Improved Proportionate NSAF Algorithms with sparseness-measured for

Acoustic Echo Cancellation

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Abstract: Compared with the normalized subband adaptive filter (NSAF) algorithm, the proportionate NSAF (PNSAF) and μ -law (MPNSAF) algorithms proposed by Abadi, *et al.* provide fast convergence for sparse impulse responses. However, both algorithms will suffer from the poor convergence performance, when the sparseness of impulse response varies over time or/and context which is often encountered in practical applications such as acoustic echo cancellation (AEC). Hence, we propose their improved versions that can adapt to the sparseness variation by incorporating the estimated sparseness into the PNSAF and MPNSAF, respectively. In addition, based on the energy conservation argument, we provide a unified formula to predict the steady-state mean-square performance of any PNSAF algorithm, which is supported by simulations. Simulation results in the context of AEC show that the proposed algorithms not only exhibit faster convergence rate than their competitors in sparse, quasi-sparse and dispersive environments, but also are robust to the variation in the sparseness of impulse responses.

Keywords: Acoustic echo cancellation; proportionate normalized subband adaptive filter algorithm; sparseness-measured; sparse impulse responses; dispersive impulse response.

1 Introduction

In many real-world applications, sparse system with the property that most of the coefficients of its impulse response are equal/close to zero (called *inactive coefficients*) and only a few coefficients have large magnitude (called *active coefficients*) are often encountered, typically such as the acoustic echo path between the terminal's loudspeaker and microphone and network echo path in echo cancellation scenarios [1]. Compared with the classic normalized least mean square (NLMS) algorithm, the proportionate NLMS (PNLMS) algorithm [2] developed by Duttweiler provides faster convergence rate, since it can exploit the sparsity of impulse responses. After that, to further improve the convergence performance of the PNLMS algorithm, various modified proportionate techniques were proposed [3]-[5]. However, these algorithms will suffer from slow convergence when the input signals are colored, especially for speech input signals. For such input signals, the family of affine projection algorithms (APAs) provide a significant improvement in convergence rate (e.g., see [6]-[8] and

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the references therein). However, they require very large computational burden due to involving the matrix inversion operation.

To speed up the convergence of the NLMS for colored input signals, an alternative approach is to use the multiband-structured of subband adaptive filter (SAF) [9], [10]. This is because the colored input signal is divided into almost mutually exclusive subband signals by the analysis filters and then decimated, so each decimated subband input signal is approximate white. In [10], Lee and Gan derived the normalized SAF (NSAF) algorithm from the least perturbation criterion, which exhibits much faster convergence rate than the NLMS algorithm for the colored input signals, due mainly to the inherent decorrelating property of SAF [11]. In addition, for a high-order adaptive filter application such as acoustic echo cancellation (AEC), the NSAF algorithm is comparable to the NLMS algorithm in the computational complexity. Following these works, Yin *et al.* developed the statistical models of the NSAF algorithm by assuming that the analysis filter bank is paraunitary [12]. Moreover, many researchers did a larger number of works to improve the NSAF's performance. For one thing, to obtain fast convergence rate and low steady-state error, several variable step-size versions [13]-[15], variable regularization version [16] and adaptive combination of the NSAF [17] were proposed. For another thing, in order to rapidly identify sparse impulse responses, a class of proportionate NSAF algorithms is developed in an analogical way by incorporating the proportionate ideas presented in the NLMS into the NSAF, e.g., the resulting proportionate NASF (PNSAF) and μ -law PNSAF (MPNSAF) algorithms [18], [19]. Notice that the derivation of the weight update formula for the PNSAF-type algorithms is not given. Interestingly, in realistic AEC within an enclosed space, the echo path is not always either sparse or dispersive, the variation in the sparseness of impulse response can also occur. Namely, its degree of sparseness can vary with the temperature, pressure, humidity, volume of the space and loudspeaker-microphone distance [5], [8], [20], [21]. Thus, both the PNSAF and MPNSAF algorithms suffer from the performance deterioration for such impulse responses with mutative sparseness, because their performances depend on the predefined parameters controlling proportionality. Moreover, numerical simulations have verified that the parameters are associated with the sparseness levels of impulse responses. For such scenario, the sparseness-measured method has applied into the NLMS-type [5] and APA-type [8] algorithms, which exhibit outstanding convergence performance regardless of sparse case or dispersive case. To the best of our knowledge, however, the effectiveness of this method in the PNSAF-type algorithms has not been investigated. Therefore, the contribution of this paper is listed as follows:

1. We provide a rigorous derivation process for the weight update of the PNSAF-type algorithms by resorting to the Riemannian distance between the old weight vector and updated one.
2. To improve the robustness of the PNSAF and MPNSAF algorithms to the variation in the sparseness of impulse responses, we develop their improvements by employing the sparseness of the estimated impulse response at each

iteration. The proposed sparseness-measured algorithms achieve faster convergence rate than their original versions whether the impulse response is sparse or dispersive.

3. More importantly, based on the well-known energy conservation argument [24], we analyze the steady-state mean-square performance of the PNSAF algorithm. Note that the theoretical result is suitable for any PNSAF-type algorithm (including our proposed algorithms) and is also verified by simulations.

