On the polarization relations of diurnal and semidiurnal tide in the mesopause region

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Abstract

The polarization relations for gravity waves are well known and have proven to be very powerful for the investigation of their dynamics. Though tides are sometimes regarded as low-frequency gravity waves, their polarization relations are hardly considered. We derive the polarization relationships of tides from the primitive equations for perturbations of a dissipationless atmosphere. The vertical wind tide is found to relate directly to temperature tide, independent of tidal structure and geometric location. On the other hand, the relation between meridional wind tide and zonal wind tide is found to depend on the horizontal derivatives of the associated geopotential perturbation, thus depending on tidal structure and geometric location. Lidar observed tides at a local station (tidal period perturbations) in the midlatitude mesopause region (80 - 105 km) and the Climatological Tidal Model of the Thermosphere (CTMT) based on Hough Mode Extension (HME) technique at the lidar site, both of which include the effect of dissipation, are then compared. These tidal amplitudes and phases are employed to discuss the results and implications of the derived tidal wave polarization relations. The dominance of the migrating tide follows from the phase relationship between zonal and meridional wind tides. By comparing the observed and derived vertical wind tides, we reveal qualitatively the altitude dependence of tidal wave dissipation.

1. Introduction

The polarization relations for gravity waves (GWs) are well known [Yeh and Liu, 1974; Fritts and Alexander, 2003]. These relations have been very useful for studying gravity wave dynamics. Such studies are innumerable in the literature; a recent example is to relate perturbations of zonal wind u' and vertical wind w' to that of temperature T' , thereby determining the momentum flux, $\langle u' w' \rangle$ of gravity waves (under certain conditions) characterized by an Advanced Mesospheric Temperature Mapper [Pautet et al., 2014] which also measures T' in the horizontal plane [Fritts et al., 2014]. Tides are undoubtedly the strongest global atmospheric perturbations. Numerical models, like Global Scale Wave Model (GSWM) [Hagan et al., 1999a], and Climatological Tidal Model of the Thermosphere (CTMT) based on the Hough Mode Extension (HME) method [Svoboda et al., 2005, Oberheide and Forbes, 2008; Oberheide et al., 2011], to be called HME model in this paper for simplicity, have greatly advanced our understanding of realistic atmospheric tides. Like GWs, the associated dynamic fields, i.e., temperature and wind tidal perturbations are also related, yielding the polarization relations among tidal fields. These polarization relations can also be used to advance our understanding of tidal dynamics. For example, to deduce one unknown tidal field from another observed tidal field, or to reveal the nature of tidal dissipation.

We organize this paper by deriving the polarization relations in Section 3 from the primitive equations for large-scale wave perturbations, which are presented in Section 2. In Section 4 we present diurnal and semidiurnal tides in temperature, zonal wind and meridional wind as a function of geometric altitude between 80 and 105 km from both lidar observations and the HME model predictions [more precisely, the Climatological Tidal Model of the Thermosphere (CTMT) based on Hough Mode Extension (HME) technique], from which tidal geopotential and its horizontal

and vertical derivatives may be deduced. These results are then investigated in terms of the derived polarization relations to reveal the characteristics of the solar tides in question. Discussion and conclusions follow in Section 5.

2. Tidal perturbations and modeling in a Hydrostatic Spherical Atmosphere

It is well known that mid- and large scale motions are hydrostatic, and the vertical momentum equation can be replaced by an equation describing hydrostatic balance. The resulting governing equations are referred to as the primitive equations [Holton, 2004]. They take a simpler form if the geopotential, Φ is introduced and log-pressure is used as the vertical coordinate, $z = -H_s \ln(p/p_s)$, where p , p_s and H_s are pressure, and pressure and scale height at a reference altitude, respectively. Assuming zero mean wind (and zero horizontal temperature gradient), the primitive equations [Andrews et al., 1987] for large-scale wave perturbations (tides and planetary waves) may be written in terms of a set of linearly coupled equations that relate the perturbations in zonal wind, meridional wind, vertical wind, temperature and geopotential, u' , v' , w' , T' and Φ' , respectively, as

$$u'_t - f v' + (a \cos \phi)^{-1} \Phi'_\lambda = X' \quad (1a)$$

$$v'_t + f u' + a^{-1} \Phi'_\phi = Y' \quad (1b)$$

$$(a \cos \phi)^{-1} [u'_\lambda + (v' \cos \phi)_\phi] + \rho_0^{-1} (\rho_0 w')_z = 0 \quad (1c)$$

$$\Phi'_{zt} + N^2 w' = \frac{\kappa J'}{H_s}; \quad N^2 = \frac{R}{H_s} \left[\frac{\partial T_0}{\partial z} + \frac{\kappa T_0}{H_s} \right] \quad (1d)$$

$$\Phi'_z = \frac{\partial \Phi'}{\partial z} = \frac{R T'}{H_s} \quad (1e)$$

where $f = 2\Omega \sin \phi$, and $t, \phi, \lambda, \Omega, a, R, c_p, \kappa$ and N are, respectively, time, latitude, longitude, earth mean rotation rate ($7.29 \times 10^{-5} \text{ s}^{-1}$), mean earth radius ($6.371 \times 10^6 \text{ m}$), specific gas constant,

1 specific heat at constant pressure, $\kappa = R/c_p = 2/7$, and buoyancy frequency (in log-pressure
 2 coordinates). Equations (1a) – (1d) are identical to (4.2.1a – 4.2.1d), and (1e) to (3.1.3c) of
 3 Andrews et al. [1987]. The functions (X' , Y') represent dissipation (rate per unit mass) of zonal
 4 and meridional wind perturbations, and J' represents the forcing (diabatic heating rate per unit
 5 mass) of the temperature perturbation. Here subscripts indicate partial derivatives, except 0 and s,
 6 which, respectively, indicate background state and reference altitude. The background state here
 7 refers to the monthly averaged diurnal (24-hour) mean density, temperature, zonal and meridional
 8 winds. Thus, the perturbations from the background state consist of tides and planetary waves,
 9 although we present only diurnal and semidiurnal tides in this paper. Equations (1a) and (1b)
 10 express conservation of zonal and meridional momentum perturbations of a moving unit-mass air
 11 parcel. The vertical momentum equation for large-scale motions becomes the statement of
 12 hydrostatic equilibrium, which is expressed as a relation between geopotential and temperature
 13 perturbation in Eq. (1e). Equation (1c) is the continuity equation, with the background atmospheric
 14 density profile, $\rho_0(z) = \rho_s \exp(-z/H_s)$, and Eq. (1d) is the energy equation, or the 1st law of
 15 thermodynamics. Here, we restrict our discussions to tidal perturbations, so that the derived
 16 relations may be compared with lidar observed and HME modeled diurnal and semidiurnal tidal
 17 perturbations in zonal wind U' , meridional wind V' , and temperature T' , from which the
 18 horizontal gradients of geopotential perturbation, Φ_λ' , Φ_ϕ' , may be deduced from Eqs. (1a) and
 19 (1b), and the vertical gradient Φ_z' from (1e). We therefore derive the polarization relations, which
 20 relate various fields of tidal perturbations in question, U' , V' , W' , T' and Φ' , from these
 21 equations by assuming a harmonic time dependence (tidal period) and setting to zero the terms,
 22 X' , Y' and J' that attenuate and excite the perturbations in question. This is a fair assumption

for altitudes between 80 and 105 km, where there are no tidal sources and the tidal waves are propagating freely.

