The Spark of Fourier Matrices: Connections to Vanishing Sums and Coprimeness [☆]

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Abstract

¹ We consider conditions under which L rows of an N point DFT matrix form a ma-
² trix with spark $L+1$, i.e. a matrix with full spark. A matrix has spark $N+1$ if all L
³ columns are linearly independent. This has application in compressed sensing for
⁴ MRI and synthetic aperture radar, where measurements are under sampled Fourier
⁵ measurements, and the observation matrix comprises certain rows of the DFT ma-
⁶ trix. It is known that contiguous rows of the DFT matrix render full spark and that
⁷ from such a base set one can build a suite of other sets of rows that maintain full
⁸ spark. We consider an alternative base set of the form $\{0, 1, \dots, K\} \setminus \{n\}$, and
⁹ derive conditions on K , n and the prime factors of N , under which full spark is
¹⁰ retained. We show that such a matrix has full spark iff there are no K distinct
¹¹ N -th roots of unity whose n -products form a vanishing sum, and leverage recent
¹² characterizations of vanishing sums of N -th roots of unity to establish the stated
¹³ conditions.

Keywords:

Coprime sensing, full spark, compressed sensing, vanishing sums.

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1. Introduction

The pioneering work of Pal and Vaidyanathan, [1], has introduced the notion of coprime sampling and revealed its importance to array processing in general and compressed sensing (CS) in particular. Broadly viewed [1] and the subsequent literature exploits certain fundamental properties of coprime integers. Inspired by these results, we consider yet another consequence of coprime conditions: their implication to the spark or Kruskal rank of a submatrix of the Discrete Fourier Transform (DFT) matrix.

The practical applications motivating our work include MRI [2, 3, 4] and synthetic aperture radar [5, 6, 7], where under-sampled Fourier measurements are used to reconstruct images. In MRI in particular such observations are expensive and need to be kept to a minimum. It is of fundamental importance in such a case to know which Fourier sampling patterns are effective in image recovery and which are not, and how to keep these samples as small as possible, without impairing the reliability of recovery. *Accordingly, this paper is dedicated to provide certain sampling patterns that permit such recovery and providing others that do not.*

To be specific the vectorized images in such applications are often sparse and are observed through observation matrices that comprise submatrices of Discrete Fourier Transform (DFT) matrices obtained by removing several rows. The rows that are retained define the sampling patterns we seek. The spark or the Kruskal rank of a matrix is the smallest number of linearly dependent columns in the matrix. It plays a fundamental role in providing the answers we seek.

Thus consider the recovery of a k -sparse vector $x \in \mathbb{C}^N$ from its measurements

$$y = Ax,$$

where a k -sparse vector is one which has at most k nonzero entries. In this case it is possible to recover x from y even if the *observation matrix* $A \in \mathbb{C}^{L \times N}$, $L < N$ is fat. Should $\mathbf{spark}(A) > 2k$ [8, 9] then in principle x can be uniquely recovered from y , though the recovery may require a combinatorically involved ℓ_0 , constrained optimization. Under the so called Restricted Isometry Property (RIP), [10],[11] recovery of x is possible using a convex, constrained ℓ_1 optimization. A necessary and sufficient condition for such ℓ_1 recovery is that the matrix A satisfy the so called Null Space Condition (NSP), [12]. Though $\mathbf{spark}(A) > 2k$ by itself may not satisfy RIP or NSP there are loose relationships between spark and RIP in NSP, [12] that permit one to determine a higher spark level for which ℓ_1 recovery is possible.

Whatever level of spark is desirable it is important that it be achieved with the smallest number of measurements, i.e. with an observation matrix with the fewest number of rows. This is so as each extra measurement comes with a penalty (e.g. acquisition time in MRI). Thus it is our goal to provide conditions under which a matrix A obtained by removing rows of a DFT matrix, has *full spark*: A matrix with L rows has full spark if $\text{spark}(A) = L + 1$.

Since the computation of the spark of a large matrix is intractable, most designs of high spark matrices use the Welch bound $\text{spark}(A) > 1/M$, where M is the maximum off-diagonal entry of $A^H A$, a number known as the mutual coherence of A [8]. Equiangular tight frame matrices achieve the Welch bound [13, 14].

As can be imagined, the Welch bound is highly conservative for DFT matrices. Many results on spark involve observation matrices whose elements are drawn from Gaussian distributions. Use of such observation matrices in applications like MRI and SAR is simply impractical. While the common practice in CS-MRI is to randomly sample the Fourier transform using a variable density sampling pattern, there are no theoretical guarantees on the recovery using these matrices. Recently, some researchers have shown that the partial Fourier matrices, obtained by picking the rows of the Discrete Fourier Transform (DFT) matrix, according to a particular difference set attains the Welch bound [15, 16].

In this paper we ask the following question: Which L rows of an $N \times N$ DFT matrix yield a submatrix that is full spark? It is well known that L -consecutive rows of the DFT matrix have this property. Further [17] shows that should L rows indexed by the set $\mathcal{M}$ have spark $L + 1$, then so do those indexed by (a) a translation, (b) or the complement of $\mathcal{M}$, or (c) rows indexed by $M\mathcal{M}$ if M is coprime with N .

Thus, starting from a *base set* comprising consecutive rows, one can generate a whole suite of row indices that yield full spark matrices. The question we address here is what happens if the base set is changed to a set of consecutive rows *with one row missing*? We argue that this augmented by (a-c) above yields a further formidable suite of full spark matrices.

