**The application of a Matsuoka-Nakai-Lade-Duncan failure criterion
to two porous sandstones**

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Abstract

We summarize two suites of recent true triaxial tests in two porous sandstones, Coconino and Benthim. One suite of tests conducted under constant σ_2 and σ_3 , which revealed the effect of σ_2 and σ_3 on rock failure. The other suite of tests, performed under constant Lode angle, determined the failure dependencies on mean stress and Lode angle. Building and modifying failure descriptions by Matsuoka-Nakai (MN) and Lade-Duncan (LD), an explicit failure criterion (modified-MNLD) that generalizes a class of failure forms is used to fit the novel loading path data of both sandstones. The determined failure descriptions fit both rocks well over a wide range of failure conditions. Then, the generated failure descriptions were further used to predict failure conditions under the constant σ_2 and σ_3 loading path. The predictions by the modified-MNLD are generally consistent with these data. The predictions also successfully replicated the variation of $\sigma_{1,peak}$ with σ_2 under constant σ_3 , suggesting the potential of the modified-MNLD to clarify the effect of σ_2 on rock failure.

1. Introduction

The field of rock mechanics is strongly concerned with the failure of rocks. A prerequisite for these analyses requires an adequate description of failure. Among others, the *Coulomb* failure theory (including its derivative, the *Mohr-Coulomb* failure criterion) is the most widely used because of its simplicity. It assumes that the major principal stress σ_1 at failure (or $\sigma_{1,peak}$) is a linear function of only σ_3 , an assumption that is consistent with numerous conventional triaxial compression experiments ($\sigma_1 > \sigma_2 = \sigma_3$) in different rocks.

However, the stress condition of $\sigma_1 > \sigma_2 = \sigma_3$ is not representative of the state of stress in the Earth's crust. Numerous in situ stress measurements have indicated that the crustal stress conditions are fully three-dimensional (where $\sigma_1 \geq \sigma_2 \geq \sigma_3$) [1-4]. Even when the crustal stress state might be approximated as $\sigma_2 \approx \sigma_3$, the stress state near perturbing structures, such as boreholes, tunnels and faults, will be fully three-dimensional. True triaxial experiments replicating realistic crustal stress conditions have revealed that σ_2 has an indisputable effect on the stress state and nearly all aspects of failure [5-7]. For this reason, Coulomb theory and other two-dimensional failure descriptions, which neglect the effect of σ_2 , are not suitable for describing rock failure in most situations. Adequate three-dimensional failure theories that incorporate three generally unequal principal stresses are needed to describe true triaxial failure data. Theoretical failure descriptions by Nadai, Mogi, Lade-Duncan and Matsuoka-Nakai are among the better known. These theories were found to suit different rock types and stress regimes to varying degrees. However, some of them are not suitable to model soft rocks, the failure of which evolve from shear-induced dilational failure to compactive failure mode as the mean stress increases [7-9].

In this study, we summarize recent true triaxial test results in two porous sandstones tested under a common true triaxial loading path (constant σ_2 and σ_3) [9] and a novel true triaxial loading path (constant Lode angle and σ_3) [10]. A recent failure criterion suggested by Rudnicki [11], modified from Lade-Duncan and Matsuoka-Nakai criteria, was employed to fit the novel test data of the two sandstones. The generated failure descriptions were further used to predict failure conditions of these rocks under true triaxial loading paths.

2. True triaxial testing using common and novel loading paths

Extensive suites of true triaxial experiments in two porous sandstones (Coconino, 17.5% porosity; Bentheim, 24% porosity) were recently conducted using a common and a novel loading path [9,10,12,13]. The common loading path tests were carried out to determine the effect of σ_2 on failure stress conditions, by maintaining σ_2 and σ_3 at preset magnitudes while raising σ_1 monotonically until failure occurred (Fig. 1). Ma and Haimson [9] demonstrated that failure of both tested sandstones is not only a function of σ_3 , but also of σ_2 (Fig. 2). For a given σ_3 , σ_1 at failure ($\sigma_{1,\text{peak}}$) increases as σ_2 is raised above σ_3 between tests. The maximum stress $\sigma_{1,\text{peak}}$ reaches a maximum when σ_2 is about halfway between σ_3 and σ_1 , and then gradually descends such that when $\sigma_2 \approx \sigma_{1,\text{peak}}$, it approaches its initial magnitude when $\sigma_2 = \sigma_3$.

In the common loading path tests, the three principal stress invariants (the octahedral shear and normal stresses, τ_{oct} and σ_{oct} , and the Lode angle Θ) constantly change during loading (where $\tau_{\text{oct}} = 1/3[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$, and $\sigma_{\text{oct}} = (\sigma_1 + \sigma_2 + \sigma_3)/3$). Here we follow Ma et al. [10] and define the Lode angle Θ as

$$\Theta = \tan^{-1}\{(\sigma_1 - 2\sigma_2 + \sigma_3) / [\sqrt{3} \cdot (\sigma_1 - \sigma_3)]\} \quad (1)$$