Strictly speaking, the result of this derivation is valid only for atmospheric tides in an atmosphere with zero background wind, zero latitudinal temperature gradient, and no dissipation. Here, the background refers to full diurnal means. In practice, the lidar tides are deduced from hourly mean time series of a multiple year dataset within a given month of the year. The mean values of the time series give rise to monthly mean background temperature and winds. By removing the monthly means, the amplitude and phase of oscillations with 24-, 12-, 8-, and 6-hour periods (only 24- and 12-hour periods are presented in this paper) are deduced from the hourly mean time series. Ideally, using the multiple year hourly mean data within a given month, the shorter period gravity waves should be averaged out. Under these assumptions, a wave number S /frequency σ pair may be specified, and the solution to Eqs. (1a) - (1e) give rise to the classical theory of tides and planetary waves [Kato, 1996; Chapman and Lindzen, 1970] as briefly summarized in Appendix. The equations may then be combined into two eigen-equations with linked eigenvalues, the Laplace equation as a function of latitude and the vertical structure equation of altitude. The solutions to these equations yields eigenfunctions, respectively, as Hough function, $\Theta_n^{s\sigma}(\phi)$, and vertical structure function, $G_n^s(z)$ with eigenvalues specified by characteristic (angular) frequency, σ , zonal wave number, S and meridional index, n . The latter is an index that labels Hough functions having different latitude dependence for the same σ and S .

More generally, the periodic steady-state solution (i.e. amplitudes constant in time) to the linearized equations, (1a) to (1e) with background wind, latitudinal temperature gradient, and dissipation included, in a spherical stratified atmosphere exist as global oscillations (for example, tides and planetary waves) consisting of independent tidal components, each characterized by a

1 wave number/frequency (S/σ) pair. The associated fields, topped by a caret and with superscript
2 S specifying the zonal wave number, may be expressed as an ansatz in the form of Eq. (2a) below.
3 For situations, in which contributions from waves with different zonal wave numbers are not
4 separable such as in tidal-period oscillations observed at a local coordinate (ϕ, λ) , one is forced to
5 deal with a superposition of tidal components with different zonal wave numbers at a given
6 frequency (tidal period). In this case, only frequency σ may be specified, and we indicate the
7 perturbation fields with tilde over them, as an ansatz in the form of Eq. (2b).

$$(u', v', w', T') = \text{Re} \left\{ \left[\hat{u}^S(\phi, z), \hat{v}^S(\phi, z), \hat{w}^S(\phi, z), \hat{T}^S(\phi, z) \right] e^{i(s\lambda - \sigma t)} \right\} \quad (2a)$$

$$(u', v', w', T') = \text{Re} \left\{ \left[\tilde{u}(\phi, \lambda, z), \tilde{v}(\phi, \lambda, z), \tilde{w}(\phi, \lambda, z), \tilde{T}(\phi, \lambda, z) \right] e^{-i\sigma t} \right\} \quad (2b)$$

9 Here $\text{Re}\{\dots\}$ is the real part of $\{\dots\}$, and the field quantities topped with either caret or tilde are
10 complex quantities each with an amplitude and a phase. We note that in an atmosphere with zero
11 wind, zero longitudinal temperature gradient and no dissipation, the altitude and latitude
12 dependence in the caret topped fields given in (2a) are separable, and the tidal equations are
13 reduced to the classical tidal equations with two (Laplace and vertical structure) equations as
14 briefly treated in Appendix.

15 With the inclusion of background zonal wind, latitudinal temperature gradient and linear
16 dissipation, Eqs. (1a), (1b) and (1d) will consist of many more terms. Though these terms are still
17 linear to the fields, u', v', w', T' and Φ' with the proportionality coefficients depending only on
18 the background state (including dissipation), there exists no analytic solution [Forbes, 1982; Hagan
19 and Forbes, 1993]. However, the polarization relations that relate a pair of fields can still be solved
20 numerically. GSWM for example, provides a numerical periodic steady-state solution to the

linearized equations for oscillations in a spherical atmosphere with prescribed zonal mean wind, temperature and dissipation distributions. Depending on the altitude and latitude dependent forcing at a particular frequency, σ , and zonal wave number, S , the GSWM computes the height and latitude dependence of the amplitudes and phases of the global oscillation fields, e.g., $\hat{u}^s(\phi, z)$, $\hat{v}^s(\phi, z)$, $\hat{w}^s(\phi, z)$ and $\hat{T}^s(\phi, z)$.

The CTMT is a TIMED-based empirical tidal model that provides realistic diurnal and semidiurnal, migrating and non-migrating tides in temperature, neutral density, and zonal, meridional and vertical winds from the ground to 400 km altitude and pole-to-pole [Oberheide et al., 2011]. It is based on fitting Hough Mode extensions (HME) to observed SABER temperature tides and TIDI wind tides in the MLT region. Specifically, a HME with given (σ, S, n) is constructed from a simplified version of GSWM that includes a vertical temperature gradient and dissipation, but no background zonal wind. The heating term in GSWM is replaced by a heating term with a latitudinal structure identical to the associated Hough function, $\Theta_n^{s,\sigma}(\phi)$. The amplitude of this heating term is arbitrarily calibrated because HMEs, as described by Svoboda et al. [2005] and Oberheide and Forbes [2008], are fitted to SABER tidal temperature and TIDI tidal wind diagnostics across a range of altitudes and heights. This determines the magnitude of each tidal mode (described by the meridional index n) and the resulting tidal component (wavenumber/frequency pair) can be reconstructed as a linear superposition. In practice, it is sufficient to use only the two lowest meridional indices (symmetric and anti-symmetric latitude structure with respect to the equator) for most tidal components. It should also be noted that the effects of background winds are inherently accounted for in the HME model (although background winds are set to zero in the HME computation), because the HME fitting to observations introduces the mode coupling (the relative strength of symmetric and anti-symmetric modes) for each tidal

component caused by background winds. See Oberheide and Forbes [2008] and Oberheide et al. [2011] for details.

The HME has been extensively validated with and compared to CHAMP, GRACE and WINDII tidal diagnostics in the upper thermosphere [Forbes et al., 2009; Häusler et al., 2013, Lieberman et al., 2013], radar and lidar observations in the mesosphere [Lu et al., 2011; Yuan et al., 2014], WACCM-X simulations [Liu et al., 2010] and other. It has been successfully applied to orbital and reentry predictions [Leonard et al., 2012] and to model tidal impacts on infrared cooling rates [Oberheide et al., 2013].

The temperature and wind tides at a local site, $\tilde{u}(\phi, \lambda, z)$, $\tilde{v}(\phi, \lambda, z)$, $\tilde{w}(\phi, \lambda, z)$ and $\tilde{T}(\phi, \lambda, z)$, can be constructed from either GSWM or HME using the superposition of dominating components having different zonal wavenumbers at the same frequency, sampled at the local coordinate (ϕ, λ) . These profiles can then be compared to lidar observed temperature and winds, as will be done below. These tilde fields given in Eq. (2b) are the main concern in this paper.

