mathbf{e}}^{\text{wind}}$ are simply the mean wave and wind directions.¹⁵ Notice that this formulation would remove the directionally-diffuse-wind-seas problem when large swell are present. In addition, it would also allow the use of different directional-Stokes-drift-components (H_*) that could be tailored (and simplified) for fully-developed conditions or bimodal wind seas. While computation would certainly increase, it would likely provide the best 1D spectral estimate of the multidirectional Stokes drift for random seas and

¹⁵This approach could also be used with (24), the depth-integrated form.

718 warrants further investigation.

719 **9. Summary**

720 Here, the directional effects of wave spreading and multidirectional waves
721 on Stokes drift have been explored in the random sea approximations. Since
722 the unidirectional $1D_h$ -SD ignores these features, a new $1D_h$ -DHH-SD ap-
723 proximation is defined to incorporate the systematic effects of wave spread-
724 ing. This approximation is based on observational studies of Donelan et al.
725 (1985) and uses a Padé approximate to simplify the solution. Both 1D fre-
726 quency spectra approximations are compared with the $2D_h$ -SD approxima-
727 tion using prescribed, observational, and model-generated wave spectra and it
728 is shown that the improved $1D_h$ -DHH-SD reduces magnitude errors in overes-
729 timation. In addition, the directional magnitude losses due to wave spreading
730 and multidirectional waves are largely uncorrelated and affect both the mag-
731 nitude and direction of Stokes drift in a nonlinear fashion that is sensitive
732 with depth. Furthermore, directional effects in depth-integrated approxima-
733 tions are also considered and results are consistent with depth-dependent
734 subsurface ones.

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944 Appendix A. The DHH-B directional-Stokes-drift-component

945 Based on observations, Banner (1990) concluded the DHH wave spread-
 946 ing function was not constant for $f/f_p \geq 1.6$ and proposed the following
 947 modification (Ewans, 1998):

$$\beta_B(f/f_p) = 10^{-0.4} \exp[0.8393 \ln[10.00] (f/f_p)^{-1.134}], \quad f/f_p \geq 1.6. \quad (\text{A.1})$$

948 Following Section 3.4, a directional-Stokes-drift-component was approximated
 949 (using a Padé approximate of order $[2/2]$) for the Banner-modified directional
 950 distribution (termed *DHH-B* here) as

$$\mathbf{H}_{\text{DHH-B}}(f; f_p, \bar{\theta}) = (\cos \bar{\theta}, \sin \bar{\theta}, 0) \frac{0.5f_p^2 + 0.22f_p f + 0.19f^2}{f_p^2 - 0.81f_p f + 0.79f^2}, \quad f/f_p \geq 1.6. \quad (\text{A.2})$$

951 The new directional-Stokes-drift-component approximation has a maximum
 952 relative error of 7×10^{-5} .

953 Appendix B. Stokes drift spectral tail calculations

954 It is necessary to estimate contributions to Stokes drift for high frequen-
 955 cies that are outside the model domain. Here, a spectral tail parameterization
 956 with cutoff frequency f_c is used to estimate 2D_h-SD tails. In addition, an
 957 integrating cutoff frequency $f_{\bar{c}}$ is introduced for flexibility in discretization.¹⁶
 958 For some positive p and f_c , let the spectral tail be governed by

$$\mathcal{S}_{f\theta}(f, \theta) = \left(\frac{f_c}{f}\right)^p \mathcal{S}_{f\theta}(f_c, \theta), \quad f \geq f_{\bar{c}} \geq f_c. \quad (\text{B.1})$$

¹⁶For practical purposes, $f_{\bar{c}}$ can be viewed as equivalent to f_c here.

