Synthetic Aperture Processing for Passive Co-prime Linear Sensor Arrays

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Abstract

In this paper, we explore the use of synthetic aperture processing for optimizing the spatial covariance estimation capabilities of a moving linear co-prime sensor array. The linear co-prime sensor array geometry is a thinned linear array that is constructed by nesting uniform linear arrays with inter-related element spacing factors. The application of synthetic aperture processing in this setting is designed to create virtual sensors at missing half-wavelength intervals up to the degree required to produce a hole-free difference co-array across the full aperture of the synthetic array. Once a full set of spatial covariances are estimated covariance matrix based array processing methods can be applied as if the synthetic array were a uniform linear array. In this sense, the synthetic array is designed to approximate a uniform linear array while it retains a thinned linear array structure. We show simulation results examining the quality of the uniform linear array approximation afforded by this application synthetic aperture processing.

Keywords: Co-prime Sensor Arrays, Co-array Processing, Synthetic Aperture Methods.

1. Introduction

Thinning a sensor array is known to lead to severe sensor array processing deficiencies. These deficiencies manifest themselves in the array pattern

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Preprint submitted to Digital Signal Processing, SI: CSA

July 27, 2016

specific to the sensor array geometry and can include periodic grating lobes, high-amplitude irregular side-lobes, as well as a loss in array gain. The severity by which these deficiencies occur is a function of the degree to which sensors are removed from the sensor array. The concept of removing sensors from an array and developing algorithms to overcome processing deficiencies has been a problem of interest in the thinned array literature [1, 2, 3]. Work in this area has been done to characterize and optimize the array patterns of thinned arrays in-terms of achieving desired sidelobe levels.

An equally important consequence of removing sensors from a sensor array can be seen via the spatial covariance estimation capability of the thinned array. Depending on the sensor placement the set of half-wavelength spatial covariances that can be estimated from sensor data can be incomplete. This leads to processing losses for signal-noise subspace and adaptive based array processing techniques. The well known Minimally Redundant Linear Array (MRLA) is an example of a thinned sensor array that preserves the half-wavelength spatial covariance sensing capabilities of the array over the entire array aperture [4]. The MRLA is however computationally difficult to design for large apertures [5].

In the work presented here, we use array motion to expand the spatial covariance sensing capabilities of a thinned sensor array. In particular, we show how Synthetic Aperture (SA) processing can be used to extend the spatial covariance lag range of a sensor array having inter-element spacings on average larger than half-wavelength. This process will provide a synthetic array with an increase in the degrees of freedom available from each synthetic array snapshot, while not requiring all missing half-wavelength sensors to be synthesized during the SA process. The benefit of SA processing in this setting is a direct consequence of the physical array geometry. The physical array geometry considered in this work is based on the notion of nesting Uniform Linear Arrays (ULA) with related inter-element spacings. In recent literature, a sampling strategy based on this type of nesting of ULA's has been studied in the setting of source localization where the number of sources to be detected is greater than the

35 number of sensors in the array [6]. In this work, the concept of the difference
 36 co-array was used to motivate the development and demonstrate the benefits of
 37 the nested array geometry concept. The difference co-array for a given sensor
 38 array characterizes the spatial covariances that can be estimated from the sensor
 39 array data. Here the difference co-array was of primary interest as it describes
 40 the degrees of freedom afforded by its underlying sensor array. It was shown
 41 that by applying the prescribed ULA nesting strategy an increase in the degrees
 42 of freedom could be realized from $\mathcal{O}(N)$ to $\mathcal{O}(N^2)$, where N is the number of
 43 sensors in the array. In particular, it was shown that from this thinned sensor
 44 array geometry the spatial covariance lag range that can be estimated is equiv-
 45 alent to that of a ULA over the same aperture. While the spatial covariance lag
 46 coverage is equivalent to that of a ULA, the nested array geometry is subopti-
 47 mal in the sensor count when compared to the MRLA. However, the simplicity
 48 in designing a nested array for large apertures provides an attractive trade-off
 49 in sensor count.

50 The array geometry of interest in this research is known as the linear Co-
 51 prime sensor array. The co-prime sensor array concept is the result of relaxing
 52 the spatial covariance lag coverage requirements of the ULA nested array geom-
 53 etry. A linear co-prime sensor array is formed by nesting two ULA's with sensor
 54 spacings given by $M\frac{\lambda}{2}$ and $N\frac{\lambda}{2}$, respectively. The inter-element scale factors M
 55 and N are co-prime integers, meaning that the greatest common divisor of the
 56 pair of integers is unity. The co-prime sensing strategy has been developed for
 57 both temporal and spatial signals [7]. In particular, for sampling spatial signals,
 58 Pal and Vaidyanathan have shown that it is possible to form a contiguous set of
 59 $MN - 1$ autocorrelation lags when M and N are co-prime and one of the sub-
 60 arrays is extended by a factor of two. It has been shown that this geometry is
 61 capable of providing $\mathcal{O}(MN)$ degrees of freedom from $\mathcal{O}(M+N)$ array elements.
 62 From this result, the co-prime array geometry can be used to approximate the
 63 sensing capabilities of a ULA with aperture $(MN - 1)\frac{\lambda}{2}$. Even though it is
 64 possible to approximate a ULA with more sensors than are physically placed
 65 in the sensing aperture, the aperture of the equivalent ULA is about half of

the physical co-prime array aperture. That is, the virtual ULA has aperture $(MN - 1)\frac{\lambda}{2}$, while the physical aperture of the co-prime array is $(2N - 1)M\frac{\lambda}{2}$. Figure 1 shows an example of the above described co-prime geometry.

Filling missing elements of the difference co-array for co-prime arrays has also been considered in the situation where the co-prime array is on a non-moving platform. BauDaher *et al.* have developed a strategy for filling in difference co-array elements based on the use of related source frequencies [8]. Here the related source frequencies are used to realize the phases of virtual sensor elements that can be used to augment the difference co-array for a co-prime array.

Figure 1: $N = 3, M = 5$ Co-prime Sensor Geometry

The focus of this paper is to address the performance gap in estimating spatial covariances inherent to the co-prime array. In particular, we wish to estimate a full set of spatial covariances over the complete aperture of the co-prime array. It has been shown that in situations where the array undergoes motion it is possible to increase the operational frequency of the array [9] [10]. Functionally this is a result of using physical sensors to sample at desired spatial locations, in the trajectory of the sensor array, to therefore virtually augment the sensor array up to the desired operational frequency. Adapting this concept, in this work we apply synthetic aperture processing to the co-prime sensor array in order to expand the spatial covariance sensing capability of the array. Here the objective is to virtually create missing sensor elements up to the degree required to produce a full set of spatial covariances over the aperture of the physical array

and the additional aperture synthesized during the SA process. In particular, we propose an SA procedure that is a function of the co-prime parameters N and M . The advantage of applying SA methods on a co-prime array is that the co-prime geometry guarantees half the physical aperture worth of spatial covariances and therefore with limited SA processing the missing spatial covariances can be realized.

In section 2 we review related co-prime and synthetic aperture work. Section 3 describes signal model and sensing environment and in section 4 we describe the synthetic aperture process for sensor synthesis and its connection to the difference co-array. In section 5 we propose our minimum temporal coherence criteria and describe its connection to the co-prime inter-element parameters N and M . In section 6 we discuss spatial covariance matrix estimation methods and in section 7 we provide simulation examples demonstrating the benefits of SA for co-prime sensor arrays. Finally in section 8 we provide a discussion and conclusion of our primary results.

2. Related Work

The primary advantage of the co-prime geometry is its simplicity in design, while providing the degrees of freedom superior to the number of sensors used to form the array. A critical challenge in achieving the performance gains afforded by co-prime arrays is how to effectively estimate a spatial covariance matrix from the spatial covariance estimates that are obtained from the array sensor channel readings. Common problems encountered are rank deficiencies and lack of positive definiteness in the covariance matrix estimate.

Methods for estimating a spatial covariance matrix from thinned sensor array readings have been proposed over several decades. From the early works of Pillai and Haber to those of Abramovich *et.al* and more recently Pal and Vaidyanathan [11, 12, 6], to name a few. These techniques vary in approach, computational cost, and performance over different signal-to-noise-ratio regimes. The common objective of these methods is to produce a covariance matrix with

116 dimension equal to the number of contiguous spatial covariances provided by
 117 the array. With a spatial covariance matrix estimate in hand one can apply
 118 array processing methods available as if the data were sensed on an equivalent
 119 ULA. The concept of ULA equivalence is based on the statistical equivalence
 120 between the thinned array snapshot and a ULA snapshot of which both can
 121 realize estimates of the same spatial covariances. In the work presented here,
 122 we apply three different spatial covariance matrix estimation techniques and
 123 explore their performance for source localization.