2 Review Proportionate-type NSAF algorithms

2.1 Basic model

Consider the desired signal $d(n)$ which arises from the output-end of echo path

$$d(n) = \mathbf{u}^T(n) \mathbf{w}_o + \eta(n), \quad (1)$$

where $(\cdot)^T$ denotes transposition, $\mathbf{w}_o$ is the unknown echo path with M taps that we want to estimate with an adaptive filter,

$\mathbf{u}(n) = [u(n), u(n-1), \dots, u(n-M+1)]^T$ is the far-end input signal vector, and $\eta(n)$ is the system noise.

Fig. 1. Block diagram of multiband-structured based SAF

Fig. 1 shows the block diagram of multiband-structured of SAF which has been used to derive the NSAF and its variants [9]-[19], where N denotes number of subbands. The desired signal $d(n)$ and the input signal $u(n)$ are partitioned into N subband signals through the analysis filters $\{H_i(z), i = 0, 1, \dots, N-1\}$, respectively. Then, the subband signals $y_i(n)$ and $d_i(n)$ for $i = 0, 1, \dots, N-1$ are critically decimated to yield $y_{i,D}(k)$ and $d_{i,D}(k)$, respectively, where n and k indicate the original sequences and decimated sequences. The filter output signal in the decimated sequences is defined as $y_{i,D}(k) = \mathbf{u}_i^T(k) \mathbf{w}(k)$, where $\mathbf{w}(k) = [w_1(k), w_2(k), \dots, w_M(k)]^T$ is the adaptive filter's weight vector, and

$\mathbf{u}_i(k) = [u_i(kN), u_i(kN-1), \dots, u_i(kN-M+1)]^T$. It is easy to know that the i th subband error signal is the difference between $d_{i,D}(k)$ and $y_{i,D}(k)$, i.e.,

$$e_{i,D}(k) = d_{i,D}(k) - \mathbf{u}_i^T(k) \mathbf{w}(k) \quad (2)$$

where $d_{i,D}(k) = d_i(kN)$.

For the PNSAF-type algorithms in [18], [19], the update of the weight vector is formulated as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \frac{\mathbf{G}(k) \mathbf{u}_i(k) e_{i,D}(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k) + \delta} \quad (3)$$

with

$$\mathbf{G}(k) = \text{diag}\{g_1(k), g_2(k), \dots, g_M(k)\} \quad (4)$$

where μ is the step-size, δ is a small positive number to avoid division by zero, and $\mathbf{G}(k)$ is a $M \times M$ diagonal matrix (called the proportionate matrix) which allocates an individual adaptation gain $g_m(k)$ for each filter coefficient $w_m(k)$ (i.e., a larger/smaller filter coefficient receives a larger/smaller gain, thus enhancing the convergence rate of the adaptive filter for sparse impulse responses). Note that, the proportionate NSAF algorithm with $\mathbf{G}(k) = \mathbf{I}_{M \times M}$ will be reduced to the NSAF, where $\mathbf{I}_{M \times M}$ is an identity matrix. In order to address the slow convergence of the NSAF for sparse impulse responses, many PNSAF-type algorithms have been developed by devising different strategies to calculate diagonal elements $g_m(k)$ for $m = 1, 2, \dots, M$, such as the PNSAF and MPNSAF algorithms which are briefly described below.

2.2 PNSAF and MPNSAF

In the PNSAF and MPNSAF [19], the diagonal elements $g_m(k)$ for $m = 1, 2, \dots, M$ can be formulated as follows

$$q_m(k) = \max\{\rho \times \max[\gamma, F(|w_1(k)|), \dots, F(|w_M(k)|)], F(|w_m(k)|)\} \quad (5)$$

$$g_m(k) = \frac{q_m(k)}{\frac{1}{M} \sum_{i=1}^M q_i(k)} \quad (6)$$

where the parameter γ with a typical value of 0.01 is to allow starting to update the weight vector $\mathbf{w}(k)$ at initialization $k = 0$, and ρ (typical value 0.01) prevents the filter coefficients from freezing when their magnitude are much less than the largest one. In (5), $F(\cdot)$ is a specified function to the algorithm. If

$$F(|w_m(k)|) = |w_m(k)|, \quad (7)$$

then the PNSAF algorithm is obtained. Here, the individual gain is proportional to the magnitude of the corresponding filter coefficient such that the *active coefficients* can obtain larger gains than the *inactive coefficients*. Consequently, the PNSAF

exhibits a faster initial convergence rate than the NSAF for sparse impulse responses. After fast initial convergence, however, the convergence of the PNSAF is slowed down dramatically. So, in order to obtain the fast initial convergence during the whole adaptation process, the MPNSAF is obtained by a new function mapping [19]

$$F(|w_m(k)|) = \ln(1 + \beta |w_m(k)|) \quad (8)$$

where $\beta = 1/\varepsilon$, ε is a very small positive number (typically, $\varepsilon = 0.001$ for echo cancellation applications).

It is worthy to note that the above PNSAF and MPNSAF algorithms suffer from the slow of convergence for dispersive impulse responses, and even becoming slower than the NSAF algorithm.

3 Sparseness-measured variants

In this section, we firstly provide a complete derivation for the weight update rule of the PNSAF-type algorithms. Then, we discuss the problems of the PNSAF and MPNSAF algorithms, and develop their improvements.