3. Polarization relations between different fields of tidal perturbations

We consider a tidal perturbation at a given (angular) frequency, σ , consists of several components, each specified by a zonal wave number S , as typically observed at a ground-based station. We are interested in the polarization relations among five different dynamical fields, u' , v' , w' , T' and Φ' . Only four homogeneous linear equations are required for ratios of these five amplitudes. The HMEs are based on the tidal equations with dissipation included which are constrained with observations and mass conservation (the continuity equation). Since there is no lidar data on horizontal winds at different latitudes and longitudes, we have chosen to use the four equations (1a, 1b, 1d, and 1e) and not to make use of the continuity equation (1c). The geopotential perturbation, Φ' appears as three gradients with respect to log-pressure spherical coordinates in

1 Eqs. (1a), (1b), (1d) and (1e); we set $X' = Y' = J' = 0$. Substituting Eq. (2b) into these equations,
 2 we can relate the complex fields, $\tilde{U}, \tilde{V}, \tilde{W}, \tilde{T}$, in terms of the derivatives of the complex
 3 geopotential, $\tilde{\Phi}_z, \tilde{\Phi}_\lambda, \tilde{\Phi}_\phi$ as:

$$-i\sigma\tilde{U} - f\tilde{V} + (a\cos\phi)^{-1}\tilde{\Phi}_\lambda = 0 \quad (3a)$$

$$-i\sigma\tilde{V} + f\tilde{U} + a^{-1}\tilde{\Phi}_\phi = 0 \quad (3b)$$

$$4 \quad \tilde{W} = \frac{i\sigma}{N^2} \tilde{\Phi}_z \quad (3c)$$

$$\tilde{T} = \frac{H_s}{R} \tilde{\Phi}_z \quad (3d)$$

5 Since both lidar and HME model report tidal fields, $\tilde{U}, \tilde{V}, \tilde{W}, \tilde{T}$ in geometric coordinates, we can
 6 derive the polarization relationships in log-pressure heights yet conveniently compare lidar and
 7 HME results in geometric height. We may derive two sets of relations from Eqs. (3). First, since
 8 the geopotential and its derivatives are the integral parts of the tidal system, we are interested in
 9 calculating $\tilde{\Phi}_\lambda, \tilde{\Phi}_\phi, \tilde{\Phi}_z$, and $\tilde{\Phi}$ in terms of fields $\tilde{U}, \tilde{V}$ and $\tilde{T}$, which are directly provided either by
 10 lidar observation or by global tidal model output such as HME. Second, we are interested in
 11 polarization relations between fields $\tilde{U}, \tilde{V}, \tilde{T}$, and $\tilde{W}$. For the first set of relations, we solve (3a)
 12 and (3b) for $\tilde{\Phi}_\lambda, \tilde{\Phi}_\phi$ in terms of $\tilde{U}$ and $\tilde{V}$. Using (3d), we express $\tilde{\Phi}_z$ in terms of $\tilde{T}$, and we
 13 relate $\tilde{\Phi}$ to $\tilde{\Phi}_z$ and in turn to $\tilde{T}$ and an empirical vertical wave number, $\tilde{m}(z)$, which as explained
 14 in Appendix, may be determined from the phase profiles of the observed $\tilde{U}$ and $\tilde{V}$. Thus, the
 15 first set of the desired equations is:

$$\tilde{\Phi}_\lambda = (a \cos \phi) (i\sigma \tilde{u} + f\tilde{v}) \quad (4a)$$

$$\tilde{\Phi}_\phi = a (i\sigma \tilde{v} - f\tilde{u}) \quad (4b)$$

$$1 \quad \tilde{\Phi}_z = \frac{R}{H_s} \tilde{T} \quad (4c)$$

$$\tilde{\Phi} = \frac{R}{H_s} \left[i\tilde{m}(z) + \frac{1}{2H} \right]^{-1} \tilde{T} \quad (4d)$$

2 For our second set of equations, we can solve Eqs. (3a) and (3b) for $\tilde{u}$ and $\tilde{v}$ in terms of $\tilde{\Phi}_\lambda$ and
3 $\tilde{\Phi}_\phi$, and then relate $\tilde{v}$ to $\tilde{u}$. We can also relate $\tilde{w}$ to $\tilde{T}$ by solving (3c) and (3d). The results
4 are:

$$\tilde{v} = \frac{-i\sigma \tilde{\Phi}_\phi - f(\cos \phi)^{-1} \tilde{\Phi}_\lambda}{f\tilde{\Phi}_\phi - i\sigma(\cos \phi)^{-1} \tilde{\Phi}_\lambda} \tilde{u} \quad (5a)$$

$$5 \quad \tilde{w} = i \frac{R\sigma}{H_s N^2} \tilde{T}, \quad \text{and} \quad \tilde{w}^* = \frac{T_0(z)}{T_s} \tilde{w} = i \frac{R\sigma H_0(z)}{(H_s N)^2} \tilde{T} \quad (5b)$$

6 where N^2 is the square of buoyancy frequency in log-pressure height z , a function of mean-state
7 temperature given in Eq. (1d). In Eq. (5b), the vertical wind of log-pressure height $\tilde{w} = Dz / Dt$ is
8 related to the vertical wind of geometric height, $\tilde{w}^* = Dz^* / Dt$ by the ratio $T_0(z) / T_s$ [Andrews et
9 al., 1987], where z^* is the geometric height, and T_s the temperature at the reference altitude.
10 Both $\tilde{w}$ and $\tilde{w}^*$ may be a function of log-pressure or geometric heights. We point out that the
11 vertical axis of all atmospheric fields from Eqs. (3a) to (5b) is in log-pressure height as this
12 simplifies the equations. To compare two quantities, A and B, that vary with altitude, we can make
13 one plot of A[z] and B[z] vs z . We can make a second plot of A[z(z*)] and B[z(z*)] vs z^* . The
14 two plots will, of course, not be the same, but both will show and compare the altitude dependences

of A and B in the chosen coordinate. This facilitates the comparison between lidar observations and deduced fields (via Eqs. (5b)), and HME reported and deduced fields, since they both are presented in geometric coordinates. For the vertical wind tide, lidar observes $\tilde{w}^*$ and HME model reports $\tilde{w}$, both as functions of geometric altitude; for comparison between the two, we convert $\tilde{w}$ of HME to $\tilde{w}^*$. We can also deduce $\tilde{w}^*$ of both lidar and HME outputs in geometric coordinates from respective $\tilde{\tau}$ in geometric coordinates, using Eq. (5b) directly.

Equations (5a) and (5b) are the main polarization relations for tides. We note first and foremost, $\tilde{w}$ and $\tilde{w}^*$ lag $\tilde{\tau}$ by 90° and the relative magnitudes may be determined by Eq. (5b). Interestingly, this relationship is quite general, and it is valid independent of the tidal structure and geographical location. The one-to-one relationship between $\tilde{w}$ and $\tilde{\tau}$ is, by and large, the consequence of adiabatic vertical displacement of an air parcel. If the tides were indeed to propagate in the idealized atmosphere with background assumed for the derivation of these equations, then the observed $\tilde{w}$ (or $\tilde{w}^*$) should relate to $\tilde{\tau}$ as given by Eq.(5b). Therefore, Eq. (5b) may be used to deduce $\tilde{w}$ (or $\tilde{w}^*$) from $\tilde{\tau}$. Any discrepancy between the observed $\tilde{w}^*$ taken from [Yuan et al., 2010, 2014], and that deduced from the observed $\tilde{\tau}$ via Eq. (5b) is then due to the effects of atmospheric mean wind, meridional temperature gradient and dissipation on tides which are not incorporated into our formulation. In particular, the amplitude of the observed vertical wind, $\tilde{w}^*$ is smaller than that deduced from Eq. (5b) as altitude increases; such a deviation is most likely the result of dissipation. Secondly, the relation between $\tilde{V}$ and $\tilde{U}$, Eq. (5a), depends on the tidal structure via $\tilde{\Phi}_\lambda$ and $\tilde{\Phi}_\phi$ at a given location. Since Eqs. (5a), (4a) and (4b) were derived from (3a) and (3b), with a given data set, Eq. (5a) does not provide new

1 information. Nonetheless, it is a valid relation, and can be used to deduce $\tilde{V}$ from $\tilde{U}$, if $\tilde{\Phi}_\lambda$ and
 2 $\tilde{\Phi}_\phi$ are independently provided or determined.