The stress invariants are considered to be closely associated with the constitutive behavior of rocks [7], and are more convenient than principal stresses when used in three-dimensional numerical calculations. For that reason, a novel loading path was designed, which maintains a fixed Θ during the test, by keeping the least principal stress (σ_3) constant and increasing incrementally the other two principal differential stresses ($\Delta\sigma_2 [= \sigma_2 - \sigma_3]$ and $\Delta\sigma_1 [= \sigma_1 - \sigma_3]$) at a fixed ratio [9,10,14] (Fig. 1). By keeping Θ constant, the novel loading path allowed the separation of the effect of σ_{oct} from that of Θ on rock failure. As shown in Fig. 3, the compressive failure (in terms of the octahedral shear stress at failure, τ_{oct}) in both Coconino and Bentheim sandstones are dependent on the mean stress at failure σ_{oct} and the Lode angle Θ . Specifically, the octahedral shear stress τ_{oct} required for failure increases with σ_{oct} but decreases with Θ when shear failure mode dominates. This is common to both sandstones. However, the dependency of failure on σ_{oct} and Θ is reversed when compactive failure mode occurs. Compactive failure mode was evident in Bentheim sandstone under high σ_{oct} , when compaction bands were observed. The Coconino sandstone did not enter the compactive failure regime within the range of stresses applied.

In both suites of tests, the applied constant σ_3 magnitudes were between 0 and 150 MPa (0, 10, 20, 50, 80, 100, 120, and 150 MPa for Coconino sandstone, and 0, 8, 15, 30, 60, 80, 120,

and 150 MPa for Bentheim sandstone. In the common loading path tests, the applied σ_2 magnitudes covered the entire range between the σ_3 and $\sigma_{1,\text{peak}}$ [9]. In the novel loading path tests of both sandstones, the applied constant Lode angle Θ values for each constant σ_3 were between $+30^\circ$ and -30° ($+30^\circ$, $+21^\circ$, $+11^\circ$, 0° , and -30° for Coconino sandstone, and $+30^\circ$, $+21^\circ$, $+11^\circ$, 0° , -16° , and -30° for Bentheim sandstone) [10].

2. True triaxial failure data in deviatoric and meridian planes

The stress conditions at failure of both sandstones can be analyzed in a coordinate system with the principal stresses as axes (σ_1 , σ_2 , σ_3). It is, however, more intuitively appealing to consider a cylindrical coordinate system with the axis of the cylinder aligned with the hydrostatic axis ($\sigma_1 = \sigma_2 = \sigma_3$). Distance along this axis is given by σ_{oct} . In this coordinate system, failure conditions can be expressed as

$$F(\tau_{\text{oct}}, \sigma_{\text{oct}}, \Theta) = 0 \quad (2)$$

The effect of σ_{oct} on the failure of both rocks has been studied with respect to rock strength, failure plane angle, and the brittle-ductile transition [8-10].

Planes perpendicular to the hydrostatic axis are deviatoric planes. The deviatoric planes are typically used to visualize the dependency of τ_{oct} on Θ since σ_{oct} is the same for all points in a given deviatoric plane. Radial distance in this plane is given by τ_{oct} . Angular position in this plane is given by the Lode angle Θ (Eq. (1)) [7].

The Lode angle is defined here so that it is zero for deviatoric pure shear ($\sigma_2 = (\sigma_1 + \sigma_3)/2$), $+30^\circ$ for axisymmetric compression ($\sigma_2 = \sigma_3$), and -30° for axisymmetric extension ($\sigma_2 = \sigma_1$). For an isotropic material, a 60° sector defines the shape of the failure surface in the entire deviatoric plane: the remaining five sectors are given by symmetry and reflection. In this way, the experimental data of both sandstones are depicted in the deviatoric planes with closed shapes. For the Coconino sandstone (Fig. 4) the shape of the failure surfaces in the deviatoric plane gradually became more circular with increasing σ_{oct} , signifying a diminishing effect of failure on the Lode angle. For the Bentheim sandstone (Fig. 5), the shape of the failure surfaces become more circular with increasing σ_{oct} until the compactive region is reached ($\sigma_{\text{oct}} > 200$ MPa), then it becomes more triangular.

Figure 6 shows a typical failure surface shape of a few common failure criteria in the deviatoric plane (τ_{oct} vs. Θ , at constant σ_{oct}) and the meridian plane (τ_{oct} vs. σ_{oct} , at constant Θ)

[15]. In the meridian plane, an examination of such failure surface shapes reveals the inadequacy of Mohr-Coulomb criterion and Drucker-Prager criterion. The linear τ_{oct} vs. σ_{oct} relationship of these two failure descriptions is not applicable to porous rocks, which in fact show a consistently declining slope with increasing σ_{oct} for any given constant Θ [8,10]. For high-porosity sandstone, the variation of τ_{oct} with σ_{oct} eventually shows a ‘cap’ on which τ_{oct} decreases for high σ_{oct} [16]. In the deviatoric plane, the circular shape of the Drucker-Prager criterion suggests no dependence of failure on Lode angle, contrary to our experimental observations [10] as well as others [7]. In the deviatoric plane, the Mogi [17] failure description has unfavorable re-entrant corners repeated every $2\pi/3$ period along the failure surface. These cause difficulty in theoretical interpretation and numerical implementation. The shape of the Lade-Duncan and Matsuoka-Nakai criteria on the deviatoric plane resembles a round-cornered triangle. The Lade-Duncan criterion differs from Matsuoka-Nakai in that it predicts a higher strength at $\sigma_2 = \sigma_1$, so that its shape on the deviatoric plane is more rounded in general.

However, the experimental observations show that failure surface shape is not fixed; in fact the shape evolves along successive deviatoric planes, i.e., it changes with increasing mean stress (Figure 4). Such evolution was also reported [7] in a 26% porosity Castlegate sandstone. In view of this, it requires that the failure criterion should not only reflect the Lode angle dependence on the deviatoric plane, but also accommodate shape change with a wide range of mean stress.