3.1 The derivation of the weight update

Although there are a lot of proportionate versions for the NSAF (see [18], [19]), a derivative process of the weight update rule (3) for this type of algorithms is not reported before. For that, we will show how to derive (3) in this section. Before starting the derivation, we define the subband input matrix, desired signal vector and error signal vector as follows:

$$\mathbf{U}(k) = [\mathbf{u}_0(k), \mathbf{u}_1(k), \dots, \mathbf{u}_{N-1}(k)] \quad (9)$$

$$\mathbf{d}_D(k) = [d_{0,D}(k), d_{1,D}(k), \dots, d_{N-1,D}(k)] \quad (10)$$

$$\mathbf{e}_D(k) = \mathbf{d}_D(k) - \mathbf{U}^T(k)\mathbf{w}(k) \triangleq [e_{0,D}(k), e_{1,D}(k), \dots, e_{N-1,D}(k)]^T \quad (11)$$

Based on the principle of minimum disturbance, we minimize a new distance between the old weight vector, $\mathbf{w}(k)$, and new one, $\mathbf{w}(k+1)$, weighted by the inverse of the diagonal matrix $\mathbf{G}(k)$, i.e.,

$$\min \{f[\mathbf{w}(k+1)]\} = \min \{[\mathbf{w}(k+1) - \mathbf{w}(k)]^T \mathbf{G}^{-1}(k) [\mathbf{w}(k+1) - \mathbf{w}(k)]\} \quad (12)$$

under the constrained condition

$$\mathbf{d}_D(k) - \mathbf{U}^T(k)\mathbf{w}(k+1) = \mathbf{0} \quad (13)$$

where the new distance is called the Riemannian distance [22] which outperforms the conventional Euclidean distance used to derive the NSAF in the sparse system. Combining (12) and (13) yields a new target function

$$\varphi(k) = f[\mathbf{w}(k+1)] + [\mathbf{d}_D(k) - \mathbf{U}^T(k)\mathbf{w}(k+1)]^T \boldsymbol{\theta} \quad (14)$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_N]^T$ is the Lagrange multiplier vector. Let the derivative of (14) with respect to $\mathbf{w}(k+1)$ equal to zero, we get

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \frac{1}{2} \mathbf{G}(k) \mathbf{U}(k) \boldsymbol{\theta} \quad (15)$$

Applying (12) and (15) into (13) yields

$$\boldsymbol{\theta} = 2 \left[\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_D(k) \quad (16)$$

Then, substituting (16) into (15) and introducing the step-size μ into (15), we can obtain the following update formal

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \mathbf{G}(k) \mathbf{U}(k) \left[\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_D(k) \quad (17)$$

Due to the decorrelating property of subband adaptive filter [9], [10] and diagonal matrix $\mathbf{G}(k)$, we can use a reasonable assumption to simplify (17) as the NSAF does. Namely, the off-diagonal elements of the matrix $\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k)$ can be neglected. With this assumption, therefore, (17) can be simplified as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \frac{\mathbf{G}(k) \mathbf{u}_i(k) e_{i,D}(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)} \quad (18)$$

In summary, the derivation of (3) has been performed in (9)-(18), since the parameter δ in (3) is artificially added in practical application to ensure numerical stability.

3.2 Problem statement

A commonly used method to quantify the sparseness of the impulse response $\mathbf{w}_o$ has been given as follows [1], [5], [8]

$$\phi(\mathbf{w}_o) = \frac{M}{M - \sqrt{M}} \left(1 - \frac{\|\mathbf{w}_o\|_1}{\sqrt{M} \|\mathbf{w}_o\|_2} \right) \quad (19)$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ indicate the l_1 - and l_2 -norm of a vector, respectively. Apparently, $0 \leq \phi(\mathbf{w}_o) \leq 1$, and if the impulse response is more sparse or dispersive, then the value of $\phi(\mathbf{w}_o)$ is more close to 0 or 1.

In this following, we discuss the performance of the PNSAF and MPNSAF algorithms for the sparseness variation of the impulse responses, aiming to get some insight for devising their modified versions. To this end, as an example, Monte Carlo (MC) simulations (average of 30 independent runs) of the PNSAF algorithm are carried out in a system identification setup, and the NMSD defined by $\text{NMSD}(k) = 10 \log_{10} \left[\|\mathbf{w}_o - \mathbf{w}(k)\|_2^2 / \|\mathbf{w}_o\|_2^2 \right]$ is used to evaluate the algorithm performance. Here, two impulse responses having $M = 512$ coefficients are identified, and their sparseness levels calculated by (19) are $\phi(\mathbf{w}_o) = 0.8083$ (sparse case) and $\phi(\mathbf{w}_o) = 0.3823$ (dispersive case), respectively. The colored input signal is an AR(1) process generated by filtering a zero-mean white Gaussian signal through a first-order autoregressive system $H(z) = 1 / (1 - 0.9z^{-1})$. A four-subband (i.e., $N = 4$) cosine modulated filter bank is used, with 60 dB stop-band attenuation, whose prototype filter has 32 coefficients [9]. The system noise $\eta(n)$ is white Gaussian process with a signal-to-noise rate

(SNR) of 30dB. Fig. 2 shows the NMSD curves of PNSAF algorithm using different values of ρ for dispersive and sparse impulse responses. As can be seen, for a sparse impulse response (e.g., $\phi(\mathbf{w}_o) = 0.8083$), the value of ρ should be small to obtain fast convergence rate, while that should be large for a dispersive impulse response (e.g., $\phi(\mathbf{w}_o) = 0.3823$). That is to say, the performance of PNSAF depends on the predefined parameter ρ which is associated with the sparseness levels of impulse responses. Therefore, the PNSAF needs to be improved in the convergence rate, to adapt to the variation in the sparseness of impulse responses which can occur in echo cancellation. This property of the PNSAF is consistent with the PNLMS, since they use the same rule to compute the proportionate matrix $\mathbf{G}(k)$ [2]. Similarly, based on the same proportionate strategy as the MPNLMS, the MPNSAF also suffer from the performance deterioration for the impulse responses with mutative sparseness.