3 Equations (5a) and (5b) seem to suggest a division of tidal fields into two groups, $(\tilde{u}, \tilde{v})$ and
 4 $(\tilde{r}, \tilde{w})$. That the fields in the two groups are related can be appreciated by realizing that $(\tilde{u}, \tilde{v})$
 5 depends on horizontal gradients of geopotential, $(\tilde{\Phi}_\lambda, \tilde{\Phi}_\phi)$, Eqs. (4a) and (4b), and $\tilde{r}$ depends on
 6 the vertical gradient of the geopotential, $\tilde{\Phi}_z$, Eq.(4c), or on the geopotential $\tilde{\Phi}$, Eq.(4d). They all
 7 depend on the same geopotential function, $\tilde{\Phi}(\lambda, \phi, z)$. As detailed in the Appendix, at a local site
 8 it is possible to determine the geopotential, at least approximately, from the observed temperature
 9 and the deduced vertical wave number, $\tilde{m}(z)$, Eq.(4d).

10 To further appreciate these polarization relations, we investigate two limiting cases. First, for
 11 migrating tides, the amplitude (magnitude) of the geopotential is independent of longitude when
 12 expressed in universal time, t . Alternatively, if viewed in local time, $t_{LT} = t + \lambda / \Omega$, it (both
 13 amplitude and phase) should be independent of longitude, i.e.,

$$14 \quad \tilde{\Phi}_\lambda e^{-i\sigma t} = \frac{\partial}{\partial \lambda} \left[\tilde{\Phi}^{s=-\sigma/\Omega}(z, \phi) e^{-i\sigma t_{LT}} \right] = 0, \text{ and then Eq.(5a) shows that } \tilde{v} = -i(\sigma / f) \tilde{u}, \text{ or } \tilde{v} \text{ leads}$$

15 $\tilde{u}$ by 90° in phase and its amplitude is (σ / f) times that of $\tilde{u}$. Second, consider tides as
 16 westward propagating low-frequency gravity waves. Since a pure gravity wave is transverse and
 17 local, we may set $\tilde{\Phi}_\phi = 0$, and then Eq.(5a) yields $\tilde{v} = -i(\sigma / f) \tilde{u}$, which is the polarization
 18 relation of low-frequency gravity waves [Fritts and Alexander, 2003]. Similarly, for a westward
 19 propagating monochromatic wave with a single zonal wave number, S , we have $\tilde{\Phi}_\lambda = iS\tilde{\Phi}$, and

1 $\tilde{\Phi}_\phi = 0$. By using these relations in Eqs. (4a) and (4b), we can relate $\tilde{\Phi}$ to $\tilde{U}$, and substituting
2 $\tilde{\Phi}$ so derived into Eq. (4d), we can then relate $\tilde{T}$ to $\tilde{U}$. By recognizing the relation between
3 horizontal wave vector k , earth radius a , and zonal wave number S , i.e., $k = S / (a \cos \phi)$, we have:

$$4 \quad \tilde{T} = \frac{H_s}{R} \left[i \tilde{m}(z) + \frac{1}{2H} \right] \frac{\sigma^2 - f^2}{\sigma k} \tilde{U}, \quad \text{with } k = \frac{2\pi S}{2\pi a \cos \phi} \quad (6)$$

5 It is reassuring to note that the relation between zonal wind and temperature tides, Eq. (6), is
6 identical to that derived from the low-frequency gravity wave polarization relation [Hu et al., 2002].

7 **4. Results: Investigation of tidal polarization relations with lidar and HME data**

8 To gain confidence in and to determine the limitations of the derived relationships of tidal
9 amplitudes and phases between different tidal fields, we first compare the monthly mean diurnal
10 and semidiurnal tidal phases and amplitudes of temperature, zonal wind, and meridional wind from
11 lidar observations, based on seven years of data from 2002 to 2008 [Yuan et al., 2010, 2014] and
12 from the HME model. Though we have done so for all 12 months, we here present the months of
13 Jan (representing the winter solstice period) and Sep (representing the equinoctial period) as
14 typical examples. In Figs (1a) and (1b), we show diurnal tidal amplitudes and phases for Jan, and
15 (1c) and (1d) for Sep as a function of geometrical height, z^* . Generally, we see good agreement
16 in altitude dependence between lidar and HME in Jan zonal wind and temperature diurnal
17 amplitudes, with lidar amplitude larger. The vertical wavelength of lidar temperature diurnal tide
18 for Jan as can be seen in Fig. (1b) is noticeably longer than that of HME, reflecting the long
19 standing discrepancy in winter temperature tidal phase between lidar observation and model
20 prediction [Yuan et al., 2010]. The lidar meridional wind tidal amplitudes, Fig. (1a) are about a

factor of 4 weaker below 95 km than HME, and are particularly small at about 86 and 93 km. The near zero amplitudes give rise to huge lidar error bar in tidal phases as shown in Fig. (1b) and discussed in a previous publication [Yuan et al., 2006]. As noted there are more "wiggles" in the lidar diurnal amplitudes than in the corresponding HME data.

While a detailed discussion of the "wiggles" is beyond the scope of this paper, they are most likely caused by short-term tidal variability [She et al., 2004] along with constructive/destructive interference between the different migrating and nonmigrating tidal components over Ft. Collins. A similar explanation was given by Ward et al. [2010] to understand differences between TIDI and lidar based tidal wind observations made during the CAWSES tidal campaigns. The 2002-2008 lidar tidal climatology is based on harmonic fits to a 24-hr composite day for each month compiled from the seven years of observations. This approach reduces the impact of short-term tidal variability (which can be of the same order as the monthly mean tides [e.g., Liu et al. 2014; Pedatella et al., 2016]) but some impacts on the lidar climatology must be expected. For example, there are about 365 (719) hours of January (September) lidar observations from 2002 to 2008, resulting in 7% (14%) temporal coverage [Yuan et al., 2010]. Short-term tidal variability on the other hand can be very different for migrating and nonmigrating tides, which have different vertical wavelengths, and as such will cause different constructive/destructive interferences from one day to another in the total tidal amplitudes observed by the lidar. The HME tidal amplitudes on the other hand are based on 60-day running mean averages of continuous SABER and TIDI zonal data, and thus not sensitive to short-term variability. Gravity waves are not expected to be a major contributor to the "wiggles". Since the lidar tide is deduced from hourly mean data, gravity waves with period shorter than 1 hr should be completely eliminated; for tidal period perturbations at a

1 local site deduced from multi-year data set, the longer period gravity wave contamination should
2 also be minimal.

3 The agreement between Sep lidar and HME temperature and zonal wind tidal amplitudes is
4 very good, especially between 85 and 100 km. The Sep diurnal tidal phases of lidar and HME are
5 in excellent agreement. It is interesting to note that the HME zonal and meridional wind amplitudes
6 are nearly the same for all heights for both Jan and Sep, while the Jan HME diurnal temperature
7 amplitudes grow gently as altitude increases, and the lidar zonal wind amplitudes exhibit saturation
8 at ~ 95 km. Both HME diurnal amplitudes of zonal and meridional tidal winds show peaks in the
9 vicinity of 90 km, a well-known effect of dissipation [Forbes and Vincent, 1989], a point to be
10 discussed later. The corresponding peaks in lidar amplitudes are somewhat obscured by the
11 wiggling patterns, as can be seen in Figs 1(a) and 1(c).

