Two-dimensional problem of thermoelectric materials with an elliptic hole or a rigid inclusion

A.B. Zhang^{1,*}, B.L. Wang², J. Wang¹, J.K. Du¹

¹Piezoelectric Device Laboratory, School of Mechanical Engineering & Mechanics,
Ningbo University, Ningbo, 315211, P.R. China

² Centre for Infrastructure Engineering, Western Sydney University, Penrith, NSW
2751, Australia

Abstract. The two-dimensional problems of an elliptic hole or a rigid inclusion embedded in a thermoelectric material subjected to uniform electric current density and energy flux at infinity are studied based on the complex variable method of Muskhelishvili and conformal mapping technique. The closed-form solutions of electric potential, temperature and stress components are presented according to electrical insulated and thermal exact boundary conditions on the rim of the hole or inclusion. Numerical results are carried out to illustrate the influence of the value of major to minor axis ratio of the elliptic geometry and heat conductivity of inhomogeneity on thermoelectric and stress fields. It is found that energy flux at surfaces of the hole or rigid inclusion does not vanish due to the Joule heat and Seebeck effect when the electric field is applied. In addition, stress induced by applied electric field has a non-linear relationship with the electric current density. The heat conductivity of the air inside the elliptic hole reduces the concentration factors of energy flux and stress. However, the concentration factors of energy flux and stress at the bonding interface increase with the increasing values of heat conductivity of the flat rigid inclusion.

Keywords: Thermoelectric material; Elliptic hole; Inclusion; Fracture mechanics

* Corresponding author. Tel.: +86 754 87600980
E-mail address: zhangaibing@nbu.edu.cn (A.B. Zhang).

1. Introduction

Thermoelectric materials for solid state energy conversion have been extensively used for electric cooling [1], electricity power generation in harvesting wasted heat [2], solar energy harvesting [3] and carbon reduction [4]. However, most thermoelectric materials are brittle in nature with low fracture strength and toughness [5]. Thermoelectric materials are subjected to significant stress induced by thermal gradient, thermal shock and externally applied mechanical loadings under in-service particularly when used in waste heat recovery [6]. Defects resulting from the manufacturing processes, such as cracks and holes, can cause electric potential and temperature discontinuities across the defects boundary and stress concentrations, then result in fracture [7-13]. Fracture problems in thermoelectric materials have been received much attention and investigated by many experts recently, like other structural materials subjected heat loadings problems [14-16]. The slow Vickers indentation crack growth behavior of Mg_2Si thermoelectric material was observed by Schmidt et al. [17]. Analytic solutions for the thermoelectric material containing a 2D crack problem were derived by Zhang and Wang [18], and Song et al. [19], respectively. It is found that electric current density, heat flux and stress exhibit traditional inverse square-root singularity at the crack front. Li et al. [20] examined the brittle failure behavior of CoSb_3 based skutterudite thermoelectric material by using large-scale molecular dynamics simulations. Based on the complex variable method, the study dealing with thermal stress concentration induced by an elliptic hole has been performed by Zhang and Wang [21]. In above-mentioned works, the boundary conditions on the hole or crack surfaces are always supposed to be electrical and thermal impermeable. The thermal conductivity of crack interior has great influences on the mode-II stress intensity factor for a thermal-medium crack in a

thermoelastic material under a thermal loading [22]. Zhang and Wang [23] discussed the applicability of crack surfaces thermal boundary condition in a layered thermoelectric or metal/thermoelectric material, and the results show that the heat conductivity of air filled in a crack cannot be neglected.

On the other hand, in order to meet the various technology applications, much effort on thermoelectric materials has been devoted towards improving their figure of merit and energy conversion efficiency [24-29]. The efficiency of thermoelectric conversion depends on a dimensionless figure of merit $ZT = \varepsilon^2 \gamma T / k$ [30], where γ , k , ε and T are the electric conductivity, heat conductivity, Seebeck coefficient and absolute temperature, respectively. Thermoelectric material with high energy conversion efficiency requires not only high Seebeck coefficient to have high voltage output, but also high electric conductivity to reduce Joule heat loss and low heat conductivity to maintain large temperature difference. However, it is rather difficult to increase ZT because of competing effects of electrical and thermal conductivities, that is, a good heat conductor is usually a good electric conductor as well. One of the main strategies in developing thermoelectric materials with high energy conversion efficiency is to engineer hybrid composites, especially with nanostructures. Comparing with the system with the single material, composite structures with the layered and/or inclusions have better thermal and mechanical behaviors [31-34]. The thermoelectric properties of semiconductor materials with nanoinclusions can be improved based on the concept of band bending at inclusion/semiconductor interfaces as an energy filter for electrons [35]. An effective path to increase the electrical conductivity while to decrease the thermal conductivity, and thus to improve the figure of merit by nano-inclusions was reported by Wang et al., [36]. Introduction of metallic inclusion can enhance the thermoelectric material performance of manganese