124 The novel contribution of this work is the non-standard application of SA
 125 processing. Typically, SA methods exploit platform motion to extend the phys-
 126 ical aperture of an array and has lead to the vast work in Synthetic Aperture
 127 Radar (SAR) and SONAR. Synthetic aperture methods use temporal signal co-
 128 herence to achieve improved array processing performance across the synthetic
 129 array aperture. Traditional SA techniques require temporal coherence that is
 130 proportional to the time required to travel the entire physical aperture. These
 131 methods have been used to improve angular resolution and signal gain over that
 132 provided by the physical array [13, 14, 15, 16].

133 While typically SA methods are used with uniformly spaced arrays, thinned
 134 array geometries have found applications in this setting. The MRLA has been
 135 used in SAR applications for the detection and localization of slow and fast
 136 moving targets [17]. In this work, Zou *et. al* proposed a MRLA-SAR system
 137 that uses the non-uniform inter-elements spacings of the MRLA to improve
 138 the maximum velocity of a target that can be detected without ambiguity. In
 139 addition, the MRLA has been used in SONAR for aperture extension [18].
 140 Here the authors combine the well know Extended Towed Array Measurements
 141 (ETAM) algorithm for phase compensation with a fourth-order cumulant to
 142 extend the array aperture with fewer sensors than required when compared to
 143 a ULA. The work presented here continues this line of research but instead of
 144 aperture extension as the primary motivation for SA processing, the focus is on
 145 expanding the spatial covariance lag estimation range. Co-prime sensor arrays
 146 have been recently studied by the authors in the SA framework [19]. It was

147 shown by using SA methods the achievable resolution of the co-prime array can
 148 be increased from the typical half-aperture resolution limit to the full extent of
 149 the physical aperture. In addition, as a function of the co-prime parameters, N
 150 M , a SA strategy that produces a sensor array with a filled difference co-array
 151 has been developed [20]. The work presented here summarizes and expands
 152 upon these results as well as considers source localization performance when
 153 ideal sensing conditions are relaxed.

154 **3. Sensing Environment & Signal Model**

155 The primary focus of this research is to develop an SA algorithm to estimate
 156 the spatial covariates of a ULA over a given aperture, but with less sensors than
 157 a ULA. In this situation, we consider a simplistic signal environment where a
 158 sensor array is towed with constant velocity with no perturbations from the
 159 medium. In addition, signal multi-path between source and receiver are not
 160 considered. These simplifications allow us to consider the SA algorithm with-
 161 out added complications. However, perturbations to the virtual sensors are
 162 considered in section 7. In future work, these factors will be studied in more
 163 detail with real data experiments.

164 Consider a co-prime array with a physical aperture of L meters composed
 165 of $Q = M + 2N - 1$ sensors. Let $d_k = \gamma_k \frac{\lambda_0}{2}$, with $k \in \{1, \dots, Q\}$ be the sensor
 166 spacing measured relative to the left most sensor. The shift factors γ_k are
 167 positive integers representing the integral number of half-wavelength spacings
 168 between sensor elements. Assume the array moves with a constant velocity v
 169 along a straight-line course and that K static incoherent radiating sources with
 170 common frequency Ω_0 and bearings $\theta_1, \dots, \theta_K$ are in the far-field of the array.
 171 In the work presented in this paper it is assumed that the source frequency Ω_0
 172 and platform velocity v m/s are known. Under these assumptions, the wave-field
 173 can be modeled as the sum of plane-waves.

Let the plane-wave signal received at the array from the k -th source be given

by, $\mathbf{s}_k(t) = \alpha_k \exp(j\Omega t) \mathbf{b}(\theta_k; \Omega)$ where $\alpha_k \sim \mathcal{CN}(0, \sigma_k^2)$, and

$$\mathbf{b}(\theta_k; \Omega) = [1, \exp(-j\Omega \frac{d_1 \sin(\theta_k)}{c}), \dots, \exp(-j\Omega \frac{d_{Q-1} \sin(\theta_k)}{c})]^\top \quad (1)$$

174 is the array manifold vector that encodes to co-prime array geometry. The
 175 factor α_k is a variable used to model the coherence time of the wave propagation
 176 medium and is further discussed in section 4.1. The frequency Ω includes the
 177 Doppler shift resulting from the relative motion between the source and the
 178 array given by $\Omega = \Omega_0(1 - \frac{v \sin(\theta)}{c})$. The Doppler shift frequency, Ω , and array
 179 element spacing terms, d_l with $l \in \{0, 1, \dots, Q-1\}$, in eq 1, represent a source-
 180 frequency-element-spacing mismatch. It is well known when the array inter-
 181 element spacing and frequency being sensed are not equivalent, $\lambda = \frac{c}{f}$, the
 182 array pattern is impacted by a widening/narrowing of the mainlobe. In addition,
 183 grating lobes appear when the source frequency is greater then the operational
 184 frequency of the array. Under certain conditions this mismatch can be negligible,
 185 to determine this condition we consider the argument of the array manifold
 186 vector described in eq 1. The argument of the l -th array manifold vector is
 187 given by,

$$\begin{aligned} \Omega \frac{d_l \sin(\theta_k)}{c} &= \Omega_0(1 - \frac{v \sin(\theta_k)}{c}) \frac{d_l \sin(\theta_k)}{c} \\ &= \Omega_0 \frac{d_l \sin(\theta_k)}{c} - \Omega_0 d_l \frac{v \sin^2(\theta_k)}{c^2} \end{aligned} \quad (2)$$

188 To enable an approximation the removes the source-frequency-element spac-
 189 ing mismatch from the array manifold vector model we require $\Omega_0 d_l \frac{v \sin^2(\theta_k)}{c^2} \approx 0$,
 190 expanding this term we have,

$$\begin{aligned} \Omega_0 d_l \frac{v \sin^2(\theta_k)}{c^2} &= \frac{c}{\lambda_0} \gamma_l \frac{\lambda_0}{2} \frac{v \sin^2(\theta_k)}{c^2} \\ &= \gamma_l \frac{1}{2} \frac{v \sin^2(\theta_k)}{c} \\ &< \gamma_l \frac{1}{2} \frac{v}{c} \end{aligned} \quad (3)$$

191 The final inequality in eq 3 suggests that when $v \ll c$, the argument of
 192 the array manifold vector terms reduces to that of a matched source-frequency-
 193 element spacing array. In addition, the dependance on the source angle θ_k on

the Doppler shift frequency vanishes under this assumption. When $v \ll c$, the k -th received signal is,

$$\begin{aligned} \mathbf{s}_k(t) &= \alpha_k \exp(j\Omega t) \mathbf{b}(\theta_k; \Omega_0) \\ &= \alpha_k \exp(j\Omega_0 t) \exp(-j\Omega_0 \frac{vt \sin(\theta_k)}{c}) \mathbf{b}(\theta_k; \Omega_0) \end{aligned} \quad (4)$$

where the array manifold vector now only depends on the source frequency Ω_0 . The snapshot collected at the array at time t from a K -source field is given by,

$$\mathbf{x}_{\text{cp}}(t) = \sum_{l=1}^K \mathbf{s}_l(t) + \mathbf{w}(t) \quad (5)$$

where $\mathbf{w}(t)$ is the receive noise vector is zero-mean complex additive Gaussian Noise with covariance $\sigma_w^2 \mathbf{I}_M$, $\mathbf{w}(t) \sim \mathcal{CN}(0, \sigma_w^2 \mathbf{I}_Q)$. The probabilistic distribution of $\mathbf{x}_{\text{cp}}(t)$ is complex Gaussian with mean and covariance given by,

$$\begin{aligned} \mathbf{E}(\mathbf{x}_{\text{cp}}(t)) &= \mathbf{0} \\ \mathbf{E}(\mathbf{x}_{\text{cp}}(t) \mathbf{x}_{\text{cp}}(t)^H) &= \mathbf{B} \mathbf{\Lambda} \mathbf{B}^H + \sigma_w^2 \mathbf{I}_{Q \times Q} \end{aligned} \quad (6)$$

where $\mathbf{B} = [\mathbf{b}(\theta_1; \Omega_0) \mathbf{b}(\theta_2; \Omega_0) \cdots \mathbf{b}(\theta_K; \Omega_0)]$ encodes the directions of arrival of the signals received at the co-prime array and $\mathbf{\Lambda} = \text{Diag}(\sigma_1^2, \cdots, \sigma_K^2)$ encodes the corresponding signal powers.