12 The corresponding comparisons for semidiurnal tides are shown in Fig.2. Note that there are
13 less “wiggles” in the lidar amplitudes. This is consistent with the short-term tidal variability and
14 interference explanation given for the diurnal tide. Semidiurnal nonmigrating tides are
15 comparatively less important in the MLT region and the overall semidiurnal tidal field observed
16 over Ft. Collins is dominated by the migrating semidiurnal tide [Yuan et al., 2014]. Indeed, we see
17 good agreement in amplitudes and excellent agreement in phases in all three fields in both Jan and
18 Sep. We also note the near equality between zonal and meridional wind amplitudes in HME;
19 interestingly, other than in Jan above ~ 97 km, this near equality is also apparent in the lidar
20 semidiurnal tide. The HME semidiurnal temperature amplitude profiles exhibit a mild exponential
21 growth as expected. Like HME zonal and meridional wind amplitudes, the lidar zonal (but not
22 meridional) wind amplitude exhibits a mild saturation at ~ 100 km in Jan and the lidar zonal and
23 meridional winds exhibit maxima at ~ 97 km in Sep.

Another relation which stands out is that the meridional wind phase appears to lead the zonal wind phase by $\sim 90^\circ$ for all altitudes for both diurnal and semidiurnal tides. This is true for all HME tides (Jan and Sep) shown in Figs. 1 and 2. The same can be found in lidar observed tides, except for Jan diurnal meridional wind phase, where the error bars are substantial. As noted earlier, for the migrating tide, the amplitude of the meridional wind is (σ / f) (0.768 for diurnal tide and 1.54 for semidiurnal tide) times that of the zonal wind, with the meridional wind phase leading by 90° .

Thus, the relative phases between $\tilde{V}$ and $\tilde{U}$ for HME and lidar appear to be consistent with the migrating tide dominance. However, their relative amplitude, i.e., ratio of V over U deviates from that of a pure migrating tide; instead of being 0.768 for diurnal tide and 1.54 for semidiurnal tide, the ratio is close to unity for HME (both diurnal and semidiurnal components) and lidar (only semidiurnal component) tides. Lidar diurnal zonal wind amplitudes however behave differently as it is considerably larger than the meridional amplitude for Jan, and between 87 and 96 km for Sep. For temperature and meridional wind amplitudes at the same altitude range, though the HME model underestimates the lidar-observed tidal amplitudes somewhat, their general profile shapes are in good agreement. That the HME under-estimates the tidal amplitude is understandable since it is based on SABER and TIDI tidal diagnostics of 60-day running means of observations. Noticeably, the HME semidiurnal temperature amplitudes grow exponentially as altitude for both Jan and Sep, while the lidar temperature is in agreement in Jan and shows a mild maximum at ~ 90 km in Sep. Other than the meridional amplitude in Jan, which continues to increase, both HME and lidar zonal and meridional semidiurnal tidal amplitudes increase to 100 km or higher, where they show saturation and dissipation. The longer vertical wavelength semidiurnal tide suffers much less dissipation than the shorter vertical wavelength diurnal tide which saturates at a lower altitude.

Amplitudes and phases of $\tilde{\Phi}$, $\tilde{\Phi}_\lambda$, and $\tilde{\Phi}_\phi$ are deduced, respectively from (4d), (4a) and (4b). Here we use lidar observed and NRLMSISE-00 [Picone et al., 2002] monthly mean temperatures between 77 and 105 km for calculating the reference level scale heights H_s in Eqs. (4), respectively, for lidar and HME tides. The results for diurnal and semidiurnal tides are shown respectively in Figs. 3 and Figs. 4 with lidar data and HME data, again using Jan and Sep as examples. The amplitude of $\tilde{\Phi}$ in diurnal Jan and in semidiurnal Sep show a factor of ~ 2 difference with HME smaller than lidar, while diurnal Sep and semidiurnal Jan are in good agreement. For the phase of $\tilde{\Phi}$, other than diurnal Jan below 90 km, Lidar and HME are in general agreement within ~ 1 hr. For $\tilde{\Phi}_\lambda$ and $\tilde{\Phi}_\phi$, the HME tides under estimated the lidar amplitudes by between 25% to a factor of 5.

Other than Jan lidar diurnal tide which has large error bar in meridional phase, $\tilde{V}$ leads $\tilde{U}$ by $\sim 90^\circ$ for both lidar and HME tides, as can be seen in Figs, 1(d), 2(b) and 2(d). We may simplify the discussion by substituting $-i(V/U)\tilde{U}$ for $\tilde{V}$ into Eqs. 4(a) and 4(b), where U and V are, respectively the magnitude of $\tilde{U}$ and $\tilde{V}$, and we obtain, respectively, $\tilde{\Phi}_\lambda = i(a \cos \phi)(\sigma - fV/U)\tilde{U}$ and $\tilde{\Phi}_\phi = a(\sigma V/U - f)\tilde{U}$. Because the amplitudes (magnitudes) of $\tilde{U}$ and $\tilde{V}$ are comparable for HME tides as well as for lidar semidiurnal tides, i.e., $V \sim U$, then $(\sigma - fV/U)$ and $(\sigma V/U - f)$ are positive for semidiurnal tide. Correspondingly, $\tilde{\Phi}_\lambda$ lags $\tilde{U}$ by $\sim 90^\circ$ and $\tilde{\Phi}_\phi$ is in phase with $\tilde{U}$ resulting in $\tilde{\Phi}_\phi$ that leads $\tilde{\Phi}_\lambda$ by $\sim 90^\circ$; other than lidar Jan below 95 km, this is in reasonable agreement with observations for HME and lidar semidiurnal tides as can be seen in Figs. (4b) and (4d). For HME diurnal tides, $(\sigma - f)$ is negative, so $\tilde{\Phi}_\lambda$

1 leads $\tilde{U}$ by $\sim 90^\circ$ and $\tilde{\Phi}_\phi$ is out-of-phase with $\tilde{U}$, again leading to $\tilde{\Phi}_\phi$ leads $\tilde{\Phi}_\lambda$ by $\sim 90^\circ$, in
 2 disagreement with the observed which is nearly out-of-phase for HME diurnal tides as seen in Figs.
 3 (3b) and (3d), apparently showing that this simplified relationship fails for HME diurnal tides. On
 4 the other hand, unlike the HME tide, the lidar diurnal amplitude U is much larger than V for Jan
 5 (and for Sep between 87 and 97 km), and Eqs. (4a) and (4b) become $\tilde{\Phi}_\lambda = i(\alpha \cos \phi) \tilde{\sigma} \tilde{U}$ and
 6 $\tilde{\Phi}_\phi = -f \alpha \tilde{U}$, suggesting that $\tilde{U}$ leads $\tilde{\Phi}_\lambda$ by 90° and is out of phase with $\tilde{\Phi}_\phi$, resulting in $\tilde{\Phi}_\lambda$
 7 leading $\tilde{\Phi}_\phi$ by 90° , in agreement with Figs. (3b) and (3d). The overall success of the simplistic
 8 relations suggest that the tidal period perturbations at the lidar site are dominated by the migrating
 9 component, more so for the semidiurnal tide than the diurnal tide. This is consistent with our
 10 previous publications [Yuan et al., 2008; 2014]. We point out that the field variables in Eqs. (4),
 11 $\tilde{\Phi}, \tilde{\Phi}_\lambda, \tilde{\Phi}_\phi, \tilde{\Phi}_z, \tilde{U}, \tilde{V}, \tilde{T}$ and $\tilde{n}(z)$ are all expressed in log-pressure height. Since the observed
 12 quantities, $\tilde{U}, \tilde{V}, \tilde{T}$ and $\tilde{n}(z)$ are available in geometric height, their substitutions in Eqs. (4) leads
 13 to $\tilde{\Phi}, \tilde{\Phi}_\lambda$, and $\tilde{\Phi}_\phi$ plotted in geometric height as shown in Figs (3) and (4).