silicide nanocomposites [37]. The presence of inclusions affects the energy conversion efficiency, strength and reliability of thermoelectric materials, and it is therefore important to study the local fields induced by the inclusions. Thermoelectric materials have increasing applications in the engineering, however, to our best knowledge relatively little work has been done for the inclusion problem based on the continuum mechanics theory.

In view of the above literature analysis, the purpose of this paper is to seek a general solution to the problem of a 2D thermoelectric material with an elliptic hole or rigid inclusion. The influence of heat conductivity of air inside the hole and inclusion on the distribution of electric field, temperature and stress is also investigated. The paper is organized as follows. Firstly, basic equations for thermoelectric material are outlined in Section 1. Next, closed-form solutions of the electric potential, temperature and stress fields for an elliptic hole and rigid inclusion are derived in Sections 3 and 4, respectively. Some numerical results are given in Section 5. Finally, Concluding remarks are made.

2. Basic equations for thermoelectric materials

We consider an isotropic thermoelectric material in which the electric potential is V and the absolute temperature is T . Such a material is characterized by the electric conductivity γ , thermal conductivity k and Seebeck coefficient ε . The governing equations for a thermoelectric material in the absence of electric charges and heat sources can be presented in the form [38, 39],

$$\begin{aligned}\nabla \cdot \mathbf{j}_e &= 0 \\ \nabla \cdot \mathbf{q} + \mathbf{j}_e \cdot \nabla V &= 0\end{aligned}\tag{1}$$

and the transport of electric current density vector $\mathbf{j}_e = [j_{ex}, j_{ey}]^T$ and heat flux

vector $\mathbf{q} = [q_x, q_y]^T$ is given as

$$\begin{aligned}\mathbf{j}_e &= -\gamma \nabla V - \gamma \varepsilon \nabla T \\ \mathbf{q} &= -\gamma \varepsilon T \nabla V - (k + \gamma \varepsilon^2 T) \nabla T\end{aligned}\quad (2)$$

Notice that the uncoupled transport equations of electricity and heat are recovered when $\varepsilon = 0$, and the temperature function T and its gradient ∇T enter into the heat transport Eq. (2)₂, making the coupling nonlinear. Since energy is transported by both electrons and heat, the energy flux vector $\mathbf{j}_u = [j_{ux}, j_{uy}]^T$ can be derived from the electric current density and heat flux as $\mathbf{j}_u = \mathbf{q} + \mathbf{j}_e V$. For the sake of convenience, an analytic function F is defined as $F = V + \varepsilon T$ [18], then we have

$$\begin{aligned}\mathbf{j}_e &= -\gamma \nabla F \\ \mathbf{j}_u &= -\gamma F \nabla F - k \nabla T\end{aligned}\quad (3)$$

Combining Eqs. (1) and (3), the constitutive equations become,

$$\begin{aligned}\nabla^2 F &= 0 \\ k \nabla^2 T + \gamma (\nabla F)^2 &= 0\end{aligned}\quad (4)$$

For the two-dimensional thermoelectric problem considered here, the general solutions of thermo-electro-elastic fields are derived detailed in our previous work [21], and summarized briefly as follows. The solutions of F and T can be expressed as,

$$\begin{aligned}F &= \text{Re}[f_1(z)] \\ T &= \text{Re}[g(z)] - \frac{\gamma}{4k} f_1(z) \overline{f_1(z)}\end{aligned}\quad (5)$$

where $z = x + iy$, $f_1(z)$ and $g(z)$ are unknown analytic functions and “Re” stands for the real part of a complex number. Re-writing Eq. (3) gives:

$$\begin{aligned}j_{ex} - ij_{ey} &= -\gamma f_1'(z) \\ j_{ux} - ij_{uy} &= -\frac{\gamma}{2} f_1(z) f_1'(z) - k g'(z)\end{aligned}\quad (6)$$