4. Joint Sensor-Covariance Domain Augmentation

Array motion allows for the increase of the dimension of the array manifold vector leading to an increase in the degrees of freedom of the array [6]. Let τ be the time for the array to travel a distance of $\frac{\lambda}{2}$ meters; then a synthetic array snapshot can be formed by appropriately interleaving the co-prime array snapshots from time instants $t, t + \tau, \cdots, t + \eta_0 \tau$. The $t + l\tau$ portion of the synthetic snapshot, for $l \in \{1, \cdots, \eta_0\}$ is given by,

$$\mathbf{s}_{l,k}(t) = \exp(-j\Psi_l) \mathbf{s}_k(t + l\tau) \quad (7)$$

where Ψ_l is a phase correction factor required to time-align the array snapshot from time $t + l\tau$ to time t . By evaluating $\mathbf{s}_k(t)$, defined above, at the augmentation instant $t + l\tau$, we see that

$$\begin{aligned} \mathbf{s}_k(t + l\tau) &= \alpha_k \exp(j\Omega_0 l\tau) \exp(j(\Omega_0 t)) \cdot \\ &\exp(-j\Omega_0 \frac{vt \sin(\theta_k)}{c}) \cdot \exp(-j\Omega_0 \frac{vl\tau \sin(\theta_k)}{c}) \mathbf{b}(\theta_k; \Omega_0) \end{aligned} \quad (8)$$

From this expression, we see that the phase correction factor is given by $\Psi_l = \Omega_0 l\tau$. Since $vl\tau = l\frac{\lambda}{2}$, elements of the synthetic array manifold vector are formed by the product $\exp(-j\Omega_0 \frac{vl\tau \sin(\theta_k)}{c}) \mathbf{b}(\theta_k)$. Let $\tilde{\mathbf{b}}(\theta_k; \Omega_0)$ represent a component of the synthetic array manifold vector given by,

$$\begin{aligned} \tilde{\mathbf{b}}(\theta_k; \Omega_0) &= \exp(-j\Omega_0 l\frac{\lambda}{2} \frac{\sin(\theta_k)}{c}) \mathbf{b}(\theta_k; \Omega_0) \\ &= [\exp(-j\Omega_0 l\frac{\lambda}{2} \frac{\sin(\theta_k)}{c}), \exp(-j\Omega_0 \frac{(\gamma_1 + l)\frac{\lambda}{2} \sin(\theta_k)}{c}), \\ &\quad \dots, \exp(-j\Omega_0 \frac{(\gamma_M + l)\frac{\lambda}{2} \sin(\theta_k)}{c})]^\top \end{aligned} \quad (9)$$

with half-wavelength sensor position shifts $\gamma_{1'}(l\tau) = \gamma_1 + l$, $\gamma_{2'}(l\tau) = \gamma_2 + l, \dots, \gamma_{M'}(l\tau) = \gamma_M + l$. By interleaving the physical and synthetic snapshot vectors, the augmented signal received at the array is given by,

$$\mathbf{s}'_k(t) = \mathbf{T}_0 \mathbf{s}_k(t) + \sum_{l=1}^{\eta_0} \exp(-j\Psi_l) \mathbf{T}_l \mathbf{s}_k(t + l\tau) \quad (10)$$

where $\mathbf{T}_i : \mathcal{C}^Q \rightarrow \mathcal{C}^{Q'}$ is 1's and 0's selection matrix that extends the Q -dimensional physical snapshot to a Q' -dimensional snapshot vector. The product $\exp(-j\Psi_l) \mathbf{T}_l \mathbf{s}_k(t + l\tau)$ produces a time-aligned vector of length Q' that inserts the entries of $\mathbf{s}_k(t + l\tau)$ in the appropriate synthesis positions. The snapshot vector $\mathbf{s}'_k(t)$ represents a wave-field sensed at an array with Q' sensors, with sensor spacing given by $\tilde{d}_i = \beta_i \frac{\lambda}{2}$ with,

$$\beta_i \in \{\gamma_m\}_{m=1}^Q \cup_{l=1}^L \{\gamma_{m'}(l\tau)\}_{m=1}^Q \quad (11)$$

The subsequent snapshot collected at the synthetic array is given by,

$$\mathbf{x}_{\text{sa}}(t) = \sum_{l=1}^K \mathbf{s}'_l(t) + \mathbf{w}'(t) \quad (12)$$

where $\mathbf{w}'(t) \sim \mathcal{CN}(0, \sigma_w^2 \mathbf{I}_{Q'})$. The probabilistic distribution of the synthetic array snapshot is a complex Gaussian with mean and covariance,

$$\begin{aligned} \mathbf{E}(\mathbf{x}_{\text{sa}}(t)) &= \mathbf{0} \\ \mathbf{E}(\mathbf{x}_{\text{sa}}(t)\mathbf{x}_{\text{sa}}(t)^H) &= \tilde{\mathbf{B}}\mathbf{\Lambda}\tilde{\mathbf{B}}^H + \sigma_w^2\mathbf{I}_{Q'\times Q'} \end{aligned} \quad (13)$$

where $\tilde{\mathbf{B}} = [\tilde{\mathbf{b}}(\theta_1; \Omega_0) \ \tilde{\mathbf{b}}(\theta_2; \Omega_0) \cdots \tilde{\mathbf{b}}(\theta_k; \Omega_0)]$ encodes the directions of arrival of far-field signals received at the synthetic array and $\mathbf{\Lambda} = \text{Diag}(\sigma_1^2, \dots, \sigma_k^2)$ encodes the corresponding signal powers.

The sensor synthesis process increases both the number of “sensors” in the array and the array aperture. The sensor count increases since virtual sensors are inserted into the array after each synthesis step. The aperture of the array is increased by $\frac{\lambda}{2}$ meters after each synthesis step and after η_0 synthesis steps the aperture of the array is increased by $\eta_0 \frac{\lambda}{2}$ meters. However, the number of “sensors” after synthesis is dependent on the initial sensor distribution of the array. This discrepancy is due to the possibility of sensor overlap between snapshots at different synthesis instants. This overlap can be advantageous as it can allow for estimation of the phase correction factor [21]. In the work presented here, the phase correction factor is assumed to be known.

4.1. Coherent Space-Time Modeling

To have spatial coherence across the synthetic array we require temporal coherence within the time window required to form the synthetic array snapshot. This can be captured in the above array motion signal model by letting α_k , for each source, be constant, (i.e. coherent) for the time required for the array to travel a distance of $\eta_0 \frac{\lambda}{2}$ meters. In this application of SA, we require spatial coherence only over a fraction of the array aperture since we are interested in expanding the spatial covariance estimation capability of the array physical array. Since the physical array is co-prime it inherently has an extended range of spatial covariance lag coverage thereby requiring limited coherence time to realize all missing covariance estimates. Traditional SA techniques require coherence over much longer distances typically on the order of the array aperture.

255 To model this requirement, we form one synthetic array snapshot every
 256 $n_0\tau + T_s$ seconds, where T_s is the system sampling time, and associate an *i.i.d.*
 257 signal amplitude vector $\vec{\alpha}(t_m) = [\alpha_1, \dots, \alpha_k]^\top$ for each snapshot interval. Each
 258 element of $\vec{\alpha}(t_m)$ is drawn $\sim \mathcal{CN}(0, \sigma_k^2)$. Under this coherence model we obtain a
 259 synthetic array snapshot where across the array the complex random amplitude
 260 is constant for each spatial signal present in the field. This model preserves the
 261 plane-wave structure of the wave-field over the aperture of the synthetic array
 262 snapshot.

