Novel Relay Node Placement Algorithms for Establishing Connected Topologies

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Abstract

This paper addresses the problem of placing the least number of fixed-range relay nodes (RNs) in order to establish multi-hop paths between every pair of terminals. We derive an optimal solution for the case of three terminals and for the cases of more than 3 terminals, we present three novel heuristics, namely, Optimized Triangle Selection based on Minimum Spanning Tree Triangulation (OTS-MST), Incremental Optimization based on Delaunay Triangulation (IO-DT) and hybrid approach. These heuristics take advantage of the optimal three-terminal based solution by forming connected sub-graphs for steinerized sets of three terminals and then connecting these sub-graphs via steinerized edges. OTS-MST considers triangles that have two mst edges and picks the subset of these triangles which provides the highest reduction in the total number of required RNs as compared to a solution that is based on steinerized mst edges. IO-DT calculates the Delaunay triangulation of terminals and iterates over the formed triangles. In each iteration, the algorithm steinerizes a triangle as part of the final topology if selecting such a triangle reduces the RN count. Finally we consider a hybrid approach, which combines the strengths of OTS-MST and IO-DT. The performance of the proposed algorithms is validated through simulation.

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1. Introduction

There has been a massive growth in the use of networked devices in civil, scientific and military applications in recent years. In some of these applications need arises to interconnect a set of terminals to serve a certain mission while
5 avoiding major infrastructure investment. An example is when multiple data sources or standalone networks need to be federated in order to aggregate their services and efficiently handle emerging events such as search and rescue, disaster management, criminal hunting, etc. The federation is often enabled by placing relays that form paths between these sources/networks. Another example is
10 when a wireless sensor network (WSN) gets partitioned due to a breakdown or damage of a subset of its nodes. WSNs often serve in inhospitable environments which make nodes susceptible to failure, e.g., due to detonation of explosives in a battle field, natural calamities, etc. [1]. The failed nodes may cause the WSN to be split into disjoint segments. Restoring connectivity among these WSN
15 segments would be necessary for the network to become operational again. Deploying relay nodes (RNs) is the most popular approach [2, 3] for repairing the network topology. Given the limited accessibility, the least resources should be used/employed.

On the other hand, many applications serve in controlled setups in which
20 deterministic positioning of nodes is pursued to meet the design requirements and provide the desired quality of service [4]. Examples of these applications include security surveillance, habitat monitoring, safety assessment of factory floor, hazard detection in urban area, etc. The controlled nature of the application allows pre-planning and careful selection of where to place nodes and
25 how they may communicate. Usually area or landmark coverage is the main criterion for determining where sensor nodes should be. For establishing connectivity, relay nodes (RNs) are deployed to ensure that data can be shared.

Determining the number and the position of RNs in the aforementioned application scenarios is usually subject to optimization. The application designer
 30 often likes to employ the least RNs to achieve the desired connectivity goal. In addition, the positioning of these RNs should establish, possibly, a multi-hop path between the terminals, e.g., data sources or network segments, subject to communication range constraints. Our problem can be thus formally defined as follows: “Given n disjoint WSN segments what are minimum number and
 35 position of the relay nodes for establishing inter-segment connectivity?”. This problem is equivalent to the Steiner Minimum Tree with Minimal Steiner Points and Bounded Edge-Length (SMT-MSPBEL), which is shown to be NP-Hard by Lin and Xue [5].

A number of heuristics have been proposed in the literature for solving the
 40 SMT-MSPBEL problem. Some of these heuristics focus on reducing the number of RNs and form a minimum spanning tree (mst), where the node degree of the individual RNs is two [6, 7, 5]. Meanwhile, the focus of other work has been on the degree of connectivity of the formed WSN topology with less emphasis on the RN count [8, 9]. In this paper we present an algorithm which finds the least
 45 RN count for forming SMT-MSPBEL for the special case of three terminals (i.e., a triangle) by either: (1) populating the RNs along mst edges, i.e. two side of the triangle, with a distance of R units apart, where R is the communication range of RNs, or (2) finding a point x inside the triangle, which we call Discrete Fermat Point, and connecting the vertices of the triangle to x . Next
 50 we study the generalized problem for the case of n terminals and present three novel algorithms, namely Optimized Triangle Selection based on mst triangulation (OTS-MST), Incremental Optimization based on Delaunay Triangulation (IO-DT) and a hybrid heuristic. These algorithms opt to minimize the required RN count and vary in the solution quality and runtime complexities.

55 The main idea behind the proposed algorithms is to optimally solve the SMT-MSPBEL problem for the case of three terminals and try to find the minimal RN count for larger networks using subsets of three terminals. OTS-MST finds the mst of the terminals and identifies the set of triangles which contains two

mst edges. The algorithm then finds adjacent triangles (i.e. the triangles shar-
 ing one mst edge) and forms a graph of triangles called Triangular Adjacency
 Graph (TAG) whose vertices represent triangles and edges denote adjacency
 relationships. Using TAG, the OTS-MST algorithm finds the best subset of
 non-adjacent triangles which provides the highest reduction in the total number
 of required relays as compared to an mst -based solution. The runtime com-
 plexity of OTS-MST is $O(n^2)$. Meanwhile, the IO-DT algorithm calculates the
 Delaunay triangulation (DT) of terminals and iterates over the formed triangles
 with the time complexity of $O(n^2)$. In each iteration, the algorithm steinerizes
 a triangle, i.e., form a Steiner tree of 3 vertices, as part of the final topology if
 selecting such a triangle provides a reduction in the total number of required
 relays as compared to an mst -based solution. Unlike OTS-MST, IO-DT does
 not restrict itself to triangles having two mst edges. The third algorithm is a
 hybrid heuristic that combines the strengths of both OTS-MST and IO-DT.
 The performance of all three heuristics is also validated through simulation and
 is shown to exceed contemporary heuristics in the literature.

The next section discusses related work. Section III describes the optimal
 solution for three terminals. Sections IV, V and VI describe the OTS-MST,
 IO-DT and hybrid approaches in detail, respectively. The runtime complexity
 and performance of the algorithms are analyzed in section VII. Section VIII
 presents the simulation results. Finally, section IX concludes the paper.

2. Related Work

The goal of the proposed algorithms is to place the minimum number of
 RNs in such a way that there exists a path between every pair of terminals
 and each hop in the path is less than or equal to the transmission range of the
 individual nodes. As we mentioned, this problem is shown to be NP-Hard by
 Lin and Xue [5]. They have further presented a polynomial time approximation
 algorithm that deploys RNs along the mst edges of terminals. The algorithm
 first constructs the complete graph of terminals and form mst using Kruskal's

algorithm. RNs are populated along the tree edges at a distance of at most R apart, where R is the communication range of RNs. In [10], the authors proved
90 that the algorithm yields a solution that is at most 4 times as much as the optimal value. In [6], the authors employ a three-step heuristic. In the first step, they connect the nodes where the distance between the nodes is less than or equal to R . In the second step, 3-stars are formed for each subset of three nodes u, v, w , for which there exists a point s such that s is at most R units away
95 from u, v , and w . Assuming that $\overline{uv}$ and $\overline{uw}$ are *mst* edges, a 3-star modifies the *mst* by replacing $\overline{uv}$ and $\overline{uw}$ with $\overline{su}$, $\overline{sv}$ and $\overline{sw}$. In the last step the algorithm populates RNs along modified *mst* edges that are not substituted with 3-stars. In [6], the authors also presented a randomized algorithm that yields better results, yet it has 50% probability of convergence. A randomized algorithm does
100 not necessarily guarantee the solution consistently. In this paper, we focus on the same problem studied in [5, 6, 10]. Unlike these publications, the proposed algorithms strive to find a subset of triangles and form triangular Steiner trees by connecting the vertices of each triangle at a point inside the triangle which will be called as Discrete Fermat Point. As we validate through simulation,
105 the proposed algorithms significantly outperform those of [5, 6, 10]. In [11], we have studied the problem in the context of federating disjoint network segments and presented an algorithm called FeSTA with $O(n^4)$ runtime complexity. The algorithm iterates over all possible triangles and greedily selects triangles to apply Discrete Fermat Point optimization. The algorithms proposed in this paper,
110 namely OTS-MST, IO-DT and Hybrid, are significantly less complex than FeSTA. A preliminary version of IO-DT algorithm was reported in [12]. This paper presents an additional scheme and extend IO-DT.

The focus of [8] and [9] is on a variant of the SMT-MSPBEL problem where additional metric is to be optimized. In [8], Lee and Younis strive to connect the
115 terminals inward so that the inter-terminal topology has a high node degree. They have proposed an algorithm called CORP that models the deployment area as equal-sized grid cells. CORP has two phases. In the first phase, the algorithm iteratively identifies border terminals and determines the best cell

for deploying an RN based on the distance between the farthest terminals (or
 120 RN) from the centroid. The cells connecting multiple terminals are called junction cells. In the second phase, after all terminals are connected, the algorithm prunes redundant RNs. In [9] a bio-inspired heuristic called SpiderWeb has been proposed to form topologies that not only exhibit stronger connectivity but also achieve better sensor coverage and enable balanced distribution of traffic load
 125 on the employed relays. Unlike these algorithms, minimizing the number of RNs is the main objective in this paper.

In addition to establishing connected topologies, relay node placement has been studied for improving the performance in terms of other metrics [4]. Network lifetime has received the most attention among them. Depending on the
 130 transmission media, energy consumption for transmitting data is directly proportional to d^j where d is the distance and $j \geq 2$ [7]. Due to this fact, long distance data transmission is very costly and stabilizing multi-RN paths is advantageous. In [13] the authors have studied this problem for both single-tiered and two-tiered WSN architectures. In single-tiered architectures, sensors transmit
 135 their own messages and also forward messages from other nodes. Meanwhile in two-tiered architectures nodes transmit only their own messages to a relay node or the base-station. Distinct polynomial-time approximation RN placement algorithms for one- and two-tiered architectures have been developed by modeling the network as a unit disc graph and identifying the best position of
 140 RNs based on overlapping unit-disc regions. In [7], the authors have studied the same problem assuming that relays and sensors have different transmission ranges. [14] opted to reduce the communication energy consumed by a gateway node in sending data to the base-station. The authors consider a two-tiered network architecture where mobile sensors are assigned to a cluster that
 145 is led by an aggregation-and-forwarding node. The RN placement and energy provisioning problem is formulated as a mixed-integer nonlinear programming optimization. In [15], the authors studied a similar problem with the goal of reducing the system overhead in two-tiered network architectures. In order to minimize the energy consumption, the problem is first formulated as Relay Node

150 Single Cover problem which is mathematically equivalent to the Geometric Disc
Covering problem, and then they placed additional relays to guarantee that the
final network topology is connected. Unlike these approaches, we focus on min-
imizing the RN count rather than energy reduction.