14 Next, we present the relationship between “observed” vertical wind (VW) tide $\tilde{w}^*$ in
 15 geometric coordinate and that deduced from the polarization relation, Eq. (5b), for diurnal and
 16 semidiurnal components, respectively in Fig. 5 and Fig. 6. We showed in a recent paper [Yuan et
 17 al., 2014] that the VW tide is directly proportional to the time derivative of either the fractional Na
 18 density tide or the fractional temperature tide, all expressed in geometric coordinates.
 19 Consequently, unlike the observed $\tilde{T}, \tilde{U}$, and $\tilde{V}$, the lidar observed $\tilde{w}^*$ we invoked here is
 20 derived indirectly from lidar observed $\tilde{T}$ along with information on background state, i.e., W_3 in
 21 Yuan et al. [2014]. Note that the Na lidar “observed” $\tilde{w}^*$ was derived with the assumption of a

1 dissipation-less environment in that Na density perturbation follows atmospheric perturbation
 2 which is in turn related to temperature perturbation via the ideal gas law. Since the lidar
 3 temperature perturbation $\tilde{T}$ used to derive $\tilde{w}^*$ is observed from the real atmosphere, the lidar
 4 “observed” $\tilde{w}^*$ contains dissipation or saturation as in the observed $\tilde{T}$. We also note, though
 5 atmospheric density decreases as altitude increases, the Na density peaks near the middle of the
 6 mesopause region [Gardner and Shelton, 1985], leading to a divergence at 2-4 km above the Na
 7 layer peak and a narrow “invalid zone” in $\tilde{w}^*$ [Yuan et al., 2014]. The HME “observed” $\tilde{w}^*$ on
 8 the other hand is numerically derived from the temperature perturbation via relations imbedded in
 9 the GSWM model in a way similar to Eq. (5b) but with atmospheric dissipation included. Thus,
 10 the “observed” HME $\tilde{w}^*$ reflects additional dissipation inherent in the GSWM beyond that in
 11 $\tilde{T}$, used to deduce $\tilde{w}^*$ via Eq. (5b). From Figs. 5 and 6, we see there is an excellent agreement
 12 in both diurnal and semidiurnal phases between lidar and HME results. It is clear that the lidar
 13 observed phases and those deduced are identical, because both lidar observed and deduced $\tilde{w}^*$
 14 are directly proportional to temperature tide, $\tilde{T}$, which leads $\tilde{w}^*$ by 90° in both observed [Yuan
 15 et al., 2014] and that deduced from Eq. 5(b). It should be noted that the agreement between
 16 observed and deduced phases is seen to be very good for HME VW tides as well. Other than lidar
 17 temperature Jan diurnal phase, which has a long-standing discrepancy [Yuan et al., 2010], the
 18 agreements between lidar and HME phases are also very good. The agreements between lidar and
 19 HME diurnal amplitudes are not as good as in phases. Neither are the agreements between
 20 observed and deduced HME amplitudes. Other than in the “invalid zone”, typically ~ 2 km above
 21 95 km [Yuan et al., 2014], the lidar deduced Jan diurnal $\tilde{w}^*$ tracks that observed with somewhat
 22 larger amplitude. The $\tilde{w}^*$ profiles in Sep have a more complex altitude dependence with the lidar

1 amplitude considerably larger than HME above 90 km, Fig. 5(c), mirroring the observed
2 temperature tidal amplitude shown in Fig. (1c). Here, though the lidar deduced amplitude still
3 tracks that of observed, the HME deduced $\tilde{w}^*$ amplitude is considerably larger than the observed
4 HME profile, above 90 km. The phases of lidar and HME observed and deduced semidiurnal $\tilde{w}^*$
5 are all in agreement, as shown in Figs. (6b) and (6d). The lidar “observed” $\tilde{w}^*$ in Jan semidiurnal
6 amplitude is somewhat larger than that of HME, with the respective deduced amplitude somewhat
7 larger than the observed. The lidar “observed” semidiurnal $\tilde{w}^*$ amplitudes generally track those
8 of deduced profiles, while HME “observed” amplitudes are somewhat smaller than those deduced
9 as can be seen in Figs. (6a) and (6c). The Sep lidar semidiurnal amplitude is considerably larger
10 than the HME counterpart between 85 and 95 km and shows saturation above 90 km as in the
11 observed temperature profile shown in Fig. (2c), while HME profiles show mostly exponential
12 growth with different rates, Figs. (6a) and (6c).

13 To shed some light on the difference between observed and deduced amplitudes of $\tilde{w}^*$, we
14 re-plot these HME diurnal and semidiurnal amplitudes between 60 and 120 km, respectively, in
15 Figs. (7a) and (7b) to better reveal their altitude dependence. It is evident that their differences are
16 negligible below 70 km for all plots as eddy diffusion is negligible below ~ 70 km. Forbes and
17 Vincent (1989) investigated the effect of mean wind and dissipation on diurnal tide analytically by
18 introducing a Doppler-shifted complex frequency σ^* , corresponding to a complex vertical wave
19 number, $k_z^* = k_r + ik_i$. The real part k_r represents the Doppler-shift resulting from mean zonal wind,
20 and its imaginary part k_i represents the effect of diffusion of momentum (mechanical dissipation)
21 and heat (thermal diffusion). The tidal amplitude grows exponentially [Forbes and Vincent, 1989;
22 Hagan et al., 1999b; Yuan et al., 2008] as $\exp([-k_i + 1/2H]z)$, where H is the local scale height.

Below $\sim 110 - 120$ km, eddy diffusion dominates, and a critical altitude exists when the damping due to k_i exactly cancels the $1/2H$ growth factor. Depending on the background state, this typically occurs around 90 km for diurnal tides. As shown in Fig. 7(a), the HME modeled diurnal tidal amplitudes peak at 90 km for Jan and 85 km for Sep. The diurnal tidal amplitudes deduced from the polarization relation with the assumption of no damping are clearly larger than those “observed” between the peak altitude and ~ 115 km, above which molecular diffusion becomes dominant. Due to the longer vertical wavelength of semidiurnal tides, the growth factor ($-k_i + 1/2H$) is positive between 60 and 110 km. We see in Fig. 7(b), a gradual increase in semidiurnal tidal amplitudes along with an appreciable difference between observed and deduced w^* ; for Jan, both deduced and observed show saturation near 110 km. It is also interesting to note that near 60 km, where dissipation is negligible, the semidiurnal tidal amplitudes are essentially the same for Jan and Sep, while the Sep diurnal amplitude is significantly larger, likely due to stronger diurnal thermal forcing in Sep.