3. Modified Matsuoka-Nakai-Lade-Duncan (MNLD) failure description

According to [18,19], failure criteria proposed by Matsuoka and Nakai [20] and Lade and Duncan [21] can be expressed as:

$$\text{Matsuoka- Nakai:} \quad I_1 I_2 / I_3 = k_{MN} \quad (3a)$$

$$\text{Lade-Duncan:} \quad I_1^3 / I_3 = k_{LD} \quad (3b)$$

where k_{MN} and k_{LD} (≥ 27) are criterion-specific material parameters. The terms I_1 , I_2 , I_3 are principal stress invariants and they are also related to τ_{oct} , σ_{oct} , and Θ (for details see [22, pp. 38-41]):

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_{\text{oct}} \quad (4a)$$

$$I_2 = -(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3) = 3/2 \tau_{oct}^2 - 3\sigma_{oct}^2 \quad (4b)$$

$$I_3 = \sigma_1 \sigma_2 \sigma_3 = -(1/2^{0.5}) \tau_{oct}^3 \sin\Theta - (3/2) \sigma_{oct} \tau_{oct}^2 + \sigma_{oct}^3 \quad (4c)$$

Here we adopt a failure function $F(\tau_{oct}, \sigma_{oct}, \Theta) = 0$ in the form

$$F(t_{oct}, S_{oct}, Q) = -\sqrt{\frac{4}{27}} \cdot A \cdot \sin(3Q) \cdot \left(\frac{t_{oct}}{t_{oct,0}}\right)^3 + \left(\frac{t_{oct}}{t_{oct,0}}\right)^2 - 1 = 0 \quad (5)$$

which was introduced in [11], and was modified from the Matsuoka and Nakai [20] and Lade and Duncan [21] criteria. Equation (5) (or ‘modified-MNLD’) had been previously employed as a yield function for predicting the failure-plane angle [10,23].

In Eq. (5), A and $\tau_{oct,0}$ are parameters that can depend on σ_{oct} . For deviatoric pure shear ($\Theta = 0$) the first term in Eq. (5) vanishes and Eq. (5) becomes:

$$\tau_{oct} = \tau_{oct,0}(\sigma_{oct}) \quad (6)$$

Thus, $\tau_{oct,0}$ describes the mean stress dependence of τ_{oct} in deviatoric pure shear. The parameter A controls the shape of the failure surface in the deviatoric plane (Fig. 7). In particular, it sets the difference in the values of τ_{oct} for axisymmetric compression and axisymmetric extension (at the same mean stress). For $A = 0$, Eq. (5) reduces to Eq. (6) and the shape is simply a circle. For $A = 1$, the shape is triangle, as for a Rankine material for which failure occurs at a critical value of the least compressive stress σ_3 . For $A = -1$ the shape is also a triangle but inverted from that for $A = 1$. For A between -1 and 1 , the roots of Eq. (5) (for $\tau_{oct}/\tau_{oct,0}$) are real and the relevant root is that closest to one. For A outside this range, two of the roots are complex conjugates and the failure surface is non-convex.

For particular choices of A and $\tau_{oct,0}$, Eq. (5) reduces to the forms for Matsuoka and Nakai [20] or Lade and Duncan [21] (for details see [11]). Choosing

$$t_{oct,0} = \frac{S_{oct}}{3} \sqrt{\frac{k_{LD} - 27}{k_{LD}}} \quad (7a)$$

$$A = \sqrt{\frac{k_{LD} - 27}{k_{LD}}} \quad (7b)$$

reduces Eq. (5) to the formulation by Lade and Duncan [21] and choosing

$$t_{oct,0} = \frac{S_{oct}}{3} \sqrt{\frac{k_{MN} - 9}{k_{MN} - 3}} \quad (8a)$$

$$A = \sqrt{\frac{3k_{MN}^2 (k_{MN} - 9)}{(k_{MN} - 3)^3}} \quad (8b)$$

reduces Eq. (5) to the Matsuoka and Nakai [20] formulation. As shown by Eq. (7) and (8), Matsuoka and Nakai (MN) and Lade and Duncan (LD) assume constant A and linear dependency of $\tau_{oct,0}$ on mean stress. Therefore, MN and LD dictate fixed shape on the deviatoric plane and meridian plane with no evolution with mean stress (although they could be modified to accommodate non-linear σ_{oct} -dependency [18,19]). By comparison, Eq. (5) is more flexible in that A and $\tau_{oct,0}$ can be chosen to fit experimental data. This is the procedure that we adhere to in succeeding sections.

We follow [11] in referring to Eq. (5) as a modified Matsuoka-Nakai & Lade-Duncan (hereafter modified-MNLD) criterion. An equation of this form was used in [23] as a yield condition for predicting the failure angle in a siltstone, but here we use it as a condition for the stress state at failure.

