A nonhydrostatic, isopycnal-coordinate ocean model for internal waves

Sean Vitousek^{a,*}, Oliver B. Fringer^a

^a*Environmental Fluid Mechanics Laboratory, Stanford University, Stanford, CA, 94305*

Abstract

We present a nonhydrostatic ocean model with an isopycnal (density-following) vertical coordinate system. The primary motivation for the model is the proper treatment of nonhydrostatic dispersion and the formation of nonlinear internal solitary waves. The nonhydrostatic, isopycnal-coordinate formulation may be preferable to nonhydrostatic formulations in z - and σ -coordinates because it improves computational efficiency by reducing the number of vertical grid points and eliminates spurious diapycnal mixing and solitary-wave amplitude loss due to numerical diffusion of scalars. The model equations invoke a mild isopycnal-slope approximation to remove small metric terms associated with diffusion and nonhydrostatic pressure from the momentum equations and to reduce the pressure Poisson equation to a symmetric linear system. Avoiding this approximation requires a costlier inversion of a non-symmetric linear system. We demonstrate that the model is capable of simulating nonlinear internal solitary waves for simplified and

*Corresponding author

Email addresses: `seanv@stanford.edu` (Sean Vitousek), `fringer@stanford.edu` (Oliver B. Fringer)

physically-realistic ocean-scale problems with a reduced number of layers.

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1. Introduction

1.1. Literature review

Simulating internal waves is one of the most computationally challenging tasks in ocean modeling. Large-scale processes such as internal tides can be modeled reasonably well with computationally-inexpensive hydrostatic models (Kantha and Clayson, 2000). Simulations of nonlinear internal solitary waves, on the other hand, require computationally-expensive nonhydrostatic models to represent dispersive behavior. Nonhydrostatic models can incur an order of magnitude increase in computational time relative to hydrostatic models due to the elliptic solver for the nonhydrostatic or dynamic pressure (Fringer et al., 2006). Furthermore, simulations of nonlinear internal solitary waves require high horizontal grid resolution to ensure that numerically-induced dispersion¹ is small relative to physical dispersion (Vitousek and Fringer, 2011).

The vertical coordinate system is often reported as the most important aspect in the design of an ocean model (Griffies et al., 2000; Chassignet et al., 2000; Willebrand et al., 2001; Chassignet, 2011). The three vertical coordinate systems typically used in ocean models are: (1) Height or

¹from truncation error of the discretized model equations

19 z -coordinates, (2) Terrain-following or σ -coordinates, and (3) Isopycnal or
 20 ρ -coordinates. Each approach has numerous advantages and disadvantages
 21 as outlined in Griffies et al. (2000). Existing nonhydrostatic models employ
 22 z - or σ -coordinates [e.g. (Mahadevan et al., 1996a,b), MITgcm (Marshall
 23 et al., 1997b,a), SUNTANS (Fringer et al., 2006) for z -coordinates, POM
 24 (Kanarska and Maderich, 2003), BOM (Heggelund et al., 2004), ROMS (Ka-
 25 narska et al., 2007), FVCOM (Lai et al., 2010a) for σ -coordinates and ICOM
 26 (Ford et al., 2004a,b) for vertically unstructured coordinates]. z - and σ -
 27 coordinate models are capable of representing overturning motions and ed-
 28 dies (e.g. Kelvin-Helmholtz, Rayleigh-Taylor, and other instabilities) that
 29 are associated with many small-scale nonhydrostatic processes. Isopycnal-
 30 coordinate models, on the other hand, cannot represent overturning motions
 31 or unstable stratification. This deficiency leads to the notion that isopycnal
 32 coordinates are not suitable for modeling nonhydrostatic processes (Adcroft
 33 and Hallberg, 2006). Consequently, existing isopycnal models such as MI-
 34 COM/HYCOM (Bleck et al., 1992; Bleck, 2002), HIM (Hallberg, 1995, 1997;
 35 Hallberg and Rhines, 1996), POSEIDON (Schopf and Lough, 1995), PO-
 36 SUM (Higdon and de Szoeke, 1997; de Szoeke, 2000) so far exclusively em-
 37 ploy the hydrostatic approximation. While clearly a deficiency of isopycnal
 38 models, the inability to represent unstable stratification is an issue for hydro-
 39 static and nonhydrostatic isopycnal formulations alike. In this paper, we do
 40 not propose a means for isopycnal-coordinate models to represent unstable
 41 stratification — this task is clearly suited to z - and σ -coordinate models.

42 Instead, the primary motivation behind the model presented in this paper is
 43 the proper treatment of dispersion and the formation of nonlinear internal
 44 solitary waves in the context of an isopycnal-coordinate model. Internal soli-
 45 tary waves are clearly nonhydrostatic and not associated with overturning
 46 motions. Although overturning structures may exist in the vicinity of internal
 47 wave generation sites that might preclude the use of an isopycnal-coordinate
 48 model, a model is not required to resolve these structures to obtain a good
 49 prediction of the internal wave generation. For example, Klymak and Legg
 50 (2010) and Klymak et al. (2010) developed a simple scheme that faithfully
 51 captures dissipation and mixing related to internal wave generation at a ridge
 52 and show that their hydrostatic model was almost identical to the nonhydro-
 53 static model in predicting the generation dynamics. This suggests that, while
 54 small-scale, nonhydrostatic processes related to internal wave generation are
 55 indeed complex, they can be parameterized appropriately in large-scale z -,
 56 σ -, and isopycnal-coordinate models that do not resolve them. Hence, a non-
 57 hydrostatic, isopycnal-coordinate formulation may be suitable for modeling
 58 internal or interfacial waves.

59 In the context of internal wave modeling, isopycnal-coordinates may pro-
 60 vide some advantages over z - and σ -coordinates. Isopycnal or density-following
 61 coordinates provide natural representations of (stably) stratified fluids. This
 62 reduces the number of vertical grid points from $\mathcal{O}(100)$ in z - and σ -coordinate
 63 models to $\mathcal{O}(1)$ in isopycnal coordinates (Bleck and Boudra, 1981). Ideally,
 64 in locations where the internal wave structure is predominantly mode-1, an

65 isopycnal model with only two layers may suffice (Simmons et al., 2011). The
 66 primary disadvantage of modeling nonhydrostatic pressure is that it requires
 67 solution of a three-dimensional elliptic (Poisson) equation for the nonhydro-
 68 static pressure which significantly increases the computational cost relative
 69 to the hydrostatic model. Solving this elliptic equation requires optimally
 70 $\mathcal{O}(N)$ operations (Briggs et al., 2000) where N is the number of grid cells.
 71 Isopycnal coordinates can improve the efficiency of nonhydrostatic methods
 72 by reducing the required number of vertical grid points by an order of mag-
 73 nitude relative to z - and σ -coordinate models. Thus, reducing the number of
 74 vertical layers and thus the overall number of grid cells by an order of magni-
 75 tude can result in at least one order of magnitude reduction in computational
 76 cost.

77 Another advantage of isopycnal-coordinates is the reduction or elimina-
 78 tion of spurious diapycnal mixing. Transport in the ocean predominantly
 79 occurs along rather than across isopycnal surfaces (Iselin, 1939; Montgomery,
 80 1940). In many applications, spurious diapycnal mixing, arising from numerically-
 81 diffusive truncation error in scalar transport schemes in z - and σ -coordinate
 82 models (Fringer et al., 2005), can be larger than physical diapycnal mixing
 83 (Griffies et al., 2000). Isopycnal coordinates, on the other hand, are not sus-
 84 ceptible to spurious diapycnal mixing because the governing equations are
 85 constructed to directly control the amount of diapycnal transport – if any
 86 (Bleck and Boudra, 1981; Griffies et al., 2000). Hence, the problem of energy
 87 loss due to spurious diapycnal mixing that occurs in numerical models dur-

ing the formation and propagation of internal solitary waves (Hodges et al., 2006) may be reduced or eliminated with isopycnal coordinates.

1.2. Outline of the proposed model

Existing approaches for simulating nonhydrostatic internal waves include z - and σ -coordinate models applied at high resolution in 3-D (Fringer et al., 2006; Vlasenko and Stashchuk, 2007; Vlasenko et al., 2009, 2010; Lai et al., 2010b; Zhang et al., 2011; Guo et al., 2011) or 2-D slices (Scotti et al., 2007, 2008; Buijsman et al., 2010) or asymptotic/Boussinesq-type approaches using 2-layer (Brandt et al., 1997; Choi and Camassa, 1999; Lynett and Liu, 2002; de la Fuente et al., 2008; Steinmoeller et al., 2012) or multi-layer models (Liu and Wang, 2012). Asymptotic or Boussinesq-type approaches do not require a pressure projection method. Instead, they include higher-order derivatives to account for the nonlinear and dispersive behavior. Boussinesq-type models have a limited range of applicability that is often the weakly nonlinear, weakly nonhydrostatic regime. To extend this range of applicability, more terms may be included or advanced formulations may be introduced. However, this can lead to an unwieldy set of governing equations containing high-order, mixed time-and-space derivatives.

The formulation presented here is intended to be flexible (using an arbitrary number of layers) and straightforward (resembling existing ocean models). The numerical method uses a pressure projection method which results in an elliptic equation for the dynamic pressure (as in z - and σ -coordinate

110 models). The elliptic equation in isopycnal coordinates results in a non-
 111 symmetric system of linear equations. However, by invoking a mild-slope
 112 approximation, the system becomes symmetric and remarkably similar to
 113 the elliptic equation in z -coordinates.

114 Another significant difference between existing isopycnal models and the
 115 model presented here (besides the treatment of nonhydrostatic pressure) is
 116 the time-stepping procedure. Most isopycnal models use mode-splitting to
 117 treat fast free-surface gravity waves (Bleck and Smith, 1990; Higdon and Ben-
 118 nett, 1996; Higdon and de Szoeke, 1997; Hallberg, 1997). The current model
 119 uses an implicit time-stepping procedure for the free surface following Casulli
 120 (1999) which is common in nonhydrostatic models in z - and σ -coordinates
 121 (a list of nonhydrostatic models using implicit time-stepping procedures is
 122 given in Vitousek and Fringer (2013)). Casulli (1997) developed a hydro-
 123 static, isopycnal model with an implicit time-stepping procedure for the free
 124 surface and layer heights. In his approach, the gradient of the Montgomery
 125 potential (M), which represents the hydrostatic pressure in isopycnal co-
 126 ordinates, is discretized implicitly. Thus, computing the free-surface and
 127 interface heights requires the inversion of a large (3-D) system of equations
 128 which is comparable in cost to the solution of the (3-D) elliptic equation for
 129 the nonhydrostatic pressure. The current model is similar to the approach of
 130 Casulli (1997). However, we split the Montgomery potential into barotropic

131 $(M^{(\text{bt})})$ and baroclinic $(M^{(\text{bc})})$ parts according to

$$M = \rho_0^{-1}(p_h + \rho g z) = \underbrace{\rho_0^{-1}(p_s + \rho_0 g \eta)}_{=M^{(\text{bt})}} + M^{(\text{bc})} , \quad (1)$$

132 where ρ is the density (ρ_0 is the reference density), p_h is the hydrostatic pres-
 133 sure, p_s is the surface (atmospheric) pressure, η is the free-surface height, z is
 134 the interface location, and the term $\rho g z$ originates from the transformation to
 135 isopycnal coordinates. Because the barotropic and baroclinic portions of the
 136 Montgomery potential represent fast free-surface and slow internal-gravity
 137 waves, they are discretized implicitly and explicitly, respectively. This dis-
 138 cretization requires inversion of a 2-D system in the horizontal to compute
 139 the free-surface height as is the case with implicit free-surface models in
 140 z - and σ -coordinates. The computational cost of the 2-D inversion for the
 141 free-surface height is minimal compared to the 3-D inversion for the nonhy-
 142 drostatic pressure.

143 *1.3. Use of Lagrangian coordinates for the nonhydrostatic equations*

144 Adcroft and Hallberg (2006) conclude that the nonhydrostatic projection
 145 method and use of Lagrangian vertical coordinates are mutually exclusive.
 146 Their argument is based on how the Lagrangian algorithm prescribes the
 147 material derivative of the (general) vertical coordinate, $\dot{r}$, in the continuity
 148 equation while in the nonhydrostatic projection method this term should
 149 instead come from the vertical momentum equation. This leads to a conun-
 150 drum in which one cannot simultaneously supply and diagnose a quantity

151 in an equation (Adcroft and Hallberg, 2006). The present study does not
 152 seem limited by this issue. When using pure isopycnal coordinates, $r \equiv \rho$
 153 and $\dot{r} \equiv \dot{\rho} \equiv \frac{D\rho}{Dt} \equiv 0$ (since density is treated as a materially conserved
 154 quantity). Consequently, $\dot{r} \equiv 0$ is prescribed in the continuity equation and
 155 this leads to an evolution equation for the isopycnal layer heights that is con-
 156 sistent with the material conservation of density. After the layer heights are
 157 evolved, the projection method uses the vertical velocity from the vertical
 158 momentum equation in the divergence-free form of the continuity equation.
 159 We note that the two forms of the continuity equation, namely the layer-
 160 height equation and the divergence-free continuity equation, are, in fact, the
 161 same equation and the condition $\dot{r} \equiv 0$ gives rise to the layer-height equa-
 162 tion. After the nonhydrostatic projection step, a final vertical velocity is
 163 computed using the divergence-free form of the continuity equation. Our
 164 interpretation of the Adcroft and Hallberg (2006) conundrum is that it im-
 165 plies that the vertical velocity computed with the projection step cannot be
 166 consistent with that which is implied by the material conservation of density
 167 in the layer-height equation. This does not appear to be an issue in our
 168 approach nor in any of the commonly used nonhydrostatic methods, likely
 169 because maintaining absolute consistency is unnecessary, but instead must
 170 be maintained to some order of accuracy. Specifically, when the free surface
 171 is evolved in typical nonhydrostatic models, this implies a vertical velocity
 172 from a discrete kinematic condition which may not necessarily be the same as
 173 that which is computed with the projection method. As discussed in Section

174 3.3, the popular approach of evolving the free-surface using the hydrostatic
 175 provisional velocities rather than the final nonhydrostatic velocities still re-
 176 sults in a method that is second-order accurate (Kang and Fringer, 2005;
 177 Vitousek and Fringer, 2013). This suggests that the effective vertical veloc-
 178 ity arising from applying the kinematic condition to the continuity equation
 179 is relatively unimportant. Further support for the method developed here
 180 comes from the fact that the numerical algorithm (presented in Section 3)
 181 simplifies to popular methods used in nonhydrostatic z -coordinate models
 182 with a free surface (e.g. Fringer et al. (2006)) in the case that only one
 183 vertical layer is used. Recently, Klingbeil and Burchard (2013) extended a
 184 generalized vertical-coordinate model (GETM) to include the nonhydrostatic
 185 pressure without using the projection method, which improves performance
 186 owing to the lack of an elliptic equation for the nonhydrostatic pressure. Like
 187 our method, the approach of Klingbeil and Burchard (2013) does not suffer
 188 from the conundrum of Adcroft and Hallberg (2006). Therefore, we do not
 189 suggest that our method or that of Klingbeil and Burchard (2013) present
 190 solutions to the conundrum, but instead that its potential limitations can be
 191 circumvented.

192 1.4. *Synopsis*

193 The remainder of this paper is divided into five sections. Section 2
 194 presents the derivation of the governing equations in continuous isopycnal
 195 coordinates. Section 3 presents the numerical method used to solve the gov-

196 urning equations. Section 4 discusses stability and convergence of the model.
 197 Section 5 presents model validation with several idealized test cases. Finally,
 198 Section 6 presents the conclusions. Lengthy derivations relevant to the model
 199 formulation are presented in appendices.

200 **2. Governing equations**

201 *2.1. Equations in Cartesian coordinates*

202 The governing equations are the Reynolds-Averaged Navier-Stokes (RANS)
 203 equations with the Boussinesq approximation in a rotating reference frame.
 204 In Cartesian coordinates, the equations are given by

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{u}u) - fv + \tilde{f}w = -\frac{1}{\rho_0} \frac{\partial p_h}{\partial x} - \frac{\partial q}{\partial x} + \nabla_H \cdot (\nu_H \nabla_H u) + \frac{\partial}{\partial z} \left(\nu_v \frac{\partial u}{\partial z} \right), \quad (2a)$$

$$\frac{\partial v}{\partial t} + \nabla \cdot (\mathbf{u}v) + fu = -\frac{1}{\rho_0} \frac{\partial p_h}{\partial y} - \frac{\partial q}{\partial y} + \nabla_H \cdot (\nu_H \nabla_H v) + \frac{\partial}{\partial z} \left(\nu_v \frac{\partial v}{\partial z} \right), \quad (2b)$$

$$\frac{\partial w}{\partial t} + \nabla \cdot (\mathbf{u}w) - \tilde{f}u = -\frac{\partial q}{\partial z} + \nabla_H \cdot (\nu_H \nabla_H w) + \frac{\partial}{\partial z} \left(\nu_v \frac{\partial w}{\partial z} \right), \quad (2c)$$

205 where $\mathbf{u} = [u, v, w]^T$ is the velocity vector, $f = 2\Omega \sin(\phi)$ is the traditional
 206 Coriolis parameter and $\tilde{f} = 2\Omega \cos(\phi)$ is the nontraditional Coriolis param-
 207 eter (see e.g. White and Bromley (1995); Marshall et al. (1997b); Gerkema

et al. (2008)), where ϕ is the latitude and Ω is the Earth's angular velocity,
 $p_h(z) = p_s + \rho_0 g(\eta - z) + g \int_z^\eta (\rho - \rho_0) dz$ is the hydrostatic pressure (de-
 rived from integrating the hydrostatic balance $\frac{\partial p_h}{\partial z} = -\rho g$), where p_s is the
 atmospheric pressure at the surface, g is gravity, η is the free-surface height,
 ρ is the density and ρ_0 is the (constant) reference density, and ν_H and ν_v
 are the horizontal and vertical eddy-viscosities, respectively. The difference
 between the nonhydrostatic equation set and the hydrostatic equation set is
 the treatment of the full vertical momentum balance (2c) and the presence of
 the nonhydrostatic pressure terms involving q on the right-hand side (RHS)
 of the momentum equations (2). The kinematic nonhydrostatic pressure is
 given by $q = p_{nh}/\rho_0$ in the notation following Casulli (1999), where p_{nh} is
 the nonhydrostatic or dynamic pressure due to the vertical momentum or
 acceleration of the fluid.

Equations (2) are subject to the incompressibility constraint

$$\nabla \cdot \mathbf{u} = 0 , \quad (3)$$

and conservation of density

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\rho) = 0 , \quad (4)$$

where we ignore diffusion of density although it can be added following the
 methods in standard isopycnal models (Hallberg, 2000). Diffusion of density
 leads to $\dot{\rho} = \frac{D\rho}{Dt} \neq 0$ and to the conundrum of Adcroft and Hallberg (2006)

226 related to the treatment of nonhydrostatic pressure. We present possible
 227 solutions to this issue in Section 3.3.

228 2.2. Mild-slope equations in isopycnal coordinates

229 The momentum equations in isopycnal coordinates result from applying
 230 the coordinate transformation

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{z_{x^*}}{z_\rho} & 0 \\ 0 & 1 & -\frac{z_{y^*}}{z_\rho} & 0 \\ 0 & 0 & \frac{1}{z_\rho} & 0 \\ 0 & 0 & -\frac{z_{t^*}}{z_\rho} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial t^*} \end{bmatrix}, \quad (5)$$

231 (see e.g. Müller (2006)) to the momentum equations in Cartesian coordi-
 232 nates (2). Next, as demonstrated in Appendix A, the resulting equations
 233 are subjected to an mild isopycnal-slope approximation to remove small met-
 234 ric terms associated with diffusion and the nonhydrostatic pressure from the
 235 momentum equations. Neglecting the small metric terms associated with dif-
 236 fusion based on a mild isopycnal-slope approximation is perhaps unnecessary.
 237 Solomon (1971) and Redi (1982) argued that the representation of diffusion
 238 in an isopycnal coordinate system (i.e. using the diffusion operators in the
 239 equations below without the transformation metric terms) is preferable to
 240 the representation in a Cartesian coordinate system in order to faithfully
 241 capture the tendency for along-isopycnal mixing of scalars and momentum

242 (Montgomery, 1940). The mild-slope equations in isopycnal coordinates are
 243 given by

$$\frac{\partial u}{\partial t} + \mathbf{u}_H \cdot \nabla_H u - f v + \tilde{f} w = -\frac{\partial M}{\partial x} - \frac{\partial q}{\partial x} + \nabla_H \cdot (\nu_H \nabla_H u) + \frac{1}{z_\rho} \frac{\partial}{\partial \rho} \left(\nu_v \frac{1}{z_\rho} \frac{\partial u}{\partial \rho} \right) , \quad (6a)$$

$$\frac{\partial v}{\partial t} + \mathbf{u}_H \cdot \nabla_H v + f u = -\frac{\partial M}{\partial y} - \frac{\partial q}{\partial y} + \nabla_H \cdot (\nu_H \nabla_H v) + \frac{1}{z_\rho} \frac{\partial}{\partial \rho} \left(\nu_v \frac{1}{z_\rho} \frac{\partial v}{\partial \rho} \right) , \quad (6b)$$

$$\frac{\partial w}{\partial t} + \mathbf{u}_H \cdot \nabla_H w - \tilde{f} u = -\frac{1}{z_\rho} \frac{\partial q}{\partial \rho} + \nabla_H \cdot (\nu_H \nabla_H w) + \frac{1}{z_\rho} \frac{\partial}{\partial \rho} \left(\nu_v \frac{1}{z_\rho} \frac{\partial w}{\partial \rho} \right) , \quad (6c)$$

244 where the star (*) notation has been dropped for convenience and M is the
 245 Montgomery Potential given in Equation (1). The main difference between
 246 the equations in Cartesian coordinates (2) and isopycnal coordinates (6),
 247 besides the appearance of the Montgomery potential, is the absence of vertical
 248 advection of momentum terms in Equations (6).