Fig. 2. The NMSD performance of the PNSAF (with $\mu = 0.25$, $\gamma = 0.01$) for different values of ρ using AR(1) input signal. (a) Sparse case. (b) Dispersive case.

3.3 Sparseness-measured versions

According to the discussion in the previous section, we know that the PNSAF and MPNSAF algorithms are not suitable to identify such impulse responses with variable sparseness. This is because that the proper values of predefined parameter ρ in both the algorithms are related to the value of $\phi(\mathbf{w}_o)$. In this section, therefore, the robustness of these algorithms for the variation of sparseness is improved by incorporating the sparseness measure $\phi(\mathbf{w}_o)$ into the adaptation process. Since the true impulse response $\mathbf{w}_o$ is unknown during the adaptation process, we employ the estimated weight vector $\mathbf{w}(k)$ to obtain the sparseness of an impulse response at each iteration, i.e.,

$$\hat{\phi}(k) = \frac{M}{M - \sqrt{M}} \left(1 - \frac{\|\mathbf{w}(k)\|_1}{\sqrt{M} \|\mathbf{w}(k)\|_2} \right). \quad (20)$$

Note that, as the iteration goes on, the estimated value of sparseness $\hat{\phi}(k)$ can gradually converge to its true value $\phi(\mathbf{w}_o)$ computed by (19), namely, $\hat{\phi}(k) = \phi(\mathbf{w}_o)$ when $k \rightarrow \infty$.

In order to enhance the convergence rate of PNSAF algorithm for dispersive impulse responses, a sparseness-measured PNSAF (SM-PNSAF) algorithm is proposed, in which the predefined parameter ρ is time-varying and denoted as $\rho(k)$. Similar to [5], $\rho(k)$ is controlled by the following function

$$\rho(k) = e^{-\lambda \hat{\phi}(k)}. \quad (21)$$

Fig. 3 illustrate the relation in (21) between $\rho(k)$ and $\hat{\phi}(k)$ for four values of $\lambda = 4, 6, 8, 10$. It is clear that the value of $\rho(k)$ in SM-PNSAF can be automatically adjusted according to the variation of the sparseness $\hat{\phi}(k)$. On the one hand, for sparse impulse responses (e.g., $\hat{\phi}(k) > 0.5$), the proposed SM-PNSAF algorithm with smaller values of $\rho(k)$ exhibits more like the PNSAF algorithm in performance. On the other hand, for dispersive impulse responses (e.g., $\hat{\phi}(k) < 0.5$), the values of $\rho(k)$ are larger such that the proposed SM-PNSAF algorithm inherit the convergence property of the NSAF algorithm. Consequently, the variable parameter $\rho(k)$ calculated by (21) can bring about our overall desired effect of the robustness to sparse and dispersive impulse responses. Interestingly, the parameter λ in (21) must be chosen properly, which will be shown in Section 4.1. Overall, the proposed SM-PNSAF can obtain a good compromise between sparse and dispersive cases by choosing λ in the range of $3 \leq \lambda \leq 5$.

Fig. 3. Variation of $\rho(k)$ versus the sparseness degree $\hat{\phi}(k)$ of impulse response.

Without loss of generality, an improvement version of the MPNSAF algorithm can be obtained in a similar manner like the SM-PNSAF does, the resulting sparseness-measured MPNSAF algorithm (SM-MPNSAF) possesses good robustness for

sparse and dispersive impulse responses. In addition, to avoid division by zero in (21) at the early stages of the adaptation caused by the initialization $\mathbf{w}(0) = \mathbf{0}$, $\hat{\phi}(k)$ can be computed only when $k \geq M$ in both SM-PNSAF and SM-MPNSAF. Otherwise, we set $\rho(k) = 5/M$ when $k < M$ [5]. As a result, the proposed SM-PNSAF and SM-MPNSAF algorithms are summarized in Table I.

Table 1. Summary of the SM-PNSAF and SM-MPNSAF algorithms

Initialization	$\mathbf{w}(0) = \mathbf{0}, 0 < \mu \leq 1$ $\gamma = 0.01, \lambda = 4, \text{ and } \rho(k) = 5/M, k < M$ $\beta = 400$ (SM-MPNSAF)
General formulations	$e_{i,D}(k) = d_{i,D}(k) - \mathbf{u}_i^T(k)\mathbf{w}(k), i = 0, 1, \dots, N-1$ $\hat{\phi}(k) = \frac{M}{M - \sqrt{M}} \left[1 - \frac{\ \mathbf{w}(k)\ _1}{\sqrt{M} \ \mathbf{w}(k)\ _2} \right], k \geq M$ $\rho(k) = e^{-\lambda \hat{\phi}(k)}, k \geq M$ $q_m(k) = \max \{ \rho(k) \times \max [\gamma, F(w_1(k)), \dots, F(w_M(k))], F(w_m(k)) \}$ $g_m(k) = \frac{q_m(k)}{\frac{1}{M} \sum_{i=1}^M q_i(k)}$ $\mathbf{G}(k) = \text{diag} \{ g_1(k), g_2(k), \dots, g_M(k) \}$ $\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{N-1} \frac{\mathbf{G}(k) \mathbf{u}_i(k) e_{i,D}(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k) + \delta}$
SM-PNSAF	$F(w_m(k)) = w_m(k) $
SM-MPNSAF	$F(w_m(k)) = \ln(1 + \beta w_m(k))$