263 4.2. $v \ll c$ Approximation Implications for SA Processing

264 The SA process is centered on creating the array manifold components for
 265 missing sensors from appropriate time shifts of the received data. In this section,
 266 we consider the implications of the source-frequency-element spacing mismatch
 267 after the synthetic aperture process. Suppose the distance between the l -th
 268 and $l + 1$ -th sensors is λ_0 (m) then SA can be used to produce a virtual sensor
 269 between the l -th and $l + 1$ -th physical sensors. In this case, the virtual sensor
 270 is produced by sampling channel data from the l -th sensor when it traverses a
 271 distance of $\frac{\lambda_0}{2}$ (m). With τ representing the time required for the array to travel
 272 $\frac{\lambda_0}{2}$ (m) we have virtual channel data derived from the SA process is given by,

$$\begin{aligned}
 s_l(t + \tau) &= \alpha_k \exp(j\Omega(t + \tau))b_l(\theta; \Omega) \\
 &= \alpha_k \exp(j\Omega t) \exp(j\Omega\tau)b_l(\theta; \Omega) \\
 &= \alpha_k \exp(j\Omega t) \exp(j\Omega_0(1 - \frac{v \sin(\theta)}{c})\tau)b_l(\theta; \Omega) \\
 &= \alpha_k \exp(j\Omega t) \exp(j\Omega_0\tau) \exp(-j\Omega_0 \frac{v\tau \sin(\theta)}{c})b_l(\theta; \Omega)
 \end{aligned} \tag{14}$$

273 Here $\exp(j\Omega_0\tau)$ is the phase term that is corrected during the SA process.
 274 Here we consider the l -th component of the array manifold vector defined in
 275 eq 1. The virtual component of the array manifold vector is derived from the
 276 product, $\exp(-j\Omega_0 \frac{v\tau \sin(\theta)}{c})b_l(\theta; \Omega)$. Expanding this term and recognizing that
 277 $v\tau = \frac{\lambda_0}{2}$ we have that

$$\exp(-j\Omega_0 \frac{v\tau \sin(\theta)}{c}) b_l(\theta; \Omega) \quad (15)$$

$$= \exp(-j\Omega_0 \frac{\frac{\lambda_0}{2} \sin(\theta)}{c}) \exp(-j\Omega_0 d_l \frac{\sin(\theta)}{c}) \exp(-j\Omega_0 \frac{v \sin^2(\theta)}{c^2} d_l) \quad (16)$$

$$= \exp(-j\Omega_0 (\frac{\lambda_0}{2} + d_l) \frac{\sin(\theta)}{c}) \exp(-j\Omega_0 \frac{v \sin^2(\theta)}{c^2} d_l) \quad (17)$$

278 The quantity $\frac{\lambda_0}{2} + d_l$ represents the distance off-set for a sensor between the
 279 l -th and $l+1$ -th physical sensors. The term $\exp(-j\Omega_0 (\frac{\lambda_0}{2} + d_l) \frac{\sin(\theta)}{c})$ represents
 280 the phase shift of a sensor placed between the l -th and the $l+1$ -th sensor for
 281 an array with operational wavelength λ_0 sensing a frequency Ω_0 . The term
 282 $\exp(-j\Omega_0 \frac{v \sin^2(\theta)}{c^2} d_l)$ is an arrival angle dependent phase offset which reduces to
 283 $\exp(-j \frac{v \sin^2(\theta)}{c} \frac{l}{2})$ since $\Omega_0 = c/\lambda_0$ and $d_l = l\lambda_0/2$. It is clear that the primary
 284 factors that influence this phase term is the ratio of tow velocity to sound
 285 propagation speed.

286 The array manifold vector after SA processing is given by,

$$\tilde{\mathbf{b}}(\theta; \Omega, \Omega_0) = \begin{bmatrix} 1 \\ \vdots \\ \exp(-j\Omega d_l \frac{\sin(\theta)}{c}) \\ \exp(-j\Omega_0 (\frac{\lambda_0}{2} + d_l) \frac{\sin(\theta)}{c}) \exp(-j \frac{v \sin^2(\theta)}{c} \frac{l}{2}) \\ \exp(-j\Omega d_{l+1} \frac{\sin(\theta)}{c}) \\ \vdots \\ \exp(-j\Omega d_{Q-1} \frac{\sin(\theta)}{c}) \end{bmatrix} \quad (18)$$

287 Here $\tilde{\mathbf{b}}(\theta; \Omega, \Omega_0)$ has an additional sensor when compared to $\mathbf{b}(\theta; \Omega)$, how-
 288 ever the frequency of the SA processed channel does not match the Doppler shift
 289 frequency. The mismatch in phase delay over the aperture of the array deterio-
 290 rates the spatial coherence of the synthetic array. The degree of degradation of
 291 the spatial coherence is a function of the ratio of the platform velocity to sound
 292 speed of the medium. This ratio will influence the Doppler Shift frequency mis-
 293 match and the phase offset introduced during the SA process. To measure the

294 loss of spatial coherence over the array we consider the ratio of the area under
 295 the array pattern formed from an array with SA processing and a ULA with no
 296 SA processing. The area under the ULA array pattern serves as a reference for
 297 perfect spatial coherence, while the area under the SA array pattern captures
 298 the effects of phase offsets. This metric is defined as

$$A = \frac{\int |\langle h, \mathbf{b}_{\text{SA}}(\theta, \Omega) \rangle| d\theta}{\int |\langle h, \mathbf{b}_{\text{ULA}}(\theta, \Omega) \rangle| d\theta} \quad (19)$$

299 where h is an all ones vector of length equal to the number of sensors in the
 300 array and $\mathbf{b}_{\text{SA}}(\theta, \Omega)$ and $\mathbf{b}_{\text{ULA}}(\theta, \Omega)$ are the array manifold vectors for the SA
 301 processed array and Uniform Line Array, respectively. When $A > 1$ phase
 302 coherence on the synthetic array has deteriorated to the degree large side-lobe
 303 distortions are present in the corresponding array pattern when compared to
 304 the ULA.

305 We consider an array aperture of $26\lambda_0/2$, a ULA over this aperture consists
 306 of 27 sensors spaced $\lambda_0/2$ (m) apart. Suppose the sensor locations of this ULA
 307 are indexed $\{0, 1, 2, \dots, 25, 26\}$ and that the ULA will have virtual sensors at
 308 index positions $l \in \{1, 3, 6, 10, 12, 15, 20, 25\}$. In this situation each of these
 309 virtual sensors contribute to the spatial coherence mismatch generated during
 310 the SA process. Figure 2 (a) shows the array pattern area ratio as a function of
 311 the v/c . Here it is shown that for $v/c < 10^{-2}$, the area ratios are nearly unity.
 312 This implies that the impact of the spatial coherence mismatch discussed above
 313 produces little to no distortion in the array pattern of the synthetic array. Figure
 314 2 (b) shows a sample array pattern. Here both the ULA and synthetic array
 315 patterns are shown. In perfect SA processing conditions, Figure 2 (b) shows that
 316 for 10 knots the loss of coherence produces a minor side-lobe increase. This
 317 measure implies that while the Doppler-frequency-element-spacing mismatch
 318 can impact the spatial coherence over the array, the $v \ll c$ assumption allows
 319 for removing this mismatch from the signal model as done in eq 4 is justified.

(a) Area Ratio

(b) Array Pattern: $v = 10$ Knots

Figure 2: Area Under Array Pattern

320 5. Minimum Temporal Coherence Period

321 Our objective is to form a synthetic array snapshot for an array with denser
 322 sensor sampling than what is available from the co-prime array. Our main
 323 strategy is to exploit the motion of the array to form the denser snapshot. As
 324 described in detail above, when the co-prime array is moved along a straight line
 325 path, measurements from different times can be fused together to fill-in desired
 326 sensor spacings. In order to accomplish this we must consider the coherence
 327 time of the spatial signals and the noise measured at the array.

SA processing requires that the coherence time be longer than the time
 needed for the array to move the required distance to form a given snapshot.

Let C_t be the coherence time, then we need to satisfy,

$$\frac{D}{v} \leq C_t \quad (20)$$

where D is to total distance traveled by the array during the synthetic aperture integration period and v is the velocity the array system is moving. Our goal is to determine how to minimize the total distance traveled by the co-prime array in order to maximize the spatial covariance lag range available from the synthetic array snapshot.

The optimization criteria is to produce a full set of spatial covariance measurements from each synthetic array snapshot, while minimizing the total synthetic aperture integration time window. In addition, this optimization criteria can be formalized in terms of the co-prime inter-element pair N, M . That is to say, as a function of the co-prime geometry we can determine the minimum number of $\frac{\lambda}{2}$ synthetic aperture integration iterations needed to fill the difference co-array over the aperture of the co-prime array.

The degrees of freedom of a line array are related to the number of unique spatial covariances that can be observed from the array snapshot. For co-prime arrays, Pal and Vaidyanathan have shown the co-array has a hole-free contiguous region of approximately half of the array aperture [6] [7]. This hole-free region is directly related to the degrees of freedom present in the array and is the primary reason the co-array plays a central role in SA methods [22]. Ramirez Jr. *et al.* have developed an upper bound on the Temporal-Coherence-Period (TCP) required to produce a filled co-array without synthesizing every missing sensor over the array aperture in a moving co-prime sensor array system [19]. This however, only provides a sufficient condition for filling the co-array. The necessary and sufficient condition for minimizing TCP while maximizing the degrees of freedom over the array aperture of a moving co-prime array has been formalized and is published in [20] and is restated here. The mathematical details are provided in [20].

Lemma 1. *The minimum Temporal-Coherence-Period for a (N, M) $\lambda/2$ co-prime array with $M + 2N - 1$ sensor elements and $N < M$ to have a hole-free*

co-array across the array aperture is given by $TCP = \eta_0 \tau$ where,

$$\eta_0 = \begin{cases} \frac{N}{2} & \text{if } N \text{ is even} \\ \frac{N-1}{2} & \text{if } N \text{ is odd} \end{cases} \quad (21)$$

and τ is the time required for the array to travel a distance of $\lambda/2$ meters.