249 The continuity equation (3) can be written in two different ways. In
 250 existing isopycnal models, the continuity equation is often written as

$$\frac{\partial h}{\partial t} = -\nabla_H \cdot (\mathbf{u}_H h) , \quad (7)$$

251 where $h = -\Delta \rho \frac{\partial z}{\partial \rho} = -\Delta \rho z_\rho$ is the thickness of an isopycnal layer. Alterna-

252 tively, the divergence-free form of the continuity equation is written as

$$\frac{\partial}{\partial x}(z_\rho u) + \frac{\partial}{\partial y}(z_\rho v) + \frac{\partial w}{\partial \rho} = 0 . \quad (8)$$

253 We note that Equation (7) is used to evolve the thickness of isopycnal layers
 254 and Equation (8) is used to calculate the nonhydrostatic pressure via an el-
 255 liptic equation that enforces continuity. This procedure is similar to methods
 256 that are typically employed in nonhydrostatic, z - and σ -coordinate models
 257 with a free surface and is discussed in Section 3. The continuity equation (8)
 258 is also subject to the mild-slope approximation as discussed in Appendix A.
 259 By invoking this approximation, the discrete elliptic equation for the nonhy-
 260 drostatic pressure reduces to a symmetric, 5-diagonal linear system in 2-D
 261 (7-diagonal in 3-D) as demonstrated in Appendix E. Berntsen and Furnes
 262 (2005) invoked a similar approximation by neglecting metric terms appearing
 263 from the transformation to σ -coordinates. Their approximation also results
 264 in a 5-diagonal linear system of equations in 2-D for the nonhydrostatic pres-
 265 sure and produces simulation results that are nearly identical to the full
 266 (unapproximated) elliptic equation (Keilegavlen and Berntsen, 2009).

267 **3. Numerical Method**

268 The numerical method used to solve the governing equations in isopycnal
 269 coordinates must be capable of handling three specific numerical challenges:
 270 [1] treatment of stiffness caused by fast free-surface gravity waves, [2] wetting

271 and drying of isopycnal layers, [3] solution of the nonhydrostatic pressure. As
 272 discussed in Section 3.1, to treat fast free-surface gravity waves, we split the
 273 barotropic and baroclinic parts of the Montgomery potential and discretize
 274 them implicitly and explicitly, respectively. Drying of isopycnal layers is
 275 handled by using upwind or MPDATA schemes to evolve the layer heights as
 276 discussed in Section 3.2. Finally, the solution procedure for the nonhydro-
 277 static pressure in isopycnal coordinates, as discussed in Section 3.3, is similar
 278 to procedures in z -coordinates (such as Casulli (1999); Fringer et al. (2006)).
 279 For clarity, we present the numerical formulation in 2-D (in an x - z plane).
 280 The method is easily extended to 3-D.

281 *3.1. Discretized momentum equations*

282 The momentum equations (6) are discretized using a second-order accu-
 283 rate finite-difference method on a staggered C-grid in which the velocities
 284 are defined at cell faces and quantities such as the layer heights and nonhy-
 285 drostatic pressure are defined at cell centers. The model setup and location
 286 of the variables of interest are shown on Figure 1. In order to discretize the
 287 governing equations in continuous isopycnal coordinates (6) and obtain the
 288 model equations with a discrete number of layers, we have used a second-
 289 order accurate finite-difference approximation given by

$$\frac{1}{z_\rho} \frac{\partial \phi}{\partial \rho} \approx \frac{\phi_{k-1/2} - \phi_{k+1/2}}{h_k} . \quad (9)$$

Figure 1: The isopycnal model setup.

290 The layer index, k , increases downward as is conventional in isopycnal models
 291 (Hallberg, 1997; Bleck, 2002). Use of Equation (9) for the vertical derivatives
 292 and an evolution equation for the layer height h_k makes the numerical method
 293 a LVD algorithm (Adcroft and Hallberg, 2006; Hallberg and Adcroft, 2009)
 294 and also allows simulation of fluids with constant density.

295 Using the spatial discretization above and following existing nonhydro-
 296 static operator-splitting procedures, the discrete momentum equations be-
 297 come

$$\frac{u_{i+\frac{1}{2},k}^* - u_{i+\frac{1}{2},k}^n}{\Delta t} = -g \frac{\eta_{i+1}^{(n+1)\text{Im}} - \eta_i^{(n+1)\text{Im}}}{\Delta x} \quad (10a)$$

$$+ F_{i+\frac{1}{2},k}^{(n)\text{Ex}} + \frac{(\tau^x)_{i+\frac{1}{2},k-\frac{1}{2}}^{(n+\theta^*)} - (\tau^x)_{i+\frac{1}{2},k+\frac{1}{2}}^{(n+\theta^*)}}{h_{i+\frac{1}{2},k}^n} - \alpha \frac{q_{i+1,k}^{n-\frac{1}{2}} - q_{i,k}^{n-\frac{1}{2}}}{\Delta x},$$

$$\frac{w_{i,k+\frac{1}{2}}^* - w_{i,k+\frac{1}{2}}^n}{\Delta t} = G_{i,k+\frac{1}{2}}^{(n)\text{Ex}} + \frac{(\tau^z)_{i,k}^{(n+\theta^*)} - (\tau^z)_{i,k+1}^{(n+\theta^*)}}{h_{i,k+\frac{1}{2}}^n} - \alpha \frac{q_{i,k}^{n-\frac{1}{2}} - q_{i,k+1}^{n-\frac{1}{2}}}{h_{i,k+\frac{1}{2}}^n}, \quad (10b)$$

298 representing the hydrostatic prediction step, and

$$\frac{u_{i+\frac{1}{2},k}^{n+1} - u_{i+\frac{1}{2},k}^*}{\Delta t} = - \frac{(q_c)_{i+1,k}^{n+\frac{1}{2}} - (q_c)_{i,k}^{n+\frac{1}{2}}}{\Delta x}, \quad (11a)$$

$$\frac{w_{i,k+\frac{1}{2}}^{n+1} - w_{i,k+\frac{1}{2}}^*}{\Delta t} = - \frac{(q_c)_{i,k}^{n+\frac{1}{2}} - (q_c)_{i,k+1}^{n+\frac{1}{2}}}{h_{i,k+\frac{1}{2}}^n}, \quad (11b)$$

299 representing the nonhydrostatic correction step. The variables u^* and w^* ,
 300 representing the hydrostatic provisional velocities, are calculated by including
 301 all terms in the governing equations except the nonhydrostatic correction
 302 pressure, q_c . In Equations (10), the superscripts enclosed in parenthesis
 303 represent

$$(\cdot)_{i',k'}^{(n)\text{Ex}} = \frac{1}{2}(3+b)(\cdot)_{i',k'}^n - \frac{1}{2}(1+2b)(\cdot)_{i',k'}^{n-1} + \frac{b}{2}(\cdot)_{i',k'}^{n-2} \quad (12)$$

$$(\cdot)_{i',k'}^{(n+1)\text{Im}} = \frac{1}{2}(1+c)(\cdot)_{i',k'}^{n+1} + \frac{1}{2}(1-2c)(\cdot)_{i',k'}^n + \frac{c}{2}(\cdot)_{i',k'}^{n-1} \quad (13)$$

$$(\cdot)_{i',k'}^{(n+\theta^*)} = \theta(\cdot)_{i',k'}^* + (1-\theta)(\cdot)_{i',k'}^n \quad (14)$$

304 where superscript n represents the time step in the multistep scheme and
 305 subscripts i' and k' represent dummy indices which can become any loca-
 306 tion on the staggered grid. Here, the terms discretized with $(\cdot)_{i',k'}^{(n)\text{Ex}}$ and
 307 $(\cdot)_{i',k'}^{(n+1)\text{Im}}$ in Equations (12) and (13), respectively, represent the explicit and
 308 implicit time stepping procedures described in Durran and Blossey (2012).
 309 The explicit (12) and implicit (13) terms in the momentum equations are
 310 weighted using parameters b and c , respectively. For the explicit portion of
 311 the method, $b = 0$ corresponds to the second-order Adams-Bashforth (AB2)
 312 method, $b = 5/6$ is the AB3 method and $b = 1/2$ is the AX2* method
 313 (Durran and Blossey, 2012). On the other hand, for the implicit portion
 314 of the method, $c = 0$ corresponds to the trapezoidal or Crank-Nicolson
 315 (CN) method (Crank and Nicolson, 1947), $c = 1/2$ is the AM2* method and
 316 $c = 3/2$ is the AI2* method (Durran and Blossey, 2012). The discretization
 317 of $(\cdot)_{i',k'}^{(n+\theta^*)}$ given in Equation (14) represents the popular semi-implicit time
 318 stepping procedure used in Casulli and Cattani (1994). Here, θ represents the
 319 implicitness parameter, where $\theta = 0$ is the forward Euler method, $\theta = 1/2$ is
 320 the CN method, and $\theta = 1$ is the backward Euler method. Finally, the split-

321 ting of the governing equations into Equations (10) and (11) represents the
 322 nonhydrostatic projection method of Armfield and Street (2000); Armfield
 323 and Street (2003) where the nonhydrostatic pressure variables are defined at
 324 the half time-steps, $n \pm \frac{1}{2}$. For the nonhydrostatic portion of the method, the
 325 parameter $\alpha = 0$ or 1 designates the use of the pressure projection (1st-order
 326 accurate) or pressure correction (2nd-order accurate) method, respectively
 327 (Armfield and Street, 2002; Kang and Fringer, 2005; Vitousek and Fringer,
 328 2013).

329 The advection, diffusion, Coriolis, and surface pressure² are discretized
 330 with the explicit method given in Equation (12). In Equations (10), $F_{i+\frac{1}{2},k}^n$
 331 and $G_{i,k+\frac{1}{2}}^n$ represent the terms discretized using the generalized, explicit AB
 332 method (12) and are given by

$$\begin{aligned}
 F_{i+\frac{1}{2},k}^n = & -C_H \left(u_{i+\frac{1}{2},k}^n \right) + D_H \left(u_{i+\frac{1}{2},k}^n \right) + f v_{i+\frac{1}{2},k}^n - \tilde{f} w_{i+\frac{1}{2},k} \\
 & - \frac{1}{\rho_0} \frac{(p_s)_{i+1}^n - (p_s)_i^n}{\Delta x} - \frac{(M^{(bc)})_{i+1,k}^n - (M^{(bc)})_{i,k}^n}{\Delta x}, \quad (15a)
 \end{aligned}$$

$$G_{i,k+\frac{1}{2}}^n = -C_H \left(w_{i,k+\frac{1}{2}}^n \right) + D_H \left(w_{i,k+\frac{1}{2}}^n \right) + \tilde{f} u_{i,k+\frac{1}{2}}^n, \quad (15b)$$

²Although the surface pressure is, technically, part of the barotropic pressure or barotropic Montgomery potential, explicit treatment of this term is common and sufficient because it does not lead to problem stiffness. In contrast, the $g\eta$ term in the barotropic Montgomery potential leads to stiffness associated with fast free-surface gravity waves and thus requires implicit discretization.

333 where $M^{(bc)}$ is the baroclinic Montgomery potential which is given by

$$M_1^{(bc)} = 0 , \quad (16)$$

$$M_{k+1}^{(bc)} = M_k^{(bc)} + \frac{(\rho_{k+1} - \rho_k)gz_k}{\rho_0} . \quad (17)$$

334 Equation (17) is a downward recursive relationship that allows the baroclinic
 335 Montgomery potential to be calculated from the layers above. In Equation
 336 (17), the interface locations are calculated from an upward recursive rela-
 337 tionship given by

$$z_{N_{\text{layers}}} = -H , \quad (18)$$

$$z_{k-1} = z_k + h_k , \quad (19)$$

338 where H is the depth to the seabed. $C_H()$ and $D_H()$ represent the horizontal
 339 advection and diffusion operators, respectively. Horizontal advection can be
 340 discretized with several different schemes – the simplest option being the cen-
 341 tral differencing scheme given by $C_H \left(u_{i+\frac{1}{2},k}^n \right) = u_{i+\frac{1}{2},k}^n \frac{u_{i+\frac{3}{2},k}^n - u_{i-\frac{1}{2},k}^n}{2\Delta x}$. For hori-
 342 zontal diffusion the operator is given by $D_H \left(u_{i+\frac{1}{2},k}^n \right) = \nu_H \frac{u_{i+\frac{3}{2},k}^n - 2u_{i+\frac{1}{2},k}^n + u_{i-\frac{1}{2},k}^n}{\Delta x^2}$,
 343 where ν_H is the horizontal eddy-viscosity.

344 The barotropic portion of the Montgomery potential is discretized using
 345 the generalized, semi-implicit Adams-Moulton (AM) method (13) of Durran
 346 and Blossey (2012). This method is reminiscent of the θ -method of Casulli
 347 (1990) and is intended to eliminate the time step restriction associated with

348 fast free-surface gravity waves. The generalized, explicit/implicit method
 349 used here is preferable to popular methods such as Leapfrog-Trapezoidal³
 350 because it does not require filtering. It is also preferable to AB2(3)- θ method⁴
 351 which is unstable for stiff problems in the inviscid limit when $\theta = 0.5$ (Durran,
 352 1991). By using the generalized, explicit/implicit method, the current model
 353 can run stably without horizontal or vertical viscosity.

354 Isopycnal or layered models must use implicit discretization for vertical
 355 diffusion of momentum when the isopycnal layers are allowed to become arbi-
 356 trarily thin (Hallberg, 2000). Following Casulli (1997), the viscous boundary
 357 condition at the free surface is specified by applying a wind stress given as

$$(\tau^x)_{i+\frac{1}{2},1/2}^* = \gamma_w \left((u_w)^{n+1}_{i+\frac{1}{2}} - u_{i+\frac{1}{2},1}^* \right) , \quad (20a)$$

$$(\tau^x)_{i+\frac{1}{2},1/2}^n = \gamma_w \left((u_w)^n_{i+\frac{1}{2}} - u_{i+\frac{1}{2},1}^n \right) , \quad (20b)$$

358 where γ_w is the wind stress coefficient and u_w is the prescribed wind speed.
 359 The boundary condition at the seabed is specified by applying the quadratic
 360 drag law

$$(\tau^x)_{i+\frac{1}{2},N_{\text{layers}}+1/2}^* = C_d |u_{i+\frac{1}{2},N_{\text{layers}}}^n| u_{i+\frac{1}{2},N_{\text{layers}}}^* , \quad (21a)$$

$$(\tau^x)_{i+\frac{1}{2},N_{\text{layers}}+1/2}^n = C_d |u_{i+\frac{1}{2},N_{\text{layers}}}^n| u_{i+\frac{1}{2},N_{\text{layers}}}^n , \quad (21b)$$

³Leapfrog is used for the explicit terms and the trapezoidal method is used for the implicit terms.

⁴AB2 or AB3 is used for the explicit terms and the θ -method is used for the implicit terms.

361 where C_d is the drag coefficient. In Equation (21a), the term representing
 362 the velocity magnitude is taken at time step n as is common for implicit
 363 discretization of the drag term. This time-lagging approach removes the
 364 need to solve a nonlinear system of equations for the horizontal velocity. For
 365 the vertical momentum equation, there is no stress applied in the vertical
 366 direction at the surface, $\tau_{i,-\frac{1}{2}}^z = 0$, and at the bottom, $\tau_{i,N_{\text{layers}}+\frac{1}{2}}^z = 0$. The
 367 shear stress between layers in the horizontal momentum equation is given by
 368

$$(\tau^x)_{i+\frac{1}{2},k+\frac{1}{2}}^* = (\nu_v)_{k+\frac{1}{2}} \frac{u_{i+\frac{1}{2},k}^* - u_{i+\frac{1}{2},k+1}^*}{h_{i+\frac{1}{2},k+\frac{1}{2}}^n}, \quad (22a)$$

$$(\tau^x)_{i+\frac{1}{2},k+\frac{1}{2}}^n = (\nu_v)_{k+\frac{1}{2}} \frac{u_{i+\frac{1}{2},k}^n - u_{i+\frac{1}{2},k+1}^n}{h_{i+\frac{1}{2},k+\frac{1}{2}}^n}. \quad (22b)$$

369 In the vertical momentum equation, the shear stress between layers is given
 370 by

$$(\tau^z)_{i,k}^* = (\nu_v)_k \frac{w_{i,k-\frac{1}{2}}^* - w_{i,k+\frac{1}{2}}^*}{h_{i,k}^n}, \quad (23a)$$

$$(\tau^z)_{i,k}^n = (\nu_v)_k \frac{w_{i,k-\frac{1}{2}}^n - w_{i,k+\frac{1}{2}}^n}{h_{i,k}^n}. \quad (23b)$$

371 In Equations (22a) and (23a), the layer height h is taken at time step n to
 372 eliminate the need to solve a fully nonlinear system.