In [16], the authors studied a constrained version of RN placement prob-
155 lem to satisfy network connectivity requirement assuming that RNs can not be
populated in dangerous regions. The authors assigned a safety weight for each
deployment point such that a larger weight indicates more dangerous point to
deploy a RN. Unlike our work, the goal of the work presented in [16] is to mini-
mize the total weights of the points where RNs are placed, instead of minimizing
160 the total number of RNs deployed.

On the other hand, In [17] the authors have studied the node placement
problem in WSNs while assuming that RNs can directly reach the base-station.
Given a deployment of sensor nodes, the problem is to find the positions of
RNs in order to meet the constraints on network lifetime and sensor-relay node
165 connectivity. The optimization is shown to be equivalent to minimum set cov-
ering problem which is an NP-Hard problem. Hence, a recursive heuristic is
proposed to find an approximate solution. This work has been further extended
in [18, 19, 20, 21] in order to address the problem of deploying multi-tier RNs
while balancing the traffic load with these additional resources. The focus of this
170 paper is on establishing connected topologies rather than prolonging the net-
work lifetime. In [22], the authors have studied a similar problem with a different
network model assuming heterogeneous WSN with multiple base-stations. [23]
have investigated the effectiveness of several heuristics such Binary integer lin-
ear programming and genetic algorithms in solving the RN placement problem.
175 However, no comparison with well-established techniques has been made. As
opposed to these publications, we focus on the connectivity of several terminals
and not just sensor-relay connectivity.

3. Solving SMT-MSPBEL of 3-Terminals

As mentioned earlier the problem that we address in this paper is how to connect n disjoint WSN segments using the minimum number of relay nodes. Such a problem is equivalent to SMT-MSPBEL problem [5]. In this section we provide an optimal solution for forming SMT-MSPBEL for the special case of three terminals. In the following sections we will direct out attention to the generalized case for n terminals. Before presenting the 3-terminals algorithm we give some important definitions that cover the basis of the solution. Note that we assume that all relay and terminal nodes have the same communication range “ R ”.

Definition 1. The minimum number of Steiner Points (SPs) required for forming SMT-MSPBEL for two terminals u and v is called the *sp-weight* of the edge $\overline{uv}$ and denoted as $W_{sp}(u, v) = \lceil \frac{|uv|}{R} \rceil - 1$, where $|uv|$ is the Euclidean distance between u and v .

Definition 2. Let T_i be a triangle where the vertices are (u, v, w) . The Fermat Point (FP) f_i (also called Torricelli Point [24]) for the triangle T_i is the point that minimizes the sum of Euclidean distances from the vertices of T_i to f_i , i.e., $f_i = \min_q \{|uq| + |vq| + |wq|\}$ where q is any arbitrary point in triangle T_i .

Lemma 1. If a triangle T_i contains an angle greater than or equal to 120° , then FP of T_i is located at the corner with the biggest angle. [24]

Definition 3. Let T_i be the triangle (u, v, w) and q be a point inside T_i . The number of relays required for connecting the vertices of T_i at an arbitrary point q is called the *fp-weight* of T_i at q and denoted as $W_{fp}(T_i, q)$. Basically, a relay will be placed at q and then connected to the vertices, u , v and w , by placing the relays on the edges $\overline{qu}$, $\overline{qv}$, and $\overline{qw}$, respectively. Thus, $W_{fp}(T_i, q) = W_{sp}(u, q) + W_{sp}(v, q) + W_{sp}(w, q) + 1$, or

$$W_{fp}(T_i, q) = (\lceil \frac{|uq|}{R} \rceil - 1) + (\lceil \frac{|vq|}{R} \rceil - 1) + (\lceil \frac{|wq|}{R} \rceil - 1) + 1$$

$$W_{fp}(T_i, q) = \lceil \frac{|uq|}{R} \rceil + \lceil \frac{|vq|}{R} \rceil + \lceil \frac{|wq|}{R} \rceil - 2 \quad (1)$$

The point φ_i that minimizes the *fp-weight* of triangle T_i is called Discrete Fermat Point (DFP) and the weight at DFP is called the *dfp-weight* of the triangles and denoted as $W_{fp}(T_i, \varphi_i)$, or simply $W_{dfp}(T_i)$. DFP can be formally defined as follows:

$$\varphi_i = \min_q \left\{ \lceil \frac{|uq|}{R} \rceil + \lceil \frac{|vq|}{R} \rceil + \lceil \frac{|wq|}{R} \rceil - 2 \right\} \quad (2)$$

205 If we remove the ceiling brackets and multiply by “ R ”, the minimization becomes linear in the distance and the point φ_i would be the Fermat Point. The DFP is the discretized version of Fermat Point, which is the basis for the name. There are only two possible ways to form optimal SMT-MSPBEL for the three terminals u , v , and w : (1) via steinerizing the two smallest edges of the triangle $T_i(u, v, w)$, as illustrated in Figure 1(a). This is also equivalent to forming steinerized *mst* for three terminals; (2) via finding DFP φ_i of T_i and steinerizing the edges $\overline{u\varphi_i}$, $\overline{v\varphi_i}$, $\overline{w\varphi_i}$ as illustrated in Figure 1(b). As we are going to explain later, the solution of steinerizing the two smallest edges, i.e., Figure 1(a), corresponds to having DFP located on one of the sides on the triangle.

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Figure 1: Two possible solutions for forming SMT-MSPBEL for a triangle $T_i = (u, v, w)$, (a) via steinerizing the two smallest edges of the triangle, (b) via DFP φ_i

Theorem 1. *The optimal solution for forming SMT-MSPBEL for three terminals u , v , and w can be achieved either via steinerizing the two smallest edges*

Figure 2: Illustrating how to form SMT-MSPBEL via DFP for a sub-triangle $T_i'(u, \varphi_i, w)$

of the triangle $T_i(u, v, w)$ or steinerizing the edges from its vertices to the DFP φ_i .

220 PROOF. We prove the theorem in two cases:

Case 1: In the optimal solution all RNs lie on the straight line joining pairs of terminals. This case is straightforward and applicable if there is no such point T_i inside the triangle which provides a better solution.

225 Case 2: In the optimal solution there is a RN inside the triangle which connects all pairs of terminals. By definition 3 we know that such an RN is located at the DFP of the triangle. As a result of finding the DFP φ_i of a triangle, three sub-triangles are formed namely, $T_i'(u, \varphi_i, w)$, $T_i''(u, \varphi_i, v)$ and $T_i'''(v, \varphi_i, w)$.

To form SMT-MSPBEL via DFP for T_i , we form steinerized mst for each of these sub-triangles. To prove the optimality, we need to show that forming the steinerized mst is the optimal solution for the sub-triangles. We pursue a proof by contradiction. Suppose forming SMT-MSPBEL for a sub-triangle, say $T_i'(u, \varphi_i, w)$, via DFP yields a better solution than forming steinerized mst using φ_i' , the DFP of the sub-triangle $T_i'(u, \varphi_i, w)$, as illustrated in Figure 2. Based on that assumption we can write the following: $W_{fp}(T_i', \varphi_i') < W_{sp}(u, \varphi_i) + W_{sp}(w, \varphi_i)$

$$\lceil \frac{|u\varphi_i'|}{R} \rceil + \lceil \frac{|w\varphi_i'|}{R} \rceil - 2 < \lceil \frac{|u\varphi_i|}{R} \rceil + \lceil \frac{|w\varphi_i|}{R} \rceil - 2 \quad (3)$$

In addition applying triangular inequality on $T(v, \varphi_i, \varphi_i')$, we have: $|v\varphi_i| +$

$|\varphi_i \varphi'_i| \geq |v \varphi'_i|$. Then, $\frac{|v \varphi_i|}{R} + \frac{|\varphi_i \varphi'_i|}{R} \geq \frac{|v \varphi'_i|}{R}$ and thus,

$$\lceil \frac{|v \varphi_i|}{R} \rceil + \lceil \frac{|\varphi_i \varphi'_i|}{R} \rceil \geq \lceil \frac{|v \varphi'_i|}{R} \rceil \quad (4)$$

After forming SMT-MSPBEL for the sub-triangle $T'_i(u, \varphi_i, w)$ via DFP, the total number of relay nodes required for forming SMT-MSPBEL for triangle $T_i(u, v, w)$ is the sum of *sp-weights* of the edges $\overline{v \varphi'_i}$, $\overline{\varphi_i \varphi'_i}$, $\overline{u \varphi'_i}$, $\overline{w \varphi'_i}$ plus 2 for the points φ_i and φ'_i

$$\begin{aligned} & (\lceil \frac{|v \varphi_i|}{R} \rceil - 1) + (\lceil \frac{|\varphi_i \varphi'_i|}{R} \rceil - 1) + (\lceil \frac{|u \varphi'_i|}{R} \rceil - 1) + (\lceil \frac{|w \varphi'_i|}{R} \rceil - 1) + 2 \\ &= \lceil \frac{|v \varphi_i|}{R} \rceil + \lceil \frac{|\varphi_i \varphi'_i|}{R} \rceil + \lceil \frac{|u \varphi'_i|}{R} \rceil + \lceil \frac{|w \varphi'_i|}{R} \rceil - 2 \end{aligned}$$

Using (4) we have,

$$\begin{aligned} & \lceil \frac{|v \varphi_i|}{R} \rceil + \lceil \frac{|\varphi_i \varphi'_i|}{R} \rceil + \lceil \frac{|u \varphi'_i|}{R} \rceil + \lceil \frac{|w \varphi'_i|}{R} \rceil - 2 \geq \\ & \lceil \frac{|v \varphi'_i|}{R} \rceil + \lceil \frac{|u \varphi'_i|}{R} \rceil + \lceil \frac{|w \varphi'_i|}{R} \rceil - 2 \end{aligned}$$