The boundary conditions of electric current density and energy flux are,

$$\begin{aligned} f_1(z) - \overline{f_1(z)} &= -\frac{2i}{\gamma} \int j_{en}(s) ds + \text{constant} \\ \text{Im} \left[\frac{\gamma}{4} f_1^2(z) + kg(z) \right] &= -\int j_{un}(s) ds + \text{constant} \end{aligned} \quad (7)$$

where “Im” stands for the imaginary part of a complex number, $j_{en}(s)$ and $j_{un}(s)$ represent the normal electric current density and normal energy flux for the point s along the boundary, respectively.

The corresponding thermal stresses and displacement induced by the temperature field T can be obtained as [21, 40]:

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4 \text{Re} [\varphi'(z)] + \frac{2\mu\alpha^*\gamma}{k(\kappa+1)} f_1(z) \overline{f_1(z)} \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2 [\bar{z}\varphi''(z) + \phi'(z)] + \frac{2\mu\alpha^*\gamma}{k(\kappa+1)} f_2(z) \overline{f_1'(z)} \end{aligned} \quad (8)$$

$$2\mu [u_x + iu_y] = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\phi(z)} + 2\mu\alpha^* \int g(z) dz - \frac{\mu\alpha^*\gamma}{k(\kappa+1)} f_2(z) \overline{f_1(z)} \quad (9)$$

where $f_2'(z) = f_1(z)$, $\varphi(z)$ and $\phi(z)$ are complex stress potential functions to be determined, μ is the shear modulus, and κ and α^* are defined as follows,

$$\kappa = \begin{cases} \frac{3-\nu}{1+\nu}, & \text{for plane stress state} \\ 3-4\nu, & \text{for plane strain state} \end{cases} \quad (10)$$

$$\alpha^* = \begin{cases} \alpha, & \text{for plane stress state} \\ (1+\nu)\alpha, & \text{for plane strain state} \end{cases} \quad (11)$$

ν and α are the Poisson's ratio and the linear thermal expansion, respectively. The resultant force between two points in an elastic body has the form of

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\phi(z)} = i \int (p_x + ip_y) ds - \frac{\mu\alpha^*\gamma}{k(\kappa+1)} f_2(z) \overline{f_1(z)} + \text{constant} \quad (12)$$

where p_x and p_y are the external forces applied to the boundary. It should be noted that the main aim of this paper is to discuss mechanical behavior of thermoelectric materials subjected to thermoelectric loads, therefore, the constant terms p_x and p_y are taken as zero.

3. The solution to an elliptic hole

As shown in Fig.1, the infinite thermoelectric material with an elliptic hole in the z -plane can be mapped into the infinite region inside of a unit circle in the ζ -plane ($\zeta = \xi + i\eta$) by the complex mapping function $z = w(\zeta)$:

$$w(\zeta) = R \left(m\zeta + \frac{1}{\zeta} \right) \quad (13)$$

with $R = \frac{a+b}{2}$, $m = \frac{a-b}{a+b}$, where a and b are length of the semiaxes of the ellipse on the x and y axes, respectively. The thermoelectric material is subjected to combined an uniform electric current density j_{e0} and an uniform energy flux j_{u0} at infinite, i.e.,

$$\begin{aligned} j_{ex} = 0, j_{ey} = j_{e0}, \\ j_{ux} = 0, j_{uy} = j_{u0}, \end{aligned} \quad \text{as } \sqrt{x^2 + y^2} \rightarrow \infty \quad (14)$$

In addition, the hole is assumed to be free of tractions and filled with air. The electric conductivity of air inside the hole can be neglected since it is almost $10^{16} \sim 10^{18}$ orders of magnitude less than that of thermoelectric material. Therefore the boundary condition of electric current density at the rim of the hole is assumed to be impermeable. However, the heat conductivity of thermoelectric material is only 10~100 times more than that of air, and the influence of thermal conductivity of air inside the elliptic hole should be assessed.