4. The application of modified-MNLD to Coconino and Bentheim sandstones

The novel loading path true triaxial test data obtained by Ma et al. [10] was used to determine the expressions of $\tau_{oct,0}$, and A as functions of σ_{oct} . The specific form of $\tau_{oct,0} = \tau_{oct,0}(\sigma_{oct})$ was simply derived from the variation of τ_{oct} with σ_{oct} under deviatoric pure shear ($\Theta = 0^\circ$). Once $\tau_{oct,0}$ was determined A was the only unknown parameter. The values of parameter A corresponding to each failure datum (τ_{oct} , σ_{oct}) were calculated via Eq. (5), except for the pure shear test series (because when $\Theta = 0^\circ$, the first term that contains A vanishes). Since A is prescribed as a function of σ_{oct} , i.e., $A = A(\sigma_{oct})$, the variation of the calculated values of A with σ_{oct} was empirically fitted to yield the specific expression $A = A(\sigma_{oct})$. The obtained $A(\sigma_{oct})$ and $\tau_{oct,0}(\sigma_{oct})$ were substituted into Eq. (5), which was then deemed complete and ready for

application. A flow chart illustrating the failure prediction process for both common and novel loading path tests is provided in Fig. 8.

4.1.1 Coconino sandstone

In Coconino sandstone, the specific form of $\tau_{\text{oct},0}$ represents the variation of τ_{oct} with σ_{oct} under pure shear ($\Theta = 0^\circ$) [10]. The trend is well fitted by a second-order polynomial equation:

$$\tau_{\text{oct},0}(\sigma_{\text{oct}}) = 8.377 + 0.697 \sigma_{\text{oct}} - 0.00054 \sigma_{\text{oct}}^2 \quad (R^2 \approx 1) \quad (9)$$

Values of A were calculated by inserting the specific form of $\tau_{\text{oct},0}$ (Eq. (9)) into Eq. (5). Figure 9 shows the variation of calculated A with σ_{oct} . Each data point corresponds to a particular stress state at failure in the novel loading path tests. A steadily declines with the rise in σ_{oct} , though at low σ_{oct} levels an initial minor increase was noticeable for $\Theta = +30^\circ$ ($\sigma_2 = \sigma_3$). This overall decline of A occurs regardless of the respective Lode angle Θ . The extent of decrease in A with σ_{oct} appears to become more significant from $\Theta = +30^\circ$ to -30° . For a constant σ_{oct} , A slightly decreases by about 0.2 as Θ varies from $+30^\circ$ ($\sigma_2 = \sigma_3$) to -30° ($\sigma_2 = \sigma_1$), albeit with substantial dispersion. The mean value of A is 0.6. Nevertheless, a linear variation of A with σ_{oct} (Fig. 9) fits the trend of the data adequately:

$$A(\sigma_{\text{oct}}) = 1.019 - 0.002 \cdot \sigma_{\text{oct}} \quad (R^2 \approx 0.54) \quad (10)$$

However, extrapolation of the linear fit to low values of σ_{oct} , outside the range of the data, leads to values of $A > 1$. Although this feature could be eliminated by using a cut-off or a more elaborated fitting for low values of σ_{oct} , for simplicity we use Eq. (10). This does, however, lead to some unrealistic predictions for low values of σ_{oct} .

4.1.2 Bentheim sandstone

For Bentheim sandstone, $\tau_{\text{oct},0}(\sigma_{\text{oct}})$ was also derived from the deviatoric pure shear data series ($\Theta = 0^\circ$) in the novel loading path tests [10]. A second-order polynomial equation was fitted to the data set:

$$\tau_{\text{oct},0}(\sigma_{\text{oct}}) = -1.445 + 0.833 \sigma_{\text{oct}} - 0.002 \sigma_{\text{oct}}^2 \quad (R^2 \approx 0.86) \quad (11)$$

The calculated values of A as a function of σ_{oct} for each constant Θ are shown in Fig. 9. A typically declined with the rise in σ_{oct} , regardless of the specific Lode angle; between constant Lode angles, for a constant σ_{oct} , A appears to decrease as Θ evolves from $+30^\circ$ ($\sigma_2 = \sigma_3$) to $\Theta = -$

30° ($\sigma_2 = \sigma_1$). This trend is similar to that found in Coconino sandstone. The mean value of A is 0.33, and a linear equation can be fit coarsely to A (σ_{oct}):

$$A(\sigma_{\text{oct}}) = 1.245 - 0.006\sigma_{\text{oct}} \quad (R^2 \approx 0.41) \quad (12)$$

Again, extrapolation to low values of σ_{oct} leads to $A > 1$ and some unrealistic predictions, but as this lies outside the range of the data, we simply use the linear form.

4.2 Application to the novel loading path data

4.2.1 Coconino sandstone

Given the input values of Θ and σ_3 , the prediction of $\sigma_{1,\text{peak}}$ returns the theoretical values when the condition prescribed by Eq. (5) is satisfied. The predicted $\sigma_{1,\text{peak}}$ values are shown in Fig. 10 for each constant Lode angle. Predicted $\sigma_{1,\text{peak}}$ underestimates slightly the experimental magnitudes at $\sigma_2 = \sigma_3$ ($\Theta = +30^\circ$) and $\sigma_2 = \sigma_1$ ($\Theta = -30^\circ$). For $\sigma_2 = \sigma_3$, the prediction consistently generates lower $\sigma_{1,\text{peak}}$ than the experimental value by about 50 MPa, except for $\sigma_3 = 0$ (unconfined compression state). The rise of predicted $\sigma_{1,\text{peak}}$ with σ_3 at $\sigma_2 = \sigma_3$ is less pronounced for $\sigma_3 \leq 50$ MPa as compared to the experimental data. At $\sigma_2 = \sigma_1$, the predicted $\sigma_{1,\text{peak}}$ is only slightly less than the experimental value, by no more than 20 MPa (Fig. 10). Again, the only exception is at $\sigma_3 = 0$ (biaxial compression state), where the experimental $\sigma_{1,\text{peak}}$ exceeds the predicted value by 20 MPa.