Based on definition 3, $\lceil \frac{|v \varphi'_i|}{R} \rceil + \lceil \frac{|u \varphi'_i|}{R} \rceil + \lceil \frac{|w \varphi'_i|}{R} \rceil - 2$ is the *fp-weight* of the triangle $T_i(u, v, w)$ at point φ'_i , which can be shortly written as $W_{fp}(T_i, \varphi'_i)$

$$\lceil \frac{|v \varphi_i|}{R} \rceil + \lceil \frac{|\varphi_i \varphi'_i|}{R} \rceil + \lceil \frac{|u \varphi'_i|}{R} \rceil + \lceil \frac{|w \varphi'_i|}{R} \rceil - 2 \geq W_{fp}(T_i, \varphi'_i) \quad (5)$$

Using (3) we can write

$$W_{dfp}(T_i) = \lceil \frac{|u \varphi_i|}{R} \rceil + \lceil \frac{|v \varphi_i|}{R} \rceil + \lceil \frac{|w \varphi_i|}{R} \rceil - 2 >$$

$$\lceil \frac{|v \varphi_i|}{R} \rceil + \lceil \frac{|\varphi_i \varphi'_i|}{R} \rceil + \lceil \frac{|u \varphi'_i|}{R} \rceil + \lceil \frac{|w \varphi'_i|}{R} \rceil - 2 \quad (6)$$

From (5) and (6), we have $W_{dfp}(T_i) > W_{fp}(T_i, \varphi'_i)$. However this is a contradiction since the definition of DFP implies that $W_{dfp}(T_i)$ is minimal and cannot be greater than $W_{fp}(T_i, \varphi'_i)$. Therefore φ'_i of $T'_i(u, \varphi_i, w)$ must be equal to φ_i , which is equivalent for forming steinerized *mst* for the sub-triangle $T'_i(u, \varphi_i, w)$. The same proof can be applied to $T''_i(u, \varphi_i, v)$ and $T'''_i(v, \varphi_i, w)$. For complete-

ness, it is worth noting that if we consider simultaneously steinerizing two of the
 240 sub-triangles via DFP, the resulting topology will not be cycle-free. Therefore,
 considering two (or three) triangles simultaneously will increase the required
 number of RNs and provide no benefit to the solution.

Per theorem 1, assuming that steinerizing the two smallest *mst* edges does not
 provide a better solution, if we find the DFP φ_i of a triangle T_i , we find the
 245 optimal SMT-MSPBEL for T_i . The rest of this section describes how to find
 the DFP φ_i of a triangle T_i .

Theorem 2. *Let $T_i(u, v, w)$ be a triangle, f_i be the Fermat Point of T_i . For
 any arbitrary point q , $W_{fp}(T_i, q) \geq W_{fp}(T_i, f_i) - 2$*

PROOF. Let $D(T_i, q) = \frac{|uq|+|vq|+|wq|}{R}$, and $E(T_i, q) = \lceil \frac{|uq|}{R} \rceil + \lceil \frac{|vq|}{R} \rceil + \lceil \frac{|wq|}{R} \rceil$
 where R is constant. Per definition 3 we can write

$$D(T_i, q) \leq E(T_i, q)$$

Moreover we have,

$$0 \leq E(T_i, q) - D(T_i, q) < 3$$

Per definition 2 we know, $D(T_i, q) \geq D(T_i, f_i)$ and the equality only holds when
 $q = f_i$ which is a unique point. So let's say $q \neq f_i$, then $D(T_i, q) > D(T_i, f_i)$.
 Now let's say $D(T_i, q) = E(T_i, q) - \epsilon_0$, and $D(T_i, f_i) = E(T_i, f_i) - \epsilon_1$, where
 $0 \leq \epsilon_0, \epsilon_1 < 3$. Since $D(T_i, q) > D(T_i, f_i)$, we can write

$$E(T_i, q) - \epsilon_0 > E(T_i, f_i) - \epsilon_1$$

Thus

$$E(T_i, q) > E(T_i, f_i) - (\epsilon_1 - \epsilon_0)$$

We know that $0 \leq \epsilon_0, \epsilon_1 < 3$, it is clear that $3 > \epsilon_1 - \epsilon_0$. Therefore we have
 250 $E(T_i, q) > E(T_i, f_i) - 3$. Since $E(T_i, q) = W_{fp}(T_i, q) + 2$ and W_{fp} is an integer-
 valued function, we can write $W_{fp}(T_i, q) \geq W_{fp}(T_i, f_i) - 2$

In theorem 2, we proved a lower bound for the *fp-weight* of any arbitrary point q . The theorem 2 implies that the *dfp-weight* of a triangle cannot be less than $W_{fp}(T_i, f_i) - 2$. We can write the following: $|uq| = k_1R + \epsilon_1$, $|vq| = k_2R + \epsilon_2$,
 255 and $|wq| = k_3R + \epsilon_3$ where $k_i \in \mathbb{Z}$ and $\epsilon_i < R$, $\forall i \in 1, 2, 3$. We thus can re-write (2) as:

$$\varphi_i = \min_q k_1 + k_2 + k_3 + \left\{ \left\lceil \frac{\epsilon_1}{R} \right\rceil + \left\lceil \frac{\epsilon_2}{R} \right\rceil + \left\lceil \frac{\epsilon_3}{R} \right\rceil - 2 \right\} \quad (7)$$

The optimal choice of φ_i corresponds to $\epsilon_i = 0$, $\forall i \in 1, 2, 3$, which will be equivalent to the Fermat Point of the triangle. If φ_i does not match the Fermat point, we should have either $\epsilon_1 = \epsilon_2 = 0$, or $\epsilon_1 = \epsilon_3 = 0$, or $\epsilon_2 = \epsilon_3 = 0$, in
 260 order to minimize (7). Given that $|uq| = k_1R + \epsilon_1$ and $|vq| = k_2R + \epsilon_2$, if a choice for the point q that makes $\epsilon_1 = \epsilon_2 = 0$ will be located at the intersection of two circles; one is centered at u with radius k_1R , the other is centered at v with radius k_2R . The same applies for $\epsilon_1 = \epsilon_3 = 0$, and $\epsilon_2 = \epsilon_3 = 0$.

At this point we are dealing with how to find DFP, in essence how many RNs
 265 can be saved by moving f_i to another location. Due to theorem 2 our saving can be at most 2. Figure 3 (a), (b) and (c) analyzes the cases for how DFP is obtained starting from Fermat Point f_i and moving f_i to the circle intersections. In Figure 3 two circles are drawn from each corner in such a way that FP f_i is located inside the bigger circle, but outside the smaller circle. For example
 270 in Figure 3(a), assuming that $|uf_i| = k_1R + \epsilon_1$ where $0 \leq \epsilon_1 < R$, two circles are drawn which are both centered at u where the bigger circle have a radius $(k_1 + 1)R$ and smaller circle have a radius k_1R . The other circles are drawn in the same manner from other corners v and w . In Figure 3(a) the points in the shaded region have a *dfp-weight* that is 2 less than the weight at f_i which is optimal according to theorem 2. If there is no region where 2 RNs can be saved
 275 in that case we will look whether we can save 1 RN. Figure 3(b) illustrates the case where at most 1 RN can be saved which is the intersection points of circles. Figure 3(c) illustrates the case when no RN can be saved. In Figure 3(c) since the angle $\widehat{acb}$ is greater than 120° , f_i of the triangle is located on the corner

Figure 3: Case-by-case analysis of the affect of moving f_i to circle intersections. (a) The *dfp-weights* of the points in the shaded region are 2 less than the weight of f_i . The shaded region is the intersection of circles centered at u, v and w having radii k_1R, k_2R and $(k_3+1)R$, respectively. (b) The *dfp-weights* of the intersections points are 1 less than the weight f_i , (c) No possible gain.

280 of the largest angle (i.e., $f_i = w$). In this case when we draw circles centered at the corners u and v none of the intersections provide a better solution than f_i . At this point one can ask the following question: How can we make sure that these 3 cases reflect all possibilities? In other words, in the case that we save 1 RN as illustrated in Figure 3 (b) (or no RNs as in Figure 3 (c), is there
 285 another point somewhere else which is far from f_i where we can save 2 RNs? As we discussed above the points that make $\epsilon_1 = \epsilon_2 = 0$ are located at the intersection of two circles; one is centered at u with radius k_1R , the other is centered at v with radius k_2R . To find all these points which make $\epsilon_1 = \epsilon_2 = 0$

Figure 4: The intersection points of the circles which are centered at the corners with incremental radii $(R; 2R; 3R; ; kR)$

we draw circles with incremental radii $(R; 2R; 3R; ; kR)$ from each corner, as
 290 illustrated in Figure 4. As shown in the figure, the DFP can be located at one
 of the edges. The intersection points which are far from f_i and close to one of
 the corners (lets say u), also satisfy $\epsilon_1 = \epsilon_2 = 0$ property. In this case, however,
 k_1 decreases while k_2 and k_3 increases. In other words, we save from one of the
 corners at an increased cost from the other two corners. Thus the overall saving
 295 will converge to one of the cases which is illustrated in Figure 3. Therefore, we
 conclude that no further optimization is possible. In addition, since two circles
 are drawn for each corner of the triangle, the maximum number of intersection
 points of the circles will be constant. Hence the time complexity of finding DFP
 for a triangle will be $O(1)$. The pseudo-code for finding the DFP of triangle is
 300 provided in Algorithm 1.

Theorem 1 provides the base for finding the optimal solution for the SMT-
 MSPBEL for the case of 3 terminals. Since SMT-MSPBEL is fundamentally
 NP-hard for the general case of n terminals, we propose three novel heuris-
 tics by grouping the n terminals into a set of non-collinear 3-tuples, forming
 305 a SMT-MSPBEL for each 3-tuple, and then establishing a connected topology
 by federating these formed 3-tuples based SMT-MSPBELs. The differences be-
 tween the proposed heuristics are in how the set of 3-tuples are picked among all
 C_3^n possible combinations and how the formed 3-tuples based SMT-MSPBELs
 are interconnected. The proposed heuristics are presented in the next sections.