This Lemma suggests that the minimum TCP is a function of the co-prime factor N and independent of the larger co-prime factor M . From this perspective, we can see that the minimum TCP is proportional to the time required for the array to travel $\frac{N}{2} \frac{\lambda}{2}$ (m) of the physical aperture.

The significance of the minimum TCP is that it describes the minimum distance required for a co-prime sensor array to travel for SA methods to produce a filled co-array. The hole-free co-array over the array aperture provides a full set of spatial covariances required to maximize the degrees of freedom available from the array aperture needed for signal subspace based source localization. The advantage of the work presented here is that via SA methods we are able to maximize the degrees of freedom over a given aperture with minimum synthesis time.

Figure 3 shows examples of the difference co-array for various amounts of SA processing on a particular linear co-prime sensor array. Here we consider a co-prime sensor array with parameters $N = 5, M = 7$ with aperture $63\frac{\lambda}{2}$, and the minimum number of SA iterations required to fill the difference co-array is $\eta_0 = 2$. In addition, the locations of the physical and virtual sensors are indicated at the top of each figure. From Figure 3 we see that at the above described minimum temporal coherence period we have a difference co-array that is hole free. In principle one could continue the SA process until the synthetic array has no missing sensors and produce the difference co-array corresponding to a ULA. However the price of this is an increase in the total SA process time to form one synthetic snapshot. The advantage of utilizing SA over the minimum coherence temporal period is that one has access to spatial covariance estimates from a thinned array as if these estimates were formed from a ULA. In principle, SA processing can be applied to any thinned array geometry to augment the

Table 1: Co-prime Array Integration Time Lengths

$c = 1500 \frac{m}{s}, v = 1.3 \frac{m}{s}, f = 440\text{Hz}$		
N, M	Physical Aperture $(2N - 1)M \frac{\lambda}{2}(m)$	$\frac{\eta_0 \frac{\lambda}{2}}{v}$ (s)
3,5	$25 \frac{\lambda}{2} = 42.61$	1.31
4,9	$63 \frac{\lambda}{2} = 107.3$	2.62
5,9	$81 \frac{\lambda}{2} = 138.0$	2.62
6,11	$121 \frac{\lambda}{2} = 206.2$	3.93
7,13	$169 \frac{\lambda}{2} = 288.0$	3.93
8,15	$225 \frac{\lambda}{2} = 383.5$	5.24
9,17	$289 \frac{\lambda}{2} = 492.6$	5.24

384 difference co-array. In particular, a Minimum-Hole array can be designed and
385 SA processing can be used to fill-in missing sensors such that the difference
386 co-array becomes hole-free.

387 Table 1 and 2 show typical co-prime array geometries along with the time
388 required to form one synthetic array snapshot in seconds for two different fre-
389 quency regimes. In these situations minimum required temporal coherence to
390 produce an synthetic array with a hole-free co-array is on the order of a few sec-
391 onds. While these estimates represent the theoretical requirements to apply SA
392 in our situations typical temporal coherence can range on the order of several to
393 tens of minutes. Williams and Battestin have reported coherence times of 8-12
394 minutes at 400 Hz, in the deep ocean [23], [24]. Similarly, Yang has studied
395 the temporal coherence of sound through shallow water over a wide range of
396 frequencies [25]. In this work, Yang found coherence time estimates of between
397 5 and 20 minutes. While temporal coherence can vary widely over signal fre-
398 quency and range the estimates found in the literature support the use of SA
399 in our application.

Table 2: Co-prime Array Integration Time Lengths

$c = 1500 \frac{m}{s}, v = 1.3 \frac{m}{s}, f = 1\text{kHz}$		
N, M	Physical Aperture $(2N - 1)M \frac{\lambda}{2}(m)$	$\frac{\eta_0 \frac{\lambda}{2}}{v}$ (s)
3,5	$25 \frac{\lambda}{2} = 18.75$	0.5769
4,9	$63 \frac{\lambda}{2} = 47.25$	1.15
5,9	$81 \frac{\lambda}{2} = 60.75$	1.15
6,11	$121 \frac{\lambda}{2} = 90.75$	1.73
7,13	$169 \frac{\lambda}{2} = 126.75$	1.73
8,15	$225 \frac{\lambda}{2} = 168.75$	2.30
9,17	$289 \frac{\lambda}{2} = 216.75$	2.30

6. Spatial Covariance Matrix Estimation

From a synthetic array snapshot, our goal is to form a positive-definite spatial covariance matrix that can be used for subspace based source localization with characteristics equivalent to a ULA. The synthetic array constructed from a co-prime array with the minimum Temporal Coherence Period criteria is capable of estimating a full set of spatial covariances over the aperture of the array. However, the synthetic array remains a thinned sensor array over the composite synthetic aperture since all missing sensor elements from the co-prime array are not virtually formed during the SA process. This implies that an outer product of synthetic array snapshots will not produce a spatial covariance matrix with the desired ULA characteristics. Instead, the spatial covariances that can be estimated from the synthetic array must be manipulated to form the desired spatial covariance matrix that approximates the ULA over the same aperture. In this section, we consider three different methods for producing a spatial covariance matrix from sensor measurements, first is the Direct Augmentation Algorithm (DAA), second is a method based minimizing the projection of the DAA estimate onto the space of positive-definite matrices, and finally the Rank Enhanced Spatial Smoothing (RESS) algorithm proposed by Pal and

418 Vaidyanathan.

419 Using the Direct Augmentation Algorithm (DAA), the co-array and the
 420 available spatial covariances provide an estimate of the l -th lag of the spatial
 421 covariance given by,

$$\hat{R}_x(l, t) = \frac{1}{c(l)} \sum_{\substack{(m_1, m_2) \in \mathcal{V}(l) \\ m_1 \leq m_2}} (\mathbf{x}(t))_{m_1} (\mathbf{x}(t))_{m_2}^* \quad (22)$$

422 where $\mathcal{V}(l)$ is the set of array snapshot element index pairs whose difference is l .
 423 By collecting the samples into a Toeplitz matrix, we can obtain an estimate of
 424 the augmented spatial-covariance matrix. However, estimating the covariance
 425 matrix in this manner does not guarantee positive-definiteness and may lead to
 426 negative eigenvalues in the spatial covariance matrix estimate.

427 Abramovich *et al.* has proposed methods for Toeplitz Positive-Definite co-
 428 variance matrix completion from the DAA spatial covariance matrix estimate
 429 [12]. However, these methods can lead to unstable localization results and
 430 are computationally expensive. A more tractable approach was introduced by
 431 Kazanci and Krolik [26]. In this work, they developed a technique known as
 432 Adaptive Channel Compensation (ACC) for dealing with sensor arrays with
 433 faulty sensors within the array aperture. The main idea of this technique is
 434 to use the reliable sensors along with an estimate of the full array covariance
 435 matrix to produce an estimate of sensor readings at the missing sensors. In this
 436 technique an estimate of a hole-free sensor array snapshot is produced via an
 437 optimization procedure. We have explored applying this technique to overcome
 438 positive definite deficiency encountered from estimating the spatial covariance
 439 matrix using the Direct Augmentation Approach. The optimization problem
 440 can be formulated as the following: we seek to find an array snapshot $\hat{\mathbf{q}}$, that
 441 will produce an outer product that is “closest“ to the DAA covariance matrix
 442 estimate of a single snapshot from the synthetic array, while minimizing its
 443 projection onto the subspace spanned by the eigenvectors corresponding to the
 444 negative eigenvalues of the DAA estimate. The array snapshot $\hat{\mathbf{q}}$ is a hole-free
 445 channel data vector over the aperture spanned by the synthetic array. This is

446 formalized mathematically by,

$$\hat{\mathbf{q}} = \min_{\mathbf{q}} \|\Phi_{\text{neg}}^H \mathbf{q}\|_F^2 \quad (23)$$

447 subject to the constraint $\|\mathbf{L}^H \mathbf{q} - \mathbf{x}_g\| < \epsilon$.