3.4 Computational complexity

Table 2 provides the computational complexity of NSAF, PNSAF, MPNSAF, SM-PNSAF, and SM-MPNSAF algorithms in terms of the total number of additions, multiplications, division, logarithm and comparisons for each fullband input sample, where the integer L is the length of the analysis filters and/or synthesis filters. Clearly, the additional complexity of the proposed sparseness-measured algorithms in comparison with their counterparts stems from the computation of sparseness $\hat{\phi}(k)$ given by (20), which requires $2M/N$ additions and M/N multiplications for each fullband input sample since $M/(M - \sqrt{M})$ can be off-line obtained. In addition, for the proposed SM-PNSAF and SM-MPNSAF

algorithms, the computation of $\rho(k)$ in (21) can be implemented using a look-up table. Fortunately, the increase of the proposed algorithms in the computational complexity is worth, since they enhance significantly performance.

Table 2. Computational complexity of various subband algorithms for each fullband input sample.

Algorithms	Multiplications	Additions	Divisions	Logarithm	Comparisons
NSAF	$3M+3NL+1$	$3M+3NL-3N$	1	0	0
PNSAF	$5M+3NL+1/N+1$	$3M+3NL+(M-1)/N-3N$	$1+M/N$	0	$2M/N$
SM-PNSAF	$5M+3NL+(M+3)/N+1$	$3M+3NL+3M/N-3N$	$1+M/N$	0	$2M/N$
MPNSAF	$5M+3NL+(M+1)/N+1$	$3M+3NL+(2M-1)/N-3N$	$1+M/N$	M/N	$2M/N$
SM-MPNSAF	$5M+3NL+(2M+3)/N+1$	$3M+3NL+4M/N-3N$	$1+M/N$	M/N	$2M/N$

4 Steady-state analysis

In this section, we will perform the steady-state performance analysis of the PNSAF-type algorithms by means of the well-known energy conservation argument [24]. Recently, this argument is widely used to analyze the performance of adaptive filter algorithms [25]-[30]. Before starting, some assumptions for simplifying analysis are listed as:

Assumption A1: The system noise $\eta(n)$ is a zero-mean white process with variance σ_η^2 .

Assumption A2: The analysis filter banks for band-decomposing signals $u(n)$ and $d(n)$ are identical and paraunitary [12], [27]-[29]. This assumption can results in the following relation

$$d_{i,D}(k) = \mathbf{u}_i^T(k) \mathbf{w}_o + \eta_i(k) \quad (22)$$

where $\eta_i(k)$, $i = 0, 1, \dots, N-1$, represents the subband noises which are generated by decomposing the system noise $\eta(n)$ through the analysis filter bank. In addition, the subband noises $\eta_i(k)$ are also zero-mean white processes with variances $\sigma_{\eta_i}^2$ [12], [27]-[29].

Assumption A3: $\eta_i(k)$, $\mathbf{w}(l)$ for $l \leq k$, $\mathbf{u}_i(k)$ are statistically independent. This assumption is very common in almost all performance analyses of adaptive algorithms, called the independence assumption [25]-[31].

Using (9) and (11), the weight update of PNSAF-type algorithms can be rewritten in matrix form as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \mathbf{G}(k) \mathbf{U}(k) \mathbf{\Lambda}^{-1}(k) \mathbf{e}_D(k) \quad (23)$$

where

$$\mathbf{\Lambda}(k) = \text{diag} \left\{ \mathbf{u}_0^T(k) \mathbf{G}(k) \mathbf{u}_0(k), \mathbf{u}_1^T(k) \mathbf{G}(k) \mathbf{u}_1(k), \dots, \mathbf{u}_{N-1}^T(k) \mathbf{G}(k) \mathbf{u}_{N-1}(k) \right\}. \quad (24)$$

Denoting the weight error vector by $\tilde{\mathbf{w}}(k) = \mathbf{w}_o - \mathbf{w}(k)$, then (23) is changed as

$$\tilde{\mathbf{w}}(k+1) = \tilde{\mathbf{w}}(k) - \mu \mathbf{G}(k) \mathbf{U}(k) \mathbf{\Lambda}^{-1}(k) \mathbf{e}_D(k). \quad (25)$$

To proceed, the *a priori* subband error vector and *a posterior* vector are defined respectively as

$$\mathbf{e}_a(k) = \mathbf{U}^T(k) \tilde{\mathbf{w}}(k) \triangleq [e_{a,0}(k), e_{a,1}(k), \dots, e_{a,N-1}(k)]^T, \quad (26)$$

$$\mathbf{e}_p(k) = \mathbf{U}^T(k) \tilde{\mathbf{w}}(k+1) \triangleq [e_{p,0}(k), e_{p,1}(k), \dots, e_{p,N-1}(k)]^T. \quad (27)$$

where the i th elements of $\mathbf{e}_a(k)$ and $\mathbf{e}_p(k)$ are expressed as $e_{a,i}(k) = \mathbf{u}_i^T(k) \tilde{\mathbf{w}}(k)$ and $e_{p,i}(k) = \mathbf{u}_i^T(k) \tilde{\mathbf{w}}(k)$, respectively.