Since the electric impermeable elliptic hole boundary condition is adopted, the potential function of electric field $f_1(z)$ can be used the result in [21],

$$f_1(z) = \frac{ij_{e0}R}{\gamma} \left(\frac{1}{\xi} - \xi \right) \quad (15)$$

Integrating Eq. (15) gives

$$f_2(z) = \frac{ij_{e0}R^2}{\gamma} \left[(m+1)\ln \xi + \frac{1}{2\xi^2} - \frac{m\xi^2}{2} \right] \quad (16)$$

3.1 The temperature field inside the hole

The temperature field inside the elliptic hole has the form of

$$T_h = \text{Re} [g_h(z)] \quad (17)$$

where $g_h(z)$ is an analytic function. Then the heat flux inside the hole can be obtained as

$$q_{hx} - iq_{hy} = -k_h g_h'(z) \quad (18)$$

where k_h is the heat conductivity of air. Using Gauss' law, we have

$$\text{Im} [k_h g_h(z)] = -\int q_{hn}(s) ds \quad (19)$$

where $q_{hn}(s)$ represents the normal heat flux for the point s along the boundary. The function $g_h(z)$ can be expressed in the form of the Faber series as [42, 43],

$$g_h(z) = \sum_{n=1}^{\infty} a_n^h (m^n \xi^n + \xi^{-n}) \quad (20)$$

where a_n^h are complex coefficients to be determined.

3.2 The temperature field in the thermoelectric material

Inside the thermoelectric material, the complex temperature function $g(z)$ takes the form

$$g(z) = c_2^\infty z^2 + c_3^\infty z + g_0(z) \quad (21)$$

where c_2^∞ and c_3^∞ are two complex constants related to the loading condition at infinity, and $g_0'(\infty) = 0$.

Substituting Eqs. (15) and (21) into Eq. (6)₂, and then taking the limit $z \rightarrow \infty$ leads to:

$$c_2^\infty = \frac{j_{e0}^2}{4k\gamma}, \quad c_3^\infty = i \frac{j_{u0}}{k} \quad (22)$$

To find $g_0(z)$, one has to use the continuous conditions of the normal component of heat flux and temperature on the hole. Noting that on the hole, $j_{en}(s) = 0$ and $\zeta = \sigma = e^{i\theta}$, these conditions require from Eqs. (5)₂, (7)₂, (17) and (19) that

$$\text{Im} \left[\frac{\gamma}{4} f_1^2(\sigma) + kg(\sigma) \right] = \text{Im} [k_h g_h(\sigma)] \quad (23)$$

$$-\frac{\gamma}{4k} f_1(\sigma) \overline{f_1(\sigma)} + \text{Re} [g(\sigma)] = \text{Re} [g_h(\sigma)] \quad (24)$$

Substituting Eqs. (15), (20) and (21) into Eqs. (23) and (24) yields, and then multiplying both sides of them by $d\sigma / [2\pi i (\sigma - \zeta)]$, one can obtain after calculating the Cauchy integration that [41]

$$g_0(\zeta) - \overline{g_0(0)} = -\frac{j_{e0}^2 R^2}{4k\gamma} (m^2 - 1) \zeta^2 - \frac{j_{u0} R}{k} (m + 1) \zeta + \frac{k_h}{k} \sum_{n=1}^{\infty} [a_n^h m^n \zeta^n - \overline{a_n^h} \zeta^n] \quad (25)$$

$$g_0(\zeta) + \overline{g_0(0)} = -\frac{j_{e0}^2 R^2}{4k\gamma} [(m^2 + 3) \zeta^2 + 4(m - 1)] - \frac{j_{u0} R}{k} (m - 1) \zeta + \sum_{n=1}^{\infty} [a_n^h m^n \zeta^n + \overline{a_n^h} \zeta^n] \quad (26)$$

To find a_n^h and $\overline{g_0(0)}$, substituting Eq. (25) into (26), and then equating the coefficients of the same power ζ^n in both sides of (26), one has

$$\overline{g_0(0)} = \frac{j_{e0}^2 R^2}{2k\gamma} (1-m) \quad (27)$$

$$a_2^h = \frac{j_{e0}^2 R^2 (m^2 + 1)}{2\gamma [k_h (m^2 - 1) - k (m^2 + 1)]} \quad (28)$$

$$a_1^h = \frac{2ij_{u0}R}{k_h (m+1) - k (m-1)} \quad (29)$$

$$a_n^h = 0, \quad (n \geq 3) \quad (30)$$

As a consequence, the solution of $g_0(\xi)$ can be written as

$$g_0(\xi) = \left[k_h a_2^h - \frac{j_{e0}^2 R^2}{4\gamma} \right] \frac{(m^2 - 1)}{k} \xi^2 + [k_h a_1^h - ij_{u0}R] \frac{(m+1)}{k} \xi - \frac{j_{e0}^2 R^2}{2\gamma} (m-1) \quad (31)$$