373 In order to demonstrate the numerical procedure used with implicit ver-
 374 tical diffusion, it is helpful to use matrix notation. After substituting Equa-

375 tions (22) and (23) and the boundary conditions for the shear stress, τ , the
 376 hydrostatic portion of the discrete momentum equations (10) can be written
 377 in matrix form as

$$\mathbf{A}_{i+\frac{1}{2}} \mathbf{u}_{i+\frac{1}{2}}^* = \mathbf{T}_{i+\frac{1}{2}}^n - \frac{1}{2}(1+c)g \frac{\Delta t}{\Delta x} (\eta_{i+1}^{n+1} - \eta_i^{n+1}) \mathbf{e}, \quad (24a)$$

$$\mathbf{B}_i \mathbf{w}_i^* = \mathbf{S}_i^n, \quad (24b)$$

378 where $\mathbf{u}_{i+\frac{1}{2}}^* = [u_{i+\frac{1}{2},1}^*, u_{i+\frac{1}{2},2}^*, \dots, u_{i+\frac{1}{2},k}^*, \dots, u_{i+\frac{1}{2},N_{\text{layers}}}^*]^T$ and
 379 $\mathbf{w}_i^* = [w_{i,\frac{1}{2}}^*, w_{i,\frac{3}{2}}^*, \dots, w_{i,k+\frac{1}{2}}^*, \dots, w_{i,N_{\text{layers}}+\frac{1}{2}}^*]^T$ are the hydrostatic predictor (*)
 380 velocities throughout a column of grid points. The equations for $\mathbf{u}_{i+\frac{1}{2}}^{n+1}$ and
 381 $\mathbf{w}_i^{n+1}$ based on Equations (11) are detailed in Section 3.3. Vectors $\mathbf{T}_{i+\frac{1}{2}}^n$ and
 382 $\mathbf{S}_i^n$ contain all explicit terms of the momentum equations (10) which include
 383 the AB terms (at time levels n , $n-1$, and $n-2$) as well as the explicit parts
 384 of the time derivative, semi-implicit free-surface method (13), θ -method (14),
 385 and nonhydrostatic pressure, viz.

$$\begin{aligned} \mathbf{T}_{i+\frac{1}{2}}^n = & \mathbf{u}_{i+\frac{1}{2}}^n - \frac{1}{2}(1-2c)g \frac{\Delta t}{\Delta x} (\eta_{i+1}^n - \eta_i^n) \mathbf{e} - \frac{c}{2}g \frac{\Delta t}{\Delta x} (\eta_{i+1}^{n-1} - \eta_i^{n-1}) \mathbf{e} \\ & + \Delta t \mathbf{F}_{i+\frac{1}{2}}^{(n)\text{Ex}} - \alpha \frac{\Delta t}{\Delta x} \left(\mathbf{q}_{i+1}^{n-\frac{1}{2}} - \mathbf{q}_i^{n-\frac{1}{2}} \right) + \Delta t(1-\theta)(\mathbf{D}_{\mathbf{A}})_{i+\frac{1}{2}} \mathbf{u}_{i+\frac{1}{2}}^n, \end{aligned} \quad (25a)$$

$$\mathbf{S}_i^n = \mathbf{w}_i^n + \Delta t \mathbf{G}_i^{(n)\text{Ex}} - \alpha \Delta t \mathbf{D}_z \left(\mathbf{q}_i^{n-\frac{1}{2}} \right) + \Delta t(1-\theta)(\mathbf{D}_{\mathbf{B}})_i \mathbf{w}_i^n, \quad (25b)$$

386 where $\mathbf{u}_{i+\frac{1}{2}}^n$, $\mathbf{F}_{i+\frac{1}{2}}^n$ and $\mathbf{w}_i^n$, $\mathbf{G}_i^n$ are defined similar to $\mathbf{u}_{i+\frac{1}{2}}^*$ and $\mathbf{w}_i^*$ (i.e. they

387 represent the variables of a column of grid points) and $\mathbf{e} = [1, 1, \dots, 1]^T \in$
 388 $\Re^{N_{\text{layers}} \times 1}$ and $\mathbf{D}_z \left(\mathbf{q}_i^{n-\frac{1}{2}} \right) = \left(q_{i,k}^{n-\frac{1}{2}} - q_{i,k+1}^{n-\frac{1}{2}} \right) / h_{i,k}^n \quad \forall k$ represents the vertical
 389 first derivative. The matrices $\mathbf{A}$ and $\mathbf{B}$, given by

$$\mathbf{A}_{i+\frac{1}{2}} = \mathbf{I} - \Delta t \theta (\mathbf{D}_{\mathbf{A}})_{i+\frac{1}{2}} , \quad (26a)$$

$$\mathbf{B}_i = \mathbf{I} - \Delta t \theta (\mathbf{D}_{\mathbf{B}})_i , \quad (26b)$$

390 with $\mathbf{D}_{\mathbf{A}}$ and $\mathbf{D}_{\mathbf{B}}$ being tridiagonal matrices of the form

$$(\mathbf{D}_{\mathbf{A}})_{i+\frac{1}{2}} = \text{tridiag} \left[\frac{(\nu_v)_{k-\frac{1}{2}}}{h_{i+\frac{1}{2},k}^n h_{i+\frac{1}{2},k-\frac{1}{2}}^n}, - \left(\frac{(\nu_v)_{k-\frac{1}{2}}}{h_{i+\frac{1}{2},k}^n h_{i+\frac{1}{2},k-\frac{1}{2}}^n} + \frac{(\nu_v)_{k+\frac{1}{2}}}{h_{i+\frac{1}{2},k}^n h_{i+\frac{1}{2},k+\frac{1}{2}}^n} \right), \frac{(\nu_v)_{k+\frac{1}{2}}}{h_{i+\frac{1}{2},k}^n h_{i+\frac{1}{2},k+\frac{1}{2}}^n} \right] , \quad (27a)$$

$$(\mathbf{D}_{\mathbf{B}})_i = \text{tridiag} \left[\frac{(\nu_v)_k}{h_{i,k+\frac{1}{2}}^n h_{i,k}^n}, - \left(\frac{(\nu_v)_k}{h_{i,k+\frac{1}{2}}^n h_{i,k}^n} + \frac{(\nu_v)_{k+1}}{h_{i,k+\frac{1}{2}}^n h_{i,k+1}^n} \right), \frac{(\nu_v)_{k+1}}{h_{i,k+\frac{1}{2}}^n h_{i,k+1}^n} \right] , \quad (27b)$$

391 represent the implicit discretization of vertical diffusion of momentum. Note
 392 that the first and last diagonal elements of the tridiagonal matrix (27a) must
 393 be modified to account for the proper boundary conditions in Equations (20)
 394 and (21), respectively. When a variable is needed that is not located at the
 395 position on the grid where it is defined (such as $h_{i\pm\frac{1}{2},k}$, $h_{i,k\pm\frac{1}{2}}$, or $h_{i\pm\frac{1}{2},k\pm\frac{1}{2}}$),
 396 then it is approximated with linear interpolation. Finally, the momentum

equations (24) can be written as

$$\mathbf{u}_{i+\frac{1}{2}}^* = \tilde{\mathbf{T}}_{i+\frac{1}{2}}^n - \frac{1}{2}(1+c)g\frac{\Delta t}{\Delta x}(\eta_{i+1}^{n+1} - \eta_i^{n+1})\tilde{\mathbf{e}}_{i+\frac{1}{2}}, \quad (28a)$$

$$\mathbf{w}_i^* = \tilde{\mathbf{S}}_i^n, \quad (28b)$$

where

$$\tilde{\mathbf{T}}_{i+\frac{1}{2}}^n = \mathbf{A}_{i+\frac{1}{2}}^{-1} \mathbf{T}_{i+\frac{1}{2}}^n, \quad (29a)$$

$$\tilde{\mathbf{e}}_{i+\frac{1}{2}} = \mathbf{A}_{i+\frac{1}{2}}^{-1} \mathbf{e}, \quad (29b)$$

$$\tilde{\mathbf{S}}_i^n = \mathbf{B}_i^{-1} \mathbf{S}_i^n, \quad (29c)$$

represent the solutions following the tridiagonal inversion for each column of grid points in the horizontal.

3.2. Discretized free-surface and layer-height equations

In isopycnal models the layer height continuity equation (7) is used to evolve both the free surface and the internal interface (Adcroft and Hallberg, 2006). In the present model, we differentiate the free surface from internal interfaces. Accordingly, we derive separate but consistent model equations for the evolution of the internal layers heights h and free surface, η , both of which use the generalized, semi-implicit AM method of Durran and Blossey (2012).

The layer height continuity equation (7) is discretized with the generalized

410 semi-implicit AM method and is given by

$$\frac{h_{i,k}^{n+1} - h_{i,k}^n}{\Delta t} = -\frac{1}{\Delta x} \left(h_{i+\frac{1}{2},k}^n u_{i+\frac{1}{2},k}^{(n+1)\text{Im}^*} - h_{i-\frac{1}{2},k}^n u_{i-\frac{1}{2},k}^{(n+1)\text{Im}^*} \right), \quad (30)$$

411 where $u_{i\pm\frac{1}{2}}^{(n+1)\text{Im}^*} = \frac{1}{2}(1+c)u_{i\pm\frac{1}{2}}^* + \frac{1}{2}(1-2c)u_{i\pm\frac{1}{2}}^n + \frac{c}{2}u_{i\pm\frac{1}{2}}^{n-1}$ following Equation (13)
 412 and using the hydrostatic predictor velocity u^* instead of the final velocity
 413 u^{n+1} . Using a first-order upwind approach to calculate $h_{i\pm\frac{1}{2},k}^n$ and assuming
 414 a positive flow direction, Equation (30) (following the approach of Stelling
 415 and Duinmeijer (2003)) can be written as

$$h_{i,k}^{n+1} = \left(1 - \frac{u_{i+\frac{1}{2},k}^{(n+1)\text{Im}^*} \Delta t}{\Delta x} \right) h_{i,k}^n + \frac{u_{i-\frac{1}{2},k}^{(n+1)\text{Im}^*} \Delta t}{\Delta x} h_{i-1,k}^n, \quad (31)$$

416 with $u_{i\pm\frac{1}{2},k}^{(n+1)\text{Im}^*} \geq 0$. A similar form can be obtained for $u_{i\pm\frac{1}{2},k}^{(n+1)\text{Im}^*} \leq 0$. As
 417 seen in Equation (31), as long as the advective Courant number $U_{i+\frac{1}{2},k}^{(n+1)\text{Im}^*} =$
 418 $\frac{u_{i+\frac{1}{2},k}^{(n+1)\text{Im}^*} \Delta t}{\Delta x} \leq 1$ then positive water depths are obtained without any special
 419 treatment for wetting and drying (Stelling and Duinmeijer, 2003). We can
 420 apply MPDATA corrections (outlined in Appendix C) to the first-order
 421 upwinding approach to ensure that the method is second-order accurate in
 422 time and space and the layer heights remain positive.

423 The equation governing the evolution of the free surface is derived by
 424 summing Equation (30) over the layer index k . The resulting free-surface

equation is given by

$$-(\alpha_1)_{i-\frac{1}{2}}^n \eta_{i-1}^{n+1} + \left[1 + (\alpha_1)_{i-\frac{1}{2}}^n + (\alpha_1)_{i+\frac{1}{2}}^n\right] \eta_i^{n+1} - (\alpha_1)_{i+\frac{1}{2}}^n \eta_{i+1}^{n+1} = R_i^n, \quad (32)$$

which is a tridiagonal equation in one horizontal dimension or a block tridiagonal in 3-D. The coefficients $(\alpha_1)_{i\pm\frac{1}{2}}^n = g \left[\frac{1}{2}(1+c)\frac{\Delta t}{\Delta x}\right]^2 \mathbf{e}^T(\mathbf{h}_{i\pm\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i\pm\frac{1}{2}})$ (where $\circ$ is the element-wise product) and the right-hand side R_i^n are derived in Appendix B.

Because both the free-surface equation (32) and the sum of the layer heights given by Equation (30) provide an estimate of the free-surface height, η , these estimates can become inconsistent over the course of a simulation. Models that use a mode-split time-stepping procedure are susceptible to this inconsistency, as discussed at length by Hallberg and Adcroft (2009). In the present model, consistency between the free-surface equation (32) and the layer-height equation (30) is ensured as long as the same flux face heights, $h_{i\pm\frac{1}{2},k}^n$, are used in each equation⁵. However, when using upwinding or MP-DATA for Equation (30) or solving the free-surface equation (32) (in 2-D) using a non-exact (iterative) method such as conjugate gradients, this is generally not the case. Notice how according to Equation (30) and (31) we seek to upwind the flux face heights, $h_{i\pm\frac{1}{2},k}^n$, using the velocity $u_{i\pm\frac{1}{2},k}^{(n+1)\text{Im}^*}$. This velocity and thus $h_{i\pm\frac{1}{2},k}^n$ are unknown⁶ when the free-surface equation (32) is

⁵naturally, since the free-surface equation is derived from the layer-height equation.

⁶The initial flux face heights, $h_{i\pm\frac{1}{2},k}^n$, and thus α and R in the free-surface equation (32)

443 solved because the velocity $u^{(n+1)_{\text{Im}^*}}$ depends on u^* which depends on η^{n+1}
 444 according to Equation (28a). There are a few methods that can be used
 445 to remove the inconsistency between the free-surface equation (32) and the
 446 layer-height equation (30). One option is to use an iterative procedure be-
 447 tween the free-surface equation (32) and the layer-height equation (30). The
 448 values of $h_{i\pm\frac{1}{2},k}^n$ (based on upwinding of the velocity $u_{i\pm\frac{1}{2},k}^{(n+1)_{\text{Im}^*}}$) can be reused
 449 in the free-surface equation and the entire procedure can be iterated until
 450 convergence is reached. This procedure is outlined in detail in Section 3.4. In
 451 general, we have found that this iterative procedure converges quite rapidly
 452 in $\mathcal{O}(1)$ iterations. Other non-iterative options to remove this inconsistency
 453 include: [1] upwind the flux face heights based on the u^n velocity; [2] directly
 454 set the free surface η to $\sum h_k - H_i$, where H_i is the still water depth at loca-
 455 tion i ; or [3] apply a uniform expansion/contraction of the isopycnal layers
 456 to fit the total depth according to $h_k = \left(\frac{\eta+H}{\sum h_k}\right) h_k$ (Bleck and Smith, 1990;
 457 Hofmeister et al., 2010). Method [3] is generally the preferred method used
 458 in existing isopycnal models.

459 3.3. Nonhydrostatic pressure

460 Equations (28) provide the hydrostatic predictor portion of the numeri-
 461 cal solution. To obtain the nonhydrostatic solution, the hydrostatic predic-
 462 tor velocities are corrected according to Equations (11) in order to satisfy

are approximated using the upwinded layer heights based on the $u_{i\pm\frac{1}{2},k}^n$ velocity instead
 of the $u_{i\pm\frac{1}{2},k}^{(n+1)_{\text{Im}^*}}$ velocity which is calculated later.

463 the divergence-free constraint. The nonhydrostatic correction pressure q_c
 464 is determined from the solution of a Poisson equation which is derived by
 465 substituting the nonhydrostatic correction velocities, $u_{i\pm\frac{1}{2},k}^{n+1}$ and $w_{i,k\pm\frac{1}{2}}^{n+1}$, from
 466 Equations (11) into the discrete divergence-free condition which, as derived
 467 in Appendix D, is subject to a mild-slope approximation and is given by

$$\frac{1}{\Delta x} \left(h_{i+\frac{1}{2},k}^{n+1} u_{i+\frac{1}{2},k}^{n+1} - h_{i-\frac{1}{2},k}^{n+1} u_{i-\frac{1}{2},k}^{n+1} \right) + w_{i,k-\frac{1}{2}}^{n+1} - w_{i,k+\frac{1}{2}}^{n+1} = 0 . \quad (33)$$

468 Substituting (11) into (33) results in a symmetric linear system of equations
 469 for the nonhydrostatic pressure q_c that is given by

$$\begin{aligned} & \lambda_{i,k-\frac{1}{2}}^{n+1} (q_c)_{i,k-1}^{n+\frac{1}{2}} + \frac{1}{\lambda_{i-\frac{1}{2},k}^{n+1}} (q_c)_{i-1,k}^{n+\frac{1}{2}} \\ & - \left(\lambda_{i,k-\frac{1}{2}}^{n+1} + \lambda_{i,k+\frac{1}{2}}^{n+1} + \frac{1}{\lambda_{i-\frac{1}{2},k}^{n+1}} + \frac{1}{\lambda_{i+\frac{1}{2},k}^{n+1}} \right) (q_c)_{i,k}^{n+\frac{1}{2}} \\ & + \frac{1}{\lambda_{i+\frac{1}{2},k}^{n+1}} (q_c)_{i+1,k}^{n+\frac{1}{2}} + \lambda_{i,k+\frac{1}{2}}^{n+1} (q_c)_{i,k+1}^{n+\frac{1}{2}} \\ & = \frac{\Delta x}{\Delta t} \left[\frac{1}{\Delta x} \left(h_{i+\frac{1}{2},k}^{n+1} u_{i+\frac{1}{2},k}^* - h_{i-\frac{1}{2},k}^{n+1} u_{i-\frac{1}{2},k}^* \right) + w_{i,k-\frac{1}{2}}^* - w_{i,k+\frac{1}{2}}^* \right] . \end{aligned} \quad (34)$$

470 In Equation (34), $\lambda = \Delta x/h$ is the grid leptic ratio or lepticity (Scotti and
 471 Mitran, 2008) of the isopycnal layer. This parameter is a measure of the
 472 ratio of numerical to physical dispersion (Vitousek and Fringer, 2011) and
 473 sets the anisotropy of the elliptic equation (34). Derivation of the full elliptic
 474 equation in isopycnal coordinates (without the mild-slope approximation)

475 is presented in Appendix E. Equation (34) is quite similar to the elliptic
 476 equation derived from z -coordinates. The only significant difference is that,
 477 here, the coefficients are based on the time-variable grid aspect-ratio $\lambda =$
 478 $\Delta x/h$ where as in z -coordinates the grid aspect ratio ($\Delta x/\Delta z$) is fixed in
 479 time. Since the condition number of the linear system is related to the grid
 480 aspect ratio (Marshall et al., 1997b; Kramer et al., 2010; Fiebig-Wittmaack
 481 et al., 2011), time variability of the coefficients of Equation (34) results in
 482 time variability of the condition number of the linear system. However, we
 483 find that popular elliptic solvers, such as preconditioned conjugate gradients
 484 or multigrid with semi-coarsening and line-relaxation⁷ (Briggs et al., 2000),
 485 are suitable to solving Equation (34) efficiently.

486 As discussed previously, Adcroft and Hallberg (2006) conclude that the
 487 nonhydrostatic equations and the LVD approach (using a prognostic equa-
 488 tion for h^{n+1} as in Equation (30)) are mutually exclusive. The present study
 489 is not limited by this issue. As mentioned in Section 1.3, the two forms of
 490 the continuity equation, the layer-height equation (30) and the divergence-
 491 free equation (33), assume different roles in the numerical method. In fact,
 492 the numerical method in current approach is remarkably similar to nonhy-
 493 drostatic methods with a free surface in z -coordinates. Here, the hydrostatic
 494 portion of the method advances the free surface and layer heights (or the free
 495 surface alone when using z -coordinates). Next, the nonhydrostatic method is

⁷a method developed for anisotropic problems

496 used to correct the velocities while holding the layers fixed. As a result, the
 497 final free surface and layer heights are not consistent with the layer-height
 498 continuity equation in terms of the final velocities (at time $n + 1$). A “fully”
 499 nonhydrostatic method would require iteration of the elliptic and layer-height
 500 equations. However, holding the free surface fixed during the nonhydrostatic
 501 portion of the pressure correction method in z -coordinates does not impact
 502 the order of accuracy (Armfield and Street, 2000; Kang and Fringer, 2005;
 503 Vitousek and Fringer, 2013). The convergence analysis presented in Section
 504 4.2 demonstrates that this is also the case when fixing the layer heights during
 505 the nonhydrostatic portion of the method in isopycnal coordinates.

506 3.4. *Solution procedure*

507 The solution procedure for one time step of the present model is given
 508 by:

- 509 1. Calculate the explicit portions of the horizontal and vertical momentum
 510 equations, $\mathbf{T}^n$ and $\mathbf{S}^n$, using Equations (25).
- 511 2. Solve the implicit vertical diffusion equations (29) to compute $\tilde{\mathbf{T}}_{i+\frac{1}{2}}^n$,
 512 $\tilde{\mathbf{e}}_{i+\frac{1}{2}}$, and $\tilde{\mathbf{S}}_i^n$ for each column of grid points in the horizontal where the
 513 tridiagonal matrices $\mathbf{A}_{i+\frac{1}{2}}$ and $\mathbf{B}_i$, given Equations (26a) and (26b),
 514 respectively, represent implicit discretization of vertical diffusion.
- 515 3. Solve the implicit free-surface equation (32) to obtain the free surface
 516 height at the next time step, η^{n+1} .

- 517 4. Compute the hydrostatic predictor velocities $\mathbf{u}_{i+\frac{1}{2}}^*$ and $\mathbf{w}_i^*$ using Equa-
518 tions (28).
- 519 5. Compute the layer heights, h^{n+1} using Equation (30) and (optionally)
520 the MPDATA procedure described in Appendix C.
- 521 6. If an iterative procedure is used to obtain consistency between η^{n+1} and
522 h^{n+1} , return to step 3, where the α_1 coefficients in the free-surface equa-
523 tion (32) are updated based on upwinding the velocities ($u^{(n+1)\text{Im}*} = f(u^*, u^n, u^{n-1})$)
524 obtained in step 4. If the iterate values for h^{n+1} have converged to some
525 tolerance, continue to step 7. If a non-iterative procedure is used, set
526 $h_k^{n+1} = \left(\frac{\eta^{n+1} + H}{\sum h_k^{n+1}} \right) h_k^{n+1}$ and continue.
- 527 7. Update the interface locations, z (see Figure 1), and the baroclinic
528 Montgomery potentials, $M^{(\text{bc})}$, using Equations (19) and (17), respec-
529 tively.
- 530 8. For the nonhydrostatic method, solve the Poisson equation (34) for the
531 nonhydrostatic correction pressure ($q_c^{n+\frac{1}{2}}$), correct the horizontal pre-
532 dictor (hydrostatic) velocities using the nonhydrostatic pressure (11a),
533 and set $q^{n+\frac{1}{2}} = \alpha q_c^{n+\frac{1}{2}} + q^{n-\frac{1}{2}}$. To run the model in hydrostatic mode,
534 this step is ignored; $q_c = 0$ and $u^{n+1} = u^*$.
- 535 9. Finally, compute the vertical velocity by integrating the continuity
536 equation (33) from the bottom grid cell (where $w_{i, N_{\text{layers}} + \frac{1}{2}} = 0$) up-
537 ward.

538 4. Stability and convergence

539 4.1. Stability

540 The current model is susceptible to both physical and numerical insta-
541 bilities. The physical instability is due to the formation of Kelvin-Helmholtz
542 (KH) billows that may occur when the Richardson number (Ri) is less than
543 $1/4$ (Miles, 1961; Howard, 1961). Numerical instability, on the other hand,
544 is related to the Courant numbers for the time stepping of terms such as the
545 baroclinic pressure gradient, advection of momentum, horizontal viscosity or
546 updating the layer heights.