448 Where $\hat{\mathbf{q}}$ is the hole-free array snapshot estimate, Φ_{neg} characterizes the
 449 subspace spanned by the negative eigenvectors of the single snapshot DAA co-
 450 variance matrix estimate, $\mathbf{x}_g$ is a vector of sensor readings from the synthetic
 451 array, $\mathbf{L}$ identifies the sensors in $\hat{\mathbf{q}}$ corresponding to the same channels in the
 452 synthetic array, and ϵ is a closeness tolerance on elements of the snapshot es-
 453 timate that correspond to synthetic array sensor readings. From this point
 454 of view, the snapshot estimate $\hat{\mathbf{q}}$ preserves channel data corresponding to the
 455 synthetic array channels, while the optimization process produces estimates of
 456 channel data for the remaining channels of $\hat{\mathbf{q}}$. The closed form solution to this
 457 optimization problem can be found by an application of Lagrangian multipliers
 458 and is given by,

$$\hat{\mathbf{q}} = (\Phi_{\text{neg}} \Phi_{\text{neg}}^H + \mu \mathbf{L} \mathbf{L}^H)^{-1} \mu \mathbf{L} \mathbf{x}_g \quad (24)$$

459 where μ is a regularization parameter used to tune the optimization process.
 460 In this approach, we seek to find an array snapshot estimate $\hat{\mathbf{q}}$ that produces
 461 a spatial covariance matrix estimate that is the closest positive-definite matrix
 462 to the DAA covariance matrix estimate derived from a single synthetic array
 463 snapshot. For a given synthetic array snapshot we can produce a virtual array
 464 snapshot that has no missing sensors over the aperture of the array that can
 465 be used to produce a covariance matrix estimate by averaging outer-products.
 466 This modification to the ACC covariance estimation method will be denoted
 467 as the Subspace-Projection Method in the remainder of this paper.

468 An alternate approach to exploit the difference co-array has been developed
 469 by Pal and Vaidyanathan and is known as Rank Enhanced Spatial Smoothing
 470 (RESS). This algorithm exploits the hole-free co-array section of the co-prime
 471 array [6, 27]. In these works, they have shown how to develop a positive-

definite spatial covariance matrix for source localization from the array snapshot vector. The RESS algorithm can be applied to other array geometries that have contiguous sections of hole-free co-array. In addition this technique can be applied to array geometries that produce hole-free difference co-arrays over the entire aperture of the array. Shakeri *et al.* have applied the RESS algorithm to Sparse Ruler Array designs for the localization of more sources than sensors [28]. The development of the RESS algorithm is based on work of Ma *et.al*, where Khatri-Rao subspace methods were used for DOA estimation [29].

The RESS algorithm for the minimum temporal coherence period synthetic array can be described in the following major stages:

Stage 1: Co-array

The co-array is hole-free for spatial lags in the range, $0 < l < (2N-1)M + \eta_0$. This corresponds to an underlying ULA with $J = (2N-1)M + \eta_0 + 1$ sensor elements.

Stage 2: Direct Data Covariance (DDC)

$$\mathbf{R}_{\mathbf{x}'(t)} = \mathbf{x}'(t)\mathbf{x}'^H(t) = \sum_{i=1}^K \mathbf{v}_s(\theta_i)\mathbf{v}_s^H(\theta_i)\sigma_i^2 + \sigma_n^2\mathbf{I} \quad (25)$$

Since the co-array for the synthetic array is hole-free, spatial covariances for lags $0 < l < (2N-1)M + \eta_0$ are present in the DDC matrix.

Stage 3: Vectorize DDC

$$\mathbf{y} = \text{vec}(\mathbf{R}_{\mathbf{x}'(t)}) = \mathbf{B}\mathbf{p} + \sigma_n^2\hat{\mathbf{e}} \quad (26)$$

Where $\mathbf{B}$ is a $Q'^2 \times K$ complex matrix and each column of $\mathbf{B}$ is the Kronecker product between array manifold vectors of the sparse array and their conjugates. Namely, $\mathbf{B}$ can be written as,

$$\mathbf{B} = \left[\mathbf{v}_s^*(\theta_1) \otimes \mathbf{v}_s(\theta_1), \dots, \mathbf{v}_s^*(\theta_K) \otimes \mathbf{v}_s(\theta_K) \right] \quad (27)$$

with $\mathbf{p} = [\sigma_1^2, \dots, \sigma_K^2]^\top$, $\hat{\mathbf{e}} = [\mathbf{e}_1^\top, \mathbf{e}_2^\top, \dots, \mathbf{e}_{Q'}^\top]$, and $\mathbf{e}_i$ a vector of all zeros except for the i -th element.

Stage 4: $(2J-1)$ -ULA

Each column of $\mathbf{B}$ has $2J - 1$ distinct values. These distinct values can be used to form the spatial covariances from a uniform linear array with $2J - 1$ sensors. The longer ULA is developed by extracting the distinct rows of $\mathbf{B}$, this results in a new matrix $\mathbf{B}_1$ with dimensions $(2J - 1) \times K$. The columns of $\mathbf{B}_1$ form the array manifold vectors for a uniform line array with $2J - 1$ sensors with a phase center at the J -th sensor. The array snapshot for the $(2J - 1)$ -ULA is,

$$\mathbf{y}_1 = \mathbf{B}_1 \mathbf{p} + \sigma_n^2 \mathbf{e}' \in \mathbf{C}^{(2J-1) \times 1} \quad (28)$$

491 with $\mathbf{e}'$ is a vector of all zeros except for a 1 at the J -th position.

492 **Stage 5:** J -ULA Subarray

The $(2J - 1)$ -ULA we can subdivide into overlapping J -ULA subarrays. The elements of each subarray are taken from the larger ULA as, $\{(-i + 1 + n) \frac{\lambda}{2}\}_{n=0}^{J-1}$. The J -ULA subarray snapshot can be written as,

$$\mathbf{y}_{1,i} = \mathbf{B}_{1i} \mathbf{p} + \sigma_n^2 \mathbf{e}'_i \quad (29)$$

493 where $\mathbf{B}_{1i}$ is the $J \times K$ array manifold vector matrix for the i -th subarray and
 494 $\mathbf{e}'_i$ is a zeros vector except for a 1 is the i -th position. For each subarray we can
 495 compute the $J \times J$ Rank-1 spatial covariance matrix as, $\mathbf{R}_{1i} = \mathbf{y}_{1,i} \mathbf{y}_{1,i}^H$.

Stage 6: Rank Enhancement

$$\mathbf{R}_{s\mathbf{x}'(t)} = \frac{1}{J} \sum_{i=1}^J \mathbf{R}_{1i} \quad (30)$$

496 Theorem 1 in [27] proves that $\mathbf{R}_{s\mathbf{x}'(t)}$ has the same form, but with squared
 497 eigenvalues, as the positive-definite spatial covariance matrix of a ULA with J
 498 sensors. This matrix provides access to the degrees of freedom on the order of
 499 the synthetic array aperture. In addition, Liu and Vaidyanathan have devel-
 500 oped a lower complexity method for co-array based processing. This method
 501 produces a covariance matrix estimate that can be used for source localiza-
 502 tion while relaxing the need for spatial smoothing in a signal/noise subspace
 503 based source localization technique [31]. In the following section we analyze the
 504 source localization performance of DAA, Subspace projection, and RESS spatial
 505 covariance estimation methods.

506 7. Source Localization Simulation Results

507 In this section, we explore the use of the co-prime array and SA processing
508 for source localization. We consider the co-prime ($N = 3, M = 5$) sensor array
509 that consists of a total of 10 physical sensors spanning a $25\frac{\lambda}{2}$ (m) aperture. To
510 enable co-array processing this array requires 1τ augmentation step, where τ is
511 the time required to travel a total distance of $\lambda_0/2$ (m). For the simulations
512 presented here, the wave propagation speed, array velocity, and source frequency
513 f_0 are chosen to be $1500\frac{\text{m}}{\text{s}}, 1.3\frac{\text{m}}{\text{s}}$, and 440Hz, respectively.

514 In particular, we examine the Direct Augmentation Algorithm (DAA), Sub-
515 space Projection, and Rank Enhanced Spatial Smoothing (RESS) techniques
516 for spatial covariance estimation. Our objective is to gain insights into these
517 approaches and contrast these covariance matrix technique under a common
518 wave-field. In addition, we consider the performance of each covariance matrix
519 estimation technique over SNR and number of temporal array snapshots used
520 in the covariance estimation process.

521 7.1. Field-Directionality Mapping

522 To evaluate the source localization performance of the co-prime array with
523 SA processing we examine the spatial spectrum derived from the above spatial
524 covariance matrix estimation techniques. We consider two different approaches
525 to accomplish this task. The first approach is to use beamforming to compute
526 the received power over a discrete set of angles θ across the field of view of
527 the array. From this set of power readings we are able to detect sources in the
528 far-field of the array. The second approach is to apply a well know algorithm for
529 source localization, the MUSIC algorithm. MUSIC is the acronym for Multiple
530 Signal Classification and is a high-resolution method based on the signal/noise
531 subspace projections.