959 Then the depth-dependent and depth-integrated 2D_h-SD tail contributions
 960 for $f \geq f_{\tilde{c}}$ are

$$\begin{aligned} \mathbf{u}_{2D,\text{tail}}^S &= \frac{16\pi^3}{g} f_c^p \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) \\ &\quad \times \left\{ \int_{f_{\tilde{c}}}^{\infty} f^{3-p} \exp\left[-\frac{8\pi^2 |z|}{g} f^2\right] df \right\} d\theta \end{aligned} \quad (\text{B.2})$$

961 and

$$\mathbf{u}_{2D,\text{tail}}^{S\text{-int}} = 2\pi f_c^p \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) \left\{ \int_{f_{\tilde{c}}}^{\infty} f^{1-p} df \right\} d\theta. \quad (\text{B.3})$$

962 *Appendix B.1. Subsurface Stokes drift tail*

963 Note that for $\alpha, f_{\tilde{c}} > 0$ ($\alpha \in \mathbb{R}$),

$$\begin{aligned} I &= \int_{f_{\tilde{c}}}^{\infty} f^{3-p} \exp[-\alpha f^2] df = \frac{\alpha^{\frac{p}{2}-2}}{2} \int_{\alpha f_{\tilde{c}}^2}^{\infty} t^{2-\frac{p}{2}} \exp[-t] dt \\ &= \frac{\alpha^{\frac{p}{2}-2}}{2} \Gamma\left[2 - \frac{p}{2}, \alpha f_{\tilde{c}}^2\right], \end{aligned}$$

964 where $\Gamma[s, x]$ is the upper incomplete gamma function. Then for $\alpha =$
 965 $8\pi^2 |z|/g$,

$$\begin{aligned} \mathbf{u}_{2D,\text{tail}}^S|_{z \neq 0} &= \frac{8\pi^3}{g} \alpha^{\frac{p}{2}-2} f_c^p \Gamma\left[2 - \frac{p}{2}, \alpha f_{\tilde{c}}^2\right] \\ &\quad \times \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) d\theta. \end{aligned} \quad (\text{B.4})$$

966 Three common values of p are 4, 4.5, and 5. These reduce to

$$\mathbf{u}_{2D, \text{tail}, p=4}^S|_{z \neq 0} = \frac{8\pi^3}{g} f_c^4 \Gamma[0, \alpha f_c^2] \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) d\theta, \quad (\text{B.5})$$

$$\begin{aligned} \mathbf{u}_{2D, \text{tail}, p=4.5}^S|_{z \neq 0} &= \frac{8\pi^3}{g} \alpha^{\frac{1}{4}} f_c^{\frac{9}{2}} \Gamma\left[-\frac{1}{4}, \alpha f_c^2\right] \\ &\times \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) d\theta, \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \mathbf{u}_{2D, \text{tail}, p=5}^S|_{z \neq 0} &= \frac{8\pi^3}{g} \alpha^{\frac{1}{2}} f_c^5 \Gamma\left[-\frac{1}{2}, \alpha f_c^2\right] \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) d\theta \\ &= \frac{16\pi^3}{g} f_c^4 \left[\exp[-\alpha f_c^2] - \sqrt{\pi \alpha f_c^2} \left(1 - \operatorname{erf}\left[\sqrt{\alpha f_c^2}\right] \right) \right] \\ &\times \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) d\theta, \end{aligned} \quad (\text{B.7})$$

967 where the error function ($\operatorname{erf} : \mathbb{R}_+^* \rightarrow [0, 1]$) is defined as

$$\operatorname{erf}[x] = \frac{2}{\sqrt{\pi}} \int_0^x \exp[-t^2] dt. \quad (\text{B.8})$$

968 *Appendix B.2. Surface Stokes drift tail*

969 If $z = 0$, then the Stokes drift tail is restricted to $p > 4$ and simplifies to

$$\mathbf{u}_{2D, \text{tail}, p>4}^S|_{z=0} = \frac{16\pi^3 f_c^4}{g(p-4)} \left(\frac{f_c}{f_c}\right)^p \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) d\theta. \quad (\text{B.9})$$

970 *Appendix B.3. Depth-integrated Stokes drift tail*

971 Similarly for $p > 2$, the tail contribution to the depth-integrated Stokes
972 drift is

$$\mathbf{u}_{2D, \text{tail}, p>2}^{S\text{-int}} = \frac{2\pi f_c^2}{p-2} \left(\frac{f_c}{f_c}\right)^p \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0) \mathcal{S}_{f\theta}(f_c, \theta) d\theta. \quad (\text{B.10})$$

973 The reader may note that a similar form to some of these spectral tail equa-
974 tions have previously appeared (Webb, 2013; Breivik et al., 2014).