547 Although nonhydrostatic models using z - and σ -coordinates are capable
548 of resolving KH billows, layered and isopycnal coordinate formulations are
549 not. The formation of KH instabilities renders layered models ill-posed and
550 unstable (Grue et al., 1997; Jo and Choi, 2002; Castro-Díaz et al., 2011;
551 Mandli, 2013). Chumakova et al. (2009) showed that the condition for in-
552 stability of the inviscid, multilayer shallow water (i.e. hydrostatic) equations
553 is $Ri < 1/4$. Barad and Fringer (2010) found that KH billows develop on
554 internal solitary waves when $Ri < 0.1$. The appearance of KH instabilities
555 in layered models can be suppressed by adding viscosity (Castro-Díaz et al.,
556 2011), filtering (Jo and Choi, 2008) or adopting a different numerical formu-
557 lation (Choi et al., 2009). The current model does not take active measures
558 to remove this physical instability. Instead, the model is typically applied in
559 regimes where $Ri > 1/4$ and KH billows do not exist. However, in general,
560 we do not attempt to proactively avoid dynamical situations where KH bil-

561 lows develop. As is typically the case in hydrostatic isopycnal models, we
 562 have found that the small amount of damping due to viscosity, drag, and the
 563 numerical schemes is usually sufficient to suppress instability due to the KH
 564 mechanism.

565 The numerical stability of the time-stepping method used in the current
 566 model is investigated in Durran and Blossey (2012). For a 1-D stiff, linear
 567 model problem, Durran and Blossey (2012) determined a maximum Courant
 568 number of 0.76 and 0.72 for the AM2*-AX2* and AI2*-AB3 methods, re-
 569 spectively. We generally find that the 2-D model is stable for nonlinear sim-
 570 ulations when the internal Courant number, $C_i = \frac{c_i \Delta t}{\Delta x} < 0.5$ where c_i is the
 571 maximum (mode-1) internal wave speed. We also find that the model is sta-
 572 ble when the horizontal diffusion Courant number $C_{\nu_H} = \frac{2\nu_H \Delta t}{\Delta x^2} < 0.25$ when
 573 using AB3. Additionally, we require that the horizontal advective Courant
 574 number $C_u = \frac{u \Delta t}{\Delta x} \leq 1$ to ensure positive layer heights according to Equation
 575 (31). There is no time step restriction associated with vertical advection
 576 since such terms do not appear in the governing equations (6). In 3-D, the
 577 maximum Courant numbers may be reduced even further. In general, there
 578 is no restriction on the free-surface Courant number since the time stepping
 579 method for the surface gravity-wave terms is A-stable (Durran and Blossey,
 580 2012).

581 4.2. Convergence

582 As mentioned in Section 3.1, the numerical method is second-order ac-
 583 curate in time, space, and the number of layers. To demonstrate this con-
 584 vergence rate, the model is run with an idealized, inviscid two-layer internal
 585 seiche in an enclosed domain as the temporal, horizontal, and vertical model
 586 resolution is increased. In the following convergence tests, the measure of
 587 convergence of the solution is obtained by

$$\text{Error} = \text{solution}(\Delta t, \Delta x, \text{ or } \Delta h) - \text{solution}(\Delta t/2, \Delta x/2, \text{ or } \Delta h/2) , \quad (35)$$

588 where “solution” means the numerical solution of variables of interest such
 589 as η, ξ, u, w and q at the final time $t_{\max}$ with a time step of Δt , a grid spacing
 590 of Δx , or a layer spacing of Δh , respectively.

591 For the convergence study and the following numerical experiments, we
 592 use the AM2*-AB3 numerical method, for which $c = 1/2$ and $b = 5/6$ (Dur-
 593 ran and Blossey, 2012). The convergence analysis shown in Figure 2 is per-
 594 formed for the evolution of an internal seiche initialized with a cosine wave of
 595 the form $\xi = z_1 + H/2 = a_i \cos(\pi x/L)$ (z_1 is the location of the internal inter-
 596 face, H is the depth, and L is the domain length) with upper- and lower-layer
 597 densities given by $\rho_1 = 1000 \text{ kg m}^{-3}$ and $\rho_2 = 1001 \text{ kg m}^{-3}$, respectively.
 598 The problem setup is similar to the configuration shown in Figure 3 D. We
 599 use the pressure correction method, $\alpha = 1$, and an iterative procedure for
 600 the free surface with MPDATA.

Figure 2: The temporal, horizontal and vertical resolution convergence analysis in panels A,B and C, respectively. The convergence analysis shows that the method converges with second-order accuracy in both time and space.

601 The temporal convergence experiment, shown in Figure 2 A, is run with
602 $a_i = 1$ m, $L = 1000$ m, and $H = 100$ m and with $N_x = 128$ grid points
603 in the horizontal and 2 layers in the vertical for various time step sizes
604 $\Delta t = [\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}]$ s with $t_{\max} = 50$ s. As shown in Figure 2 A, the
605 numerical solution gives $\mathcal{O}(\Delta t^2)$ convergence with refinement of the time
606 step size, Δt , as expected. In Figure 2 A, the first-order convergence rate
607 of the nonhydrostatic pressure, q , arises because the nonhydrostatic pres-
608 sure is defined at time $n + \frac{1}{2}$ (Armfield and Street, 2003; Fringer et al.,
609 2006). If this pressure is extrapolated to the final time $t_{\max} = n_{\max}\Delta t$ using
610 $q_{\text{extrap}} = \frac{3}{2}q^{n_{\max}-\frac{1}{2}} - \frac{1}{2}q^{n_{\max}-\frac{3}{2}} + \mathcal{O}(\Delta t^2)$, then $\mathcal{O}(\Delta t^2)$ convergence is achieved
611 as demonstrated by the slope of q_{extrap} in Figure 2 A.

612 The horizontal grid refinement convergence experiment, shown in Figure
613 2 B, is run with $a_i = 1$ m, $L = 100$ m and $H = 100$ m and a small fixed time

614 step of $\Delta t = 0.01$ s for 100 time steps. The model is run with 2 vertical layers
 615 and a variable number of horizontal grid points $N_x = [4, 8, 16, 32, 64, 128]$.
 616 Because the model uses a staggered grid, variables located at cell centers
 617 such as η are interpolated to the location $x = 0$ using AB2 $\eta_{(x=0)} = \frac{3}{2}\eta_1 -$
 618 $\frac{1}{2}\eta_2 + \mathcal{O}(\Delta x^2)$. As shown in Figure 2 B, the numerical solution gives $\mathcal{O}(\Delta x^2)$
 619 convergence with refinement of the horizontal grid spacing, Δx , as expected.

620 The vertical layer refinement convergence experiment, shown in Figure
 621 2 C, is run with $a_i = 1$ m, $L = 100$ m and $H = 100$ m and a small fixed
 622 time step of $\Delta t = 0.01$ s for 100 time steps. The model is run with a fixed
 623 horizontal resolution of $N_x = 512$ and variable number of layers $N_{\text{layers}} =$
 624 $[2, 4, 8, 16, 32]$. We note that although the model may have more than two
 625 layers, the density only changes across the internal interface. The density
 626 of the upper half of the layers is $\rho_1 = 1000$ kg m⁻³ and the density of the
 627 lower half of the layers is $\rho_2 = 1001$ kg m⁻³. The variables of interest are
 628 interpolated to the location of the free surface η or internal interface ξ at
 629 the final time t_{max} with a vertical layer resolution of $\Delta h = H/N_{\text{layers}}$. As
 630 shown in Figure 2 C, the numerical solution gives $\mathcal{O}(\Delta h^2)$ convergence with
 631 refinement of the layer heights, Δh , as expected.

632 **5. Numerical experiments**

633 In this section, numerical experiments are conducted to verify the model.
 634 The test cases employed here are: (1) free-surface, (2) two-layer, and (3)
 635 continuously stratified seiches in enclosed domains; the formation of internal

solitary waves (4) following KdV theory, (5) following Vitousek and Fringer (2011), and (6) following the laboratory experiments of Horn et al. (2001); (7) internal wave generation over a ridge in the South China Sea after Buijsman et al. (2010); (8) the formation of internal wave beams generated from oscillatory flow over a sill [see e.g. Kundu (1990)].

All of the numerical experiments use the AM2*-AB3 numerical method (for which $c = 1/2$ and $b = 5/6$ (Durran and Blossey, 2012)), the pressure correction method ($\alpha = 1$), the iterative procedure for the free surface and the upwind method to compute the flux-face heights unless stated otherwise. All of the numerical experiments are run with a free surface and without surface pressure, $p_s = 0$, or Coriolis force, $f = \tilde{f} = 0$.

5.1. Free-surface seiche

In the following sections (5.1-5.3), the (inviscid) isopycnal model is validated against linear gravity-wave theory. The dispersion relations used in the numerical experiments in Sections 5.1-5.3 are listed in Table 1.

Table 1: Dispersion relations for numerical experiments in Sections 5.1-5.3.

Section	Seiche test case	C (wave speed)	$C_h = \lim_{kH \rightarrow 0} C$ (shallow-water limit)	$C_d = \lim_{kH \rightarrow \infty} C$ (deep-water limit)
5.1	Free-surface	$\sqrt{\frac{g}{k} \tanh(kH)}$	$\sqrt{gH}$	$\sqrt{\frac{g}{k}}$
5.2	Two-layer internal	$\sqrt{\frac{g'}{2k} \tanh\left(\frac{kH}{2}\right)}$	$\sqrt{\frac{g'H}{4}}$	$\sqrt{\frac{g'}{2k}}$
5.3	Continuously stratified	$\sqrt{\frac{g'}{2k} \tanh\left(\frac{kH}{2}\right) f_i(k\delta_\rho)}$	$\sqrt{\frac{g'H}{4} f_i(k\delta_\rho)}$	$\sqrt{\frac{g'}{2k} f_i(k\delta_\rho)}$

In Table 1, $k = 2\pi/\lambda_w$ is the wavenumber where $\lambda_w = 2L$ is the fun-

652 damental wavelength in an enclosed domain of length L , H is the depth, g
653 is gravity, $g' = \frac{\rho - \rho_0}{\rho_0}g$ is the reduced gravity, $kH \rightarrow 0$ is the (hydrostatic)
654 shallow-water limit, $kH \rightarrow \infty$ is the (nonhydrostatic) deep-water limit and
655 $f_i(k\delta_\rho) = (1 + k\delta_\rho/2)^{-1}$ accounts for the effect of the finite-thickness interface
656 (Kundu, 1990) where δ_ρ is the pycnocline thickness as discussed in Section
657 5.3.

658 In Sections 5.1-5.3, the modeled wave speeds are determined by estimating
659 the period of oscillation, T_{modeled} , from the simulation start time to the time
660 of the next maximum in the free surface or isopycnal at $x = 0$ and then
661 calculating the wave speed based on $c_{\text{modeled}} = \omega_{\text{modeled}}/k = 2\pi/(kT_{\text{modeled}})$.

662 In Table 1, the ratio of the hydrostatic wave speed to the theoretical
663 (nonhydrostatic) wave speed for a free-surface seiche is given by

$$\frac{c_h}{c} = \sqrt{\frac{kH}{\tanh(kH)}}. \quad (36)$$

664 In this numerical experiment, we compare the wave speeds of the nonhydro-
665 static and hydrostatic models to the theoretical wave speed. Equation (36)
666 shows that the hydrostatic model always overpredicts the wave speed except
667 in the limit that $kH \rightarrow 0$. In this numerical experiment, we also compare
668 depth-profiles of the horizontal and vertical velocity in the nonhydrostatic
669 and hydrostatic models to linear theory. The analytical solutions for the
670 horizontal and vertical velocity of a nonhydrostatic free-surface seiche are

671 given by

$$u = ag \frac{k \cosh(k(z+H))}{\omega \cosh(kH)} \sin(kx) \sin(\omega t) , \quad (37a)$$

672

$$w = -ag \frac{k \sinh(k(z+H))}{\omega \cosh(kH)} \cos(kx) \sin(\omega t) , \quad (37b)$$

673 where a is the initial amplitude of the free-surface seiche with analytical
 674 solution $\eta = a \cos(kx) \cos(\omega t)$. The corresponding analytical solutions for
 675 the horizontal and vertical velocity profiles of a hydrostatic free-surface seiche
 676 are given by

$$u = ag \frac{k}{\omega} \sin(kx) \sin(\omega t) , \quad (38a)$$

677

$$w = -ag \frac{k}{\omega} (k(z+H)) \cos(kx) \sin(\omega t) , \quad (38b)$$

678 which can be found by taking Equations (37) in the limit that $kH \rightarrow 0$.

679 The initial configuration of this numerical experiment is depicted in Fig-
 680 ure 3 (A,B,C), but with varying domain sizes, number of layers, and a smaller
 681 seiche amplitude⁸. We note that the densities of each layer do *not* necessarily
 682 need to change from one layer to the next as in the case of a free-surface se-
 683 iche. The model is initialized with a cosine wave of the form $\eta = a \cos(\pi x/L)$
 684 where the seiche amplitude is given by $a = 0.1$ m and the depth is given by
 685 $H = 10$ m on a grid with $N_x = 64$ grid points in the horizontal and a vary-
 686 ing number of layers in the vertical each with density $\rho_0 = 1000$ kg m⁻³.
 687 To change the degree of nonhydrostasy, kH , the model is run with varying

⁸Note: The seiche amplitude is chosen to be small to minimize the free-surface nonlin-
 earity, $\delta = a/H$, so that linear wave theory is valid.

688 domain lengths while holding the depth fixed. The model is run with a free-
689 surface Courant number $C = \frac{c\Delta t}{\Delta x} = 1$ for 1.5 seiche periods, $t_{\max} = 1.5T$
690 where $T = 2\pi/\omega$.

691 The modeled wave speeds for the nonhydrostatic and hydrostatic models
692 normalized by the theoretical wave speed agree with the relationships given
693 by $c_{\text{nh}}/c = 1$ and Equation (36), respectively, as shown in Figure 4 A. This
694 indicates that the model achieves the correct dispersive behavior. When the
695 nonhydrostasy parameter kH is small, the hydrostatic and nonhydrostatic
696 models are similar. However, when there is an appreciable degree of nonhy-
697 drostasy, the hydrostatic model overpredicts the true wave speed. Figure 4 A
698 also demonstrates how the number of isopycnal layers affects the dispersive
699 properties of the model. The effect of the nonhydrostatic pressure is to intro-
700 duce depth-variability in the velocity field that decays like k^{-1} . Nonhydro-
701 static models with a limited number of layers underpredict the wave speeds,
702 particularly for large values of kH . However, as layers are added to the non-
703 hydrostatic model, the modeled wave speeds converge rapidly to the theory.
704 For problems with low to moderate kH , a nonhydrostatic model with two
705 layers is sufficient to accurately capture the dispersion (Stelling and Zijlema,
706 2003; Zijlema and Stelling, 2005). On the other hand, hydrostatic waves are
707 non-dispersive and induce horizontal velocities that are depth-invariant (see
708 Equation 38a). Consequently, the wave speeds in the hydrostatic model are
709 uninfluenced by the number of layers.

710 The test case for the velocity profiles of a free-surface seiche uses the

711 same parameters as above except with $H = 16$ m, $L = 10$ m and $N_{\text{layers}} = 20$
 712 which are chosen so that the experiment is strongly nonhydrostatic. Figure
 713 5 A shows the normalized horizontal velocity profiles for the nonhydrostatic
 714 and hydrostatic models at location $x = L/2$ as compared to the theoretical
 715 profiles given in Equations (37a) and (38a), respectively, at time $t = T/4$.
 716 Likewise, Figure 5 B shows the normalized vertical velocity profiles from
 717 the nonhydrostatic and hydrostatic model at location $x = 0$ as compared to
 718 the theoretical profiles given in Equations (37b) and (38b), respectively, at
 719 time $t = T/4$. In each case the model accurately reproduces the theoretical
 720 behavior. The most significant difference between the hydrostatic and non-
 721 hydrostatic velocity profiles is the depth-variation of the horizontal velocity.
 722 As seen in Figure 5 A, the hydrostatic horizontal velocity profile does not
 723 vary with depth. In contrast, the nonhydrostatic horizontal velocity profile
 724 decays rapidly with depth from its maximum value at the free surface.

725 5.2. *Two-layer internal seiche*

726 In this test case, the (inviscid) isopycnal model is validated against linear
 727 theory describing internal gravity waves in a two-layer system. From Table
 728 1, the ratio of the hydrostatic wave speed to the theoretical wave speed for
 729 a two-layer seiche is given by

$$\frac{c_h}{c} = \sqrt{\frac{kH/2}{\tanh(kH/2)}}. \quad (39)$$

Equation (39) exhibits similar behavior to that of the free-surface seiche, (36), but differs in that the nonhydrostasy parameter is effectively halved for the internal seiche because the interface is located in the middle of water column which is effectively half as deep. In this test case, we also compare depth profiles of the horizontal and vertical velocity from the nonhydrostatic and hydrostatic models to theory which is computed with the first-mode linearized eigenfunction analysis of Fringer and Street (2003).

The initial configuration of this numerical experiment on the behavior of a two-layer internal seiche is shown in Figure 3 (D,E,F), but with varying domain sizes, number of layers, and a smaller constant seiche amplitude. The model is initialized with an internal cosine wave of the form $z_1 = -H/2 + a \cos(\pi x/L)$, where $a = 0.1$ m and $H = 10$ m on a grid with $N_x = 64$ grid points in the horizontal and a varying number of layers in the vertical. We note that although the model may have more than two layers, the density only changes across the internal interface. The density in the upper half of the layers is $\rho_1 = 1000$ kg m⁻³ and the density in the lower half of the layers is $\rho_2 = 1010$ kg m⁻³. To change the degree of nonhydrostasy, kH , the model is again run with varying domain lengths as in the free-surface seiche test case.

As shown in Figure 4 B, the model achieves the correct dispersive behavior. The model of the two-layer internal seiche behaves in a very similar manner to the model of the free-surface seiche in terms of its performance as a function of kH and the improvement of the dispersive properties as layers

753 are added.

754 The test case for the velocity profile for the two-layer internal seiche
755 uses the same parameters as above except with $H = 16$ m, $L = 10$ m and
756 $N_{\text{layers}} = 40$ which are chosen so that the experiment is strongly nonhydro-
757 static. Figures 5 C and D show the horizontal and vertical velocity profiles
758 normalized by their respective maxima at locations $x = L/2$ and $x = 0$,
759 respectively, at time $t = 1/4T$. In each case the model accurately reproduces
760 the theoretical behavior. As seen in Figure 5 C, the hydrostatic horizontal
761 velocity profile does not vary with depth above and below the interface. In
762 contrast, the nonhydrostatic horizontal velocity profile decays rapidly away
763 from its maximum value at the interface. Thus the ability of the nonhy-
764 drostatic model to accurately reproduce the modal solution, particularly one
765 that is strongly nonhydrostatic, depends on the number of model layers used
766 to represent the vertical variability of the solution even in regions of constant
767 density.

768 5.3. *Continuously-stratified internal seiche*

769 In this test case, the (inviscid) isopycnal model is validated against linear
770 theory describing internal gravity waves in a continuously-stratified system.
771 From Table 1, the ratio of the hydrostatic wave speed to the theoretical wave
772 speed, c_h/c , for a continuously-stratified internal seiche is identical to Equa-
773 tion (39). The theoretical wave speed for a continuously-stratified internal
774 seiche is identical to the theoretical wave speed for a 2-layer internal seiche

775 since the influence of the finite thickness interface (given in Table 1) is re-
776 moved during the normalization. As in the previous test case, we compare
777 depth-profiles of the horizontal and vertical velocity in the nonhydrostatic
778 and hydrostatic models to theory from the first-mode linearized eigenfunc-
779 tion analysis of Fringer and Street (2003).

780 The initial configuration of this numerical experiment on the behavior
781 of a continuously-stratified internal seiche is shown in Figure 3 (G,H,I), but
782 with varying domain sizes, number of layers, and a smaller constant seiche
783 amplitude. The model parameters are identical to those in the two-layer
784 seiche case, with the exception of the density profile. The density in this
785 case is given by

$$\rho(x, z, t = 0) = \rho_0 + \frac{1}{2}\Delta\rho \left[1 - \tanh \left(\frac{2 \tanh^{-1} \alpha_s}{\delta_\rho} (z + H/2 - \xi) \right) \right] \quad (40)$$

786 where the depth is $H = 10$ m, $\alpha_s = 0.99$ [see Fringer and Street (2003)],
787 $\delta_\rho = 1$ m, $\rho_0 = 1000$ kg/m³ and the density difference is given by $\Delta\rho/\rho_0 =$
788 0.01 . The interface height $\xi = a_i \cos(kx)$ and $a_i = 0.1$ m as in the previous
789 test case. The isopycnal layers are initialized to density contours that are
790 linearly spaced in density space. Using this approach, the layers are localized
791 at the stratified interface as shown in Figure 3 H. This approach resolves
792 the stratification, but does not resolve the vertical variability caused by the
793 nonhydrostatic effects (since no additional layers are present away from the
794 interface). Thus we also consider a so-called “hybrid” approach, where layers

are added to resolve the stratified interface as well as the velocity profile in regions above and below the interface as shown in Figure 3 I.