Therefore, pre-multiplying (25) by $\mathbf{U}^T(k)$, we have

$$\mathbf{e}_p(k) = \mathbf{e}_a(k) - \mu \mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \mathbf{\Lambda}^{-1}(k) \mathbf{e}_D(k). \quad (28)$$

Assuming that the matrix $\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k)$ is inverse, (28) becomes

$$\left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} (\mathbf{e}_a(k) - \mathbf{e}_p(k)) = \mu \mathbf{\Lambda}^{-1}(k) \mathbf{e}_D(k). \quad (29)$$

Substituting (29) into (25), we obtain

$$\begin{aligned} \tilde{\mathbf{w}}(k+1) + \mathbf{G}(k) \mathbf{U}(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_a(k) = \\ \tilde{\mathbf{w}}(k) + \mathbf{G}(k) \mathbf{U}(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_p(k) \end{aligned} \quad (30)$$

Introducing the squared weighted Euclidean norm of a vector $\mathbf{x}$, denoted by $\|\mathbf{x}\|_{\Sigma}^2 = \mathbf{x}^T \Sigma \mathbf{x}$, with Σ being a symmetric positive definite matrix of compatible dimensions, and then applying it into (30), we arrive at

$$\begin{aligned} \|\tilde{\mathbf{w}}(k+1)\|_{\mathbf{G}^{-1}(k)}^2 + \mathbf{e}_a^T(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_a(k) = \\ \|\tilde{\mathbf{w}}(k)\|_{\mathbf{G}^{-1}(k)}^2 + \mathbf{e}_p^T(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_p(k) \end{aligned} \quad (31)$$

Notice that (31) is a deterministic energy relation without using any assumptions. Taking the expectation on both sides of (31) yields

$$\begin{aligned} E \left\{ \|\tilde{\mathbf{w}}(k+1)\|_{\mathbf{G}^{-1}(k)}^2 \right\} + E \left\{ \mathbf{e}_a^T(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_a(k) \right\} = \\ E \left\{ \|\tilde{\mathbf{w}}(k)\|_{\mathbf{G}^{-1}(k)}^2 \right\} + E \left\{ \mathbf{e}_p^T(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_p(k) \right\} \end{aligned} \quad (32)$$

where $E\{\cdot\}$ denotes the expectation of random variable. As long as the algorithm goes into the steady-state under the convergence condition of $0 < \mu < 2$ (which has been given in [23]), we will be able to assume

$\lim_{k \rightarrow \infty} \|\tilde{\mathbf{w}}(k+1)\|_{\mathbf{G}^{-1}(k)}^2 = \lim_{k \rightarrow \infty} \|\tilde{\mathbf{w}}(k)\|_{\mathbf{G}^{-1}(k)}^2$ so that (32) can be simplified as

$$\begin{aligned} \lim_{k \rightarrow \infty} E \left\{ \mathbf{e}_a^T(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_a(k) \right\} = \\ \lim_{k \rightarrow \infty} E \left\{ \mathbf{e}_p^T(k) \left(\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k) \right)^{-1} \mathbf{e}_p(k) \right\}. \end{aligned} \quad (33)$$

Substituting (28) into (33), and again using the diagonal assumption that neglects the off-diagonal elements of the matrix $\mathbf{U}^T(k) \mathbf{G}(k) \mathbf{U}(k)$, we get

$$2 \lim_{k \rightarrow \infty} E \left\{ \mathbf{e}_a^T(k) \mathbf{\Lambda}^{-1}(k) \mathbf{e}_D(k) \right\} = \mu \lim_{k \rightarrow \infty} E \left\{ \mathbf{e}_D^T(k) \mathbf{\Lambda}^{-1}(k) \mathbf{e}_D(k) \right\}. \quad (34)$$

Combining (11), (22), and (26), and applying assumption A3, thus (34) becomes in summation form

$$(2 - \mu) \sum_{i=0}^{N-1} \lim_{k \rightarrow \infty} E \left\{ \frac{e_{a,i}^2(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)} \right\} = \mu \sum_{i=0}^{N-1} \lim_{k \rightarrow \infty} E \left\{ \frac{\eta_i^2(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)} \right\}. \quad (35)$$

To further process, we assume that signals in each subband are uncorrelated so that we may deduce that the i th term of the summations on both sides of (35) corresponds to each other, namely,

$$(2 - \mu) \lim_{k \rightarrow \infty} E \left\{ \frac{e_{a,i}^2(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)} \right\} = \mu \lim_{k \rightarrow \infty} E \left\{ \frac{\eta_i^2(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)} \right\}, \quad (36)$$

for $i = 0, 1, \dots, N-1$.