532 In both of the above approaches, we will hold the convention of scanning the
533 array relative to broadside where angles are measured from the center of the
534 aperture on the range of $\theta \in [-90^\circ, 90^\circ]$.

The first approach for field-directionality mapping is obtained by computing the power received at the synthetic array from angle θ . The mathematical model for this approach is given by,

$$P(\theta) = \frac{1}{Q'^2} \mathbf{V}(\theta)^H \hat{\mathbf{R}}_a \mathbf{V}(\theta) \quad (31)$$

where $\mathbf{V}(\theta)$ is the synthetic array steering vector given by,

$$\mathbf{V}(\theta) = [1, \exp(-j\Omega_0 \frac{d_1 \sin(\theta)}{c}), \dots, \exp(-j\Omega_0 \frac{d_{Q'-1} \sin(\theta)}{c})]^\top \quad (32)$$

535 and $\hat{\mathbf{R}}_a$ is the spatial covariance matrix estimate from the synthetic array com-
 536 puted via one of the above described estimation methods. This method produces
 537 what is commonly referred to as the Spatial Power Spectrum.

538 The second approach is the well known Multiple Signal Classification algo-
 539 rithm (MUSIC), and is a subspace method where the spatial covariance matrix
 540 estimate, $\hat{\mathbf{R}}_a$, is decomposed into its signal and noise subspaces. Here

$$\hat{\mathbf{R}}_a = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^{-1} \quad (33)$$

where $\mathbf{U}$ is a matrix of the eigenvectors and $\mathbf{\Lambda}$ is a diagonal matrix of the corre-
 sponding eigenvalues of $\hat{\mathbf{R}}_a$. The MUSIC spectrum is computed by evaluating
 the matrix polynomial

$$f(\theta) = \frac{1}{\mathbf{V}(\theta)^H \hat{\mathbf{R}}_{a,\text{noise}} \mathbf{V}(\theta)} \quad (34)$$

541 where $\mathbf{V}(\theta)$ is as defined above, and $\hat{\mathbf{R}}_{a,\text{noise}}$ represents the noise subspace of the
 542 spatial covariance matrix. The dimension of the noise subspace is a parameter
 543 that needs to be estimated and is typically taken to be the number of the smallest
 544 eigenvalues with approximately the same power. Here the idea is to evaluate
 545 the polynomial $\mathbf{V}(\theta)^H \hat{\mathbf{R}}_{a,\text{noise}} \mathbf{V}(\theta)$ and if this quantity is small that the steering
 546 vector $\mathbf{V}(\theta)$ lies in the compliment of the noise subspace. From this perspective
 547 one would say that $\mathbf{V}(\theta)$ lies in the signal subspace and therefore represents
 548 an angle at which a signal is present. For a more complete development of the
 549 MUSIC algorithm see [30].

Table 3: Covariance Matrix Estimation Summary

Technique	Summary	Algorithm Parameters
RESS	Spatial smoothing technique applied to the hole-free component of the difference co-array	None
Subspace Projection	Estimate array channel data via subspace minimization	Regularization parameter μ
DAA	Form a Toeplitz matrix from spatial covariance estimates	Array snapshot support

550 In the remainder of this section we will compare and contrast the perfor-
551 mance of the three different covariance matrix estimation techniques presented
552 in section 4. The common objective in the use of each of the different tech-
553 niques is to form an estimate of a spatial covariance matrix that approximates
554 what would have been obtained from a ULA system, but with fewer sensors.
555 In this sense, we are interested in the source localization and side-lobe perfor-
556 mance of each technique. Table 3 summarizes each technique covariance matrix
557 estimation technique.

558 The Subspace Projection method used the single snapshot DAA covariance
559 matrix approximation to produce an array snapshot where all post synthetic
560 aperture missing channels of data are estimated. In this respect, the Subspace
561 Projection method produces an array snapshot that has active channels at all
562 $\lambda/2$ spacings over the aperture of the array, $\hat{q}$ is of the same dimension as a ULA
563 over the same aperture. The regularization parameter μ is a tuning parameter in
564 the subspace-projection optimization process that has an impact on the level of
565 agreement between the channels produced during the SA process and the same
566 channels produced after Subspace Projection. Figure 4 shows the norm-squared
567 error between $\hat{q}(t)$ and the corresponding ULA snapshot $\mathbf{x}_{\text{ULA}}(t)$. In this sense,

Table 4: Source Location Ground Truth

Angle θ°	$-60^\circ, -50^\circ, -40^\circ, -30^\circ, -20^\circ, -10^\circ, -2.5^\circ$ $0^\circ, 5^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ, 50^\circ, 60^\circ$
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we obtain an estimate of how close the output of the optimization process is to producing accurate channel data on any missing channels after the SA process is complete. The impact of μ on the Average norm-squared error can be seen in Figure 4 (a-b). Here the data suggests that for $10^{-4} < \mu < 10^{-2}$ the mean error is minimum as well as has a minimum variance.

The DAA covariance matrix estimation method is sensitive to the number of snapshot used to average spatial covariance estimates. Figure 5 shows the eigenvalue spectra for a single source scenario with SNR of 0 (dB). Here we see that we require at least 55 array snapshots to produce non-negative eigenvalues.

Using this understanding of the effect of snapshot support for the DAA estimation technique and the effect of μ on the snapshot estimate for the Subspace Projection technique, we compare the source localization performance across the different techniques.

7.2. Multiple Source Localization

Consider the wave-field that is formed by 15 far-field sources, where the source angles relative to array broadside with an SNR of 0 dB are given in Table 4. Here sources are spaced uniformly over the intervals of $[-60^\circ, -10^\circ]$ and $[10^\circ, 60^\circ]$ at increments of 10° . However, within the interval $(-10^\circ, 10^\circ)$ sources are spaced non-uniformly at -2.5° , 0° , and 5° .

Figure 6 shows the Spatial Power Spectrum and MUSIC Spectrum derived from each of the spatial covariance estimation methods. A total of 500 array snapshots are used to estimate the spatial covariance matrix for each estimation method and the SNR of each source is constant at 0 (dB). In these plots we compare the spatial spectrums of each covariance estimation method against that of the ULA. Figure 6(a) shows the Spatial Power Spectrum, in this plot we

593 see that the spectrum derived from the RESS and DAA approximation methods
 594 are in good agreement with the ULA. In particular, these methods produce
 595 side-lobe levels equivalent to a ULA. However, the Subspace Projection methods
 596 suffers from higher side-lobes. The side-lobes of the Subspace Projection method
 597 are a result of the agreement between the estimated channel data produce by
 598 the choice of regularization parameter μ . Here μ was chosen to minimize the
 599 error in the estimated channel data when compared to the ULA. Figure 6(a)
 600 also shows the Spatial Power Spectrum from the physical co-prime array with
 601 no SA processing. In this case, the side-lobes are the highest due to the non-
 602 uniformity of the sensor placement within the array. For each of the arrays
 603 and estimation methods the source at -2.5° and 0° are not resolvable. Figure
 604 6(b) shows the MUSIC spectrum derived form each estimation technique and
 605 the true ULA. Here in all but the Subspace Projection Method we are able to
 606 resolve all 15 sources.

607 7.3. Performance Analysis

608 We consider a wave-field with a single far-field source located at 0° relative to
 609 array broadside. Here we perform Monte Carlo (MC) analysis over the number
 610 of snapshots used to estimate a given spatial covariance matrix and the SNR of
 611 the far-field source. In addition, we show the Mean and Variance of the Peak
 612 Side-lobe over both the number of snapshots and SNR parameters. The peak
 613 side-lobe informs the ability of the covariance matrix estimation techniques have
 614 to approach that of the ULA. A total of 100 noise realizations are used for each
 615 MC analysis.

616 Figure 7 shows the MC results over the number of array snapshots used to es-
 617 timate a given covariance matrix. The RMSE of the bearing estimate are in good
 618 agreement between each of the covariance matrix estimation methods. However,
 619 the Peak Side-lobe level reveals performance differences in each estimation tech-
 620 nique. Figure 7 (b-c) shows the mean and variance of the peak side-lobe level
 621 over a range of array snapshots. Here we see that as the number of snapshots
 622 increases the side-lobes from the DAA and RESS covariance estimates follow

the trend of the ULA while the side-lobes of the Subspace Projection method do not. This discrepancy is the result of the choice of regularization parameter μ , which controls the level of agreement between the estimated channel data. This limiting peak side-lobe level of the Subspace Projection method is the result of mismatch over the channel data.

Figure 8 shows the MC results over SNR. Here we see again that the RMSE of the bearing estimate are in good agreement with that of the ULA over the SNR range. However the Average Peak side-lobe level for both the RESS and DAA covariance matrix estimation methods follow that of the ULA. Again the effect of μ is seen in the average peak side-lobe measurement from the Subspace Projection method.