At intermediate Lode angles ($\Theta = +21^\circ$, $+11^\circ$, and 0°), the predictions of $\sigma_{1,\text{peak}}$ agree well with experiments (Fig. 10). The best match is found at deviatoric pure shear ($\Theta = 0^\circ$). This is obvious since pure shear test data were used to derive the relationship $\tau_{\text{oct},0} = \tau_{\text{oct},0}(\sigma_{\text{oct}})$.

The deviation of the predicted $\sigma_{1,\text{peak}}$ from the experimental data given the same (Θ , σ_3) induces changes in both τ_{oct} and σ_{oct} , which are not maintained constant under the novel loading path. As a result, a point-to-point comparison in the τ_{oct} vs. σ_{oct} domain is obscure and therefore not pursued. Instead, a continuous variation of τ_{oct} with σ_{oct} is predicted and compared against the experimental data (Fig. 3). The predicted τ_{oct} increases monotonically with the rise in $\sigma_{\text{oct},f}$, in agreement with the experimental results. Similar to the comparison in the $\sigma_{1,\text{peak}} - \sigma_3$ domain (Fig. 10), the best correlation is obtained under pure shear ($\Theta = 0^\circ$), which is naturally the case since the prediction are based on $\Theta = 0^\circ$ data series. The difference between the experimental results and the theoretical prediction at other Lode angles increased as the deviatoric stress state approached the axisymmetric compression ($\Theta = +30^\circ$, i.e., $\sigma_2 = \sigma_3$).

The effect of Lode angle on failure is also illustrated in the deviatoric plane ($\tau_{\text{oct}} - \Theta$ domain with constant σ_{oct}). Deviatoric planes of various constant σ_{oct} ($= 50, 150, 250, 350, 450$ MPa) were selected, as was shown in the novel test analysis. In each deviatoric plane, the predicted τ_{oct} covers the deviatoric stress states from $\sigma_2 = \sigma_3$ to $\sigma_2 = \sigma_1$ (corresponding to Lode angle in the range $-30^\circ \leq \Theta \leq +30^\circ$), and was further extended to cover $-180^\circ \leq \Theta \leq +180^\circ$ via symmetry. This results in a continuous, closed failure envelope in the deviatoric plane (Fig. 4). Because the novel loading path did not maintain a constant σ_{oct} throughout the test, the experimental data could not be directly compared with the theoretical predictions for the same σ_{oct} (i.e., on the same deviatoric plane). To address this, based on the τ_{oct} vs. σ_{oct} fitting curves to the novel loading path test results [10], the values of τ_{oct} were extrapolated at $\sigma_{\text{oct}} = 50, 150, 250, 350, 450$ MPa. The extrapolated τ_{oct} are compared with the predicted failure envelope on those deviatoric planes. The predicted failure envelopes agree well with the extrapolated results. However, a noticeable discrepancy at $\sigma_2 = \sigma_3$ was found in the deviatoric plane, in that τ_{oct} was overestimated at $\sigma_{\text{oct},f} = 150$ and 250 MPa, and underestimated at $\sigma_{\text{oct}} = 450$ MPa.

The predictions complement the experiments by demonstrating an unambiguous effect of σ_{oct} on the variations of τ_{oct} with Θ . This effect is best illustrated by projecting these predicted failure envelopes in one single deviatoric plane. Shown in Fig. 11, the failure envelope confined area consistently grows as σ_{oct} is raised, though at a decreasing rate. This rate change is consistent with the declining slope of the τ_{oct} vs. σ_{oct} trend (Fig. 3). The shape of the failure envelopes becomes increasingly circular, signifying a diminishing effect of the Lode angle on τ_{oct} .

4.2.2 Benthem sandstone

The prediction procedure of true triaxial failure under the novel loading path in Benthem sandstone was identical to that followed in Coconino sandstone. For the test series of constant Θ , σ_3 was varied from test to test, and the experimental $\sigma_{1,\text{peak}}$ was compared with the theoretical prediction at the same σ_3 level. At both axisymmetric stress conditions ($\Theta = \pm 30^\circ$), the prediction yields results that closely resemble the experimental ones (Fig. 10), with the experimental $\sigma_{1,\text{peak}}$ being greater than the predicted by no more than 25 MPa for σ_3 between 30 and 100 MPa. The predictions at $\sigma_3 = 0$ MPa for $\sigma_2 = \sigma_3$ and $\sigma_2 = \sigma_1$ and, yield unrealistic solutions and are not shown. As noted earlier, this occurs because linear extrapolation of Eq. (12) causes the values of

A for $\sigma_3 = 0$ MPa to exceed +1. Consequently, stress conditions at failure for $\sigma_3 = 0$ MPa series were not pursued in the subsequent analysis.

The predicted stress conditions at failure were further analyzed in the τ_{oct} vs. σ_{oct} domain, for each constant Lode angle. A continuous variation of the predicted τ_{oct} with σ_{oct} is shown in Fig. 3. Generally, the predicted τ_{oct} vs. σ_{oct} trends follow the experimental data in that the rise in $\sigma_{\text{oct},f}$ brings about the increase of τ_{oct} at first, reaches a peak at some level of σ_{oct} , and then declines for larger σ_{oct} . The exception is that the prediction yields abnormal results for $\sigma_2 = \sigma_3$ series when $\sigma_{\text{oct}} \leq 50$ MPa (the unusual whip shown in Fig. 3). Again, this occurs because extrapolation of parameter A using Eq. (12) for low σ_{oct} values exceeds the limit of validity of parameter A. This unrealistic prediction could be improved given a better-constrained dependency on σ_{oct} .