5. Discussion and conclusion

In this paper, we have derived the polarization relations for tidal waves from the primitive equations for large scale waves. These are given in Eqs. (5a) and (5b). The former, Eq. (5a), relates tidal meridional wind $\tilde{V}$, to tidal zonal wind $\tilde{U}$ via horizontal derivatives of the associated geopotential perturbations, $\tilde{\Phi}_\lambda$ and $\tilde{\Phi}_\phi$; thus unlike local gravity waves, the relation between $\tilde{V}$ and $\tilde{U}$ depends on tidal structure, which is non-local. For migrating tides, $\tilde{\Phi}_\lambda = 0$ in local time, $\tilde{V}$ is found to lead $\tilde{U}$ by 90° . The latter, Eq. (5b), relates vertical wind tide $\tilde{w}^*$ and temperature tide $\tilde{T}$ in a way independent of tidal structure and geographic location, reflecting adiabatic

vertical motion of air parcel. These relationships and their implications are discussed using the Na lidar observed tides and the HME modeled tides at 40.6°N and 105°W. Since tidal fields $\tilde{U}$, $\tilde{V}$, and $\tilde{T}$ are directly observed by lidar and reported by HME, we can also derive tidal geopotential $\tilde{\Phi}$ and its derivatives $\tilde{\Phi}_\lambda$, $\tilde{\Phi}_\phi$, and $\tilde{\Phi}_z$ in terms of these quantities, as given in Eqs. 4(d), 4(a), 4(b) and 4(c), respectively.

Comparing $\tilde{U}$, $\tilde{V}$ and $\tilde{T}$ in geometric coordinates between lidar observation and HME modeled data at the lidar site, 40.6°N and 105°W, we find general agreement. Since the phase of $\tilde{V}$ leads that of $\tilde{U}$ by approximately 90° for both lidar observed and HME modeled tides as shown in Figs. 1 and 2, the tidal wave at this midlatitude site is dominated by the migrating tide. Correspondingly, this dominance is also evident in the relationship between $\tilde{\Phi}_\lambda$ and $\tilde{\Phi}_\phi$, i.e., $\tilde{\Phi}_\phi$ leads $\tilde{\Phi}_\lambda$ by $\sim 90^\circ$ for semidiurnal tides as shown in Figs. 4(b) and 4(d). Since the HME modeled tide has the capability to resolve each tidal component (as denoted by zonal wave number) separately, one can compare the amplitude of total tide to that of migrating and non-migrating tidal contributions with HME modeled data. The month-altitude plots of the vertical wind diurnal and semidiurnal tides and their migrating and non-migrating contributions were shown in Fig. 4 of a recent publication [Yuan et al., 2014]. From this figure, it is clear that the non-migrating tide made only a minor contribution to the total tide.

According to Eq. (5b), the vertical wind tide $\tilde{w}^*$ and temperature tide $\tilde{T}$ has a one-to-one relationship which depends only on the background atmosphere, and is independent of tidal structure and geographic location. Therefore, in theory, one could use Eq. (5b) and temperature tide $\tilde{T}$ to determine the vertical wind tide $\tilde{w}^*$, whose direct measurement is known to be very challenging. Though the background (diurnal mean) wind has not been included in the derivation,

the resulting polarization relations may be used for the investigation of thermal tides in the MLT, since the background wind filtering below the MLT are accounted for in both lidar observation and HME model, as they both invoke data from observations in the MLT. However, since Eq. (5b) was derived for dissipation-less background atmosphere, the deduced $\tilde{w}^*$ is likely to be valid below 80 km, or lower as discussed, and it is an overestimation above 80 km, where dissipation is significant. The effect due to dissipation in this connection is clearly shown in Figs. (5a), (5c), (6a) and (6c) as the HME modeled amplitudes above 80 km are significantly smaller those deduced from polarization relation (Eq. (5b)); this is particularly obvious for diurnal $\tilde{w}^*$ for Sep, in Fig. (5c). In the same figures, outside the invalid zone [Yuan et al., 2014] typically 95 - 97 km, the “observed” lidar amplitudes appear to follow those deduced from Eq. (5b) via lidar temperature tides. This is because the indirectly observed lidar vertical wind tide was based on the observed Na density perturbation, assumed to follow the vertical perturbation of atmosphere adiabatically, i.e., without dissipation. The altitude dependence of dissipation may be seen more clearly by comparing HME modeled vertical wind output to that deduced from Eq. (5b) as shown in Fig. 7.

Linear dissipation will not only attenuate tidal amplitudes, but it could also shift the tidal phases. The comparisons in these figures show very good agreement between the observed and deduced phases, suggesting that the dissipation in question hardly affects the tidal phases of vertical wind in the mesopause region. Further improvements in tidal polarization relations, Eqs. (5a) and (5b), can be made by numerically incorporating meridional background temperature gradient and dissipation along with non-zero background zonal wind into the governing equations (1a) to (1e), in a way similar to those in the stripped-down GSWM and HME models [Hagan et al., 1999a; Oberheide and Forbes, 2008]. In this regard, both GSWM and HME model outputs may be used to exhibit the numerical relationships between $\tilde{u}$, $\tilde{v}$, $\tilde{T}$, and $\tilde{w}^*$. However, analytical polarization

relations between these tidal fields along with geopotential and derivatives facilitate physical discussions and understanding via tidal polarization relations as is done in this paper.

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Appendix: Classical tidal theory and the effective vertical wave number $\tilde{m}(z)$

An upward conservative wave in a stratified atmosphere grows exponentially as $e^{z/2H}$ because the background atmospheric density decreases exponentially by the factor $e^{-z/H}$. Since waves at a given frequency with different zonal wave numbers are independent, the latitudinal and altitude dependence of the wave function are separable in an atmosphere with zero wind, zero longitudinal temperature gradient and no dissipation. We can express the perturbation fields in terms of complex field topped with a caret and in turn with a bar as follows,

$$w' = \text{Re} \left\{ \widehat{W}^s(\phi, z) e^{i(s\lambda - \sigma t)} \right\}, \text{ with } \widehat{W}^s(\phi, z) = e^{z/2H} W(z) \overline{W(\phi)}. \quad (\text{A1})$$

$$(u', v', \Phi') = \text{Re} \left\{ \left[\widehat{u}^s(\phi, z), \widehat{v}^s(\phi, z), \widehat{\Phi}^s(\phi, z) \right] e^{i(s\lambda - \sigma t)} \right\},$$

$$\left[\widehat{u}^s(\phi, z), \widehat{v}^s(\phi, z), \widehat{\Phi}^s(\phi, z) \right] = e^{z/2H} G(z) [\bar{u}(\phi), \bar{v}(\phi), \bar{\Phi}(\phi)] \quad (\text{A2})$$

where $G(z) = W_z - W/2H$. Substituting Eqs. (A1) and (A2) into Eqs. (1a), (1b), (1c) and (1d) with forcing and dissipation set to zero, one obtains two equations [Forbes, 1995], one as a function of latitude (Laplace equation) and the other as a function of altitude (the vertical structure equation), linked by a common constant (eigenvalue, or equivalent depth h) as follows:

$$\frac{d}{d\mu} \left(\frac{1-\mu^2}{f_r^2 - \mu^2} \frac{d\bar{\Phi}}{d\mu} \right) - \frac{1}{(f_r^2 - \mu^2)} \left[-\frac{s(f_r^2 + \mu^2)}{f_r(f_r^2 - \mu^2)} + \frac{s^2}{1-\mu^2} \right] \bar{\Phi} + \frac{(2a\Omega)^2}{gh} \bar{\Phi} = 0 \quad (\text{A3})$$