For a long adaptive filter, i.e., $M \gg 1$, the fluctuation of $\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)$ from one iteration to next iteration can be assumed to be very small, since the sum of the diagonal elements of $\mathbf{G}(k)$ equals to one or M for any proportionate algorithm, thus the following approximations are expressed as

$$\lim_{k \rightarrow \infty} E \left\{ \frac{e_{a,i}^2(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)} \right\} \simeq \lim_{k \rightarrow \infty} \frac{E \left\{ e_{a,i}^2(k) \right\}}{E \left\{ \mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k) \right\}} \quad (37)$$

and

$$\lim_{k \rightarrow \infty} E \left\{ \frac{\eta_i^2(k)}{\mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k)} \right\} \simeq \lim_{k \rightarrow \infty} \frac{E \left\{ \eta_i^2(k) \right\}}{E \left\{ \mathbf{u}_i^T(k) \mathbf{G}(k) \mathbf{u}_i(k) \right\}}. \quad (38)$$

Inserting (37) and (38) into (39), the following relation holds

$$E \left\{ e_{a,i}^2(\infty) \right\} \simeq \frac{\mu}{(2 - \mu)} E \left\{ \eta_i^2(\infty) \right\}. \quad (39)$$

According to [9], [28], the mean square error (MSE) of subband algorithms is defined as

$$\begin{aligned} \text{MSE}(k) &= \frac{1}{N} \sum_{i=0}^{N-1} E \left\{ e_{i,D}^2(k) \right\} \\ &= \text{EMSE}(k) + \frac{1}{N} \sum_{i=0}^{N-1} \sigma_{\eta_i}^2, \end{aligned} \quad (40)$$

where $\text{EMSE}(k) = \frac{1}{N} \sum_{i=0}^{N-1} E \{e_{a,i}^2(k)\}$ denotes the excess MSE (EMSE).

Therefore, for any PNSAF algorithm, its theoretical steady-state EMSE and MSE can be given respectively as

$$\text{EMSE}(\infty) \approx \frac{\mu}{(2-\mu)} \frac{1}{N} \sum_{i=0}^{N-1} \sigma_{\eta_i}^2, \quad (41)$$

and

$$\text{MSE}(\infty) \approx \frac{\mu}{(2-\mu)} \frac{1}{N} \sum_{i=0}^{N-1} \sigma_{\eta_i}^2 + \frac{1}{N} \sum_{i=0}^{N-1} \sigma_{\eta_i}^2. \quad (42)$$

The equations (41) and (42) show that the steady-state EMSE or MSE of any PNSAF family is independent of the proportionate matrix $\mathbf{G}(k)$. Different choices of this matrix influence significantly only the convergence rate of the algorithms. Besides, since $\text{EMSE}(\infty)$ is always larger than zero, we can also conclude that the range of the step size is $0 < \mu < 2$.

5 Simulation results

In this section, the performance of the proposed SM-PNSAF and SM-MPNSAF algorithms are evaluated in the context of AEC. Fig. 4 shows the impulse responses of three typical echo paths having $M = 512$ coefficients (the sampling frequency is 8 kHz), and their sparseness are $\phi(\mathbf{w}_o) = 0.8083$ (sparse case), $\phi(\mathbf{w}_o) = 0.5366$ (quasi-sparse case) and $\phi(\mathbf{w}_o) = 0.3823$ (dispersive case), respectively. Throughout our simulations: the colored input signal is either an AR(1) process or a speech signal; the Gaussian white noise $\eta(n)$ is added to give a SNR of 30dB; and a cosine modulated filter bank with four-band $N = 4$ is used. All the results are averaged over 30 independent trials.

Fig. 4. Three acoustic echo paths. (a) Sparse path. (b) Quasi-sparse path. (c) Dispersive path.

5.1 Choice of λ

In this section, we investigate the effect of the parameter λ on the convergence performance of SM-PNSAF and SM-MPNSAF in different sparseness cases. Fig. 5 illustrates that the SM-PNSAF using different values of λ require the number of iterations to reach -30 dB NMSD for AR(1) input. In Fig. 5, we can see that, compared with a dispersive impulse response, the SM-PNSAF using a specific value of λ require fewer number of iterations for a sparse impulse response, i.e., its convergence rate is faster. The reason behind this result is that the weight vector is initialized as $\mathbf{w}(0) = \mathbf{0}$, leading to the initialization values of most filter coefficients are close to their optimal values in sparse case. In addition, when λ is larger than a certain value (in this case, $\lambda = 5$), the SM-PNSAF slows convergence as the value of λ increases. Fig. 6 shows the NMSD performance of SM-PNSAF using values of $\lambda = 1, 4, 6$ and 8 for AR(1) input, with the changes of echo path at sample index 50, 000 and 100, 000. In Fig. 6, the SM-PNSAF with $\lambda = 1, 4$ has a faster convergence rate for a sparse case and the SM-PNSAF with $\lambda = 6$ has a faster convergence rate for quasi-sparse and dispersive cases. As a consequence of Figs. 5 and 6, a suitable range of λ in the SM-PNSAF is $3 \leq \lambda \leq 5$ to obtain a balance between sparse and dispersive cases, and $\lambda = 4$ is used in the following tests. Similarly, the range of λ is applicable to the SM-MPNSAF, due to the fact that it also employs (21) to improve performance.

Fig. 5. The number of iterations to reach -30 dB NMSD for different values of λ in SM-PNSAF (with $\mu = 0.25$ and $\delta = 0.01$) using AR(1) input signal.