In contrast to the results for the Coconino sandstone, for which τ_{oct} rises continuously with σ_{oct} , the results for the Bentheim sandstone shown in Fig. 3 indicate a region of decreasing τ_{oct} with σ_{oct} . Although based on relatively few data points, this suggests the emergence of a ‘cap’ that would be more prominent with larger values of σ_{oct} . The ‘cap’ regime suggests the failure mode is primarily by compaction, in which the high confinement induces high stress concentration to break the sandstone grains and high shear stress is no longer critical. As it implies, a limiting case is that sufficiently high confinement may cause failure in a porous sandstone without any applied shear stress, i.e., hydrostatic stress condition. A complete ‘cap’ under conventional triaxial condition has been mapped in several porous sandstones with porosities comparable to that of Bentheim sandstone [8], and the emergence of ‘cap’ in the Bentheim sandstone is expected at the applied stress level.

A clear variation of the predicted τ_{oct} vs. σ_{oct} trend with Lode angle Θ is observed (Fig. 3). For a constant τ_{oct} , the σ_{oct} required for failure unequivocally increases when Θ varies from $+30^\circ$ to -30° . This trend is in qualitative agreement with the experimental results [10], though the experimental trend is obscured by significant dispersion. In Bentheim sandstone, the critical σ_{oct} level where τ_{oct} reaches its maximum, rises consistently as the corresponding constant Θ approaches -30° ($\sigma_2 = \sigma_1$), suggesting an increased σ_{oct} magnitude at which the cap was developed.

It is interesting to note that in Fig. 3, when $\sigma_{\text{oct}} \approx 200$ MPa, all predicted curves converge to $\tau_{\text{oct}} \approx 90$ MPa, regardless of the specific Lode angle. The convergence indicates that the Lode

angle has no influence on failure (in terms of τ_{oct}) at $\sigma_{\text{oct}} \approx 200$ MPa. This phenomenon is also illustrated in the deviatoric plane ($\tau_{\text{oct}} - \Theta$). As shown in Fig. 5, the failure surface at $\sigma_{\text{oct}} = 200$ MPa takes a circular shape, indicating no variation τ_{oct} on Θ ; this trend could be best described by the Drucker-Prager failure criterion [24]. For constant $\sigma_{\text{oct}} < 200$ MPa, τ_{oct} at failure when $\sigma_2 = \sigma_3$ is greater than that at $\sigma_2 = \sigma_1$, so that the failure surface shape protrudes at $\Theta = +30^\circ, +150^\circ$, and -90° , resembling a round-cornered triangle. The triangular shape could be adequately described by the Matsuoka and Nakai [20] or Lade and Duncan [21] criterion. As σ_{oct} approaches 200 MPa, the dependence of τ_{oct} on Θ diminishes; thus the failure surface shape gradually becomes equi-dimensional. For $\sigma_{\text{oct}} > 200$ MPa, the τ_{oct} vs. σ_{oct} curves diverge again but take a different trend as compared to that at $\sigma_{\text{oct}} < 200$ MPa: the shape of the failure surface in the deviatoric plane elongates at $\Theta = -150^\circ, -30^\circ$ ($\sigma_2 = \sigma_1$) and $+90^\circ$. Such variation of the failure surface shape in the deviatoric plane with increasing σ_{oct} is recaptured in Fig. 11, showing the predicted τ_{oct} only. The variation of the failure surface shape with σ_{oct} generally agrees with the novel loading path experimental results. An apparent discrepancy is found at $\sigma_{\text{oct}} = 300$ MPa and Θ between pure shear and axisymmetric compression. Because the applied σ_{oct} in the experiments at these Lode angles only reaches a maximum of 250 MPa, the extrapolated τ_{oct} at $\sigma_{\text{oct}} = 300$ MPa requires verification with additional experimental data.

4.3 Application to the common loading path data

The application of the modified-MNLD failure description to the novel loading path test data is essentially data fitting rather than prediction. Because the modified-MNLD fits Coconino and Bentheim sandstones data under the novel loading path, we test its applicability in both sandstones to results under the common loading path.

4.3.1 Coconino sandstone

Under the common loading path, $\sigma_{1,\text{peak}}$ was calculated using Eq. (5) applied to constant σ_2 and σ_3 conditions [9]. For each constant σ_3 series, $\sigma_{1,\text{peak}}$ was calculated by increasing the variable σ_2 in small increments from its lower limit ($= \sigma_3$) to the upper end ($= \sigma_1$). The predicted continuous $\sigma_{1,\text{peak}}$, and the experimental data points were then compared (Fig. 2). The failure prediction in Coconino sandstone generally replicates the variation of $\sigma_{1,\text{peak}}$ with σ_2 observed experimentally. For constant σ_3 , the increase of σ_2 first brings about the rise in $\sigma_{1,\text{peak}}$, which

reaches its maximum at an intermediate level of σ_2 , then consistently declines for larger σ_2 . $\sigma_{1,\text{peak}}$ for a constant σ_3 level (i.e., $\sigma_{1,\text{peak}}^{\text{MAX}}$) is compared with its base value at $\sigma_2 = \sigma_3$.