975 Appendix C. Numerical Stokes drift calculations

976 Discretization of Stokes drift for model inclusion is outlined in this sec-
 977 tion. Let the number of spectral bins, centers, and widths be represented by
 978 N_* , $*_i$, $\Delta*_i$ respectively. In addition, all spectral tail and depth-dependent
 979 subsurface calculations use $p = 5$ and $\alpha = 8\pi^2 |z|/g$ respectively. There are
 980 two natural choices here for the integrating cutoff frequency $f_{\tilde{c}}$. The first
 981 choice, $f_{\tilde{c}} := f_c + \Delta f_{N_f}/2$, does not require any modification to the dis-
 982 cretizations presented. The second choice, $f_{\tilde{c}} := f_c$, uses only half of the last
 983 frequency bin (i.e., $f_{N_f} - \Delta f_{N_f}/2 \leq f \leq f_{N_f}$) for all numerical calculations.
 984 This latter method was employed in this manuscript.

985 Appendix C.1. 2D_h-SD

986 The *subsurface* and *surface* 2D_h-SD numerical approximations are

$$\begin{aligned} \mathbf{u}_{2D}^S \approx & \frac{2\pi\alpha}{|z|} \sum_{j=1}^{N_\theta} \Delta\theta_j (\cos\theta_j, \sin\theta_j, 0) \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i f_i^3 \exp[-\alpha f_i^2] S_{f\theta}(f_i, \theta_j) \right] \right. \\ & \left. + f_c^4 \left[\exp[-\alpha f_{\tilde{c}}^2] - \sqrt{\pi\alpha f_{\tilde{c}}^2} \left(1 - \operatorname{erf} \left[\sqrt{\alpha f_{\tilde{c}}^2} \right] \right) \right] S_{f\theta}(f_c, \theta_j) \right\}, \end{aligned} \quad (C.1)$$

$$\begin{aligned} \mathbf{u}_{2D}^S|_{z=0} \approx & \frac{16\pi^3}{g} \sum_{j=1}^{N_\theta} \Delta\theta_j (\cos\theta_j, \sin\theta_j, 0) \\ & \times \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i f_i^3 S_{f\theta}(f_i, \theta_j) \right] + \frac{f_c^5}{f_{\tilde{c}}} S_{f\theta}(f_c, \theta_j) \right\}. \end{aligned} \quad (C.2)$$

987 *Appendix C.2. Directional 1D_h-SD*

988 Recall that for a generic $\|\mathbf{H}\| \neq 1$, the directional 1D_h-SD can be written
 989 as

$$\mathbf{u}_{\text{1D-dir}}^S = \hat{\mathbf{e}}^w \frac{16\pi^3}{g} \int_0^\infty H(f) f^3 S_f(f) \exp\left[\frac{8\pi^2 f^2}{g} z\right] df, \quad (\text{C.3})$$

990 where $\hat{\mathbf{e}}^w$ is the assumed direction and $H(f) = \|\mathbf{H}(f)\|$ is the *directional-*
 991 *Stokes-drift-component magnitude*. The resultant *subsurface* and *surface* di-
 992 rectional 1D_h-SD numerical approximations are

$$\begin{aligned} \|\mathbf{u}_{\text{1D-dir}}^S\| \approx \frac{2\pi\alpha}{|z|} \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i H(f_i) f_i^3 \exp[-\alpha f_i^2] S_f(f_i) \right] \right. \\ \left. + f_c^4 \left[\exp[-\alpha f_c^2] - \sqrt{\pi\alpha f_c^2} \left(1 - \operatorname{erf}\left[\sqrt{\alpha f_c^2}\right] \right) \right] H(f_c) S_f(f_c) \right\}, \end{aligned} \quad (\text{C.4})$$

$$\|\mathbf{u}_{\text{1D-dir}}^S|_{z=0}\| \approx \frac{16\pi^3}{g} \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i H(f_i) f_i^3 S_f(f_i) \right] + \frac{f_c^5}{f_c} H(f_c) S_f(f_c) \right\}. \quad (\text{C.5})$$