Finally, we have

$$g(z) = \frac{j_{e0}^2 R^2}{4k\gamma} \left(\frac{1}{\xi} + \xi \right)^2 + \frac{ij_{u0}R}{k} \left(\frac{1}{\xi} - \xi \right) + \frac{k_h}{k} [(m^2 - 1)a_2^h \xi^2 + (m+1)a_1^h \xi] \quad (32)$$

3.3 Thermally and electrically induced elastic field

Considering the condition of single-valued elastic displacement and stress, the complex potentials $\varphi(z)$ and $\phi(z)$ in the thermoelectric material have the form of

$$\begin{aligned} \varphi(z) &= A(\xi) \ln \xi + \varphi_0(z) \\ \phi(z) &= B(\xi) \ln \xi + \phi_0(z) \end{aligned} \quad (33)$$

where $A(\xi)$ and $B(\xi)$ are two unknown functions. Substituting Eq. (33) into Eqs. (9) and (12), the elastic displacement and resultant force must return the starting value when moving round the elliptic hole. These single-value conditions require

$$A = (m+1)A_0 R^2 \left[1 - \frac{2k_h}{k_h (m+1) - k (m-1)} \right] \quad (34)$$

$$B(\xi) = \overline{A} + 2(m+1)R^3 P_0 \left(\frac{1}{\xi} - \xi \right) \quad (35)$$

with $A_0 = -\frac{2\mu\alpha^* ij_{u0}}{k(\kappa+1)}$ and $P_0 = \frac{\mu\alpha^* j_{e0}^2}{2k\gamma(\kappa+1)}$.

The external forces on the rim of elliptic hole may be let zero for the thermoelectric material problem since the principle of superposition is applicable. Noting that on the elliptic hole, $\xi = \sigma = e^{i\theta}$, the stress boundary condition (12) can be expressed as

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \overline{\varphi'(\sigma)} + \overline{\phi(\sigma)} = -\frac{\mu\alpha^* \gamma}{k(\kappa+1)} f_2(\sigma) \overline{f_1(\sigma)} \quad (36)$$

Substituting Eqs. (13), (15), (16) and (33)-(35) into Eq. (36), and multiplying both sides of Eq. (36) by $d\sigma/[2\pi i(\sigma - \xi)]$ and carrying out the Cauchy integration yields [41],

$$\varphi_0(\xi) = m\overline{A} + mR^3 P_0 \xi (\xi^2 - 1) - \overline{\phi_0(0)} \quad (37)$$

Taking the conjugate form of Eq. (36), multiplying by $d\sigma/[2\pi i(\sigma - \xi)]$ and carrying out the Cauchy integration, $\phi_0(z)$ is obtained as

$$\phi_0(\xi) = -\frac{m + \xi^2}{m\xi^2 - 1} A + R^3 P_0 \xi (\xi^2 - 1) - \frac{\xi(m + \xi^2)}{m\xi^2 - 1} \varphi_0'(\xi) \quad (38)$$

From Eq. (38), we have $\overline{\phi_0(0)} = m\overline{A}$. Finally, the complex stress potentials $\varphi(z)$

and $\phi(z)$ have the form of

$$\varphi(z) = A \ln \xi + mR^3 P_0 \xi (\xi^2 - 1) \quad (39)$$

$$\begin{aligned} \phi(z) = & \left[\overline{A} + 2(m+1)R^3 P_0 \left(\frac{1}{\xi} - \xi \right) \right] \ln \xi - \frac{m + \xi^2}{m\xi^2 - 1} A \\ & + \frac{R^3 P_0 \xi}{m\xi^2 - 1} [m^2 + 1 - (3m^2 + 1)\xi^2 - 2m\xi^4] \end{aligned} \quad (40)$$