Fig. 6 The NMSD of the SM-PNSAF (with $\mu = 0.25$ and $\delta = 0.01$) for different values of λ using AR(1) input signal.

5.2 Performance of SM-PNSAF

Fig. 7 compares the NMSD performance of NSAF, PNSAF and proposed SM-PNSAF algorithms for AR(1) input signal, where the changes of echo path (see Fig. 4) occurs at sample index 50, 000 and 100, 000. To obtain a fair comparison, the parameters of various algorithms are selected in such a way that all algorithms have the same steady-state NMSD. As expected, the PNSAF speeds up convergence for a sparse impulse response in comparison with the NSAF, while the convergence rate of the PNSAF is slower than that of the NSAF for a dispersive impulse response. Clearly, the convergence rate of the proposed SM-PNSAF is slightly faster than PNSAF for a sparse case but much better than the PNSAF for a quasi-sparse case, and the SM-PNSAF has the same convergence rate as the NSAF even in the case of dispersion. In other words, the SM-PNSAF achieves our objective of improving robustness to varying sparseness. This is because the SM-

PNSAF combines the beneficial properties of both NSAF and PNSAF by employing (21). Fig. 8 shows the NMSD performance of NSAF, PNSAF and proposed SM-PNSAF algorithms for a speech input signal, where the values of parameters are the same as those in Fig.7. In this case, the changes of echo path occur at sample index 60, 000 and 120, 000. As can be seen, the proposed SM-PNSAF obtains a large improvement in terms of the robustness to the sparseness variation of impulse response. Moreover, the conclusion of Fig. 8 is consistent with that of Fig. 7.

Fig. 7. The NMSD curves of various subband algorithms for AR(1) input. NSAF: $\mu = 0.25$; PNSAF: $\mu = 0.25, \gamma = 0.01$; SM-PNSAF:

$$\mu = 0.25, \gamma = 0.01, \lambda = 4 .$$

Fig. 8. The NMSD curves of various subband algorithms for speech input. The choice of the parameters is the same as Fig. 7.

5.3 Performance of SM-MPNSAF

Fig. 9 provides a comparison of NSAF, MPNSAF and proposed SM-MPNSAF algorithms in the convergence rate for AR(1) input signal. Fig. 10 shows the comparison results of these algorithms using speech as input signal. In Figs. 9 and 10, the changes of echo path are the same as Figs. 7 and 8, respectively, and the choices of the algorithms parameters are to

achieve the same steady-state NMSD. It is clear that, for a dispersive system, the MPNSAF has worse convergence performance than the NSAF, even if it works well in a sparse system. As can be seen from Figs. 9 and 10, the proposed SM-MPNSAF not only exhibits a little improvement of the convergence rate compared to the NSAF and MPNSAF for sparse and dispersive cases, respectively, but also it has the fastest convergence rate for a quasi-sparse case. It is worth to note that the proposed SM-MPNSAF can adapt to the variation in the sparseness of impulse response, owing to the fact that the estimated sparseness $\hat{\phi}(k)$ computed by (20) is used to adjust the parameter $\rho(k)$ by the exponential function in (21).

Fig. 9. The NMSD curves of various subband algorithms for AR(1) input. NSAF: $\mu = 0.25$; MPNSAF: $\mu = 0.25$, $\gamma = 0.01$, $\beta = 400$; SM-MPNSAF: $\mu = 0.25$, $\gamma = 0.01$, $\beta = 400$, $\lambda = 4$.

Fig. 10. The NMSD curves of various subband algorithms for speech input. The choice of the parameters is the same as Fig. 9.

5.4 Verification of (42)

In this experiment, the theoretical model (42) will be verified for different PNSAF algorithms under the condition of SNR=30 dB and SNR=20 dB. The input signal is an AR(1) process. The step size for all the algorithms is chosen as

$\mu = 0.25$ and other parameters setting are the same as Figs. 7 and 9. As shown in Fig. 11, the theoretical results given by (42) can match well with simulation results of these PNSAF algorithms in the steady-state. In other words, the model (42) can be used to predict the steady-state behavior of any PNSAF algorithm.

Fig. 11. The MSE performance of various PNSAF algorithms for AR(1) input. (a) SNR=30 dB, (b) SNR=20 dB.

6 Conclusions

In this paper, we develop a theoretical basis for the weight update of PNSAF-type algorithms by minimizing the Riemannian distance between the old filter coefficient vector and new one with multiple subband constraints. After that, the improved versions of the PNSAF and MPNSAF algorithms are presented, which exploit the sparseness measure of the estimated impulse response to adjust the algorithms predefined parameters. The resulting SM-PNSAF and SM-MPNSAF algorithms obtain large improvement in convergence rate for sparse, quasi-sparse and dispersive impulse responses. Importantly, the proposed algorithms can adapt the sparseness variation of impulse responses, with only a slight increase in computational complexity. In addition, the steady-state MSEs of PNSAF-type algorithms are analyzed and verified. Simulation results in AEC have demonstrated the superiority of the proposed algorithms.

7 Acknowledgments

This work was partially supported by National Science Foundation of P.R. China (Grant: 61271340, 61571374 and 61433011), and the Fundamental Research Funds for the Central Universities (Grant: SWJTU12CX026).

8 Reference

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