993 *Appendix C.3. Unidirectional 1D_h-SD*

994 For unidirectional approximations, $\|\mathbf{H}\| = 1$ and the *subsurface* and *sur-*
 995 *face* unidirectional 1D_h-SD numerical approximations simplify to

$$\begin{aligned} \|\mathbf{u}_{\text{1D-uni}}^S\| \approx \frac{2\pi\alpha}{|z|} \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i f_i^3 \exp[-\alpha f_i^2] S_f(f_i) \right] \right. \\ \left. + f_c^4 \left[\exp[-\alpha f_c^2] - \sqrt{\pi\alpha f_c^2} \left(1 - \operatorname{erf}\left[\sqrt{\alpha f_c^2}\right] \right) \right] S_f(f_c) \right\}, \end{aligned} \quad (\text{C.6})$$

$$\|\mathbf{u}_{\text{1D-uni}}^S|_{z=0}\| \approx \frac{16\pi^3}{g} \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i f_i^3 S_f(f_i) \right] + \frac{f_c^5}{f_c} S_f(f_c) \right\}. \quad (\text{C.7})$$

996 *Appendix C.4. Depth-integrated Stokes drift*

997 The numerical depth-integrated Stokes drift approximations resemble their
 998 surface counterparts and only require small modifications. They are

$$\mathbf{u}_{2D}^{S\text{-int}} \approx 2\pi \sum_{j=1}^{N_\theta} \Delta\theta_j (\cos \theta_j, \sin \theta_j, 0) \times \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i f_i S_{f\theta}(f_i, \theta_j) \right] + \frac{f_c^5}{3f_c^3} S_{f\theta}(f_c, \theta_j) \right\}, \quad (\text{C.8})$$

$$\|\mathbf{u}_{1D\text{-dir}}^{S\text{-int}}\| \approx 2\pi \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i H(f_i) f_i S_f(f_i) \right] + \frac{f_c^5}{3f_c^3} H(f_c) S_f(f_c) \right\}, \quad (\text{C.9})$$

$$\|\mathbf{u}_{1D\text{-uni}}^{S\text{-int}}\| \approx 2\pi \left\{ \left[\sum_{i=1}^{N_f} \Delta f_i f_i S_f(f_i) \right] + \frac{f_c^5}{3f_c^3} S_f(f_c) \right\}. \quad (\text{C.10})$$

999 **Appendix D. Matlab Stokes drift functions**

1000 Matlab functions for all six approximations (both depth-dependent and
 1001 depth-integrated) can be found online at www.adreanwebb.com and the MAT-
 1002 LAB Central File Exchange.

Figure 1: Here, pairs of monochromatic waves (red and blue) are shown traveling about a mean direction $\bar{\theta}=\pi/2$ with a total directional difference (for each pair) of $2\theta'$. Only the y vector components of the bichromatic waves contribute to Stokes drift.

Figure 2: Magnitudes of the **DHH** (black solid) and **DHH-B** (gray dashed) **directional-Stokes-drift-components**.

Figure 3: Ratios of $1D_h$ -DHH-SD to $1D_h$ -SD magnitudes using prescribed spectra for continuous e -folding depths: **JONSWAP** (gray solid), **PM** (gray dashed), **fetch-limited DHH** (black solid), and **fully-developed DHH** (black dashed).

Figure 4: OWS-P buoy data: a snapshot on 2012/04/12 05:32 of (a) **Stokes drift magnitudes** at depth and (b) the corresponding **directional distribution**. The effect of opposing swell and wind seas is clearly apparent in the $2D_h$ -SD approximation.

Figure 5: OWS-P buoy **mean (open circles) and median (closed circles) Stokes drift magnitude ratios** with the two-thirds centered distribution shaded for the 27 month period from 2010/06/21 to 2012/09/20.

Figure 6: Density-shaded scatter plots generated from one year of global WAVEWATCH III model data. The colors red, green, and blue indicate the highest 0–30%, 31–60%, and 61–90% mode-centered distributions respectively. Here, magnitude comparisons of depth-dependent and depth-integrated $1D_h\text{-SD}$ (x -axis) versus $2D_h\text{-SD}$ (y -axis) are normalized by the largest $2D_h\text{-SD}$ value.

Figure 7: Density-shaded scatter plots of depth-dependent and depth-integrated normalized magnitudes of $1D_h\text{-DHH-SD}$ (x -axis) versus $2D_h\text{-SD}$ (y -axis). See Fig. 6 for an explanation of colors and normalization.