4.3.2 Bentheim sandstone

The prediction in Bentheim sandstone for given constant σ_2 and σ_3 conditions simulates its common loading path tests [9]. The results are shown in the $\sigma_{1,\text{peak}} - \sigma_2$ domain, and compared with the experimental data for each constant σ_3 (Fig. 2). The failure predictions in Bentheim sandstone generally agree with the experimental observation, in which $\sigma_{1,\text{peak}}$ is clearly a function of σ_2 . For a constant σ_3 , $\sigma_{1,\text{peak}}$ first rises with σ_2 , reaching its maximum at some point, then declines for larger σ_2 until $\sigma_2 = \sigma_{1,\text{peak}}$. The correlation between the experimental results and the predictions is moderate-to-strong overall although the predictions at low σ_3 level (corresponding to $\sigma_{\text{oct}} \leq 50$ MPa) are poor. In particular, the prediction overestimates $\sigma_{1,\text{peak}}$ at $\sigma_3 = \sigma_2 = 8$ MPa by ~30%, causing a monotonic decrease of $\sigma_{1,\text{peak}}$ with σ_2 . As acknowledged earlier, predictions are poor for the $\sigma_3 = 0$ series because of the oversimplified fitting of A at low values of σ_{oct} and this also affects other predictions for low σ_3 .

5. Discussion

The good agreement between the theoretical description and the experimental results in both sandstones subjected to both loading paths demonstrated the applicability of the modified-MNLD [11] to model failure through the range of compressive stress magnitudes applied (σ_3 between 0 and 150 MPa). Its superior performance over common three-dimensional failure descriptions such as Nadai and Mogi descriptions is due to the flexibility of the Eq. (5) in describing failure in terms of the variation of τ_{oct} with Θ for constant σ_{oct} , and the variation of τ_{oct} with σ_{oct} for constant Θ . Through the explicit use of parameter A as a function of σ_{oct} , the evolution of failure surface shape in the deviatoric plane ($\tau_{\text{oct}} - \Theta$ domain) with σ_{oct} is included. This is particularly advantageous in modeling those rocks in which failure is highly sensitive to the compressive stress magnitudes. In contrast, many other failure expressions have a rigid mathematical form, and therefore yield a failure surface with little or no variation in shape. In this sense, the modified-MNLD failure theory represents a class of failure descriptions. There are other failure descriptions with similar flexibility such as Willam and Warnke criterion [25]. For example, Descamps et al. [26] used the Willam and Warnke criterion [25] to model the failure

surface for a highly porous Sorey limestone (porosity $\approx 27.5\%$) and was in good agreement of their true triaxial test data.

Advanced failure descriptions such as those by Rudnicki [11] and Willam and Warnke [25] are straightforward to use in a general three-dimensional framework [18,19,27]. These descriptions share certain common features. First is to allow failure (τ_{oct}) variation between axisymmetric compression and extension states ($\sigma_2 = \sigma_3$ and $\sigma_2 = \sigma_1$) for constant σ_{oct} . The description by Nadai [28] offers no such variation. Second is differentiability to ensure the theoretical function $F(\tau_{\text{oct}}, \sigma_{\text{oct}}, \Theta) = 0$ varies smoothly and continuously across any Lode angle. The final feature is that the failure surface in deviatoric planes should be convex [29]. The description by Mogi [17] has problems of both differentiability and non-convexity at those cusps.

Rudnicki's modified-MNLD failure description [11] also qualitatively replicates the variation of $\sigma_{1,\text{peak}}$ with σ_2 for constant σ_3 as observed in common loading path tests in Coconino and Bentheim sandstones [9]. This is a significant extension of the applicability of Eq. (5) since the failure description is independently constrained by the novel loading path test data [10]. It shows that the failure (in terms of τ_{oct}) dependence on Θ and σ_{oct} is interconnected with the dependence of failure (in terms of $\sigma_{1,\text{peak}}$) on σ_2 . Furthermore, it implies that the evolving σ_2 effect with increasing σ_3 is a result of the characteristic dependencies of τ_{oct} on Θ and σ_{oct} .

In some failure theories, the σ_2 - dependency of $\sigma_{1,\text{peak}}$ evolves negligibly with σ_3 . This could be attributed to the absence of curvature change in τ_{oct} vs. σ_{oct} curve. For example, the failure expression [24] $\tau_{\text{oct}} = A + B\sigma_{\text{oct}}$ yields no apparent change in the $\sigma_{1,\text{peak}}$ vs. σ_2 trend with σ_3 level because the three-dimensional friction coefficient B is assumed constant. Nor does the true triaxial failure criterion of Wiebols and Cook [30] impose any apparent variation of $\sigma_{1,\text{peak}}$ vs. σ_2 trend with σ_3 . This is considered to be a direct result of their basic assumption of a constant effective friction coefficient (associated with but not equal to the friction coefficient $B [= d\tau_{\text{oct}}/d\sigma_{\text{oct}}]$). Therefore both failure expressions are only applicable to specific rocks within a certain stress level. In this sense, the prediction using the modified-MNLD form [11] more realistically describes the stress conditions at failure of the two porous sandstones, since both were subjected to a wide range of stresses (σ_3 between 0 and 150 MPa).