Algorithm 1 FINDING DFP OF A TRIANGLE

Require: Triangle $T_i(u, v, w)$, constant R

Ensure: DFP of T_i φ

- 1: Calculate Fermat Point of T_i [24]
 - 2: Let f be Fermat Point of T_i
 - 3: $k_1 \leftarrow \lfloor \frac{|uf|}{R} \rfloor$
 - 4: $k_2 \leftarrow \lfloor \frac{|vf|}{R} \rfloor$
 - 5: $k_3 \leftarrow \lfloor \frac{|wf|}{R} \rfloor$
 - 6: Draw two circles centered at u with radii $k_1 R$ and $(k_1 + 1)R$
 - 7: Draw two circles centered at v with radii $k_2 R$ and $(k_2 + 1)R$
 - 8: Draw two circles centered at w with radii $k_3 R$ and $(k_3 + 1)R$
 - 9: Let P be set of intersection points of the circles drawn
 - 10: $\varphi_i = \min_{q \in P} \left\{ \lceil \frac{|uq|}{R} \rceil + \lceil \frac{|vq|}{R} \rceil + \lceil \frac{|wq|}{R} \rceil - 2 \right\}$
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4. Optimized Triangle Selection Based on *mst* Triangulation

One of the approaches for establishing inter-terminal connectivity is to form steinerized *mst* where RNs are placed R units apart from each other along the *mst* edges of the terminals [5]. However, in section 3 we have shown that steiner-
 315 izing a set of 3 terminals, i.e., a triangle, by determining the DFP minimizes the required number of RNs. In this section we present an Optimized Triangle Selection based on *mst* triangulation (OTS-MST) algorithm that leverages the DFP based steinerization. Basically, the OTS-MST algorithm identifies a subset of triangles, i.e., all possible non-collinear 3 terminals, places RNs at the
 320 DFP of the picked triangles, modifies the *mst* of terminals on-the-fly and then forms steinerized *mst* on the modified tree where the required number of RNs is reduced. The details of the OTS-MST heuristic are provided in this section.

4.1. OTS-MST Heuristic

The main idea of the OTS-MST heuristic is to apply the DFP based steiner-
 325 ization, discussed in the previous section to a steinerized *mst* based solution of the SMT-MSPBEL problem. Basically, the triangles, i.e., sets of 3 terminals, that have two *mst* edges will be good candidates for consideration for

the DFP-based optimization. In other words, these triangles become the set of 3-tuples for which optimal SMT-MSPBELs are formed, and the *mst* of the terminals will serve as the interconnection structure for these individual 3-tuple based SMT-MSPBELs. The fundamental question is whether some of these *mst*-based triangles are redundant and if so, how to identify the best choices so that the least number of relays are involved in solving the SMT-MSPBEL problem for the entire set of terminals. The following definitions assist in answering these questions.

Definition 4. Let $T_i(u, v, w)$ be a triangle with two edges $e_1 = \overline{uv}$ and $e_2 = \overline{vw}$, that are on the *mst* of the set of terminals V . The *mst*-weight of T_i is defined as $W_{mst}(T_i) = W_{sp}(e_1) + W_{sp}(e_2)$. Later we will extend this definition for the case, when T_i does not have edges on the *mst* of V .

Definition 5. The gain of T_i is defined as the difference between *mst*- and *dfp*-weights of T_i

$$Gain(T_i) = W_{mst}(T_i) - W_{dfp}(T_i)$$

Definition 6. The triangulation of a set of terminals V based on *mst* edges in the 2D plane is called *mst* triangulation and denoted as Γ_{mst} . Let $G_{mst}(V, E_{mst})$ be the minimum spanning tree of the terminals. A triangle $T_i(u, v, w) \in \Gamma_{mst}$ if it satisfies the following properties: Let α be the $\widehat{uvw}$ angle where $0^\circ < \alpha < 180^\circ$ and let z be a point where $z \in V - \{u, v, w\}$ and $\overline{vz} \in E_{mst}$. Assume that α' and α'' are the angles $\widehat{zvu}$ and $\widehat{zvw}$, respectively.

1. $\overline{uv} \in E_{mst}$ and $\overline{vw} \in E_{mst}$
2. $\alpha \neq \alpha' + \alpha''$

Figure 5 illustrates the *mst* triangulation of the set of terminals $\{u, v, w, x, y, z\}$. Each triangle has two *mst* edges. The triangles $T(u, v, w)$ and $T(y, v, z)$ are not included in Γ_{mst} since their big angles are split by two other *mst* edges $\overline{vz}$ and $\overline{vu}$, respectively, which violates the second condition in definition 6. The reason of excluding these triangles is to avoid redundant RN deployment. For

example, the triangle $T(u, v, w)$ is split by an *mst* edge, namely $\overline{vz}$, into the triangles $T(u, v, z)$ and $T(v, w, z)$ that already cover the edges $\overline{vu}$ and $\overline{vw}$.

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Figure 5: Illustrating *mst* triangulation, solid lines represent *mst* edges, dashed lines represent the additional edges introduced by *mst* triangulation

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Redundancy in covering an *mst* edge can take place when considering that edge in two triangles. For example, steinerizing the triangles $T(u, v, z)$ and $T(z, v, w)$ covers the *mst* edge $\overline{vz}$ twice. A more efficient way is to handle the edge $\overline{vw}$ in the triangle $T(u, v, y)$ and steinerize the triangle $T(z, v, w)$ to cover $\overline{vz}$.

OTS-MST strives to avoid such redundancy by forming Triangular Adjacency Graph (TAG) and then using

the TAG to find the subset of triangles which maximizes the total gain (reduction in the relay count). As explained below.

4.1.1. Forming Triangular Adjacency Graph (TAG)

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As mentioned above, two triangles which share an *mst* edge spans 4 terminals. Therefore steinerizing both triangles will introduce redundancy since some nodes will be covered twice. In other words picking one triangle T disqualifies all triangles that share an edge with T and warrants their exclusion. The interesting question is how to select a set of triangles that ought to be steinerized in order to minimize the required relay count for federating all terminals. To be able to determine such a set, we introduce the Triangular Adjacency Graph (TAG). In TAG the vertices represent the triangles and the edges represent triangular adjacency, i.e., there is an edge between two vertices in TAG if the corresponding triangles share an *mst* edge.

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Definition 7. Two triangles are called adjacent if they share an *mst* edge. Formally, let $T_i(u_i, v_i, w_i)$ and $T_j(u_j, v_j, w_j)$ be two triangles in Γ_{mst} . T_i and T_j

are called adjacent if the cardinality of the intersection of the vertices of the triangles is 2. $|\{u_i, v_i, w_i\} \cap \{u_j, v_j, w_j\}| = 2$

The process of forming TAG is as follows. First, we perform *mst* triangulation of the terminals, i.e., form Γ_{mst} , and then determine the gain for each triangle, as specified in definition 5 above. Each triangle in Γ_{mst} corresponds to a vertex in TAG and the weight of each vertex is the gain of the triangle, i.e., $c(T_i) = \text{Gain}(T_i)$. An edge between two vertices in TAG is added if the corresponding triangles in Γ_{mst} are adjacent. In other words, let $G_T(V_T, E_T)$ be the TAG of Γ_{mst} where $V_T = \forall T_i$ s.t. $T_i \in \Gamma_{mst}$ and $E = \forall (T_i, T_j)$ s.t. T_i and T_j are adjacent. The process of forming TAG is illustrated through an example in Figure 6.

Figure 6: Forming TAG from *mst*. (a) *mst* of terminals. (b) Γ_{mst} of terminals. (c) corresponding TAG

As pointed out, the TAG will be used for selecting a set of non-adjacent triangles for optimization, i.e., forming SMT-MSPBEL via DFP. The terminals that are not covered by these selected non-adjacent triangles will be intercon-

nected by steinerized mst edges. Obviously, the triangles of the highest gains will be preferred in order to reduce the overall RN count. Determining the set of non-adjacent triangles while maximizing the total gain is equivalent to finding
400 maximum-weighted independent set ($MWIS$) of vertices from TAG.

4.1.2. Finding Max Weighted Independent Set ($MWIS$)

The second step of the OTS-MST algorithm is to find the $MWIS$ from TAG, which is also an NP-Hard problem. However, optimal solution can be found with dynamic programming (DP) if the graph is a tree [25]. The TAG, however,
405 may contain cycles as seen in Figure 6(c). Some of the cycles in TAG can be eliminated by pruning zero-weighted vertices and the corresponding edges, since selecting those triangles will provide no gain at all. The remaining cycles can be broken by ignoring some adjacencies and removing some edges; in that regard OTS-MST prefers the edges between the vertices (i.e., triangles) with the least
410 weights that break the largest number of cycles where the weight of a vertex in TAG represents the Gain of the triangle which is defined in Definition 5.

After converting TAG to a tree by removing cycles, we can apply DP to find optimal $MWIS$ as presented in [25]. Let $G_T(V_T, E_T)$ be the triangular adjacency tree of Γ_{mst} . We root the tree at an arbitrary node T_k as illustrated
415 in Figure 7. The optimal substructure of the problem is as follows: If T_k is not included in the independent set the solution will be the union of the independent sets for the sub-trees rooted at $T_{k1}, T_{k2}, \dots, T_{km}$. On the other hand if T_k is

Figure 7: Illustrating triangular adjacency tree rooted at T_k .

included in the independent set, none of the children of T_k can be in optimal solution. Similarly the solution will be the union of T_k and the independent sets for the sub-trees rooted at the grandchildren of T_k . We can formulate the
420 problem as follows: Let $\mathcal{M}(T_k)$ be the optimal solution of *MWIS* problem for the sub-tree rooted at T_k . We can write the following recursive formula.

$$\mathcal{M}(T_k) = \max\left\{\left(\sum_{T_i \text{ is child of } T_k} \mathcal{M}(T_i)\right), c(T_k) + \left(\sum_{T_i \text{ is grandchild of } T_k} \mathcal{M}(T_i)\right)\right\}$$

Due to ignoring some triangular adjacencies while breaking cycles, the resulting
425 *MWIS* may not be actually independent and may still contain some adjacent triangles. Finally we identify the least weighted triangles (i.e., those providing the least gain as defined in Definition 5) which violate the independency condition in the resulting *MWIS*, and remove them from the *MWIS*.