Figure 4 C demonstrates that the model achieves the correct dispersive behavior. Interestingly, as demonstrated in Figure 4 C, adding layers to improve the representation of the stratification alone does not improve the dispersive properties. Additional layers are needed throughout the water column to improve the representation of the nonhydrostatic velocity profile with depth. The hybrid approach, which adds such layers, improves the dispersive characteristics.

The test case to compute the velocity profile of the continuously-stratified internal seiche uses the same parameters as the two-layer velocity profile test case (Section 5.2) except with 100 layers⁹ in the hybrid configuration. Figures 5 E and F show the horizontal and vertical velocity profiles normalized by their respective maxima at locations $x = L/2$ and $x = 0$, respectively, at time $t = 1/4T$. A similar test case for the velocity profile of an internal seiche is given in Fringer et al. (2006), where the velocity profiles are computed using the first-mode linearized eigenfunction analysis of Fringer and Street (2003). The eigenfunction analysis requires the wave frequency as input. In panels E and F of Figure 5, the wave frequency is computed both from the asymptotic theory in Table 1 (where $\omega = ck$) and from the model (ω_{modeled}). These are denoted as “NH Theory 1” and “NH Theory 2”, respectively, in

⁹Note that we use 100 layers for graphical representation. The analytical solution may be well represented using fewer layers.

816 Figures 5 E and F. As expected, the modal solution using the wave frequency
 817 of the model more closely follows the modeled velocity profiles owing to
 818 errors in the asymptotic theory which is only valid in the limit of small $k\delta_\rho$.
 819 As seen in Figures 5 E and F, the hydrostatic and nonhydrostatic velocity
 820 profiles are well-represented using the model. In this case, the hydrostatic
 821 velocity profile has a smooth variation near the stratified interface (as in
 822 Figure 5 E) unlike the velocity profile for the two-layer stratification which
 823 is discontinuous across the interface (as in Figure 5 C).

824 In summary of the results in Sections 5.1-5.3, we find that

- 825 1. Nonhydrostatic dispersion gives rise to depth-dependence even in un-
 826 stratified regions. Therefore, nonhydrostatic models require multiple
 827 layers to resolve both the vertical dependence of the velocity field and
 828 the dispersive properties, especially for large values of kH .
- 829 2. The horizontal velocity in unstratified regions does not vary with the
 830 vertical coordinate for hydrostatic models and thus adding layers in
 831 such regions does not affect the velocity profiles or the dispersive prop-
 832 erties.
- 833 3. In weakly nonhydrostatic regimes for which $kH < 1$, the depth-dependence
 834 and dispersion are minimal. Thus, in this regime, a model with a lim-
 835 ited number of layers may suffice.

Figure 3: Initial density configurations (in kg m^{-3}) for a free-surface seiche (panels A,B,C), a 2-layer internal seiche (panels D,E,F), and a continuously-stratified internal seiche with a tanh density profile (panels G,H,I). In Sections 5.1-5.3, we use smaller wave amplitudes than shown here so that linear theory is valid.

Figure 4: The dispersive characteristics of a free-surface seiche (Figure 3, top row), a 2-layer internal seiche (Figure 3, middle row), and a continuously stratified internal seiche (Figure 3, bottom row) in panels A,B, and C, respectively, for hydrostatic (red) and nonhydrostatic (blue) equations. The modeled and theoretical wave speeds are normalized by the theoretical wave speed, c_0 , for each scenario.

Figure 5: Normalized horizontal and vertical velocity profiles for a free-surface seiche (panels A and B), a two-layer internal seiche (panels C and D), and a continuously stratified internal seiche with a tanh density profile (panels E and F) for the hydrostatic (red) and nonhydrostatic (blue) models.

836 *5.4. Internal solitary waves: Comparison of isopycnal model to KdV theory*

837 In this test case, we compare the widths of solitary waves formed using
 838 the current model to Korteweg and de-Vries (1895) (KdV) and extended KdV
 839 (eKdV) theory. The well-known solitary-wave solution to the KdV equation
 840 is

$$\xi = a_0 \operatorname{sech}^2 \left(\frac{x - c_i t}{L_0} \right), \quad (41)$$

841 where the length scale of solitary waves in a two-layer system is given by

$$L_0 = \sqrt{\frac{4}{3} \frac{(h_1 h_2)^2}{(h_1 - h_2) a}}, \quad (42)$$

842 where a is the amplitude of the solitary wave and h_1 and h_2 are the depths of
 843 the upper and lower layers, respectively (Bogucki and Garrett, 1993). After
 844 normalizing and rearranging, we can write Equation (42) as

$$\frac{h_1}{L_0} = \frac{h_1}{h_2} \sqrt{\frac{3}{4} \left(1 - \left(\frac{h_1}{h_2} \right)^{-1} \right) \frac{a}{h_1}}, \quad (43)$$

845 which relates the magnitude of nonlinearity a/h_1 to the wave aspect ratio
 846 h_1/L_0 . The eKdV theory includes cubic nonlinearity and results in expres-
 847 sions similar to (41) and (42) although slightly more complicated as given in
 848 Helfrich and Melville (2006).

849 The simulations in this numerical experiment are performed in a domain
 850 of length $L_d = 125$ km and depth $H = 2000$ m with a two-layer stratification
 851 where $\rho_1 = \rho_0 = 999.5$ kg m⁻³ is the density of the upper layer, and $\rho_2 =$

852 $\rho_0 + \Delta\rho = 1000.5 \text{ kg m}^{-3}$ is the density of the lower layer. The simulations
 853 are performed in an enclosed domain with an initial isopycnal displacement
 854 of

$$\xi(x, t = 0) = z_1 + h_1 = -a_i \text{sech}^2 \left[\left(\frac{x}{L_\rho} \right)^2 \right], \quad (44)$$

855 with $h_1 = 250 \text{ m}$ and $L_\rho = 2L_0$, and the model is run with varying initial
 856 amplitudes a_i . The initial wave of depression evolves rapidly into a solitary
 857 wave of width given closely by L_0 with a small train of trailing waves. The
 858 model is discretized with 2, 4, and 10 vertical layers each with $N_x = 1000$ grid
 859 points in the horizontal. The model is inviscid and is run with a free-surface
 860 Courant number of 20 which corresponds to a time step of 17.85 s.

861 Figure 6 shows h_1/L_0 vs. a/h_1 of the leading solitary waves (at $t =$
 862 16.29 hrs) for the isopycnal models with 2, 4 and 10 layers in panels A, B
 863 and C, respectively. The computed solitary wave amplitude a and width L_0
 864 are determined following the procedure used in Vitousek and Fringer (2011).
 865 The modeling results are compared to the KdV and eKdV theories. The
 866 main difference between the two theories is that the eKdV theory (which
 867 is valid for higher orders of nonlinearity) captures the tendency for solitary
 868 waves to broaden and develop flat crests as the waves become more non-
 869 linear (Helfrich and Melville, 2006). Figure 6 demonstrates that for weakly
 870 nonlinear ($a/h_1 \ll 1$), weakly nonhydrostatic ($h_1/L_0 \ll 1$) waves, the model
 871 can reproduce solitary waves consistent with theory. However, for waves
 872 with an appreciable degree of nonlinearity ($a/h_1 > 0.15$), the KdV theories

Figure 6: The wave aspect ratio h_1/L_0 vs. the magnitude of nonlinearity a/h_1 of the leading solitary waves computed by the isopycnal model at $t = 81.27$ hrs compared to the KdV and eKdV theories.

are not valid; they underpredict the solitary wave widths (or equivalently overpredict h_1/L_0 relative to the modeling results). As shown in Figure 6, the eKdV theory captures the broadening of solitary waves (or equivalently decrease in h_1/L_0) as the nonlinearity increases (e.g. for $a/h_1 > 0.5$). However, the eKdV theory still underpredicts the solitary wave widths relative to the current model for high orders of nonlinearity. The slight mismatch between the model and the theoretical predictions is not due to model error but instead arises from the limited range of applicability of the weakly nonlinear, weakly nonhydrostatic KdV theory. Although in certain dynamical situations the model can revert to the simple KdV theory, it can also be applied to situations beyond weakly nonlinear, weakly nonhydrostatic regimes as demonstrated in the numerical experiments in the next section.

885 *5.5. Internal solitary waves: Comparison of isopycnal model to z-level model*

886 In this test case, we compare formation of nonlinear internal solitary-like
 887 wave trains using the current isopycnal model to the nonhydrostatic SUN-
 888 TANS z -level model (Fringer et al., 2006) in a two-dimensional x - z domain.
 889 The parameters for this test case follow the numerical experiment in Vi-
 890 tousek and Fringer (2011) using SUNTANS with 100 layers. This test case
 891 was chosen to approximate the evolution of solitary-like waves in the South
 892 China Sea following Zhang et al. (2011). The simulations are performed in
 893 a domain of length $L_d = 300$ km and depth $H = 2000$ m that is initialized
 894 with approximate two-layer stratification as an idealized representation of
 895 the South China Sea. The initial stratification is given by

$$\rho(x, z, t = 0) = \rho_0 - \frac{1}{2}\Delta\rho \tanh \left[\frac{2 \tanh^{-1} \alpha_s}{\delta_\rho} (z - \xi(x, t = 0) + h_1) \right], \quad (45)$$

896 where the upper-layer depth is $h_1 = 250$ m, $\alpha_s = 0.99$ [see Fringer and Street
 897 (2003)], $\delta_\rho = 200$ m, $\rho_0 = 1000$ kg/m³ and the density difference is given by
 898 $\Delta\rho/\rho_0 = 0.001$. The initial Gaussian of depression that evolves into a train
 899 of solitary-like waves is given by

$$\xi(x, t = 0) = -a_i \exp \left[- \left(\frac{x}{L_\rho} \right)^2 \right], \quad (46)$$

900 with $a_i = 250$ m and $L_\rho = 15$ km. The boundary conditions are closed
 901 at $x = 0$ and $x = L_d$ and the bottom at $z = -H$ is free-slip with a free
 902 surface at the upper boundary. The model is discretized with 2 and 10

903 layers in the vertical. The layer heights and densities of the 10-layer model
 904 are initialized to follow contour lines of the density in Equation (45) which
 905 are linearly-spaced in density space. This numerical experiment is also run
 906 with 2 and 10 additional hybrid layers (distributed linearly in physical space
 907 in the upper and lower stratified layers) that are added to the simulation
 908 to improve the representation of the dispersive properties as discussed in
 909 Sections 5.1-5.3. The simulations are performed with $N_x = 4800$ grid points
 910 in the horizontal. The model is run with a free-surface Courant number
 911 of 10 which corresponds to a time step of 17.85 s. There is no viscosity
 912 or drag used in the isopycnal model, although the SUNTANS model uses
 913 (constant) viscosities of $\nu_H = 0.1 \text{ m}^2 \text{ s}^{-1}$ and $\nu_V = 10^{-4} \text{ m}^2 \text{ s}^{-1}$ to stabilize
 914 its central-differencing scheme for momentum advection. No scalar diffusivity
 915 is employed in the SUNTANS model.

916 The SUNTANS model requires approximately 2.0 s per time step¹⁰ us-
 917 ing 16 cores on four quad-core Opteron 2356 QC processors (Vitousek and
 918 Fringer, 2011). The 2-layer isopycnal model with the same horizontal reso-
 919 lution on the other hand requires approximately 0.15 s per time step using
 920 only one processor. This reduction in computational time and resources rep-
 921 resents more than an order of magnitude improvement in efficiency. The
 922 10-layer isopycnal model requires approximately 0.86 s per time step using

¹⁰We note that SUNTANS requires a time step of $\Delta t = 5 \text{ s}$ for these simulations which
 leads to a significant increase in computational effort compared to the time step of $\Delta t =$
 17.85 s needed in the isopycnal model.

923 one processor and also represents a significant reduction in computational
 924 cost.

925 As shown in Figure 7, the initial Gaussian wave of depression steepens
 926 into a rank-ordered train of solitary waves. The 2- and 10-layer isopycnal
 927 models are shifted downward by -700 m and -350 m to allow comparison
 928 of the results. In Figure 7, the x - and y -axes are normalized by $L_0 = 1436$ m
 929 and $a_0 = 220$ m, respectively, which represent the final solitary wave length
 930 and amplitude scales of the simulations presented in Vitousek and Fringer
 931 (2011). The time scale is normalized by $T_s = L_0/c_1 = 1028.7$ s, where
 932 $c_1 = 1.396$ m s $^{-1}$ is the speed of a mode-1 wave given in Vitousek and Fringer
 933 (2011). Here, we adopt the same normalization to facilitate comparison with
 934 the results presented in Vitousek and Fringer (2011).

935 The speed of the internal waves varies slightly among the different mod-
 936 els. The wave speed of the SUNTANS model is slightly reduced due to
 937 vertical numerically-induced scalar diffusion of density which is not present
 938 in the isopycnal model. We note that this scalar diffusion is purely numer-
 939 ical because no scalar diffusivity is employed. Numerically-induced scalar
 940 diffusion reduces the wave amplitude and thus the wave speed (which is
 941 weakly-dependent on amplitude) and also leads to thickening of the density
 942 contours in the wake of the leading wave in the wave train. The wave speed
 943 of the 10-layer isopycnal model is slightly slower than the 2-layer isopycnal
 944 model due to the finite-thickness of the interface in the 10-layer model which
 945 reduces the wave speed by a factor of $f_i(k\delta_\rho)$ as demonstrated in Table 1.

Figure 7: The formation of internal solitary waves generated from a Gaussian-shaped initial wave of depression computed by the current isopycnal model with 2 and 10 layers compared to the z -level SUNTANS model with 100 layers following the test case of Vitousek and Fringer (2011). The 2- and 10-layer isopycnal model results are shifted downward by -700 m and -350 m, respectively, to allow comparison of the two models.

946 Overall, this numerical experiment demonstrates consistency between the
 947 number and widths of modeled solitary waves computed by the nonhydro-
 948 static isopycnal model and the nonhydrostatic z -level model. The hydrostatic
 949 versions of SUNTANS and the current isopycnal model (not shown) are also
 950 remarkably similar.

951 Finally, we present a convergence study of the solitary wave widths as
 952 a function of grid resolution. This test case follows Vitousek and Fringer

953 (2011) with the same simulation parameters as above but with a variable
 954 number of horizontal grid points, $N_x = [150, 300, 600, 1200, 2400, 4800]$ and 2
 955 and 10 additional “hybrid” layers (discussed in Section 5.3) to improve the
 956 dispersive properties. Figure 8 A shows the convergence study presented in
 957 Vitousek and Fringer (2011) using SUNTANS with 100 layers and Figure 8 B
 958 shows the convergence study of the current nonhydrostatic and hydrostatic
 959 isopycnal models with 2 and 10 layers. Figure 8 shows the normalized length
 960 scales (L/L_0) of the leading solitary wave at the final time step, $t = 40.4$ hrs
 961 ($t/T_s = 141.5$) as a function of grid lepticity $\lambda = \Delta x/h_1$. The procedure
 962 to determine solitary wave length scales is outlined in Vitousek and Fringer
 963 (2011). As demonstrated in Vitousek and Fringer (2011), the normalized
 964 length scales of the nonhydrostatic and hydrostatic models behave like

$$L/L_0 = \sqrt{1 + K\lambda^2} , \quad (47)$$

965 and

$$L_h/L_0 = \lambda\sqrt{K} , \quad (48)$$

966 respectively, as a function of lepticity λ , where K is a constant that depends
 967 on the numerical method and is obtained with a best-fit of the model results
 968 to expressions (47) and (48). Equations (47) and (48) arise from the second-
 969 order accuracy of the numerical method. Numerical dispersion, which is
 970 proportional to the grid spacing squared, mimics physical dispersion and
 971 directly contributes to the widths of modeled solitary waves. Figure 8 B

972 demonstrates that the isopycnal model has very similar convergence behavior
 973 to SUNTANS with respect to decreasing the grid spacing or grid lepticity.
 974 It is not surprising that the behavior of the two models is similar given that
 975 they employ similar numerical methods (and the same order of accuracy).
 976 However, the isopycnal model requires additional “hybrid” layers above and
 977 below the interface to better resolve the wave dispersion and thus the solitary
 978 wave length scales. Figure 8 B shows that adding layers above and below
 979 the stratification interface improves the modeled solitary wave widths. By
 980 adding 2 hybrid layers (indicated by +2 in Figure 8) the isopycnal model
 981 forms solitary waves that are approximately 10% larger than SUNTANS. By
 982 adding 10 hybrid layers (indicated by +10 in Figure 8) the isopycnal model
 983 forms solitary waves that are nearly identical to SUNTANS.

984 *5.6. Internal solitary waves: Comparison of isopycnal model to laboratory* 985 *experiments*

986 This test case compares the formation of an internal solitary-like wave
 987 train using the current isopycnal model to the laboratory experiments in
 988 Horn et al. (2001). The laboratory experiments investigate the formation of
 989 solitary waves in an approximately two-layer density profile with an interface
 990 thickness of $\delta_\rho = 0.01$ m in an enclosed tank of depth $H = 0.29$ m and length
 991 $L = 6$ m. The internal interface is initialized with a linear tilt which is given
 992 by

$$\xi(x, t = 0) = a_i \frac{x - L/2}{L/2}, \quad (49)$$

Figure 8: Convergence of leading solitary wave widths (at time $t = 40.4$ hrs) normalized by $L_0 = 1436$ m as a function of the grid lepticity $\lambda = \Delta x/h_1$. Panel A shows the results from the SUNTANS model following Vitousek and Fringer (2011). Panel B shows the results from the isopycnal model with different numbers of layers. For reference the theoretical convergence relationships for the nonhydrostatic (blue line) and hydrostatic models (red line) given in Equations (47) and (48), respectively, are shown.

Figure 9: Comparison of nonhydrostatic model-predicted and measured (from Horn et al. (2001)) interface displacement (ξ) time series at $x = L/2$.

Figure 10: Comparison of hydrostatic model-predicted and measured (from Horn et al. (2001)) interface displacement (ξ) time series at $x = L/2$. The hydrostatic model forms steep bores instead of trains of solitary waves as in Figure 9.

993 where the wave amplitude is given by $a_i=[0.01305, 0.0261, 0.03915, 0.0522,$
 994 $0.0783]$ m. The model of the laboratory experiments is run with 2 layers and
 995 $N_x = 256$. The density difference in the laboratory experiments is approxi-
 996 mately $\frac{\Delta\rho}{\rho_0} = 0.02 \pm 0.002$. The model is run with a slightly lower density dif-
 997 ference of $\frac{\Delta\rho}{\rho_0} = 0.0175$ because the finite-thickness interface in the laboratory
 998 experiment reduces the wave speed slightly when compared to the two-layer
 999 configuration used in the model with (still-water) layer heights $h_1 = 0.203$ m
 1000 and $h_2 = 0.087$ m. The model is run with a free-surface Courant number
 1001 $C = 4$ for a total time of 400 s. The model uses a drag coefficient $C_d = 0.0175$
 1002 and horizontal and vertical eddy-viscosities of $\nu_H = 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$ and
 1003 $\nu_v = 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$, respectively. These parameters are chosen to fit the
 1004 amplitude of the solitary waves formed during the laboratory experiment.
 1005 Figure 9 shows the results of the model-predicted time series of the interface
 1006 displacements (located at the center of the tank $x = L/2$) compared to the
 1007 laboratory experiments of Horn et al. (2001). Panels A,B,C,D,E of Figure
 1008 9 correspond to the increasing initial interface amplitudes of $a_i=[0.01305,$
 1009 $0.0261, 0.03915, 0.0522, 0.0783]$ m, respectively. The agreement between the
 1010 model and the laboratory experiments is excellent for the first half of the
 1011 experiment ($t < 200$ s). However, for the second half of the experiment there
 1012 is some mismatch in the phase and amplitude of the waves. Over the course
 1013 of a laboratory experiment Horn et al. (2001) reported “a gradual thickening
 1014 of the density interface over the set, typically from approximately 1 cm to 2
 1015 cm”. The finite-thickness interface and thickening are partly responsible for

1016 the mismatch with the model (which has a sharp interface and no interface
1017 thickening). Additionally, the model appears to form slightly wider solitary
1018 waves than the laboratory experiment. This indicates that the dispersive
1019 properties of the waves are not captured exactly in the model because of the
1020 two-layer approximation. Overall, however, the model adequately captures
1021 the formation of solitary waves in the laboratory experiment.