4. The solution to an elliptic rigid inclusion

Consider an infinite thermoelectric material containing an elliptic rigid inclusion with the major axis $2a$ and the minor axis $2b$ in the z -plane as shown in Fig. 1. The uniform thermoelectric loadings j_{e0} and j_{u0} are simultaneously applied at infinity. The inclusion is assumed to be electrical insulated, i.e., $\gamma_i = 0$, but its heat conductivity k_i is not zero. In addition, the bonds between the thermoelectric material and the inclusion are assumed to be perfect along the common boundary. The mapping function is expressed by Eq. (13), and the electric potential and temperature for such a thermoelectric boundary condition is given by Eqs. (15) and (32). The corresponding displacement boundary condition is,

$$\kappa\varphi(\sigma) - \frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\varphi'(\sigma)} - \overline{\phi(\sigma)} + 2\mu\alpha^* \int g(\sigma)\omega'(\sigma)d\sigma - \frac{\mu\alpha^*\gamma}{k(\kappa+1)}f_2(\sigma)\overline{f_1(\sigma)} = 0 \quad (41)$$

The stress potential functions can be determined as the similar procedures of elliptic hole problem, and the expressions of $\varphi_0(\xi)$ and $\phi_0(\xi)$ are obtained as:

$$\begin{aligned} \kappa\varphi_0(\xi) = & -(\kappa+1)P_0R^3\xi\left(\frac{m}{3}\xi^2 - 2m - 1\right) - \frac{\kappa+1}{2}A_0mR^2\xi^2 - P_0R^3m\xi(\xi^2 - 1) \\ & - 2\mu\alpha^*\frac{k_iRa_2^i}{k}(m^2 - 1)\xi\left(\frac{m}{3}\xi^2 - 1\right) - \mu\alpha^*\frac{k_iRa_1^i}{k}m(m+1)\xi^2 \end{aligned} \quad (42)$$

$$\begin{aligned} \phi_0(\xi) = & \frac{m+\xi^2}{m\xi^2-1}A - (\kappa+1)P_0R^3\xi\left(\frac{m}{3}\xi^2 - m + 2\right) - \frac{\kappa+1}{2}A_0R^2\xi^2 \\ & - P_0R^3\xi(\xi^2 - 1) + \frac{\xi(m+\xi^2)}{m\xi^2-1}\varphi_0'(\xi) \end{aligned} \quad (43)$$

where

$$a_2^i = \frac{j_{e0}^2R^2(m^2+1)}{2\gamma[k_i(m^2-1) - k(m^2+1)]} \quad (44)$$

$$a_1^i = \frac{2ij_{u0}R}{k_i(m+1) - k(m-1)} \quad (45)$$

5. Numerical results and discussion

In this section, the influences of applied electric current density and energy flux, heat conductivity of medium inside the elliptic hole/inclusion on the temperature field and stress components are investigated, respectively. The thermoelectric material is considered to be Bi_2Te_3 , and its electric conductivity $\gamma = 1.1 \times 10^5 \text{ Sm}^{-1}$, heat conductivity $k = 1.6 \text{ Wm}^{-1} \text{ K}^{-1}$, Seebeck coefficient $\varepsilon = 2 \times 10^{-4} \text{ VK}^{-1}$ [44], and Poisson's ratio $\nu = 0.4$ [45]. The effect of j_{e0} on hoop electric field $j_{e\theta}$ and $\sigma_{\theta\theta}$ around an electrical impermeable elliptic hole was studied in our previous work [21], and does not discuss it again in this paper. The energy flux and stress components σ_{rr} and $\sigma_{\theta\theta}$ in the curvilinear coordinates (r, θ) expressed by the mapping function are

$$j_{ur} - ij_{u\theta} = \frac{\xi \omega'(\xi)}{|\xi \omega'(\xi)|} [j_{ur} - ij_{u\theta}] = -\frac{\xi}{|\xi \omega'(\xi)|} \left[\frac{\gamma}{2} f_1(\xi) f_1'(\xi) + k g'(\xi) \right] \quad (46)$$

$$\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{xx} + \sigma_{yy} = 4 \operatorname{Re} \left[\frac{\varphi'(\xi)}{\omega'(\xi)} \right] + \frac{2\mu\alpha^*\gamma}{k(\kappa+1)} f_1(\xi) \overline{f_1(\xi)} \quad (47)$$