After finding an *MWIS* as described above, the OTS-MST algorithm de-
430 ploys RNs to the DFPs of the triangles in the *MWIS* and updates the *mst* of terminals on-the-fly. Let $T_i(u, v, w) \in \text{MWIS}$ where $\overline{uv}$ and $\overline{vw}$ are *mst* edges, and let φ_i be the DFP of T_i . The algorithm updates *mst* by removing the two edges $\overline{uv}$ and $\overline{vw}$ and adding the three edges $\overline{u\varphi_i}$, $\overline{v\varphi_i}$ and $\overline{w\varphi_i}$ instead.

4.2. Illustrative Example

435 The pseudo-code for OTS-MST algorithm is provided in Algorithm 2 in the appendix. We illustrate how OTS-MST works with a detailed example. Let's assume we have 15 terminals, namely $\{t_1, t_2, \dots, t_{15}\}$, whose *mst* is illustrated in Figure 8(a). The algorithm forms the triangular adjacency graph (TAG) of Γ_{mst} as shown in Figure 8(b). Note that the triangle $\{t_{14}, t_6, t_{15}\}$ is
440 not in Γ_{mst} , since the *mst* edge $\overline{t_6 t_2}$ splits the obtuse angle $\widehat{t_{14} t_6 t_{15}}$ into two. Some of the triangles in Γ_{mst} have zero-gain (dashed nodes in Figure 8(b)). Therefore the OTS-MST algorithm does not take these triangles into account, and consequently the dashed nodes and all induced edges can be pruned. The

Algorithm 2 OTS-MST

Require: Set of terminal points V .

Ensure: Set of relay nodes S and modified G_{mst}

```

1:  $S \leftarrow \emptyset$ 
2:  $V_{mst} \leftarrow V$ 
3: Let  $G_{mst}(V_{mst}, E_{mst})$  be the mst of  $V_{mst}$ 
4: Form  $\Gamma_{mst}$  and  $G_T(V_T, E_T)$  where  $V_T = \Gamma_{mst}$ 
5:  $C \leftarrow \emptyset$ 
6: if  $G_T$  contains cycle then
7:    $C \leftarrow$  the minimal set of edges that breaks all cycles, between the vertices
   having the least weight
8: end if
9: Let  $G'_T(V'_T, E'_T)$  be the tree where  $V'_T = V_T$  and  $E'_T = E_T - C$ 
10: Find MWIS using the dynamic programming formula
11: if  $C \neq \emptyset$  then
12:   if MWIS contains adjacent triangles then
13:     Remove the least weighted triangle that violates independency
14:   end if
15: end if
16: for all  $T_i(u, v, w) \in \text{MWIS}$  do
17:   put  $s_i$  to the DFP of  $T_i$ 
18:    $S \leftarrow S \cup \{s_i\}$ 
19:    $V_{mst} \leftarrow V_{mst} \cup S$ 
20:    $E_{mst} \leftarrow (E_{mst} - \{e_x, e_y\}) \cup \{\overline{us_i}, \overline{vs_i}, \overline{ws_i}\}$ 
21: end for
22: return  $S$  and  $G_{mst}$ 

```

bold nodes in Figure 8(b) reflect the positive-gain triangles. In this example
445 we assume the gain of all triangles which are shown in bold in Figure 8(b)
is equal to 1. Pruning zero-gain triangles helps significantly for breaking the
cycles in TAG. However, the remaining graph may still contain cycles. To ap-
ply dynamic programming we need to eliminate all cycles. When a cycle is
detected, the algorithm breaks the cycles by ignoring some adjacencies as de-
450 scribed above. In this example all edges are equivalent, i.e., each edge breaks
only one cycle and the weights of adjacent vertices are the same. For this
case, the algorithm randomly removes one of these edges to break the cycle.
Let's say the algorithm ignored the edge $\overline{\{t_{14}, t_6, t_2\}\{t_4, t_{14}, t_6\}}$. After applying
dynamic programming, we get $\{\{t_2, t_8, t_{12}\}, \{t_4, t_{13}, t_5\}, \{t_4, t_{14}, t_6\}, \{t_{14}, t_6, t_2\},$

Figure 8: Illustration of how OTS-MST algorithm works. (a) *mst* of terminals, where the *sp-weight* of each edge is displayed in the figure. (b) The corresponding triangular adjacency graph (TAG). Solid vertices represent the triangles having positive gain. (c) Updated *mst*. The resulting SMT-MSPBEL will be formed steinerizing the *mst* edge where the number of required RNs is annotated on the edges.

455 $\{t_6, t_{15}, t_{11}\}$ as the *MWIS*. Since $\{t_4, t_{14}, t_6\}$ and $\{t_{14}, t_6, t_2\}$ are adjacent, the algorithm randomly removes one of them from *MWIS*. Assuming the latter is removed, $MWIS = \{\{t_2, t_8, t_{12}\}, \{t_4, t_{13}, t_5\}, \{t_4, t_{14}, t_6\}, \{t_6, t_{15}, t_{11}\}\}$. OTS-MST deploys 4 RNs $\{s_1, s_2, s_3, s_4\}$ to the DFP's of the triangles in *MWIS* as illustrated in Figure 8(c). Finally the *mst* edges are steinerized in order to form
 460 SMT-MSPBEL.

Figure 9: Illustrating redundant node deployment as a result of DFP optimization of a triangle having another terminal inside. (a) 7 RNs are required (b) 6 RNs are required for connecting 4 terminals.

5. Incremental Optimization Based on Delaunay Triangulation (IO-DT)

In OTS-MST, DFP optimization works for a small subset of triangles where the triangles have two *mst* edges. The main motivation of the IO-DT algorithm comes from the question on whether there are any other triangles not having two *mst* edges that could be optimized to further reduce the total number of RNs. Since there are $\binom{n}{3} = O(n^3)$ possible triangles, where n is the number of terminals, the complexity of the solution increases significantly if all triangles are to be considered. Most of these triangles, however, are not needed since considering big triangles having multiple terminals inside may require many redundant RNs. For example in Figure 9(a), the triangle $T(u, v, w)$ contains the terminal z . The DFP optimization of the triangle $T(u, v, w)$ yields 7 RNs one of which is redundant as shown in Figure 9(b). Therefore the algorithm should avoid those redundancies. In other words we need a triangulation such that no terminal is located inside any triangle. Delaunay Triangulation (DT) meets this requirement and thus enables us to determine the set of triangles that ought to be considered. Hereafter, we use Γ_{dt} to denote the triangles in the DT graph of terminals.

5.1. The IO-DT Heuristic

The IO-DT algorithm looks for the best option for connecting the terminals of each triangle in DT such that the required RN count for the entire network is minimized. It, however, tries to do so by incrementally considering groups of

3 terminals. The first step for IO-DT is to determine Γ_{dt} and then form *mst* of all terminals. IO-DT then considers each triangle in Γ_{dt} to decide on whether *mst* edges or DFP based steinerization is better. In definition 4, we defined *mst-weight* of a triangle that contains two *mst* edges. In Delaunay Triangulation, however, a triangle may not contain two *mst* edges and thus we cannot apply definition 4 to calculate the *mst-weight* of the triangle. Definition 8, is a generalization of definition 4 to handle these cases.

Definition 8. Let T_i be a triangle with the vertices (u, v, w) . Given a minimum spanning tree G_{mst} , assume that Υ_{uv} and Υ_{uw} are two paths in G_{mst} from u to v and from u to w , respectively, such that $\Upsilon_{uv} \not\subseteq \Upsilon_{uw}$ and $\Upsilon_{uw} \not\subseteq \Upsilon_{uv}$. Let e_x and e_y be the edges having the maximum *sp-weight* in Υ_{uv} and Υ_{uw} , respectively. Also let e_z be the *mst* edge having the maximum *sp-weight* in $(\Upsilon_{uv} \cup \Upsilon_{uw}) - (\Upsilon_{uv} \cap \Upsilon_{uw})$. The *mst-weight* of T_i is denoted and computed as follows:

$$W_{mst}(T_i) = \begin{cases} W_{sp}(e_x) + W_{sp}(e_y), & \text{if } e_x \neq e_y \\ W_{sp}(e_x) + W_{sp}(e_z), & \text{otherwise} \end{cases}$$

The two *mst* edges e_x and e_y (or e_z , if $e_x = e_y$) are called the associated *mst* edges with T_i . Finding *mst-weight* of T_i is illustrated in Figure 10.

By definition 8, in order to calculate the *mst-weight* of a triangle, we need to find two *mst* paths Υ_{uv} and Υ_{uw} and associated *mst* edges. Since the *mst* changes dynamically, i.e., after each RN deployment at a DFP of one of the triangles, the *mst-weight* and the gain of a triangle may change depending on the previously deployed RNs. To tackle these challenges, we pursue a greedy heuristic. Let Γ_{dt} be the sorted list of triangles in the DT in an ascending order according to the *dfp-weight* of the individual triangles. The rationale for pursuing such an order is that we'd like to introduce edges with the smallest possible *sp-weight* to the emerging network topology, i.e., involving the least RN count, so that we can remove existing edges with large *sp-weight* in subsequent iterations.