Figure 8: Density-shaded scatter plots of normalized surface magnitudes of $1D_h\text{-SD}$ (x -axis) versus $1D_h\text{-DHH-SD}$ (y -axis) using both (a) WAVEWATCH III model and (b) OWS-P buoy data. The slope of the red line in both is $m = 0.795$. See Fig. 6 for an explanation of colors and normalization.

Figure 9: Density-shaded scatter plots of the **10 m surface wind direction** (x -axis) versus depth-dependent and depth-integrated **$2D_h$ -SD directions** (y -axis). See Fig. 6 for an explanation of colors and normalization.

Figure 10: Density-shaded scatter plots of the **mean wave direction** (x -axis) versus depth-dependent and depth-integrated **$2D_h$ -SD directions** (y -axis). See Fig. 6 for an explanation of colors and normalization.

Figure 11: A comparison of **mean magnitude ratios** using OWS-P buoy (circles) and WAVEWATCH III model (squares) data. Physical processes are differentiated by color with blue for **wave spreading** ($1D_h\text{-DHH-SD}$ and $1D_h\text{-SD}$), red for **multidirectional wave** ($2D_h\text{-SD}$ and $1D_h\text{-DHH-SD}$), and black for **combined** ($2D_h\text{-SD}$ and $1D_h\text{-SD}$) **effects**. In addition, lower and upper halves represent directional magnitude losses and overestimation errors respectively.

Figure 12: Density-shaded scatter plots of $2D_h$ -SD (y -axis) versus $1D_h$ -SD and $1D_h$ -DHH-SD (x -axes; left and right columns respectively) using two different depths ($z = 0, -3$) for OWS-P buoy data. Magnitudes are normalized by the largest corresponding WAVEWATCH III model $2D_h$ -SD value.

Table 1: Abbreviation guide for the random sea Stokes drift approximations

$1D_h, 2D_h$	Horizontally one-, two-dimensional.
$2D_h$ -SD	Directional random seas Stokes drift; uses 2D wave spectra.
$1D_h$ -SD	Unidirectional random seas Stokes drift; uses 1D wave spectra.
$1D_h$ -DHH-SD	Uses 1D wave spectra with the Donelan et al. (1985) directional distribution to correct for wave spreading.
$1D_h$ -DHH-B-SD	Uses 1D wave spectra with the Banner (1990) directional distribution to correct for wave spreading.
int-*-SD	Depth-integrated form of the depth- dependent approximations.

Table 2: Example e -folding depths $|z_n|$ (m) for the peak frequencies $f_p = 0.05, 0.16, 0.34$ (Hz). For reference, $\exp[-\{0.01, 0.5, 1, 2, 3\}] = \{0.990, 0.607, 0.368, 0.135, 0.050\}$.

f_p	T_p	$ z_{0.01} $	$ z_{0.5} $	$ z_1 $	$ z_2 $	$ z_3 $
0.34	2.94	0.0107	0.537	1.07	2.15	3.22
0.16	6.25	0.0485	2.43	4.85	9.71	14.6
0.05	20	0.497	24.8	49.7	99.4	149

Table 3: Ratio of **1D_h-DHH-SD** to **1D_h-SD** magnitudes using empirical spectra for various e-folding depths. Ratios with DHH1 (fetch-limited) and DHH2 (fully-developed) are not provided at z_0 since the spectra is undefined at the surface (see Section 4.1).

	z_0	$z_{0.01}$	$z_{0.5}$	z_1	z_2	z_3
JONSWAP	0.812	0.820	0.874	0.893	0.908	0.913
DHH1	-	0.803	0.875	0.896	0.911	0.915
PM	0.799	0.805	0.854	0.877	0.898	0.906
DHH2	-	0.795	0.853	0.876	0.897	0.905

Table 4: A comparison of **mean depth-integrated magnitude ratios** using WAVE-WATCH III model data. Mean magnitude losses and overestimation errors (mean reciprocal losses) are listed for **wave spreading** (int-1D_h-DHH-SD and int-1D_h-SD), **multidirectional wave** (int-2D_h-SD and int-1D_h-DHH-SD), and **combined** (int-2D_h-SD and int-1D_h-SD) **effects**.

	loss ratio	error ratio
wave spreading	0.841	1.191
multidirectional wave	0.830	1.410
combined	0.698	1.682