5.1 Elliptic hole case

First, a thermoelectric material with an elliptic cavity filled with air is considered, and the heat conductivity of air $k_h = 0.024 \text{ Wm}^{-1} \text{ K}^{-1}$. From Eqs. (15), (32), (39), (46) and (47), we have

$$j_{ur} - ij_{u\theta} = \frac{\xi}{R\xi(m - \xi^{-2})} \left\{ j_{u0} R(1 + \xi^{-2}) - k_h \left[2(m^2 - 1)a_2^h \xi + (m+1)a_1^h \right] \right\} \quad (48)$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 4 \operatorname{Re} \left[\frac{\xi A + mR^3 P_0 \xi^2 (3\xi^2 - 1)}{R(m\xi^2 - 1)} \right] + 4R^2 P_0 \left(\frac{1}{\xi} - \xi \right) \left(\frac{1}{\xi} - \bar{\xi} \right) \quad (49)$$

Noting that on the hole rim, $\zeta = \sigma = e^{i\theta} = \cos \theta + i \sin \theta$, $j_{u\theta}$ and $\sigma_{\theta\theta}$ can be obtained as

$$j_{u\theta} = \frac{(m^4 - 1)k_h R j_{e0}^2 \sin \theta}{\gamma \left[(m^2 - 1)k_h - (m^2 + 1)k \right] \sqrt{m^2 + 1 - 2m \cos(2\theta)}} + \frac{2(m - 1)k j_{u0} \cos \theta}{\left[(m + 1)k_h - (m - 1)k \right] \sqrt{m^2 + 1 - 2m \cos(2\theta)}} \quad (50)$$

$$\sigma_{\theta\theta} = \frac{4mR^2 P_0 \left\{ [m - \cos(2\theta)] [3 \cos(2\theta) - 1] + 3 \sin^2(2\theta) \right\}}{m^2 + 1 - 2m \cos(2\theta)} + 16R^2 P_0 \sin^2 \theta - \frac{4i(m + 1)^2 R A_0 \sin \theta}{m^2 + 1 - 2m \cos(2\theta)} \left[1 - \frac{2k_h}{k_h(m + 1) - k(m - 1)} \right] \quad (51)$$

From the definition of energy flux and potential function F , the hoop heat flux q_θ on the hole rim has the form of

$$q_\theta = j_{u\theta} + \frac{2j_{e0}^2 R \sin(2\theta)}{\gamma \sqrt{m^2 + 1 - 2m \cos(2\theta)}} - \frac{\varepsilon j_{e0}^3 R^2 (m^2 + 1)^2 \cos \theta \cos(2\theta)}{\left[(m^2 - 1)k_h - (m^2 + 1)k \right] \sqrt{m^2 + 1 - 2m \cos(2\theta)}} \quad (52)$$

From Eq. (52), we can see that $q_\theta = j_{u\theta}$ if $j_{e0} = 0$, the j_{e0}^2 and j_{e0}^3 terms are induced by the Joule heat and Seebeck effect, respectively. It is clear that constants P_0 and A_0 stand for the applied electric current density and energy flux from Eqs. (34) and (35), respectively. Stress field has a linear relationship with the remote applied energy flux, but has a non-linear relationship with the remote applied electric current density. The variations of normalized $j_{u\theta}/j_{u0}$ and $\sigma_{\theta\theta}/(iRA_0)$ along the rim of the elliptic hole with θ for different a/b are plotted in Figs. 2-3. It is found that the concentration factors of hoop energy flux and stress increase as a/b increases, no matter if the heat conductivity of air inside the hole is or not assumed to be zero. The

numerical results show that the temperature field within the elliptic hole can be neglected for the case of $a/b \leq 2$, that is, the heat impermeable boundary condition can be used to simplify the analysis. However, as a/b increases, the effect of k_h on hoop energy flux and stress should be considered by comparing the results obtained based on the heat exact boundary condition and heat insulated model, respectively. Figs. 4 and 5 show that the effect of applied electric current density j_{e0} on the normalized hoop energy flux $j_{u\theta}$ and hoop heat flux q_θ for the different value of major to minor axis ratio of the elliptic hole. Generally, the distribution of temperature field is affected by the heat conductivity of air inside the elliptic hole k_h due to thermoelectric coupling effect even if the electric current density j_{e0} is solely applied.