1022 Figure 10 shows the same numerical experiment as Figure 9 except using
1023 the developed isopycnal model in hydrostatic mode. The hydrostatic model
1024 in Figure 10 forms steep bores as opposed to trains of solitary waves as in
1025 Figure 9. This is particularly visible for the experiments with larger ini-
1026 tial amplitudes. Although nonhydrostatic dispersion does influence the wave
1027 speed, the effect is relatively weak. Consequently, the wave speed of the
1028 nonhydrostatic and hydrostatic models are consistent. This is evidenced by
1029 the remarkably similar timing of the modeled hydrostatic wave fronts com-
1030 pared to the laboratory experiments. Therefore, a hydrostatic model may
1031 suffice for predicting the arrival time of the initial wave fronts. However, the
1032 significant difference between the hydrostatic and nonhydrostatic models is
1033 the form of the trailing waves. The hydrostatic model in Figure 10 has a
1034 tendency to form narrow, numerically-induced¹¹ solitary waves on each wave
1035 front. These numerically-induced solitary waves are, in this case, much nar-
1036 rower than the waves in the laboratory data because the numerical dispersion

¹¹generated by dispersive (odd-order) truncation error

1037 is much smaller than (the unmodeled) physical dispersion due to the high
1038 grid resolution. The numerical dispersion is smaller than physical dispersion
1039 because the grid lepticity, $\lambda = \frac{\Delta x}{h_1} = 0.1155$ is less than unity (Vitousek and
1040 Fringer, 2011). In summary, these numerical experiments demonstrate the
1041 necessity of using nonhydrostatic models when simulating nonlinear internal
1042 solitary waves like those in the experiments of Horn et al. (2001).

1043 *5.7. Internal solitary waves: Internal wave generation in the South China* 1044 *Sea*

1045 In this numerical experiment, we investigate the capability to simulate
1046 solitary wave formation in realistic physical domains that are typically sim-
1047 ulated using nonhydrostatic z - or σ -coordinate models. The prospect for
1048 simulating complex internal wave generation with a limited vertical layer
1049 structure seems possible due to the work of Li and Farmer (2011) who suc-
1050 cessfully applied a fully nonlinear two-layer model to predict the evolution
1051 of the internal tide in the South China Sea.

1052 In this test case, we follow the modeling results of Buijsman et al. (2010)
1053 who studied the generation and evolution of nonlinear internal waves in the
1054 South China Sea with a σ -coordinate model using 60 layers. The model is
1055 run with bathymetry that approximates the eastern ridge of the Luzon Strait
1056 with a height of $a_b = 2600$ m in a depth of $H_0 = 3000$ m. Following Buijsman

Figure 11: The model domain and bathymetry used in the South China Sea test case following Buijsman et al. (2010). The black filled area represents the bathymetry, the blue line represents the free surface, and the black dashed lines represent the locations of the sponge layers.

1057 et al. (2010), the depth profile is given by

$$H = H_0 - a_b \exp\left(-\frac{x^2}{2L_b^2}\right), \quad (50)$$

1058 where the bathymetry length scale $L_b = 12$ km. The model domain and
1059 bathymetry are shown in Figure 11.

1060 A least squares fit to the density profile in Buijsman et al. (2010) gives

$$\rho(z) = \rho_0 + \Delta\rho \left(1 - \exp(z/h_0)^p + s_1 z + s_2 z^2\right), \quad (51)$$

1061 where $\rho_0 = 1024.75 \text{ kg m}^{-3}$, $\Delta\rho = 5.22 \text{ kg m}^{-3}$, $h_0 = 224 \text{ m}$, $p = 1.15$,
1062 $s_1 = -0.00018 \text{ m}^{-1}$, and $s_2 = -3.64 \times 10^{-8} \text{ m}^{-2}$ are the best-fit parameters.
1063 Internal waves are generated from tidal flow in the Luzon Strait over the
1064 bathymetric ridge. To capture this in the model, the boundary is forced at

1065 both sides of the domain by

$$u_{bc} = u_0 \sin(\omega_{bc}t) , \quad (52)$$

1066 where $u_0 = 0.134 \text{ m s}^{-1}$ and $\omega_{bc} = 1.41 \times 10^{-4} \text{ s}^{-1}$ as in case R1 in Buijsman
1067 et al. (2010).

1068 The model is run in hydrostatic and nonhydrostatic mode with 2 and 10
1069 layers. The initial layer configurations are set to best match the weakly non-
1070 linear and nonhydrostatic behavior of solitary waves in the stratification given
1071 by Equation (51). The still-water interface depths in the 10-layer isopycnal
1072 model are arbitrarily set to $z = -[0, 50, 100, 250, 460, 750, 1000, 1500, 2000, 2500, 3000] \text{ m}$
1073 with density values computed at the vertical center of each layer using
1074 Equation (51). Following the procedure described in Appendix F, the
1075 still-water internal interface for the 2-layer model is set to $z = -460 \text{ m}$
1076 and the upper- and lower-layer densities are set to $\rho_1 = 1028 \text{ kg m}^{-3}$ and
1077 $\rho_2 = 1030.1967 \text{ kg m}^{-3}$, respectively, in order to match the mode-1 wave
1078 speed of the 10-layer model.

1079 To properly simulate the generation and evolution of internal waves in
1080 the South China Sea, the model is run with a domain length of $L = 800 \text{ km}$
1081 and $N = 1750$ grid points so that $\Delta x \approx h_1 = 460 \text{ m}$ and the grid lepticity
1082 $\lambda = \frac{\Delta x}{h_1} \approx 1$ so that the numerical dispersion is relatively small according to
1083 Vitousek and Fringer (2011).

1084 The model is run with a horizontal eddy-viscosity of $\nu_H = 100 \text{ m}^2 \text{ s}^{-1}$ and

1085 a vertical eddy-viscosity of $\nu_v = 1 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ to characterize unresolved
 1086 mixing of momentum due to turbulence in the presence of internal waves
 1087 and a drag coefficient of $C_d = 0.0025$. The model uses the QUICK (upwind-
 1088 biased, 3rd-order spatially accurate) advection scheme of Leonard (1979)
 1089 to reduce spurious, numerically-induced oscillations associated with central-
 1090 differencing methods. The model is run with a free-surface Courant number
 1091 of $C = 10$ which gives a time step of $\Delta t = 26.5 \text{ s}$. The simulation starts from
 1092 rest and runs for 10 tidal periods. To minimize internal wave reflection from
 1093 the boundaries, the model uses sponge layers following Zhang et al. (2011)
 1094 on each boundary of the domain. These are implemented by adding a term
 1095 to the right-hand side of the horizontal momentum equation of the form

$$S(x, z, t) = -\frac{u(x, z, t) - u_{bc}(x, z, t)}{\tau_s} s(r) , \quad (53)$$

1096 where $s(r) = \exp(-4r/L_s)$ causes the influence of the sponge layer to decay
 1097 over the distance $L_s = L/20$ where r is the distance to the domain boundary
 1098 and $\tau_s = 65 \text{ s}$ is the damping time scale.

1099 Figure 12 shows the generation and evolution of internal solitary waves
 1100 in the nonhydrostatic 2-layer vs. 10-layer model. Panels A-F depict the in-
 1101 terface displacement of the isopycnal initially located at $z = -h_1 = -460 \text{ m}$
 1102 at time intervals of $0.75T_{bc}$ where $T_{bc} = 2\pi/\omega_{bc}$. The 2- and 10-layer models
 1103 form trains of nonlinear internal solitary waves with nearly identical wave
 1104 speeds, although the amplitude, length scales, and number of solitary waves

1105 differ between the models. The two-layer model only supports mode-1 waves
 1106 whereas the 10-layer model supports higher modes. Consequently, in the 2-
 1107 layer model, all of the barotropic-to-baroclinic energy is converted to mode-1
 1108 waves. In the 10-layer model, energy is converted to higher-mode waves,
 1109 the effects of which are clearly visible near the ridge. This results in larger
 1110 amplitude solitary waves in the 2-layer model relative to the 10-layer model.
 1111 Additionally, the solitary wave widths in the 2-layer model are slightly larger
 1112 than the 10-layer model due to unresolved dispersion in models with limited
 1113 vertical resolution (discussed in Section 5.5) and the larger dispersion coef-
 1114 ficient, ϵ , in the 2-layer model relative to the 10-layer model (see Appendix
 1115 F).

1116 Figure 13 shows the same test case as in Figure 12 except using the
 1117 hydrostatic model. Instead of trains of nonlinear solitary waves, the 2- and
 1118 10-layer models form steep bores with nearly identical wave speeds. The
 1119 wave amplitudes differ slightly between the 2- and 10-layer model, as in the
 1120 nonhydrostatic simulations, due the energy conversion to the higher-mode
 1121 waves supported by the 10-layer model. Comparing Figure 13 to Figure 12,
 1122 the 2- and 10-layer hydrostatic models are more similar to each other than
 1123 the 2- and 10-layer nonhydrostatic models. This result is expected because
 1124 of the conclusions presented in Sections 5.1-5.3. Specifically, there is no
 1125 difference in the dispersive properties of the hydrostatic models, i.e. they are
 1126 both physically nondispersive. Consequently, there is no depth variability
 1127 that must be resolved with additional vertical layers in order to improve the

Figure 12: Comparison of internal wave generation in the nonhydrostatic 2-layer (red) and 10-layer (blue) model at time intervals of $0.75T_{bc}$. Both models form trains of nonlinear internal solitary waves. The two-layer model only supports mode-1 waves whereas the 10-layer model supports higher modes, the effects of which are visible near the ridge.

1128 model's performance with regard to the mode-1 wave speed and shape.

1129 The model results presented in Figures 12 and 13 are used to compute
 1130 the baroclinic energy flux associated with internal waves which is given by

$$F = \sum_k u'_k M'_k h_k , \quad (54)$$

1131 where $u'_k = u_k - \frac{1}{\sum_k h_k} \sum_k u_k h_k$ is the baroclinic perturbation velocity and
 1132 $M'_k = \sum_{m=2}^k (\rho_m - \rho_{m-1}) g (z_{m-1} - (z_0)_{m-1})$ is the perturbation baroclinic
 1133 Montgomery potential, and $(z_0)_m$ represents the interface location for the
 1134 m -th still-water isopycnal (Simmons et al., 2004; Miao et al., 2011). We note
 1135 that the nonhydrostatic contributions to the energy flux, $F_{\text{nh}} = \sum_k u'_k q_k h_k$,
 1136 are negligible for the simulations presented here. The results for the max-
 1137 imum, tidally-averaged energy flux, F_{max} , and the spatial location, x_{max} ,
 1138 where the maximum occurs, are given in Table 2. F_{max} is obtained by apply-
 1139 ing a tidally-averaging filter, a moving average of size T_{bc} , over the simulation
 1140 time series and averaging the filtered value for the last 7 tidal periods (after
 1141 3 tidal periods of spin up). As indicated in Table 2, the maximum energy
 1142 flux, F_{max} , for the 2-layer hydrostatic and nonhydrostatic models are nearly
 1143 identical which indicates the relative unimportance of nonhydrostatic effects
 1144 as far as the large-scale energetics is concerned. However, nonhydrostatic ef-
 1145 fects substantially alter the amplitude and distribution of the solitary waves.
 1146 The 10-layer hydrostatic and nonhydrostatic models are also very similar
 1147 with approximately 5-10% relative difference in energy flux between the hy-

Figure 13: Comparison of internal wave generation in the hydrostatic 2-layer (red) and 10-layer (blue) model at time intervals of $0.75T_{bc}$. Instead of trains of nonlinear internal solitary waves of, both models form bores. The two-layer model only supports mode-1 waves whereas the 10-layer model supports higher modes, the effects of which are visible near the ridge.

drostatic and nonhydrostatic models. The locations of the maximum energy
flux also show good agreement — they differ in location by a maximum of
14 grid points of the total 1750. The values for the energy flux reported here
are qualitatively similar to the values in Warn-Varnas et al. (2010)¹².

Table 2: Maximum energy flux ($F_{\max}$) and its location x_0 for the South China Sea test case.

# of layers	nonhydrostatic mode	$F_{\max}$ [kW m ⁻¹]	$x_{\max}$ [km]
2	hydrostatic	328.8	29.7
2	nonhydrostatic	328.4	30.2
10	hydrostatic	354.4	31.1
10	nonhydrostatic	365.6	36.1

As mentioned earlier, the main difference between the internal waves in
the 2-layer vs multilayer systems is the presence of higher modes. Figure 14
shows a Hovmöller diagram for the interface displacement of the isopycnal
initially located at $z = -h_1 = -460$ m for the 2- and 10-layers in Panels A
and B, respectively. Hovmöller or “trough-and-ridge” diagrams (Hovmöller,
1949) depict a particular quantity as a function of x and t . These dia-
grams are informative because lines on a Hovmöller diagram correspond to
wave characteristics. Figure 14 shows the tidal velocity (black solid line -
whose magnitude is shown on the top x -axis), and the first- (black, dashed
line) and second-mode (black, dotted line) wave speeds which are given by
 $c_1 = 2.86$ m s⁻¹ and $c_2 = 1.52$ m s⁻¹, respectively. Figure 14 A shows

¹²Values for the energy flux are not reported in Buijsman et al. (2010).

Figure 14: Hovmöller diagram (x vs. t) for the South China Sea test cases with 2- and 10-layers in Panels A and B, respectively. These diagrams show the internal interface displacement of the isopycnal initially located at -460 m. Shown for reference are tidal velocity (black, solid line), and the first- (black, dashed line) and second-mode (black, dotted line).

1163 how the 2-layer model only generates mode-1 characteristics. The 10-layer
 1164 model, on the other hand, generates mode-1 and higher-mode characteris-
 1165 tics that match both the mode-1 and mode-2 characteristics in Figure 14 B.
 1166 The Hovmöller diagram shown in Figure 14 B is qualitatively similar to the
 1167 Hovmöller diagram for the σ -coordinate model results presented in Figure 5
 1168 of Buijsman et al. (2010), with respect to the solitary and higher-mode wave
 1169 generation.

1170 Overall, the results demonstrate good performance of isopycnal-coordinate
 1171 models with limited vertical resolution to represent both the small-scale dy-
 1172 namics (the formation of nonlinear internal solitary waves) and the large-scale
 1173 dynamics (the baroclinic energy flux).

1174 5.8. Internal wave beams

1175 The last test case is the formation of internal wave beams generated from
 1176 oscillatory flow over a Gaussian sill. The primary goal of the developed
 1177 model is the treatment of internal waves with a reduced number of vertical
 1178 layers. However, in this test case, we demonstrate that the current isopycnal
 1179 model is also applicable to modeling continuously stratified environments
 1180 and internal wave beams which require $\mathcal{O}(10-100)$ layers to be well resolved
 1181 in the vertical.

1182 A fluid with constant stratification of

$$N^2 = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \quad (55)$$

1183 produces internal wave beams radiating with a constant slope given by

$$\frac{dz}{dx} = \tan(\varphi) = \sqrt{\frac{\omega^2 - f^2}{N^2 - \omega^2}}, \quad (56)$$

1184 [see e.g. Kundu (1990)], where ω is the forcing frequency and f is the Coriolis
 1185 parameter.

1186 The purpose of this numerical experiment is to demonstrate that the
 1187 model is able to correctly capture the internal wave beam angle and to
 1188 demonstrate the difference between the nonhydrostatic and hydrostatic beam
 1189 angles. The nonhydrostatic internal wave beam angle in the absence of the

1190 Coriolis force ($f = 0$) is given by

$$\varphi = \tan^{-1} \left(\sqrt{\frac{(\omega/N)^2}{1 - (\omega/N)^2}} \right) . \quad (57)$$

1191 The hydrostatic beam angle, given by Equation (57) for $\omega/N \ll 1$, is

$$\varphi_h = \tan^{-1} (\omega/N) . \quad (58)$$

1192 In this numerical experiment, oscillatory flow over a Gaussian sill produces
1193 a strong signal of internal wave beams. The bathymetry is given by

$$H = H_0 - a_b \exp \left(-\frac{x^2}{2L_b^2} \right) , \quad (59)$$

1194 where the still water depth is $H_0 = 1000$ m, the sill amplitude is $a_b = 20$ m
1195 and $L_b = L/100$. The domain is given by $x \in [-L/2, L/2]$, where length of
1196 the domain, L , is varied according to $L = 4H_0/\tan(\varphi)$ so that the beams
1197 reflect off the surface before encountering the open boundaries. Tidal forcing
1198 at the boundaries is given by

$$u_{bc} = u_0 \sin(\omega t) , \quad (60)$$

1199 where $u_0 = 0.01$ m s⁻¹. The parameters are chosen to ensure small val-
1200 ues of a_b/H_0 and $u_0\omega/L_b$ (Llewellyn Smith and Young, 2002) to prevent
1201 nonlinear processes such as lee waves, higher harmonics, and associated mul-

1202 tiple beams. This numerical experiment is run with a constant stratifica-
 1203 tion of $N = 0.007 \text{ s}^{-1}$ which corresponds to linearly-varying density with
 1204 $\Delta\rho = 5 \text{ kg m}^{-3}$ and $\rho_0 = 1000 \text{ kg m}^{-3}$. The forcing frequency (ω) of the
 1205 oscillatory flow is varied to achieve different values of ω/N . The model uses a
 1206 high vertical resolution of 100 layers to resolve the higher internal wave modes
 1207 which compose the beams. The model also uses sponge layers following Equa-
 1208 tion (53) of length $L_s = L/10$ to absorb waves at the boundaries with a time
 1209 scale of $\tau_s = 100$. The model is run with $N_x = 128$ grid points and a time
 1210 step of $\Delta t = T_{bc}/500$ for a total time of 20 tidal periods ($T_{bc} = 2\pi/\omega$) so that
 1211 the model has sufficient time to spin up (i.e. sufficient time for transients
 1212 to decay and the beams to develop). Only nonhydrostatic simulations with
 1213 $\omega/N > 0.8$ require extensive spin-up time¹³ due to the nearly vertical en-
 1214 ergy propagation and slow horizontal velocity of (nonhydrostatic) high-mode
 1215 waves. The model is run with a drag coefficient of $C_d = 0.001$, a vertical
 1216 viscosity of $\nu_v = 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$, and no horizontal viscosity.

1217 Figure 15 shows the computational domain for the nonhydrostatic model
 1218 with $\omega/N = 0.8$. The figure shows a pseudo-color plot of the normalized
 1219 perturbation velocity, $u'/u_0 = (u - u_{bc})/u_0$, at peak flood, $t = 19.25T_{bc}$, which
 1220 illustrates internal wave beams radiating away from the sill. Also note that,
 1221 according to theory, the internal wave phase velocity vector is perpendicular
 1222 to the beam angle. For reference, the theoretical nonhydrostatic (thick blue

¹³Approximately 10 tidal periods are needed for the higher modes to propagate over the entire domain and for the beams to develop.

Figure 15: The setup for the internal wave beam numerical experiment. This figure shows the ridge and the normalized perturbation velocity $u'/u_0 = (u - u_{bc})/u_0$ associated with the internal wave beams at peak flood, $t = 19.25T_{bc}$, for a forcing frequency of $\omega = 0.8N$ in the nonhydrostatic model. Also shown for reference are the theoretical nonhydrostatic (thick blue dashed-dot line) and hydrostatic (thick red dashed line) beam angles given by Equations (57) and (58), respectively. The locations of the sponge layers are shown as thin black dashed lines near the domain boundaries.

1223 dashed-dot line) and hydrostatic (thick red dashed line) beam angles given
 1224 by Equations (57) and (58), respectively.

1225 Figure 16 shows the theoretical nonhydrostatic and hydrostatic internal
 1226 wave beam angles, given by Equations (57) and (58), respectively, compared
 1227 to the model as a function of ω/N . To compute the beam angle of the
 1228 model, we determine the location of the maximum of the (RMS) perturbation
 1229 velocity averaged over the last ten tidal periods for each layer in a limited
 1230 region ($x_{\min} = 2\Delta x$, $x_{\max} = 0.75L$, $z_{\min} = -0.80H_0$, $z_{\max} = -0.15H_0$)
 1231 surrounding the beam on the right-half of the sill. A linear least-squares fit

Figure 16: The internal wave beam angle φ as a function of ω/N . The theoretical curves for the internal wave beam angles are given in Equations (57) [solid blue line] and (58) [solid red line] for nonhydrostatic and hydrostatic internal waves, respectively. The beam angles determined from the model are given by the blue diamonds and red circles for the nonhydrostatic and hydrostatic models, respectively.

1232 to the $(x-z)$ location of these maxima provides an estimate of the internal
1233 wave beam angle produced by the model.

1234 As shown in Figure 16, the model correctly reproduces the hydrostatic and
1235 nonhydrostatic beam angles, which diverge when $\omega/N \gtrsim 0.3$. For $\omega/N \approx 1$,
1236 the nonhydrostatic beams become vertical, whereas the hydrostatic beams
1237 become nearly 45° (since $\tan^{-1}(1) = 45^\circ$). Ultimately, this test case demon-
1238 strates that the isopycnal model is capable of resolving internal wave behavior
1239 in smoothly stratified environments (i.e. not composed of a layered struc-
1240 ture).