Figure 10: Illustration of calculating *mst-weight* of a triangle $T(u, v, w)$. Bold edges represent maximum *sp-weighted mst* edges in the paths Υ_{uv} and Υ_{uw} . (a) $\Upsilon_{uv} = \{e_1, e_2, e_3, e_4\}$ and $\Upsilon_{uw} = \{e_5, e_6, e_7\}$ are node disjoint and e_3 and e_6 are maximum *sp-weighted mst* edges in Υ_{uv} and Υ_{uw} , respectively (i.e. $e_x = e_3$ and $e_y = e_6$). Therefore, $W_{mst}(T) = W_{sp}(e_3) + W_{sp}(e_6)$. (b) $\Upsilon_{uv} = \{e_1, e_2, e_3, e_4, e_5\}$ and $\Upsilon_{uw} = \{e_1, e_6, e_7, e_8\}$. Let e_1 is the maximum *sp-weighted mst* edge in both Υ_{uv} and Υ_{uw} (i.e. $e_x = e_y = e_1$), then we calculate maximum *sp-weighted mst* edge in $(\Upsilon_{uv} \cup \Upsilon_{uw}) - (\Upsilon_{uv} \cap \Upsilon_{uw}) = \{e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ which is e_6 in this example (i.e. $e_z = e_6$). Therefore $W_{mst}(T) = W_{sp}(e_1) + W_{sp}(e_6)$

In other words, we would like to get rid of the edges having large *sp-weight* and keep the edges having small *sp-weight* in our final topology. The algorithm iteratively processes the triangles in Γ_{dt} . In each iteration the *mst-weight* and the gain of the selected triangle $T_i(u, v, w)$ are calculated. If T_i has positive gain, the algorithm deploys an RN at the DFP of T_i , and updates the *mst* by removing the two *mst* edges that are associated with T_i and adding the three edges $\overline{u\varphi_i}$, $\overline{v\varphi_i}$ and $\overline{w\varphi_i}$ instead, where φ_i is the DFP of T_i .

5.2. Illustrative Example

The pseudo code for IO-DT algorithm is provided in Algorithm 3 in the appendix. Figure 11 illustrates how the IO-DT algorithm works through a detailed example. The configuration of the example is similar to that of Figure 8. First the algorithm calculates DT of the terminals. A subset of three terminals forms a triangle. The algorithm sorts the list of triangles in Γ_{dt} according to their *dfp-weights*. Table 2 shows the sorted list of triangles. Figure 11(a) shows the DT of terminals. In Figure 11(a) the solid lines represent the *mst* edges and dashed lines are edges that belong only to Γ_{dt} . IO-DT is a greedy algorithm and at each iteration the algorithm picks the triangle having the

Figure 11: A Step-by-step illustration of how IO-DT algorithm works

Algorithm 3 IO-DT

Require: Set of terminal points V .

Ensure: Set of relay nodes S and modified G_{mst}

```

1:  $S \leftarrow \emptyset$ 
2:  $V_{mst} \leftarrow V$ 
3: Let  $G_{mst}(V_{mst}, E_{mst})$  be the mst of  $V_{mst}$ 
4: Let  $\Gamma_{dt}$  be the sorted list of triangles in DT
5: for all  $T_i(u, v, w) \in \Gamma_{dt}$  do
6:   Let  $\Upsilon_{uv}$  be a path in  $G_{mst}$  from  $u$  to  $v$  and  $\Upsilon_{uw}$  be a path in  $G_{mst}$ 
   from  $u$  to  $w$  where  $\Upsilon_{uv} \not\subseteq \Upsilon_{uw}$  and  $\Upsilon_{uw} \not\subseteq \Upsilon_{uv}$ 
7:   Let  $e_x$  and  $e_y$  be the edges having maximum sp-weight in  $\Upsilon_{uv}$  and
    $\Upsilon_{uw}$ , respectively.
8:   Also let  $e_z$  be the edge having the maximum sp-weight in  $(\Upsilon_{uv} \cup \Upsilon_{uw}) -$ 
    $(\Upsilon_{uv} \cap \Upsilon_{uw})$ .
9:   if  $e_x = e_y$  then
10:      $e_y \leftarrow e_z$ 
11:   end if
12:    $W_{mst}(T_i) = W_{sp}(e_x) + W_{sp}(e_y)$ 
13:   if  $W_{mst}(T_i) > W_{dfp}(T_i)$  then
14:     put  $s_i$  to the DFP of  $T_i$ 
15:      $S \leftarrow S \cup \{s_i\}$ 
16:      $V_{mst} \leftarrow V_{mst} \cup S$ 
17:      $E_{mst} \leftarrow (E_{mst} - \{e_x, e_y\}) \cup \{\overline{us_i}, \overline{vs_i}, \overline{ws_i}\}$ 
18:   end if
19: end for
20: return  $S$  and  $G_{mst}$ 

```

least *dfp-weight*. In the first iteration, the algorithm picks $T(t_7, t_9, t_{10})$ and
520 calculates the two paths Υ_{uv} and Υ_{uw} where $u = t_7, v = t_9$, and $w = t_{10}$.
For this triangle, $\Upsilon_{uv} = \overline{t_7 t_9}$ and $\Upsilon_{uw} = \overline{t_7 t_{10}}$. Thus, $W_{mst}(T(t_7, t_9, t_{10})) =$
 $W_{sp}(\overline{t_7 t_9}) + W_{sp}(\overline{t_7 t_{10}}) = 2$. Since the *dfp-* and *mst-weights* of the triangle
 $T(t_7, t_9, t_{10})$ are equal, the algorithm skips this triangle. Similarly in the second
iteration the triangle $T(t_1, t_7, t_{10})$ is skipped due to the same reason. In the third
525 iteration the triangle $T(t_2, t_8, t_{12})$ is picked, where $\Upsilon_{uv} = \{t_8 t_2\}$ and $\Upsilon_{uw} =$
 $\{\overline{t_8 t_{12}}\}$ such that $u = t_8, v = t_2$, and $w = t_{12}$. Since the *dfp-weight* of the
triangle is less than the *mst-weight*, the algorithm places an RN, s_1 , at the
DFP of $T(t_2, t_8, t_{12})$ and updates the *mst* as illustrated in Figure 11(b). In the
next iteration the triangle $T(t_1, t_{10}, t_{12})$ is considered, where $\Upsilon_{uv} = \{\overline{t_1 t_{10}}\}$ and

530 $\Upsilon_{uw} = \{\overline{t_{10}t_7}, \overline{t_7t_9}, \overline{t_9t_{14}}, \overline{t_{14}t_6}, \overline{t_6t_2}, \overline{t_2s_1}, \overline{s_1t_{12}}\}$. Since $\overline{t_6t_2}$ is the edge with the largest *sp-weight* in Υ_{uw} , where $W_{mst}(T(t_1, t_{10}, t_{12})) = W_{sp}(\overline{t_1t_{10}}) + W_{sp}(\overline{t_6t_2}) = 3$, the algorithm places an RN, s_2 , to the DFP of $T(t_1, t_{10}, t_{12})$ and updates the *mst* by removing the edges $\overline{t_1t_{10}}$ and $\overline{t_6t_2}$, and adding $\overline{t_1s_2}, \overline{t_{10}s_2}$ and $\overline{t_{12}s_2}$ as illustrated in Figure 11(c). The rest of the execution steps of the algorithm is

535 shown in Table 2 and Figures 11(d) and (e). Finally the *mst* edges are steinerized to form SMT-MSPBEL in Figure 11(f).

6. A Hybrid Approach

In the previous sections we have presented two algorithms namely OTS-MST and IO-DT. Depending on the topology, both approaches may be superior to

540 one another. For example in Figure 8(c), the resulting topology of OTS-MST for a particular topology is illustrated where the algorithm required 20 RNs to establish connectivity (i.e. the sum of *sp-weights* of the edges where the edge weights are annotated plus number of triangles in *MWIS*). In Figure 11(f), however, for the same topology that is used in OTS-MST, the IO-DT required

545 19 RNs. In this example IO-DT outperforms OTS-MST in terms of the number of required RNs. The major advantage of IO-DT over OTS-MST is that unlike OTS-MST, IO-DT does not restrict itself to only the triangles having two *mst* edges and considers some triangles which do not have two *mst* edges as well. However, as illustrated in Figure 12, there are cases that some triangles might

550 be in Γ_{mst} but not in Γ_{dt} (i.e. $T(u, w, z) \in \Gamma_{mst}$ and $T(u, w, z) \notin \Gamma_{dt}$).

In other words the triangle $T(u, w, z)$ is never considered in IO-DT. If

$$\max\{GoT(u, w, z) + GoT(v, u, w)\} > \max\{GoT(v, u, w) + GoT(v, w, z)\}$$

where $GoT(u, w, z)$ is the gain of triangle whose corners are (u, w, z) , then OTS-MST would perform better than IO-DT.

The idea behind the hybrid solution is to combine the strengths of OTS-MST and IO-DT, by executing these two algorithms sequentially. The order of

555 execution is important since the RNs, which are deployed by the first algorithm,

Table 1: Execution of IO-DT algorithm

<i>Triangle T_i</i>	$W_{dfp}(T_i)$	$W_{mst}(T_i)$	Action
(t_7, t_9, t_{10})	2	2	skip
(t_1, t_7, t_{10})	2	2	skip
(t_2, t_8, t_{12})	2	3	<ul style="list-style-type: none"> • Place s_1 to DFP, Remove $\overline{t_2 t_8}$ and $\overline{t_8 t_{12}}$ from mst • Add $\overline{t_2 s_1}$, $\overline{t_8 s_1}$ and $\overline{t_{12} s_1}$ to mst
(t_1, t_{10}, t_{12})	2	3	<ul style="list-style-type: none"> • Place s_2 to DFP, Remove $\overline{t_1 t_{10}}$ and $\overline{t_2 t_6}$ from mst • Add $\overline{t_1 s_2}$, $\overline{t_{10} s_2}$ and $\overline{t_{12} s_2}$ to mst
(t_6, t_{11}, t_{15})	3	4	<ul style="list-style-type: none"> • Place s_3 to DFP, Remove $\overline{t_6 t_{15}}$ and $\overline{t_{11} t_{15}}$ from mst • Add $\overline{t_6 s_3}$, $\overline{t_{11} s_3}$ and $\overline{t_{15} s_3}$ to mst
(t_3, t_8, t_{15})	3	4	<ul style="list-style-type: none"> • Place s_4 to DFP, Remove $\overline{t_3 t_8}$ and $\overline{t_6 t_{14}}$ from mst • Add $\overline{t_3 s_4}$, $\overline{t_8 s_4}$ and $\overline{t_{15} s_4}$ to mst
(t_2, t_6, t_{15})	3	2	skip
(t_2, t_9, t_{14})	3	3	skip
(t_4, t_5, t_{13})	3	4	<ul style="list-style-type: none"> • Place s_5 to DFP, Remove $\overline{t_4 t_{13}}$ and $\overline{t_5 t_{13}}$ from mst • Add $\overline{t_4 s_5}$, $\overline{t_5 s_5}$ and $\overline{t_{13} s_5}$ to mst
(t_2, t_9, t_{10})	3	2	skip
(t_2, t_{10}, t_{12})	3	2	skip
(t_3, t_8, t_{12})	4	1	skip
(t_3, t_{11}, t_{15})	4	2	skip

may negatively affect the performance of the second algorithm and consequently the overall performance. Let V be the set of terminals. The hybrid solution

Figure 12: Illustrating the cases which is/is not handled by *mst* based triangulation and Delaunay Triangulation. Solid lines represent *mst* edges, dashed lines represent the triangulation. (a) The triangle $T(u, w, z)$ is considered but $T(v, w, z)$ is not considered in *mst* based triangulation. (b) $T(u, w, z)$ is not considered but $T(v, w, z)$ is considered in Delaunay triangulation.