$$\frac{d^2 W}{dz^2} + \left(\frac{N^2}{gh} - \frac{1}{4H^2} \right) W = 0 \quad (\text{A4})$$

where $\mu = \sin \phi$ and $f_r = \sigma/2\Omega$ (not to confused with $f = 2\Omega \sin \phi$), and the common constant h is called equivalent depth. The solution to the Laplace equation yields a set of eigenfunctions, called Hough function, $\Theta_n^{s,\sigma}(\phi)$, and associated eigenvalue, h_n^s , labeled by meridional index n . The solution to the vertical structure equation takes the familiar form:

$$W_n^s(z) = A e^{im_n^s z} + B e^{-im_n^s z} \text{ with } m_n^{s2} = \left(\frac{N^2}{gh_n^s} - \frac{1}{4H^2} \right), \text{ where } N^2 = \frac{R}{H_s} \left[\frac{\partial T}{\partial z} + \frac{\kappa T}{H_s} \right]$$

$$\text{and } m_n^{s2} = \frac{1}{h_n^s H_s} \left[\frac{\partial H}{\partial z} + \frac{\kappa H}{H_s} \right] - \frac{1}{4H^2} \xrightarrow{\text{isothermal}} \approx \frac{1}{H^2} \left(\frac{\kappa H}{h_n^s} - \frac{1}{4} \right)$$

We can now evaluate the height derivative in the log-pressure coordinate, by these manipulations:

$$\text{We first express } \widehat{\Phi}^s(\phi, z) e^{i(s-\sigma)t} = \sum_n e^{z/2H} G_n^s(z) \Theta_n^{s,\sigma}(\phi) e^{i(s-\sigma)t}, \text{ and recognize,}$$

$$\begin{aligned} \widehat{\Phi}^s(\phi, z) &= \sum_n e^{z/2H} G_n^s(z) \Theta_n^{s,\sigma}(\phi) = \sum_n e^{z/2H} \left(\frac{d}{dz} - \frac{1}{2H} \right) W_n^s(z) \Theta_n^{s,\sigma}(\phi) \\ &= \sum_n e^{z/2H} \left(im_n^s - \frac{1}{2H} \right) W_n^s(z) \Theta_n^{s,\sigma}(\phi) \rightarrow \frac{\partial \widehat{\Phi}^s(\phi, z)}{\partial z} = \sum_n \left(im_n^s + \frac{1}{2H} \right) e^{z/2H} G_n^s(z) \Theta_n^{s,\sigma}(\phi). \end{aligned}$$

1 Since m_n^s for different meridional index n are different, the observation of tidal-period
 2 perturbations is summing over n . To simplify, we introduce a mean vertical wave number, $\widehat{m}$
 3 over modes of different n but with the same S . For cases which consist of many different S , we
 4 introduce an overall mean vertical wave number, $\widetilde{m}$. Mathematically, we make the following
 5 approximation,

$$\frac{\partial \widehat{\Phi}^s(\phi, z)}{\partial z} = \sum_n \left(i m_n^s + \frac{1}{2H} \right) e^{z/2H} G_n^s(z) \Theta_n^{s,\sigma}(\phi) \rightarrow$$

$$\left(i \widehat{m}^s + \frac{1}{2H} \right) \sum_n e^{z/2H} G_n^s(z) \Theta_n^{s,\sigma}(\phi) = \left(i \widehat{m}^s + \frac{1}{2H} \right) \widehat{\Phi}^s(\phi, z),$$

6 And $\widetilde{\Phi}(\phi, \lambda, z) = \sum_s \widehat{\Phi}^s(\phi, z) e^{is\lambda} \rightarrow \frac{\partial \widetilde{\Phi}(\phi, \lambda, z)}{\partial z} = \sum_s \frac{\partial \widehat{\Phi}^s(\phi, z)}{\partial z} e^{is\lambda}$

$$= \sum_s \left(i \widehat{m}^s + \frac{1}{2H} \right) \widehat{\Phi}^s(\phi, z) e^{is\lambda} \rightarrow \left(i \widetilde{m} + \frac{1}{2H} \right) \widetilde{\Phi}(\phi, \lambda, z)$$

7 Substituting $\widetilde{\Phi}_z$ given above into Eq.(4c), we obtain $\widetilde{T} = \frac{H_s}{R} \left(i \widetilde{m} + \frac{1}{2H} \right) \widetilde{\Phi}(\phi, \lambda, z)$ which justifies
 8 Eq. (4d) in the text. The overall vertical wave number, or effective wavenumber, is altitude
 9 dependent, i.e., $\widetilde{m}(z)$, and it can be determined from the phase profiles of $\widetilde{u}$ and/or $\widetilde{v}$.

1 **Figure Captions:**

2 Fig.1 Comparison between lidar (in diamond, open circle and open triangle) and HME modeled
3 (in solid, long dash, and short dash lines) diurnal tides (respectively for temperature, zonal wind
4 and meridional wind). Upper panels for Jan and lower panels for Sep with amplitudes on the left
5 and phases on the right.

6 Fig. 2 Same as Fig. 1, except for semidiurnal tides.

7 Fig.3 Comparison between lidar (in diamond, open circle and open triangle) and HME modeled
8 (in solid, long dash, and short dash lines) diurnal tides (respectively for geopotential, Φ , and its
9 longitudinal derivative, Φ_{λ} , and latitudinal derivative, Φ_{ϕ}). Upper panels for Jan and lower panels
10 for Sep with amplitudes on the left and phases on the right.

11 Fig.4 Same as Fig. 3, except for semidiurnal tides.

12 Fig.5 Comparison between observed and deduced vertical wind diurnal tide for lidar data
13 (respectively in diamond and open circle) and HME data (respectively in solid and long dash lines).
14 Upper panels for Jan and lower panels for Sep with amplitudes on the left and phases on the right.

15 Fig.6 Same as Fig. 5, except for semidiurnal tides.

16 Fig.7 Comparison of altitude dependences (from 60 up to 120 km) between observed (solid) and
17 deduced (open) HME modeled vertical wind tidal amplitudes. Diurnal amplitudes on the left and
18 semidiurnal amplitudes on the right, triangles for Jan and circles for Sep.

19

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Fig.1 Comparison between lidar (in diamond, open circle and open triangle) and HME modeled (in solid, long dash, and short dash lines) diurnal tides (respectively for temperature, zonal wind and meridional wind). Upper panels for Jan and lower panels for Sep with amplitudes on the left and phases on the right.

Fig. 2 Same as Fig. 1, except for semidiurnal tides.

Fig.3 Comparison between lidar (in diamond, open circle and open triangle) and HME modeled (in solid, long dash, and short dash lines) diurnal tides (respectively for geopotential, Φ , and its longitudinal derivative, Φ_λ , and latitudinal derivative, Φ_ϕ). Upper panels for Jan and lower panels for Sep with amplitudes on the left and phases on the right.

Fig.4 Same as Fig. 3, except for semidiurnal tides.

Fig.5 Comparison between observed and deduced vertical wind diurnal tide for lidar data (respectively in diamond and open circle) and HME data (respectively in solid and long dash lines). Upper panels for Jan and lower panels for Sep with amplitudes on the left and phases on the right.

Fig.6 Same as Fig. 5, except for semidiurnal tides.

Fig.7 Comparison of altitude dependences (from 60 up to 120 km) between observed (solid) and deduced (open) HME modeled vertical wind tidal amplitudes. Diurnal amplitudes on the left and semidiurnal amplitudes on the right, triangles for Jan and circles for Sep.