6. Conclusions

This paper presents a nonhydrostatic ocean model with an isopycnal coordinate system. The governing model equations invoke a mild-slope approximation to the momentum equation and the divergence-free condition relating to the nonhydrostatic pressure. This approximation arises from scaling arguments and is valid for most oceanic flows of interest. The scaling analysis conducted here suggests that the nonhydrostatic pressure terms which are often neglected can be significantly larger than many metric terms that are often retained (such as those arising from viscosity). The mild-slope approximation reduces the elliptic equation for the nonhydrostatic pressure to a symmetric linear system. Ultimately, metric terms associated with nonhydrostatic pressure can be retained for flows with steep isopycnal slopes (where $z_{x^*} = \mathcal{O}(1)$) at the expense of increased computational cost of solving a nonsymmetric linear system with approximately double the bandwidth. Despite the approximations, the framework is in place to implement the nonhydrostatic pressure into existing isopycnal-coordinate ocean models. The method presented here is valid for a 2-D (x - z) system, however the method is easily extended to 3-D by treating the north-south (v) velocity in a similar manner to the east-west (u) velocity. Overall, the numerical method is second-order accurate in time and space and uses an implicit free-surface discretization that is similar to that in existing nonhydrostatic models using z - or σ -coordinates. Although the use of a Lagrangian vertical coordinate and a nonhydrostatic projection method was reported to be mutually exclusive (Adcroft and Hallberg, 2006),

1264 we find that such a numerical formulation is possible at least to second-order
1265 accuracy. It appears that a solution to the conundrum of Adcroft and Hall-
1266 berg (2006) may be possible in the context of numerical formulations of the
1267 Navier-Stokes equations in generalized vertical coordinates since it is not nec-
1268 essary to maintain absolute consistency between the layer-height continuity
1269 equation (7) and the divergence-free continuity equation (8). Instead, it is
1270 only necessary to obtain consistency between Equations (7) and (8) to a suf-
1271 ficient order of accuracy. In this case, we consider second-order accuracy to
1272 be sufficient because higher-order nonhydrostatic models are rare.

1273 The numerical method is stable provided the time step satisfies the in-
1274 ternal wave, wetting/drying, and horizontal advection and diffusion Courant
1275 numbers. The main limitation of this model, like all isopycnal-coordinate
1276 models, is the inability to represent unstable stratification and overturning
1277 motions which are effects typically associated with nonhydrostatic pressure.
1278 Overturning motions typically exist on small scales and, consequently, are
1279 often under-resolved in modeling applications. Parameterization of over-
1280 turning motions in the context of isopycnal-coordinate models, especially in
1281 nonhydrostatic formulations such as the current model, is beyond the scope
1282 of this paper. However, this may provide interesting topics for future re-
1283 search — e.g. How can we parameterize internal wave breaking in isopycnal
1284 models? Can accounting for nonhydrostatic effects improve such a parame-
1285 terization? Despite the limitation of the isopycnal-coordinate framework in
1286 that it does not allow for overturning motions, our model shows that addi-

tion of the nonhydrostatic pressure to an isopycnal-coordinate ocean model allows for computation of nonhydrostatic internal waves which account for a substantial fraction of the energy spectrum spanning between the hydrostatic low-frequency motions and the high-frequency overturning motions.

We demonstrate that the model captures dispersive wave properties and simulates nonlinear internal solitary waves in idealized test cases and in a realistic oceanographic problem of internal solitary wave generation over a ridge. The model is capable of representing internal waves with a reduced number of vertical layers (ideally two). Additional layers may be necessary to capture the vertical variability of strongly nonhydrostatic flows. Ultimately, the model may provide an efficient formulation that is well-suited to simulations of nonhydrostatic internal gravity waves that are prohibitively expensive using z -level or σ -coordinate models.

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1310 **Appendix A. Nondimensionalization and coordinate transforma-**
1311 **tion**

The governing equations (2) are nondimensionalized according to

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{H}, \quad t' = \frac{t}{T} = \frac{c_i t}{L},$$

$$u' = \frac{u}{U}, \quad v' = \frac{v}{U}, \quad w' = \frac{w}{W} = \frac{w}{\epsilon U}, \quad \rho' = \frac{\rho}{\rho_0}, \quad p'_h = \frac{p_h}{\rho_0 U c_i}, \quad q' = \frac{q}{\epsilon^2 U c_i},$$

1312 where L is the horizontal length scale, H is the vertical depth scale, T is the
1313 time scale, U is the horizontal velocity scale, W is the vertical velocity scale,
1314 c_i is the wave speed scale and $\epsilon = \frac{H}{L}$. The scale for the hydrostatic pressure is
1315 chosen to achieve a first-order balance between the unsteady term and the hy-
1316 drostatic pressure gradient in the horizontal momentum equations. Likewise,
1317 the scale for the nonhydrostatic pressure is chosen to achieve a first-order bal-
1318 ance between the inertia term and the nonhydrostatic pressure gradient in
1319 the vertical momentum equation. After applying the nondimensionalization,
1320 Equations (2) become

$$\frac{\partial u'}{\partial t'} + F \nabla \cdot (\mathbf{u}' u') - \frac{F}{\text{Ro}} v' + \frac{F \epsilon}{\widetilde{\text{Ro}}} w' = -\frac{\partial p'_h}{\partial x'} - \epsilon^2 \frac{\partial q'}{\partial x'} + \frac{F}{\text{Re}_H} \nabla_H^2 u' + \frac{F}{\text{Re}_v} \frac{\partial^2 u'}{\partial z'^2} , \quad (\text{A.1a})$$

$$\frac{\partial v'}{\partial t'} + F \nabla \cdot (\mathbf{u}' v') + \frac{F}{\text{Ro}} u' = -\frac{\partial p'_h}{\partial y'} - \epsilon^2 \frac{\partial q'}{\partial y'} + \frac{F}{\text{Re}_H} \nabla_H^2 v' + \frac{F}{\text{Re}_v} \frac{\partial^2 v'}{\partial z'^2} , \quad (\text{A.1b})$$

$$\frac{\partial w'}{\partial t'} + F \nabla \cdot (\mathbf{u}' w') - \frac{F}{\epsilon} \frac{1}{\widetilde{\text{Ro}}} u' = -\frac{\partial q'}{\partial z'} + \frac{F}{\text{Re}_H} \nabla_H^2 w' + \frac{F}{\text{Re}_v} \frac{\partial^2 w'}{\partial z'^2} , \quad (\text{A.1c})$$

1321 where $F = \frac{u}{c_i}$, $\text{Re}_H = \frac{UL}{\nu_H}$, $\text{Re}_v = \frac{WH}{\nu_v} = \epsilon^2 \frac{UL}{\nu_v}$, $\text{Ro} = \frac{U}{fL}$, and $\widetilde{\text{Ro}} = \frac{U}{\tilde{f}L}$ are
1322 the Froude, horizontal and vertical Reynolds, traditional and nontraditional
1323 Rossby numbers, respectively. In Equations (A.1), we have assumed a con-
1324 stant horizontal and vertical viscosity for convenience. These equations are
1325 subject to the nondimensional incompressibility constraint given by

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0 . \quad (\text{A.2})$$

1326 Applying the nondimensional equivalent of the transformation rules (5) re-
1327 sults in the nondimensional equations in isopycnal coordinates

$$\begin{aligned}
\frac{\partial u'}{\partial t'^*} + F(\mathbf{u}'_{H^*} \cdot \nabla_{H^*} u') - \frac{F}{\text{Ro}} v' + \frac{F\epsilon}{\text{Ro}} w' = -\frac{\partial M'}{\partial x'^*} - \epsilon^2 \frac{\partial q'}{\partial x'^*} + \frac{F}{\text{Re}_H} \nabla_{H^*}^2 u' + \frac{F}{\text{Re}_v} \left(\frac{1}{z'_{\rho'}} \right)^2 \frac{\partial^2 u'}{\partial \rho'^2} \\
+ \epsilon^2 \frac{z'_{x'^*}}{z'_{\rho'}} \frac{\partial q'}{\partial \rho'} + \frac{F}{\text{Re}_H} \left[-2 \frac{z'_{x'^*}}{z'_{\rho'}} \frac{\partial^2 u'}{\partial x'^* \partial \rho'} - 2 \frac{z'_{y'^*}}{z'_{\rho'}} \frac{\partial^2 u'}{\partial y'^* \partial \rho'} + \left(\frac{z'_{x'^*}}{z'_{\rho'}} \right)^2 \frac{\partial^2 u'}{\partial \rho'^2} + \left(\frac{z'_{y'^*}}{z'_{\rho'}} \right)^2 \frac{\partial^2 u'}{\partial \rho'^2} \right],
\end{aligned}
\tag{A.3a}$$

$$\begin{aligned}
\frac{\partial v'}{\partial t'^*} + F(\mathbf{u}'_{H^*} \cdot \nabla_{H^*} v') + \frac{F}{\text{Ro}} u' = -\frac{\partial M'}{\partial y'^*} - \epsilon^2 \frac{\partial q'}{\partial y'^*} + \frac{F}{\text{Re}_H} \nabla_{H^*}^2 v' + \frac{F}{\text{Re}_v} \left(\frac{1}{z'_{\rho'}} \right)^2 \frac{\partial^2 v'}{\partial \rho'^2} \\
+ \epsilon^2 \frac{z'_{y'^*}}{z'_{\rho'}} \frac{\partial q'}{\partial \rho'} + \frac{F}{\text{Re}_H} \left[-2 \frac{z'_{x'^*}}{z'_{\rho'}} \frac{\partial^2 v'}{\partial x'^* \partial \rho'} - 2 \frac{z'_{y'^*}}{z'_{\rho'}} \frac{\partial^2 v'}{\partial y'^* \partial \rho'} + \left(\frac{z'_{x'^*}}{z'_{\rho'}} \right)^2 \frac{\partial^2 v'}{\partial \rho'^2} + \left(\frac{z'_{y'^*}}{z'_{\rho'}} \right)^2 \frac{\partial^2 v'}{\partial \rho'^2} \right],
\end{aligned}
\tag{A.3b}$$

$$\begin{aligned}
\frac{\partial w'}{\partial t'^*} + F(\mathbf{u}'_{H^*} \cdot \nabla_{H^*} w') - \frac{F}{\epsilon} \frac{1}{\text{Ro}} u' = -\frac{1}{z'_{\rho'}} \frac{\partial q'}{\partial \rho'} + \frac{F}{\text{Re}_H} \nabla_{H^*}^2 w' + \frac{F}{\text{Re}_v} \left(\frac{1}{z'_{\rho'}} \right)^2 \frac{\partial^2 w'}{\partial \rho'^2} \\
+ \frac{F}{\text{Re}_H} \left[-2 \frac{z'_{x'^*}}{z'_{\rho'}} \frac{\partial^2 w'}{\partial x'^* \partial \rho'} - 2 \frac{z'_{y'^*}}{z'_{\rho'}} \frac{\partial^2 w'}{\partial y'^* \partial \rho'} + \left(\frac{z'_{x'^*}}{z'_{\rho'}} \right)^2 \frac{\partial^2 w'}{\partial \rho'^2} + \left(\frac{z'_{y'^*}}{z'_{\rho'}} \right)^2 \frac{\partial^2 w'}{\partial \rho'^2} \right],
\end{aligned}
\tag{A.3c}$$

1328 where $M' = p'_h + \frac{1}{F} \rho' z'$ is the nondimensional Montgomery potential. Equa-
1329 tions (A.3) include all of the metric terms associated with the coordinate
1330 transformation. Since $z'_{\rho'} = \mathcal{O}(1)$, terms containing $z'_{\rho'}$ cannot be neglected.
1331 However, many of the other metric terms can be safely neglected based
1332 on scaling arguments. For example, we can safely assume that the isopy-
1333 cnal slope terms ($z'_{x'^*}$ and $z'_{y'^*}$) are small since they scale like $\frac{a}{L}$ where a
1334 is the scale for the internal wave amplitude. We can rewrite this slope

1335 scaling as $\frac{a}{L} = \frac{a}{H} \frac{H}{L} \approx F\epsilon < \mathcal{O}(1)$, since we have assumed the first-order
 1336 gravity-wave balance $\frac{a}{H} \approx \frac{u}{c_i} = F$. These small nondimensional parameters
 1337 ($z'_{x^{*'}} \approx z'_{y^{*'}} < \mathcal{O}(1)$, $\epsilon^2 < \mathcal{O}(1)$, $\frac{F}{\text{Re}_H} < \mathcal{O}(1)$, $\frac{F}{\text{Re}_v} < \mathcal{O}(1)$) are negligible
 1338 (i.e. $\ll 1$) when they appear together in products. Additional support in fa-
 1339 vor of neglecting these metric terms is given in Vitousek and Fringer (2011),
 1340 who showed that, at coarse resolution, model truncation error from the $\mathcal{O}(1)$
 1341 terms in the governing equations can be larger than (asymptotically weak)
 1342 physical terms such as the nonhydrostatic pressure. Thus, in the current
 1343 model, we ignore the small metric terms in Equation (A.3) associated with
 1344 the nonhydrostatic pressure and viscosity (the terms on the second line of
 1345 each equation).

1346 Equations (6) which appear in Section 2.2 are the dimensional version
 1347 of Equations (A.3) which are subject to the mild-slope approximation and a
 1348 horizontally and vertically varying viscosity.

1349 Transformed to isopycnal coordinates, the nondimensional continuity equa-
 1350 tion (A.2) becomes

$$\frac{\partial}{\partial x^{*'}}(z'_{\rho'} u') + \frac{\partial}{\partial y^{*'}}(z'_{\rho'} v') + \frac{\partial W'}{\partial \rho'} = 0, \quad (\text{A.4})$$

1351 where $W' = w' - z'_{x^{*'}} u' - z'_{y^{*'}} v'$. Invoking the mild-slope approximation for
 1352 the continuity equation reduces Equation (A.4) to

$$\frac{\partial}{\partial x^{*'}}(z'_{\rho'} u') + \frac{\partial}{\partial y^{*'}}(z'_{\rho'} v') + \frac{\partial w'}{\partial \rho'} = 0, \quad (\text{A.5})$$

1353 since $w' = W' + \mathcal{O}(z'_{x^{*'}}, z'_{y^{*'}})$. In contrast to Equations (A.3), Equation (A.4)
 1354 does not contain products of small nondimensional parameters. We can still
 1355 neglect the isopycnal slope terms in Equation (A.4) (thus leading to Equation
 1356 (A.5)) because the primary purpose of this form of the continuity equation is
 1357 the calculation of the nonhydrostatic pressure q' through an elliptic equation.
 1358 Because the nonhydrostatic pressure gradient in the momentum equations
 1359 (A.3) has a small nondimensional coefficient of ϵ^2 , the isopycnal slope term
 1360 in Equation (A.4) can be safely neglected as if appearing in a product of
 1361 small parameters. Further discussion of the mild-slope approximation for
 1362 the continuity equation (A.4) and (A.5) in discrete form and the resulting
 1363 symmetry of the linear system for the nonhydrostatic pressure are presented
 1364 in Appendix D and Appendix E, respectively.

1365 **Appendix B. Implicit free-surface equation**

1366 To derive an implicit equation for the free surface we follow an ap-
 1367 proach similar to Casulli (1999) in which the hydrostatic provisional veloci-
 1368 ties, $\mathbf{u}_{i\pm 1/2}^*$, are substituted into the free-surface equation. In this approach,
 1369 rather than use an equation for the free surface directly, we derive an equa-
 1370 tion for the free surface by summing over the layer-height continuity equation
 1371 after substitution of the provisional velocities.

1372 We can write the layer-height continuity equation (30) in matrix form as

$$\begin{aligned}
\frac{\mathbf{h}_i^{n+1} - \mathbf{h}_i^n}{\Delta t} = & -\frac{1}{2}(1+c)\frac{1}{\Delta x} \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^* - \mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^* \right) \\
& -\frac{1}{2}(1-2c)\frac{1}{\Delta x} \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^n - \mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^n \right) \\
& -\frac{c}{2}\frac{1}{\Delta x} \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^{n-1} - \mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^{n-1} \right), \tag{B.1}
\end{aligned}$$

1373 where $\mathbf{h}_i = [h_{i,1}, h_{i,2}, h_{i,3}, \dots, h_{i,k}, \dots, h_{i,N_{\text{layers}}}]^T$, represents the variables of a col-
 1374 umn of grid points and the operator $\circ$ represents an element-wise product.
 1375 To derive an implicit equation for the free surface we substitute the provi-
 1376 sional velocities, $\mathbf{u}_{i\pm\frac{1}{2}}^*$, given in Equation (28a) into the layer-height continu-
 1377 ity equation (B.1) resulting in

$$\begin{aligned}
& \mathbf{h}_i^{n+1} - g \left(\frac{1}{2}(1+c)\frac{\Delta t}{\Delta x} \right)^2 \left\{ (\mathbf{h}_{i+\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i+\frac{1}{2}}) \eta_{i+1}^{n+1} - \left[(\mathbf{h}_{i+\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i+\frac{1}{2}}) + (\mathbf{h}_{i-\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i-\frac{1}{2}}) \right] \eta_i^{n+1} + (\mathbf{h}_{i-\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i-\frac{1}{2}}) \eta_{i-1}^{n+1} \right\} \\
& = \mathbf{h}_i^n - \frac{1}{2}(1+c)\frac{\Delta t}{\Delta x} \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \tilde{\mathbf{T}}_{i+\frac{1}{2}}^n - \mathbf{h}_{i-\frac{1}{2}}^n \circ \tilde{\mathbf{T}}_{i-\frac{1}{2}}^n \right) \\
& \quad - \frac{1}{2}(1-2c)\frac{\Delta t}{\Delta x} \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^n - \mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^n \right) \\
& \quad - \frac{c}{2}\frac{\Delta t}{\Delta x} \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^{n-1} - \mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^{n-1} \right). \tag{B.2}
\end{aligned}$$

1378 The free surface can be computed by summing over the layer heights as
 1379 $\eta_i = \sum_{k=1}^{N_{\text{layers}}} h_{i,k} - H_i = \mathbf{e}^T \mathbf{h}_i - H_i$. Therefore, we multiply Equation (B.2)
 1380 by $\mathbf{e}^T$ and subtract the depth, H_i from both sides to give

$$\begin{aligned}
& \eta_i^{n+1} \\
& - g \left(\frac{1}{2}(1+c) \frac{\Delta t}{\Delta x} \right)^2 \mathbf{e}^T \left\{ (\mathbf{h}_{i+\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i+\frac{1}{2}}) \eta_{i+1}^{n+1} - \left[(\mathbf{h}_{i+\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i+\frac{1}{2}}) + (\mathbf{h}_{i-\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i-\frac{1}{2}}) \right] \eta_i^{n+1} + (\mathbf{h}_{i-1/2}^n \circ \tilde{\mathbf{e}}_{i-\frac{1}{2}}) \eta_{i-1}^{n+1} \right\} \\
& = \eta_i^n - \frac{1}{2}(1+c) \frac{\Delta t}{\Delta x} \left[\mathbf{e}^T \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \tilde{\mathbf{T}}_{i+\frac{1}{2}}^n \right) - \mathbf{e}^T \left(\mathbf{h}_{i-\frac{1}{2}}^n \circ \tilde{\mathbf{T}}_{i-\frac{1}{2}}^n \right) \right] \\
& - \frac{1}{2}(1-2c) \frac{\Delta t}{\Delta x} \left[\mathbf{e}^T \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^n \right) - \mathbf{e}^T \left(\mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^n \right) \right] \\
& - \frac{c}{2} \frac{\Delta t}{\Delta x} \left[\mathbf{e}^T \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^{n-1} \right) - \mathbf{e}^T \left(\mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^{n-1} \right) \right] . \tag{B.3}
\end{aligned}$$

1381 Thus Equation (B.3) can be simplified further to

$$-(\alpha_1)_{i-\frac{1}{2}}^n \eta_{i-1}^{n+1} - \left[1 + (\alpha_1)_{i-\frac{1}{2}}^n + (\alpha_1)_{i+\frac{1}{2}}^n \right] \eta_i^{n+1} - (\alpha_1)_{i+\frac{1}{2}}^n \eta_{i+1}^{n+1} = R_i^n . \tag{B.4}$$

1382 which is a tridiagonal equation in one horizontal dimension (i.e. 2-D x - z) or

1383 a block tridiagonal in 3-D. The coefficients are

$$(\alpha_1)_{i\pm\frac{1}{2}}^n = g \left(\frac{1}{2}(1+c) \frac{\Delta t}{\Delta x} \right)^2 \mathbf{e}^T (\mathbf{h}_{i\pm\frac{1}{2}}^n \circ \tilde{\mathbf{e}}_{i\pm\frac{1}{2}}) , \tag{B.5}$$