5.2 Elliptic rigid inclusion case

Further, to study the effects of thermoelectric loadings on mechanical behavior of elliptic rigid inclusion problem, the hoop stress along the elliptic rigid inclusion boundary can be calculated from Eqs. (33)₁, (42) and (47), and it is very complicate. In the following analysis, the hoop stress is given for the case of $j_{e0} = 0$ to illustrate the influence of heat conductivity of inclusion and a/b on $\sigma_{\theta\theta}$,

$$\begin{aligned} \sigma_{\theta\theta} = & -\frac{4i(m+1)RA_0}{m^2+1-2m\cos(2\theta)} \left\{ \frac{m+1}{\kappa+1} \left[1 - \frac{2k_i}{k_i(m+1)-k(m-1)} \right] \sin\theta \right. \\ & \left. + \frac{1}{\kappa} \left[\frac{2mk_i}{k_i(m+1)-k(m-1)} - \frac{m}{m+1} \right] [\sin(3\theta) - m\sin\theta] \right\} \end{aligned} \quad (53)$$

The expression of hoop energy flux at the elliptic inclusion faces agrees with Eq. (50). Figs. 6-8 show that k_i has great influence on the hoop energy flux and stress at the inclusion faces when the remote energy flux is applied. It is found that the results

based on the exact boundary condition show the hoop component of heat flux decreases with the increasing value of k_i/k for $a/b=10$ as shown in Fig. 6. However, unlike the elliptic cavity problem, Figs. 7 and 8 show that the concentration factors of hoop stress increase as the heat conduction coefficient of inclusion increasing for a given thermoelectric loading and a/b . The reason is that the displacement on the common boundary between the thermoelectric medium and inclusion caused by the temperature change is completely prevented since the elliptic inclusion is assumed to be absolutely rigid and the common interface is perfectly bonded. It is shown from Figs. 9 and 10 that energy flux and heat flux inside the inclusion is not zero when the heat exact boundary condition is used for a given electric current density due to the Joule heat and Seebeck effect. When j_{e0} exerted at infinity, the influence of k_i/k is very small on $\sigma_{\theta\theta}$ on the rim of inclusion (Fig. 11). Especially for $a/b=1$ (a circular inclusion), the heat conductivity of the rigid inclusion can be neglected.

6. Conclusion

The exact solution for an elliptic hole or rigid inclusion embedded in an infinite thermoelectric material subjected to far-field uniform energy flux and electric current density loading is presented in this paper. The derivation is based on the complex conformal mapping technique, electric insulated and heat exact boundary conditions on the rim of the hole or inclusion. The complex potentials in the thermoelectric medium and temperature field inside the hole or inclusion are obtained in closed-form, respectively. Particular attention is given to the effect of heat conduction coefficient inside the elliptic hole or the rigid inclusion on the hoop energy flux and stress distribution. For the elliptic hole case, it is found that the influence of heat

conductivity of air is very limited. It can be neglected especially when $a/b \leq 2$. However, for the elliptic rigid inclusion case, the heat conductivity of inclusion has a great influence on the hoop energy flux and stress in thermoelectric material when the energy flux is exerted at infinity. In addition, the stress concentration factor increases as the heat conduction coefficient of rigid inclusion increases. This paper presents a theoretical basis on studying the effective behavior of thermoelectric composite materials with cracks for their design and optimization.

Finally, the effect of electric permeability of an elliptic rigid inclusion on the fracture behavior of thermoelectric materials should be considered if the metallic inclusion is introduced. Here, some brief discussions are given. The electric current density, electric potential and temperature fields across the interface between the matrix and inclusion are continuous, therefore, the electric boundary condition can be expressed as,

$$\begin{aligned} \gamma \left[f_1(z) - \overline{f_1(z)} \right] &= \gamma_i \left[f_i(z) - \overline{f_i(z)} \right] \\ \text{Re} \left[f_1(z) \right] &= \text{Re} \left[f_i(z) \right] - \frac{\varepsilon \gamma_i}{8k_i} \left[f_i(z) + \overline{f_i(z)} \right]^2 + \varepsilon \text{Re} \left[g_i(z) \right] \end{aligned} \quad (54)$$

where γ_i is the electric permeability of the inclusion, and $f_i(z)$ is the electric potential function to be determined by the boundary conditions. The subsequent task is to obtain the temperature and stress fields, which is similar to the analysis in Section 3.

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Fig. 1

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Fig. 4

Fig. 5

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