The above performance analysis suggest that the DAA and RESS covariance matrix estimation methods are capable of tracking the performance of the ULA, while the sensor array where these covariance matrix estimate are derived from contains fewer sensors. These simulation results suggest that while the Subspace Projection can produce accurate bearing estimates, the limiting factor in its performance is the Peak Side-lobe level.

7.4. Synthetic Aperture Temporal Sampling Perturbation

The synthetic aperture process presented in this work is centered on temporally sampling at desired spatial locations such that array snapshots from different time instants can be fused together to form a denser array. The synthetic array snapshot referenced at time t is formed by fusing together sensor array snapshots from sample times $t + \tau, t + 2\tau, \dots, t + \eta\tau$, where η is the number of synthesis steps required to fill the difference co-array of the synthetic sensor array. Recall that τ is the time required for the array to travel a distance of $\lambda_0/2$ (m) for a given velocity v m/s. The process of forming virtual sensors amounts to developing desired phase delays from the Doppler-shift and physical array manifold vector to account for virtual sensors at desired spatial locations. Recall that this is expressed mathematically from equation 8 and 9 of section 2. Here we see that the l -th segment of elements of the synthetic array manifold

vector, $\tilde{\mathbf{b}}_l(\theta_k; \Omega_0)$ where $l \in \{1, 2, \dots, \eta\}$, is given by,

$$\tilde{\mathbf{b}}_l(\theta_k; \Omega_0) = \exp(-j\Omega_0 v l \tau \frac{\sin(\theta_k)}{c}) \mathbf{b}(\theta_k; \Omega_0) \quad (35)$$

where $\mathbf{b}(\theta_k; \Omega_0) = [1, \exp(-j\Omega_0 \frac{d_1 \sin(\theta_k)}{c}), \dots, \exp(-j\Omega_0 \frac{d_{Q-1} \sin(\theta_k)}{c})]^\top$ is the array manifold vector of the physical array and τ is the sampling instant such that, $v\tau = \lambda_0/2$ (m).

To examine the robustness of the synthetic aperture sampling process we consider the effects of a random offset in half-wavelength sampling instant τ . Let the modified sampling instant $\tau' = \tau + \delta$, where $\delta \sim \mathcal{N}(0, \sigma_\delta^2)$ and δ carries the units of seconds. With this model we see that virtual sensors are no longer realized at half-wavelength intervals instead the virtual sensor locations are normally distributed $\lambda_0/2 + v\delta \sim \mathcal{N}(\frac{\lambda}{2}, v^2\sigma_\delta^2)$ random variables.

In order to evaluate the effect of the random offset in half-wavelength sampling instant we consider the situation of estimating a spatial covariance matrix using the RESS algorithm compared to a ULA with no synthetic aperture processing. The evaluation metric is the angle ϕ between the signal subspaces derived from each spatial covariance matrix and is given by,

$$\phi(V_1, V_2) = \cos(|V_1^\top V_2|)^{-1} \quad (36)$$

where an angle near zero implies the subspaces are nearly co-linear, while an angle $\frac{\pi}{2}$ indicates orthogonal subspaces. In particular, we measure the angle between the signal subspace derived from the RESS spatial covariance matrix with perturbed sampling realizations against the signal subspace derived from a ULA with no synthetic aperture processing.

In this simulation, the signal field is made of three far-field sources with SNR of 5 dB at arrival angles $-8^\circ, 10^\circ, 18^\circ$, referenced to array broadside. Each spatial covariance matrix is estimated from 200 array snapshots. Figure 7 shows the subspace matching performance as a function of offset variance σ_δ^2 . For each variance σ_δ^2 100 Monte Carlo iterations of subspace measurement are computed. The average subspace angle and first standard deviation are shown for each offset

679 variance. Figure 9 suggests that the synthetic aperture process is not robust
680 to offsets in the sampling times used for creating virtual sensors at the desired
681 spatial locations. Current work is underway to implement correction factors for
682 timing offsets into the processing scheme and was not the focus of the work
683 presented here. This includes applying time domain interpolation to compute
684 the sensor data at the required time instants $t+\tau, t+2\tau, \dots, t+\eta\tau$. In addition,
685 sampling errors can arise from array shape deformation due to sea conditions.
686 Addressing this issue is beyond the scope of this work.

687 The key point in the experiments presented above is the performance of
688 the co-prime array with minimal SA processing. The primary advantage of
689 using the co-prime array in the SA setting is the minimum temporal coherence
690 requirements in order to approximate the performance of a ULA in idealized
691 signal environments. However, it remains to develop an effective difference
692 co-array SA methods to deal with offsets in sampling as well more complicated
693 signal models. This is a topic of current work that will be developed in upcoming
694 publications.

695 8. Conclusion

696 In this work, we considered the use of synthetic aperture processing for im-
697 proving the performance of a thinned array system. In particular the primary
698 objective was to use a thinned array to approximate the side-lobe level and
699 source localization performance of a ULA. The fundamental trade-off between
700 this approach and processing data with a true ULA is computational complexity.
701 This complexity is derived from the SA processing along with the spatial covari-
702 ance matrix estimation requirements. However, in situations where resources
703 are limited or sensors fail, in moving array systems, these approaches can be
704 applied to mitigate performance losses. The primary focus of the application of
705 SA in this work is to address the limitations in the spatial covariance lag esti-
706 mation range of linear co-prime arrays. Co-prime arrays are a particular class of
707 thinned arrays that have spatial covariance lag coverage guarantees that make

708 them attractive candidates for SA processing. In the static setting co-prime
709 arrays provide a contiguous range of spatial covariances of $\mathcal{O}(MN)$ while the
710 array has only $\mathcal{O}(M + N)$ sensors. In addition, this property of co-prime arrays
711 scales with co-prime factors M, N allowing for large aperture arrays to be easily
712 designed. This characteristic of co-prime arrays is what makes them appealing
713 in sensing applications.

714 In particular, the goal of this work is to maximize the spatial covariance
715 sensing capability of the co-prime array system, while minimizing the synthetic
716 aperture processing time requirements. We have shown that the minimum num-
717 ber of SA iterations in order to produce a hole-free co-array is a function of the
718 co-prime parameters N, M and is approximately half of the smaller of the two
719 parameters. This implies that spatial coherence of the environment is required
720 for the amount of time required for the array to traverse only a fraction of total
721 array aperture. In addition, we have presented three methods for computing
722 spatial covariance matrices from the SA array snapshots. These methods var-
723 ied in approach and computational complexity. While each method is suitable
724 under certain conditions, they each are able to produce reliable covariance ma-
725 trix estimates. In principle, these techniques can be applied to other thinned
726 array geometries and alternative minimum coherence time requirements can be
727 developed.

728 Overall, SA processing was shown to improve the array processing perfor-
729 mance of the co-prime array. By virtue of adjoining virtual sensors at half-
730 wavelength spacings, we are able to reduce the peak side-lobe level of the physi-
731 cal array. However, the true performance gain comes from the ULA equivalence
732 derived from the spatial covariance estimates acquired from the synthetic array
733 snapshot. While the synthetic array remains a thinned array, the corresponding
734 difference co-array is hole-free. This property enables processing techniques the
735 exploit the spatial covariance lag coverage of the synthetic array. To this end we
736 have shown that it is possible to approximate the side-lobe level of a ULA from
737 synthetic array. However, when ideal conditions are relaxed these properties
738 quickly deteriorate. The focus of future work on the subject is to develop and

investigate strategies for accounting for mismatch in SA processing and allowing
for more realistic signal environments.

Acknowledgment

This work was supported by the Office of Naval Research (ONR) Code 321US
grant no. N00014-13-1020-1.

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(a) Physical Array Difference Co-array

(b) Synthetic Array Difference Co-array, $\eta_0 = 2$

(c) Synthetic Array Difference Co-array, $\eta = 6$

Figure 3: Difference Co-array for different stages of synthetic aperture processing.

(a) Average Channel Error

(b) Variance of Channel Error

Figure 4: Subspace Projection Channel Error

(a) 10 Array Snapshots

(b) 25 Array Snapshots

(c) 55 Array Snapshots

Figure 5: DAA Covariance Matrix Eigenvalues

(a) Spatial Power Spectrum

(b) MUSIC Spectrum

Figure 6: Source Localization

(a) Bearing RMSE

(b) Average Peak Side-lobe

(c) Variance of Peak Side-lobe

Figure 7: Array Snapshot Monte Carlo

(a) Bearing RMSE

(b) Average Peak Side-lobe

(c) Variance of Peak Side-lobe

Figure 8: SNR Monte Carlo

Figure 9: Subspace Angle (Rad)