1384 and the right-hand side is given by

$$\begin{aligned}
R_i^n &= \eta_i^n - \frac{1}{2}(1+c) \frac{\Delta t}{\Delta x} \left[\mathbf{e}^T \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \tilde{\mathbf{T}}_{i+\frac{1}{2}}^n \right) - \mathbf{e}^T \left(\mathbf{h}_{i-\frac{1}{2}}^n \circ \tilde{\mathbf{T}}_{i-\frac{1}{2}}^n \right) \right] \\
& - \frac{1}{2}(1-2c) \frac{\Delta t}{\Delta x} \left[\mathbf{e}^T \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^n \right) - \mathbf{e}^T \left(\mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^n \right) \right] \\
& - \frac{c}{2} \frac{\Delta t}{\Delta x} \left[\mathbf{e}^T \left(\mathbf{h}_{i+\frac{1}{2}}^n \circ \mathbf{u}_{i+\frac{1}{2}}^{n-1} \right) - \mathbf{e}^T \left(\mathbf{h}_{i-\frac{1}{2}}^n \circ \mathbf{u}_{i-\frac{1}{2}}^{n-1} \right) \right] . \tag{B.6}
\end{aligned}$$

1385 Appendix C. MPDATA to compute flux-face heights

1386 The MPDATA method to compute layer heights for volume fluxes involves
 1387 first solving the first-order upwind or donor-cell approach using

$$h_{i,k}^* = h_{i,k}^n - \left[F \left(h_{i,k}^n, h_{i+1,k}^n, U_{i+\frac{1}{2},k}^{(n+1)\text{Im}*} \right) - F \left(h_{i-1,k}^n, h_{i,k}^n, U_{i-\frac{1}{2},k}^{(n+1)\text{Im}*} \right) \right] , \quad (\text{C.1})$$

1388 where the fluxes are

$$F \left(h_{i,k}^n, h_{i+1,k}^n, U_{i+\frac{1}{2},k}^{(n+1)\text{Im}*} \right) = U_{i+\frac{1}{2},k}^+ h_{i,k}^n + U_{i+\frac{1}{2},k}^- h_{i+1,k}^n , \quad (\text{C.2})$$

1389 where

$$U_{i+\frac{1}{2},k}^\pm = \frac{1}{2} \left(U_{i+\frac{1}{2},k}^{(n+1)\text{Im}*} \pm |U_{i+\frac{1}{2},k}^{(n+1)\text{Im}*}| \right) , \quad (\text{C.3})$$

1390 and

$$U_{i+\frac{1}{2},k}^{(n+1)\text{Im}*} = \frac{u_{i+\frac{1}{2},k}^{(n+1)\text{Im}*} \Delta t}{\Delta x} , \quad (\text{C.4})$$

1391 is the advective Courant number. We can correct the numerical diffusion as-
 1392 sociated with this step by applying an anti-diffusive correction step using the
 1393 MPDATA approach of Smolarkiewicz and Margolin (1998). This correction
 1394 is given by

$$h_{i,k}^{n+1} = h_{i,k}^* - \left[F \left(h_{i,k}^*, h_{i+1,k}^*, \tilde{U}_{i+\frac{1}{2},k}^{(n+1)\text{Im}*} \right) - F \left(h_{i-1,k}^*, h_{i,k}^*, \tilde{U}_{i-\frac{1}{2},k}^{(n+1)\text{Im}*} \right) \right] , \quad (\text{C.5})$$

1395 where

$$\tilde{U}_{i+\frac{1}{2},k}^{(n+1)\text{Im}^*} = \left(|U_{i+\frac{1}{2},k}^{(n+1)\text{Im}^*}| - \left(U_{i+\frac{1}{2},k}^{(n+1)\text{Im}^*} \right)^2 \right) A_{i+\frac{1}{2},k}^* , \quad (\text{C.6})$$

1396 and

$$A_{i+\frac{1}{2},k}^* = \begin{cases} \frac{h_{i+1,k}^* - h_{i,k}^*}{h_{i+1,k}^* + h_{i,k}^*} & \text{if } h_{i+1,k}^* + h_{i,k}^* \neq 0 \\ 0 & \text{otherwise} \end{cases} . \quad (\text{C.7})$$

1397 Using MPDATA ensures second-order accuracy in time and space while also
1398 ensuring positive layer heights.

1399 **Appendix D. Divergence-free continuity equation**

1400 The continuity equation in continuous isopycnal coordinates in 2-D is
1401 given by

$$\frac{\partial}{\partial x}(z_\rho u) + \frac{\partial}{\partial \rho}(w - z_{x^*} u) = 0 . \quad (\text{D.1})$$

1402 Discretizing this equation results in

$$\begin{aligned} & \frac{1}{\Delta x} \left[\left(\frac{z_{i+\frac{1}{2},k+\frac{1}{2}}^{n+1} - z_{i+\frac{1}{2},k-\frac{1}{2}}^{n+1}}{\Delta \rho_k} \right) u_{i+\frac{1}{2},k}^{n+1} - \left(\frac{z_{i-\frac{1}{2},k+\frac{1}{2}}^{n+1} - z_{i-\frac{1}{2},k-\frac{1}{2}}^{n+1}}{\Delta \rho_k} \right) u_{i-\frac{1}{2},k}^{n+1} \right] \\ & + \frac{w_{i,k+\frac{1}{2}}^{n+1} - w_{i,k-\frac{1}{2}}^{n+1}}{\Delta \rho_k} - \frac{(z_{x^*})_{i,k+\frac{1}{2}}^{n+1} u_{i,k+\frac{1}{2}}^{n+1} - (z_{x^*})_{i,k-\frac{1}{2}}^{n+1} u_{i,k-\frac{1}{2}}^{n+1}}{\Delta \rho_k} = 0 . \end{aligned}$$

1403 Factoring the $1/\Delta \rho_k$ from this equation and writing $z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}} = z_{\text{bottom}} -$
1404 $z_{\text{top}} = -(z_{\text{top}} - z_{\text{bottom}}) = -h_k$, gives the full, divergence-free continuity
1405 equation to be

$$\begin{aligned} & \frac{1}{\Delta x} \left(h_{i+\frac{1}{2},k}^{n+1} u_{i+\frac{1}{2},k}^{n+1} - h_{i-\frac{1}{2},k}^{n+1} u_{i-\frac{1}{2},k}^{n+1} \right) + w_{i,k-\frac{1}{2}}^{n+1} - w_{i,k+\frac{1}{2}}^{n+1} \\ & - \beta \left((z_{x^*})_{i,k-\frac{1}{2}}^{n+1} u_{i,k-\frac{1}{2}}^{n+1} - (z_{x^*})_{i,k+\frac{1}{2}}^{n+1} u_{i,k+\frac{1}{2}}^{n+1} \right) = 0 . \end{aligned} \quad (\text{D.2})$$

1406 The metric term z_{x^*} is calculated as $z_{x^*} = \frac{\partial z}{\partial x^*} = \frac{z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}}{\Delta x}$ and $\beta = 1$ or 0
 1407 depending on whether the metric terms are included or not, respectively. We
 1408 generally use the simplified continuity equation, $\beta = 0$. However, the full
 1409 continuity equation is required when the isopycnal slopes are significant.

1410 **Appendix E. Nonhydrostatic elliptic equation**

1411 The full continuity equation in 2-D is given by Equation (D.2). In this
 1412 equation, the velocity $u_{i,k\pm\frac{1}{2}}^{n+1}$ appearing as a product with the slope term, z_{x^*} ,
 1413 is undefined at these locations. Hence, we obtain its value using the following
 1414 interpolation

$$u_{i,k\pm\frac{1}{2}}^{n+1} = \frac{1}{4} \left(u_{i+\frac{1}{2},k}^{n+1} + u_{i-\frac{1}{2},k}^{n+1} + u_{i+\frac{1}{2},k\pm 1}^{n+1} + u_{i-\frac{1}{2},k\pm 1}^{n+1} \right) . \quad (\text{E.1})$$

1415 Substituting Equation (E.1) into the continuity equation (D.2) gives

$$\begin{aligned}
& \frac{1}{\Delta x} \left(h_{i+\frac{1}{2},k}^{n+1} u_{i+\frac{1}{2},k}^{n+1} - h_{i-\frac{1}{2},k}^{n+1} u_{i-\frac{1}{2},k}^{n+1} \right) + w_{i,k-\frac{1}{2}}^{n+1} - w_{i,k+\frac{1}{2}}^{n+1} \\
& - \frac{\beta}{4} (z_{x^*})_{i,k-\frac{1}{2}}^{n+1} \left(u_{i+\frac{1}{2},k}^{n+1} + u_{i-\frac{1}{2},k}^{n+1} + u_{i+\frac{1}{2},k-1}^{n+1} + u_{i-\frac{1}{2},k-1}^{n+1} \right) \\
& + \frac{\beta}{4} (z_{x^*})_{i,k+\frac{1}{2}}^{n+1} \left(u_{i+\frac{1}{2},k}^{n+1} + u_{i-\frac{1}{2},k}^{n+1} + u_{i+\frac{1}{2},k+1}^{n+1} + u_{i-\frac{1}{2},k+1}^{n+1} \right) = 0 .
\end{aligned} \tag{E.2}$$

1416 The nonhydrostatic elliptic equation, derived by substituting Equations (11)
1417 into Equation (E.2), is then given by

$$\begin{aligned}
& \frac{\Delta x}{h_{i,k-\frac{1}{2}}^{n+1}} (q_c)_{i,k-\frac{1}{2}}^{n+\frac{1}{2}} + \left[\frac{h_{i-\frac{1}{2},k}^{n+1}}{\Delta x} - \frac{\beta}{4} \left((z_{x^*})_{i,k+\frac{1}{2}}^{n+1} - (z_{x^*})_{i,k-\frac{1}{2}}^{n+1} \right) \right] (q_c)_{i-1,k}^{n+\frac{1}{2}} \\
& - \left(\frac{h_{i+\frac{1}{2},k}^{n+1}}{\Delta x} + \frac{h_{i-\frac{1}{2},k}^{n+1}}{\Delta x} + \frac{\Delta x}{h_{i,k+\frac{1}{2}}^{n+1}} + \frac{\Delta x}{h_{i,k-\frac{1}{2}}^{n+1}} \right) (q_c)_{i,k}^{n+\frac{1}{2}} \\
& \frac{\Delta x}{h_{i,k+\frac{1}{2}}^{n+1}} (q_c)_{i,k+\frac{1}{2}}^{n+\frac{1}{2}} + \left[\frac{h_{i+\frac{1}{2},k}^{n+1}}{\Delta x} + \frac{\beta}{4} \left((z_{x^*})_{i,k+\frac{1}{2}}^{n+1} - (z_{x^*})_{i,k-\frac{1}{2}}^{n+1} \right) \right] (q_c)_{i+1,k}^{n+\frac{1}{2}} \\
& - \frac{\beta}{4} (z_{x^*})_{i,k-\frac{1}{2}}^{n+1} \left((q_c)_{i+1,k-1}^{n+\frac{1}{2}} - (q_c)_{i-1,k-1}^{n+\frac{1}{2}} \right) \\
& + \frac{\beta}{4} (z_{x^*})_{i,k+\frac{1}{2}}^{n+1} \left((q_c)_{i+1,k+1}^{n+\frac{1}{2}} - (q_c)_{i-1,k+1}^{n+\frac{1}{2}} \right) = R_{i,k}^* ,
\end{aligned} \tag{E.3}$$

1418 where

$$\begin{aligned}
R_{i,k}^* = & \frac{\Delta x}{\Delta t} \left[\frac{1}{\Delta x} \left(h_{i+\frac{1}{2},k}^{n+1} u_{i+\frac{1}{2},k}^* - h_{i-\frac{1}{2},k}^{n+1} u_{i-\frac{1}{2},k}^* \right) + w_{i,k-\frac{1}{2}}^* - w_{i,k+\frac{1}{2}}^* \right] \\
& - \frac{\beta}{4} \frac{\Delta x}{\Delta t} (z_{x^*})_{i,k-\frac{1}{2}}^{n+1} \left(u_{i+\frac{1}{2},k}^* + u_{i-\frac{1}{2},k}^* + u_{i+\frac{1}{2},k-1}^* + u_{i-\frac{1}{2},k-1}^* \right) \\
& + \frac{\beta}{4} \frac{\Delta x}{\Delta t} (z_{x^*})_{i,k+\frac{1}{2}}^{n+1} \left(u_{i+\frac{1}{2},k}^* + u_{i-\frac{1}{2},k}^* + u_{i+\frac{1}{2},k+1}^* + u_{i-\frac{1}{2},k+1}^* \right) .
\end{aligned}$$

Equation (E.3) constitutes a non-symmetric 9-diagonal matrix in 2-D. We
 note that Equation (E.3) represents the next order asymptotic approximation
 to the mild-slope elliptic equation and including the nonhydrostatic metric
 terms in the momentum equations (A.3) in the correction step (11) will result
 in the full elliptic equation. Equation (E.3) can be simplified to a symmetric
 5-diagonal matrix in 2-D by neglecting the terms that appear in products
 with the isopycnal slope terms, i.e. $\beta = 0$. This approximation is justified by
 scaling arguments of the isopycnal slope term and the fact that the nonhy-
 drostatic terms scale as ϵ^2 in the momentum equation (A.3) which leads to a
 product of small nondimensional parameters. If these terms were included,
 it would require extremely high grid resolution to ensure that the magnitude
 of the additional terms is larger than the truncation error associated with
 the $\mathcal{O}(1)$ terms (Vitousek and Fringer, 2011). When these approximations
 are made $\beta = 0$ and Equation (E.3) becomes Equation (34). This approx-
 imation is very similar to the approximation made in Berntsen and Furnes
 (2005) which represents a mild-slope approximation to the nonhydrostatic
 elliptic equation in σ -coordinates. As shown in Keilegavlen and Berntsen

(2009), simulations using the simplified elliptic equation of Berntsen and Furnes (2005) produce results that are nearly identical to simulations using the full elliptic equation in σ -coordinates.

Appendix F. Internal wave generation in the South China Sea: Procedure to determine initial layer heights

This appendix describes the procedure to determine the layer heights and densities in the 2-layer model by matching the mode-1 wave speed and solitary wave speed with the theoretical wave speed based on the stratification given in Equation (51).

Solitary wave behavior is approximately represented by the KdV equation

$$\frac{\partial \xi}{\partial t} + c_1 \frac{\partial \xi}{\partial x} + \delta \xi \frac{\partial \xi}{\partial x} + \epsilon \frac{\partial^3 \xi}{\partial x^3} = 0, \quad (\text{F.1})$$

where ξ is either the interface displacement or mode-1 amplitude. The solitary wave solution to Equation (F.1) is given by

$$\xi = a \operatorname{sech}^2 \left(\frac{x - ct}{L_0} \right), \quad (\text{F.2})$$

where

$$c = c_1 + \frac{\delta a}{3}, \quad (\text{F.3})$$

is the solitary wave speed with c_1 the linear, mode-1 wave speed determined

1450 from a modal analysis, and

$$L_0 = \sqrt{\frac{12 \epsilon}{a \delta}}, \quad (\text{F.4})$$

1451 is the solitary wave length (Liu, 1988; Helfrich and Melville, 2006). Following
 1452 Liu (1988), the coefficients in Equation (F.1) are given by

$$\delta = \frac{3}{2} c_1 \frac{\int_{-H_0}^0 \left(\frac{d\phi}{dz}\right)^3 dz}{\int_{-H_0}^0 \left(\frac{d\phi}{dz}\right)^2 dz}, \quad (\text{F.5})$$

1453 and

$$\epsilon = \frac{1}{2} \frac{\int_{-H_0}^0 \phi^2 dz}{\int_{-H_0}^0 \left(\frac{d\phi}{dz}\right)^2 dz}, \quad (\text{F.6})$$

1454 where $\phi = \phi(z)$ is the mode-1 eigenfunction determined from a modal anal-
 1455 ysis.

1456 Equations (F.5) and (F.6) are used to compute the values of δ and
 1457 ϵ from the stratification in Equation (51) using an eigenfunction analy-
 1458 sis (Fringer and Street, 2003) with 1000 equispaced vertical layers. These
 1459 values along with the mode-1 wave speed, c_1 , are indicated in Table F.3,
 1460 and are referred to as “Normal mode theory” values. The still-water in-
 1461 terface depths in the 10-layer isopycnal model are arbitrarily set to $z =$
 1462 $-[0, 50, 100, 250, 460, 750, 1000, 1500, 2000, 2500, 3000]$ m with density values
 1463 computed at the vertical center of each layer using equation (51). Using
 1464 these values of the layer heights and densities to compute the first-mode
 1465 eigenfunction gives c_1 , δ , and ϵ that match the theoretical values quite well

1466 as shown in Table F.3. For the two-layer isopycnal model, the layer depths
 1467 and densities are set to match the theoretical values for c_1 , δ , and ϵ from
 1468 Helfrich and Melville (2006) that are given by

$$c_1 = \sqrt{2 \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} \frac{h_1 h_2}{h_1 + h_2}}, \quad (\text{F.7})$$

$$\delta = \frac{3}{2} c_1 \frac{h_1 - h_2}{h_1 h_2}, \quad (\text{F.8})$$

$$\epsilon = \frac{1}{6} c_1 h_1 h_2, \quad (\text{F.9})$$

1469 respectively. For given values of δ and ϵ , Equations (F.8) and (F.9) can be
 1470 used to derive expressions for h_1 (since specifying h_1 also sets the value of
 1471 h_2 , i.e. $h_2 = H_0 - h_1$) which are given by

$$h_1 = \frac{1}{2} H_0 - \frac{3}{2} \frac{c_1}{\delta} \pm \frac{1}{2} \sqrt{H_0^2 - 9 \left(\frac{c_1}{\delta} \right)^2}, \quad (\text{F.10})$$

$$h_1 = \frac{1}{2} H_0 \pm \frac{1}{2} \sqrt{H_0^2 - 24 \frac{\epsilon}{c_1}}. \quad (\text{F.11})$$

1472 Based on the form of Equations (F.10) and (F.11), it is, in general, not
 1473 possible find a value of h_1 to match both the δ and ϵ coefficients with only
 1474 two layers. For example, using values of $c_1 = 2.856 \text{ m s}^{-1}$, $\delta = -0.0076 \text{ s}^{-1}$,
 1475 and $\epsilon = 2.895 \times 10^5 \text{ m}^3 \text{ s}^{-1}$ from the 10-layer model, Equation (F.10) gives
 1476 $h_1 = 3662 \text{ m}$ or $h_1 = 460 \text{ m}$ and Equation (F.11) gives $h_1 = 3191 \text{ m}$ or
 1477 $h_1 = -191 \text{ m}$. Clearly, since $H_0 = 3000 \text{ m}$, the only possible choice for the

1478 upper layer thickness, h_1 , is $h_1 = 460$ m (and accordingly $h_2 = H_0 - H_1 =$
 1479 2540 m). This choice allows the coefficient of nonlinearity, δ , to match in the
 1480 2-layer and 10-layer models. Here, we match the coefficient of nonlinearity,
 1481 δ , in the 2- and 10-layer models to best match the solitary wave speed, c ,
 1482 which is dependent on δ as shown in Equation (F.3). We note that there
 1483 are a number of viable alternative methods to compute h_1 . For example,
 1484 it is possible to select h_1 to match the ratio of $\frac{\epsilon}{\delta}$ and thereby match the
 1485 solitary wave width, L_0 , which depends on this ratio as shown in Equation
 1486 (F.4). It may be also possible to choose h_1 to match quantities of interest
 1487 such as the energy flux according to linear theory. However, development of
 1488 an ideal method to choose h_1 is beyond the scope of this paper. With values
 1489 of h_1 and h_2 known, the density difference between the layers is specified
 1490 by matching the linear 2-layer wave speed from Equation (F.7). In the 2-
 1491 layer model, the density of the upper layer, $\rho_1 = 1028 \text{ kg m}^{-3}$, is set using
 1492 Equation (51) at the center of the layer. Next, the density of the lower layer,
 1493 $\rho_2 = 1030.1967 \text{ kg m}^{-3}$, is calculated to match the mode-1 wave speed of the
 1494 10-layer model. Compilation of the KdV parameters based on normal-mode
 1495 theory and the 2- and 10-layer models is given in Table F.3. Finally, we
 1496 note that while the layer configurations should produce similar behavior for
 1497 weakly nonlinear, mode-1 solitary wave behavior, nonlinear and higher-mode
 1498 effects will produce results in the isopycnal model simulations that differ from
 1499 the normal-mode theory.

Table F.3: Matching parameters for the 2- and 10-layer models vs. normal model theory.

	c_1 [m s ⁻¹]	δ [s ⁻¹]	ϵ [m ³ s ⁻¹]
Normal mode theory	2.866	-0.0076	2.94×10^5
10-layer model	2.856	-0.0076	2.89×10^5
2-layer model	2.856	-0.0076	5.56×10^5

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