Algorithm 4 HYBRID

Require: Set of terminal points V .

Ensure: Set of relay nodes S and modified G_{mst}

```

1:  $\{S_{OTS-MST}, G_{mst}(V, E_{mst})\} \leftarrow OTS-MST(V)$ 
2:  $\{S_{IO-DT}, G_{mst}(V, E_{mst})\} \leftarrow IO-DT(V \cup S_{OTS-MST})$ 
3:  $C_1 = |S_{OTS-MST}| + |S_{IO-DT}| + \sum_{e \in E_{mst}} W_{sp}(e)$ 
4:  $G_{temp}(V_{temp}, E_{temp}) \leftarrow G_{mst}(V, E_{mst})$ 
5:  $\{S_{IO-DT}, G_{mst}(V, E_{mst})\} \leftarrow IO-DT(V)$ 
6:  $\{S_{OTS-MST}, G_{mst}(V, E_{mst})\} \leftarrow OTS-MST(V \cup S_{IO-DT})$ 
7:  $C_2 = |S_{OTS-MST}| + |S_{IO-DT}| + \sum_{e \in E_{mst}} W_{sp}(e)$ 
8: if  $C_1 < C_2$  then
9:    $G_{mst}(V, E_{mst}) \leftarrow G_{temp}(V_{temp}, E_{temp})$ 
10: end if
11: return steinerized  $G_{mst}$ 

```

compares the following two cases and selects the best order.

1. *Run OTS-MST first and then run IO-DT*: In this case the hybrid solution
560 first runs the OTS-MST algorithm without steinerizing the edges in E_{mst}
and then runs IO-DT. Let $S_{OTS-MST}$ be the set of RNs deployed by
OTS-MST. The algorithm determines Γ_{dt} for $V \cup S_{OTS-MST}$.
2. *Run IO-DT first and then run OTS-MST*: In this case the hybrid solution
565 first runs the IO-DT algorithm without steinerizing the edges in E_{mst} and
then runs OTS-MST. Let S_{IO-DT} be the set of RNs deployed by IO-DT.
The algorithm determines Γ_{mst} for $V \cup S_{IO-DT}$.

The pseudo code of the hybrid approach is provided in Algorithm 4 in the appendix. We illustrate how the hybrid approach works with the same example

that is used in figures 8 and 11. In this example we consider case (1) and first
run OTS-MST algorithm. We leave out case (2), i.e., running IO-DT first, for
space limitation and since it yields worse solution than case (1). The OTS-
MST algorithm deploys 4 RNs $S_{OTS-MST} = \{s_1, s_2, s_3, s_4\}$ to the DFP's of
the triangles in $MWIS$, as depicted in Figure 13(a). Then we apply IO-DT
algorithm over the set of $V \cup S_{OTS-MST}$ where V is the set of terminals. The
IO-DT algorithm deploys two more RNs $S_{IO-DT} = \{s_5, s_6\}$, as illustrated in
Figure 13(b). Finally the algorithm steinerizes the mst edges in the resulting
topology (Figure 13(c)). As shown in this example the hybrid approach leverages
the strengths of both OTS-MST and IO-DT and requires 18 RNs to establish
connectivity which outperforms both of them.

Theorem 3. *The hybrid algorithm always performs better than (or equal to in
the worst case) OTS-MST and IO-DT.*

PROOF. Both OTS-MST and IO-DT aims to improve the mst -based solution by
leveraging the DFP based steinerization. In the worst case where no triangles
is found to be optimized by DFP based steinerization, both algorithms end up
with the steinerized mst . In other words running either OTS-MST or IO-DT
never produces a topology which requires more RNs than mst -based solution.
As described above, the hybrid approach compares two cases and selects the one
which requires the fewest number of RNs. In the first case where OTS-MST
is executed first, if IO-DT cannot find any more triangles to optimize then the
result matches those of OTS-MST. Similarly in the second case where IO-DT
is executed first, if OTS-MST cannot find any triangles, then the result will be
equal to IO-DT. Since hybrid approach selects the best of the two cases, it will
always perform better than (or equal to) IO-DT and OTS-MST.

7. Analysis of the Algorithms

This section presents the analysis of the runtime complexity of the OTS-MST
and IO-DT algorithms. We start with the following lemmas.

Figure 13: Illustrating how hybrid approach works. (a) The resulting *mst* after OTS-MST is applied. 4 RNs are placed. (b) First iteration of IO-DT is applied, the RN s_5 is placed. (c) Second iteration of IO-DT is applied, the RN s_6 is placed and *mst* is updated. Finally the *mst* edges will be steinerized as the last step as illustrated in Figure 11(f)

Lemma 2. *The mst is a sub-graph of DT [26].*

Lemma 3. *Both the number of edges and the number of triangles in DT can be bounded by $O(n)$ where n is the number of terminals [27].*

600 **Corollary 1.** *Per lemma 2, the mst can be computed after computing the De-launay Triangulation of a set of terminals. The time complexity of finding De-launay Triangulation is $O(n \log n)$ [27]. Using Kruskal's algorithm the cost of forming mst is $O(|E| \log |V|)$ [28]. Per lemma 3, the number of edges for DT is $O(n)$ thus we can conclude that $|E| = O(n)$ and $|V| = n$. Therefore the cost of forming mst becomes $O(n \log n)$.*
605

Lemma 4. *The node degree (i.e., number of neighboring nodes) of a vertex in an Euclidean mst cannot be greater than 6 [29].*

Lemma 5. *Let $G_T(V_T, E_T)$ be the TAG of a set terminals V where the mst of V is denoted as $G_{mst}(V, E_{mst})$ and $|V| = n$. The number of vertices in G_T (i.e., the number of triangles in mst triangulation) and the number of edges in G_T (i.e., total number of triangular adjacencies) can be upper bounded by $O(n)$. In other words $|V_T| = O(n)$ and $|E_T| = O(n)$.*
610

PROOF. By definition 6, the triangles in mst triangulation contains two mst edges sharing one vertex where the shared vertex is non-leaf terminal in G_{mst} .
615 Let v be a non-leaf terminal in G_{mst} . Definition 6 implies that the number of triangles which can be formed by a pair of mst edges which share v , is a linear function of the node degree of v (i.e. number of neighboring nodes). Therefore the total number of triangles in mst triangulation is a linear function of the sum of node degrees of non-leaf terminals. Per lemma 4, we know that the
620 node degree of any node in G_{mst} can be at most 6. Therefore, $|V_T| < 6n$, in other words $|V_T| = O(n)$. Similarly, the number of triangular adjacencies in mst triangulation is also a linear function of the sum of node degrees. Hence we have $|E_T| = O(n)$.

Lemma 6. *The cost of finding a path between two vertices in a tree is $O(|V| + |E|)$, where $|V|$ and $|E|$ are the number vertices and edges, respectively. Since in an mst $|E| = |V| - 1$, the time complexity of finding a path is $O(|V|)$ [28].*

Theorem 4. *The runtime complexity of OTS-MST algorithm is $O(n^2)$, where n is the number of terminals.*

PROOF. The OTS-MST heuristic consists of a series of procedures. We will analyze each procedure individually. The algorithm computes the *mst* of terminals where the complexity is $O(n \log n)$ based on corollary 1. Next, $G_T(V_T, E_T)$ is formed. By lemma 5, we have $|V_T| = O(n)$ and $|E_T| = O(n)$. To form TAG, as described in section 4.1.1, the algorithm needs to consider every pair of edges in *mst*. Since we have $|E_T| = O(n)$ the TAG can be computed in $O(n^2)$. After forming the TAG, the algorithm finds the set of edges which breaks cycles in TAG. Identifying these edges is by running Depth First Search (lines 5 - 7 in Algo. 2), where the cost is $O(|V_T| + |E_T|) = O(n)$. Then the algorithm computes *MWIS* (lines 8-9 in Algo. 2). In [25], it was proven that the cost of finding *MWIS* is $O(n \log n)$. Since $MWIS \subset V_T$, then $|MWIS| = O(n)$. Hence the cost of finding adjacent triangles in *MWIS* is $O(n)$ and deploying relay nodes to the DFP of the triangles in *MWIS* is thus $O(n)$. Therefore the overall complexity is $O(n^2)$.

Theorem 5. *The runtime complexity of IO-DT algorithm is $O(n^2)$, where n is the number of terminals.*

PROOF. The IO-DT heuristic consists of a series of steps. We analyze each step individually. Throughout the proof we refer to the pseudo-code of Algorithm 3 in the appendix. IO-DT first calculates DT and *mst* of terminals where the cost of finding DT is $O(n \log n)$ [27] and the cost of forming *mst* using Kruskal's Algorithm is $O(n \log n)$ (by corollary 1). By lemma 3, since the number of triangles in DT is $O(n)$, the cost of sorting the triangles becomes $O(n \log n)$

(line 4 in Algo. 3). The “for” loop (lines 5 to 24) iterates $O(n)$ times. Inside the loop, the algorithm calculates two paths Υ_{uv} and Υ_{uw} (line 6), and finds the two edges e_x and e_y having the largest *sp-weights*. By lemma 6, the cost of finding a path in a tree is $O(n)$. Thus the complexity of the “for” loop is $O(n^2)$.
655 Hence the overall time complexity is $O(n^2)$.