In the prediction process, we restricted the parameter A to be a function of only σ_{oct} . The trend that A declines monotonically is generally the case revealed by experimental data. However, for a constant σ_{oct} , a slight decrease of A as Θ varies from $+30^\circ$ to -30° is noticeable

for both sandstones. The dependency of A on Θ is not included for simplicity. It is unclear whether the discrepancy between the experimental data and predictions is in fact due to our removal of the Θ -dependency, which is to be warranted in future study. Including dependence of A on Θ would change the basic shape of the surface in the deviatoric plane (Fig. 7). Any addition of this dependence would need to maintain the symmetry imposed by isotropy.

As acknowledged earlier, unsatisfactory predictions were found at high and low values of σ_{oct} where a linear-fitted A exceeds its range of validity $[-1, 1]$. A more elaborated form of parameter A could remedy this. For example, an ad hoc cut-off could be applied to restrict parameter A between -1 and $+1$. A more sophisticated representation of parameter A with theoretical meaning is expected without compromising the simplicity of Eq. (5).

6. Concluding Remarks

In this study, we reviewed true triaxial failure results in two porous sandstones using a common loading path [9] and a novel loading path [10]. An explicit failure description [11] that generalizes a class of failure forms, including Matsuoka-Nakai and Lade-Duncan, was employed to fit the novel loading path (constant Lode angle) data of two sandstones. This modified-MNLD criterion generated failure descriptions for both sandstones and proved adequate for them over the wide range of applied stress conditions. The generated failure descriptions were further used to predict failure conditions of both rocks under the common loading path (constant σ_2 on σ_3). The prediction was satisfactorily applied to the common test data [9] and successfully replicated the effect of σ_2 on $\sigma_{1,\text{peak}}$, including the typical ascending-then-descending $\sigma_{1,\text{peak}}$ under constant σ_3 , and an evolving σ_2 effect with the magnitude of σ_3 . Therefore, we recommend the application of the modified-MNLD failure description to porous sandstones over a wide range of stress magnitudes.

Acknowledgement

This research was supported by the National Science Foundation Grant EAR-0940323 (for BCH and XM) and Grant EAR-0940981 (for JWR).

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- 516

List of figures

Figure 1. Two distinct loading paths of true triaxial testing: (a) common loading path (σ_2 and σ_3 constant, σ_1 increasing monotonically), and (b) novel loading path (σ_3 constant, σ_2 and σ_1 rising while maintaining constant stress ratio $(\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3)$).

Figure 2. Variations of $\sigma_{1,peak}$ with σ_2 in Coconino and Bentheim sandstones tested under the common loading path for all constant σ_3 levels: experimental data in solid dots and predictions using the modified-MNLD failure criterion in dashed curves.

Figure 3. Variations of the octahedral shear stress at failure (τ_{oct}) with the octahedral normal stress at failure (σ_{oct}) in Coconino and Bentheim sandstones: experimental data in solid dots and the predictions using the modified-MNLD failure criterion in solid curves for each constant Lode angle.

Figure 4. Comparison of stress conditions at failure in the deviatoric planes ($\tau_{oct} - \theta$ domain) between the empirically-fitted novel loading path test data (solid dots) and the predicted failure surface (solid curve) in Coconino sandstone for constant $\sigma_{oct} = 50, 150, 250, 350$, and 450 MPa.

Figure 5. Comparison of stress conditions at failure in the deviatoric planes ($\tau_{oct} - \theta$ domain) between the empirically-fitted novel loading path test data (solid dots) and the predicted failure surface (solid curve) in Bentheim sandstone for constant $\sigma_{oct} = 50, 100, 150, 200, 250$, and 300 MPa.

Figure 6. Typical failure envelope shapes in the deviatoric plane ($\tau_{oct} - \theta$ domain) and the meridian plane ($\tau_{oct} - \sigma_{oct}$ domain) of some common failure descriptions: (a) Mohr-Coulomb; (b) Mogi; (c) Drucker-Prager; (d) Lade-Duncan; (e) Matsuoka-Nakai. The irregular hexagon failure envelop of Mohr-Coulomb (in red) in the deviatoric plane was repeated (in dash) in (b) through (e) for comparison.

Figure 7. Variation of the failure envelope shapes with parameter A on the deviatoric plane. The envelopes were set to converge at pure shear in order to facilitate comparison. Note as A moves away from 0, the envelope becomes increasingly distorted into triangular shape.

Figure 8. Flowchart illustrating the procedure of failure prediction using the modified-MNLD failure criterion against common and novel loading path test data.

545 Figure 9. Variations of inferred A value with the octahedral normal stress at failure (σ_{oct}) for all
546 constant Lode angles (excluding $\Theta = 0^\circ$) in Coconino and Bentheim sandstones: the dashed line
547 represents the best linear regression of all data points.

548 Figure 10. Variations of $\sigma_{1,\text{peak}}$ with σ_3 for each constant Lode angle Θ under the novel loading
549 path in Coconino sandstone (left) and Bentheim sandstone (right). The predicted conditions are
550 open circles and the experimental results are solid dots.

551 Figure 11. Predicted evolution of failure surface shapes of the Coconino and Bentheim
552 sandstones on successive deviatoric planes ($\tau_{\text{oct}} - \Theta$ domain) projected on the π -plane.

Figure 1. Two distinct loading paths of true triaxial testing: (a) common loading path (σ_2 and σ_3 constant, σ_1 increasing monotonically), and (b) novel loading path (σ_3 constant, σ_2 and σ_1 rising while maintaining constant stress ratio $(\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3)$).

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