Theorem 6. *The runtime complexity of hybrid algorithm is $O(n^2)$, where n is the number of terminals.*

PROOF. As explained above the hybrid approach compares two possible cases and selects the best one. These cases are 1) Run OTS-MST first and then
660 IO-DT, 2) Run IO-DT first and then OTS-MST. In either case the hybrid approach is indeed a sequential execution of IO-DT and OTS-MST approaches. By Theorems 4 and 5 we know that time complexities of OTS-MST and IO-DT are both $O(n^2)$, therefore the time complexity of the hybrid approach is $2(O(n^2) + O(n^2))$ which is $O(n^2)$.

665 In the next section we validate the performance of our heuristics via simulation and show that their average performance surpasses all the best known deterministic algorithms for solving the SMT-MSPBEL problem.

8. Performance Evaluation

This section describes the experiment setup, performance metrics and simulation results.
670

8.1. Experiment Setup and Baseline for Comparison

A simulation environment is developed using Java. In the simulation, a number of terminals are randomly placed in a 1500m x 1500m area. We compare the performance OTS-MST, IO-DT and the hybrid approach with two baseline algorithms, namely SMST [5, 10], and SMT-MSP 3-star [6]. In our simulations we assumed that the transmission ranges of terminals and RNs are the same. At this point we would like to note that in some practical applications of WSNs,

the transmission ranges of terminals and relays can be different. If the transmission range of a terminal is greater than the transmission range of a RN then we do not need to change anything in our algorithms since the terminals can communicate with relays without any modification. However if the transmission range of a terminal is less than the transmission range of a relay we can still use the proposed algorithms with a possible small modification by placing an additional RN at the terminal locations in order to enlarge the transmission range of the terminal. In this case both our algorithms and the baseline algorithms will employ N more relay nodes where N is the number of terminals. For an arbitrary topology with N terminals, let $\#RN_{OTS-MST}$, $\#RN_{IO-DT}$, $\#RN_{hybrid}$, and $\#RN_{3-star}$ be the number of relay nodes required by OTS-MST, IO-DT, Hybrid, SMT-MSP and 3-Star algorithms respectively. If we have the following ordering,

$$\#RN_{hybrid} < \#RN_{IO-DT} < \#RN_{OTS-MST} < \#RN_{3-star}$$

then the following is also true,

$$\#RN_{hybrid} + N < \#RN_{IO-DT} + N < \#RN_{OTS-MST} + N < \#RN_{3-star} + N$$

where N is the number of terminals. Therefore, for simplicity, assuming same transmission ranges for terminals and RNs does not have an effect on the performance of the algorithms.

675 In our simulations we varied two parameters, namely, number of terminals and transmission range of a RN (R). The R is fixed at 100m (both for terminals and RNs) to capture the effect of varying the number of terminals in the range [10, 50]. When studying the impact of R , it is varied between 25m to 200m while fixing the number of terminals at 20.

680 The following performance metrics are considered:

- *Number of RNs*: Minimizing the number of RNs for establishing the connectivity is the primary objective of the three proposed heuristics

- *Average Node Degree*: This is the average number of neighbors of each RN in the resulting topology. A higher node degree indicates stronger connectivity and helps in balancing the traffic load and reducing the data latency.

8.2. Performance Results

This section provides the performance results. The result for each configuration is the average of 100 different random topologies. We observed that with a %95 confidence interval, our results stayed within %6-%10 of the sample mean.

Figure 14(a) shows that all of the three proposed heuristics perform significantly better than SMST and SMT-MSP 3-star in terms of the number of used RNs. The performance gap widens as the number of terminals increases reaching about %20 for a network of 50 terminals. This is mainly because the number of triangles considered by OTS-MST, IO-DT and the hybrid heuristic increases with the growth in terminal count. On the other hand, as expected the hybrid heuristic performs better than OTS-MST and IO-DT and the performance of IO-DT is slightly better than OTS-MST.

The performance comparison of our three heuristics is further illustrated in Figure 14(b), which shows the number of RNs gained by each algorithm relative to SMST. Clearly the gain for the hybrid approach is the highest among all three, which indicates that it requires fewer RNs than OTS-MST and IO-DT. The reason of such performance is that hybrid approach takes advantage of combining the strengths of both IO-DT and OTS-MST and maximizes the total gain. In addition, the results in Figure 14(b) also show that the gain for IO-DT is higher than the gain for OTS-MST. The reason of the performance advantage of IO-DT over OTS-MST is basically the number of triangles considered by each algorithm. As mentioned earlier in the paper, IO-DT may consider triangles that do not have two *mst* edges, and thus factors in more triangles in the DFP optimization than OTS-MST which enables connecting the terminals with fewer RNs than OTS-MST. We repeated the experiments in Figure 14(a) and (b) for different values of R . We observed that for different transmission ranges the

Figure 14: Under varying # of terminals, comparison of (a) RN count, (b) gain with respect to SMST, (c) average node degree. Under varying transmission ranges, comparison of (d) RN count, (e) gain wrt. SMST, (f) average node degree. Note that for (a) and (d) a lower curve is better where for (b)(c)(e) and (f) a higher curve is better.

effect of number of terminals on the required number of relays does not change and the pattern of each plot is the same.

715 Figure 14(c) illustrates the performance comparison in terms of number required RNs under varying R . The figure indicates that all three heuristics

outperform SMST and SMT-MSP 3-star. It is clear that, the number of RNs diminishes as R increases, since fewer RNs are required for steinerizing an edge for big R values. In addition, the performance of the three heuristics is very close to one another under varying R . A better illustration of the performance comparison is depicted in Figure 14(d). Among the three heuristics, the hybrid approach always provided the highest gain and outperformed the others which is very much expected. We have further studied the effect of communication ranges on number of relay nodes for different number of terminals. We repeated the same experiment for $N = 10$, $N = 20$ and $N = 30$ where N is the number of terminals and observed that the pattern in each plot is the same which indicates the effect of transmission range to the required number of relay nodes is the same for different number of terminals.

The average node degree comparison of the algorithms is illustrated in Figures 14(e) and (f). In Figure 14(e), all three heuristics outperform the SMST and SMT-MSP 3-star algorithms when the number of terminals is small. In SMST and SMT-MSP 3-star algorithms almost all RNs have two neighbors since they are populated along *mst* edges. In contrast, all heuristics presented in this paper, place some of the RNs to the DFPs of the triangles, which yield a node degree of three. Thus if the algorithm finds more triangles to optimize, the average node degree gets closer to three, since more RNs will have a node degree of three. However, when the number of terminals exceeds 30, SMST and SMT-MSP 3-star start outperforming our proposed heuristics. This is mainly attributed to the fact that SMST and SMT-MSP 3-star requires more RNs than our heuristics for the same deployment area, and the increased RN density for large networks introduces more communication links.

Figure 14(f) compares the performance in terms of average node degree under varying transmission ranges. The average node degree of all algorithms increases as the transmission range grows. The reason is that for long transmission ranges, a relay can find more neighbors in its proximity which increases average node degree. For small transmission ranges, all three proposed heuristics perform better than SMST and SMT-MSP 3-star. Interestingly, however,

the performance of SMST in terms of average node degree grows at a higher rate and outperforms the other algorithms when R is bigger than 120m. The reason is that in OTS-MST, IO-DT and hybrid heuristics, RNs may be placed inside a triangle while in SMST RNs are always placed on the perimeter of a triangle. Due to this fact for big transmission ranges it is more likely for an RN in SMST to get connected with others on different edges and thus have more neighbors. Nonetheless, the increased RN count diminishes such an advantage for SMST and favors our heuristics.

9. Conclusion

In this paper, we have studied the problem of establishing connected topologies over wireless links through the placement of the least number of relays. This problem has significant importance in some WSN applications which operate unattended in harsh environments, e.g., battlefield surveillance and habitat monitoring. In this paper we have devised an optimal solution for the case of three terminals (i.e., WSN segments) by determining the Discrete Fermat Point (DFP), and used such a solution in developing three novel algorithms, namely OTS-MST, IO-DT, and the hybrid approach. OTS-MST applies DFP based steinerization to the triangles (i.e., sets of 3 terminals) having two minimum spanning tree (*mst*) edges and finds the subset of non-adjacent triangles which connects the terminals using the least number of relays. IO-DT computes the Delaunay Triangulation of terminals and iteratively checks if the DFP optimization of each triangle provides reduction in the total number of RNs as compared to an *mst*-based solution. After processing all triangles, the algorithm steinerizes the *mst* edges for both terminals and picked DFPs in order to form a connected topology. Finally we have presented a hybrid approach which combines the strengths of the OTS-MST and IO-DT. The simulation results have shown that all three heuristics significantly outperform competing schemes in the literature. In the future we plan to extend our work to establish connectivity in 3D settings.

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APPENDIX

Table 2: Notations and Definitions

Notation	Definition
$W_{sp}(u, v) = \lceil \frac{ uv }{R} \rceil - 1$	sp-weight of (u, v) : minimum number of relay nodes for connecting two terminals u and v (see Definition 1)
$T_i(u, v, w)$	Any arbitrary triangle whose vertices are (u, v, w)
$W_{fp}(T_i, q)$	The number of relay nodes required for connecting the vertices of T_i to an arbitrary point q . $W_{fp}(T_i, q) = \lceil \frac{ uq }{R} \rceil + \lceil \frac{ vq }{R} \rceil + \lceil \frac{ wq }{R} \rceil - 2$ (see Definition 3)
$W_{fp}(T_i, \varphi_i)$ or $W_{dfp}(T_i)$	The Discrete Fermat Point weight (dfp-weight) of T_i , where φ_i is the best point at which the vertices of T_i can be connected.
$W_{mst}(T_i)$	mst-weight of T_i , (see Definition 8 and Figure 10)
$Gain(T_i)$	The difference between mst-weight and df-weight $W_{mst}(T_i) - W_{dfp}(T_i)$
G_{mst}	Minimum Spanning Tree of set of terminals V
Γ_{mst}	mst-triangulation of a set of terminals V (see Definition 6)
Γ_{dt}	Delaunay triangulation of a set of terminals V