A counter-example in [17] shows that such a base set need not yield a full spark matrix. We first provide a necessary and sufficient condition when the base set in focus here provides full spark matrices. With the spark depending on n , the location of the missing row that prevents contiguity, and K is the total number of rows that are retained, this condition involves whether or not there are K distinct N -th roots of unity, whose n -products sum to zero, i.e. constitute a *vanishing sum*. As the example of [17] involves $N = 10$, a product of two prime numbers, 2 and 5, we also separately consider the case where N is the product of two primes and

87 leverage recent results by [18] and [19], to give conditions on these prime factors,
 88 K and n that guarantee full spark, *and as importantly those that do not*. These
 89 conditions in turn depend on whether certain integers can be expressed as a linear
 90 combination of the prime factors of N . Details are provided in the bullets below
 91 together with contrasts with a preliminary version of this paper [22], that appeared
 92 in a conference proceedings and has terse proofs, fewer results and two important
 93 mistakes that we correct here.

94 (A) For a general N , we show that full spark obtains whenever $\binom{K}{n}$ is not a
 95 nonnegative integer combination of the prime factors of N , i.e. should these
 96 prime factors be p_i , then for all nonnegative integers α_i

$$\binom{K}{n} \neq \sum_i \alpha_i p_i. \quad (1.1)$$

97 (B) Observe (A) provides a condition that is sufficient but not necessary. We
 98 show that should the missing index n be either 1 or $K - 1$, then in fact a
 99 *necessary and sufficient* condition can be found for full spark: that at least
 100 one among K or $N - K$ not be a nonnegative integer combination of the
 101 prime factors of N . This corrects and *strengthens the erroneous assertion*
 102 *in [22]* that in this case a necessary and sufficient condition for full spark is
 103 that K , not be a nonnegative integer combination of the prime factors of N .

104 (C) The rest of the paper refines these when $N = pq$ is the product of two
 105 distinct primes p and q . The conference paper incorrectly asserts that in such
 106 a case a necessary and sufficient condition for full spark is that K not be a
 107 nonnegative integer combination of the prime factors p and q , irrespective
 108 of the missing index n . In fact, at this point for $n \notin \{1, K - 1\}$ a necessary
 109 and sufficient condition is not known. In *a new set of results we explore*
 110 *the tightness of the sufficient condition in (1.1)* and provide a *much stronger*
 111 *condition even when $n \in \{1, K - 1\}$.*

112 (i) We show that in this case when $n \in \{1, K - 1\}$, (B) can be vastly
 113 strengthened. That a necessary and sufficient condition for full spark
 114 is that K be neither a multiple of p nor of q . Observe this is much
 115 easier to check than checking whether K and $N - K$ are nonnegative
 116 integer combinations of p and q .

117 (ii) We next show that (1.1) is not in fact necessary.

118 (iii) Just as providing conditions under which full spark is important, so is
 119 it important to provide conditions under which full spark *cannot hold*.
 120 Such conditions indicate which sampling patterns should be avoided.
 121 In this spirit we show that full spark does not hold if any among the
 122 integers $\{K, K - 1, \dots, K - n + 1\}$ is a multiple of either p or q .

123 The rest of the paper is structured as follows. Section 2 recounts relevant
 124 results, refines a result of [17], and sets the context of the remaining paper. Sec-
 125 tion 3 establishes the connection between spark and vanishing sums. Section 4
 126 recounts recent characterizations of vanishing sums of roots of unity. Section 5
 127 applies these to the spark problem. Section 6 provides conditions for which the
 128 full spark condition fails to hold. *We observe that only one theorem presented in*
 129 *Section 5 and Section 6 is in [22].* Section 7 provides simulations. Section 8 is
 130 the conclusion.

131 2. Context and Problem Statement

132 Our observation matrices are submatrices of the $N \times N$ DFT matrix W_N .
 133 Specifically, with row and column indices taking values from the set

$$\mathbb{Z}_N = \{0, 1, \dots, N - 1\}, \quad (2.1)$$

134 the il -th element of W_N is $e^{\frac{j2\pi il}{N}}$. These are of course N -th roots of unity. We will
 135 assume that the observation matrix $A(\mathcal{M}, N)$ only retains the rows of W_N that are
 136 indexed by $\mathcal{M} \subset \mathbb{Z}_N$. The central question of this paper is: *What conditions on*
 137 *$\mathcal{M}$ and N ensure that $A(\mathcal{M}, N)$ has full spark?* We now recount some important
 138 results addressing this question.

139 Suppose for some i and μ , $\mathcal{M} = \{i, i + \mu, \dots, i + \mu - 1\}$ i.e. contains μ
 140 consecutive elements of $\mathbb{Z}_N$. Define

$$z_n = e^{j\frac{2\pi n}{N}} \quad (2.2)$$

141 Then the matrix comprising any l columns of $A(\mathcal{M}, N)$, indexed by the integers
 142 $i_1, \dots, i_\mu$ comprises contiguous rows of W_N , and can be expressed as

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ z_{i_1} & z_{i_2} & \dots & z_{i_\mu} \\ \vdots & \vdots & \vdots & \vdots \\ z_{i_1}^{\mu-1} & z_{i_2}^{\mu-1} & \dots & z_{i_\mu}^{\mu-1} \end{bmatrix} \text{diag} \{z_{i_1}^i, z_{i_2}^i, \dots, z_{i_\mu}^i\}.$$

and being a product of a Vandermonde and a nonsingular diagonal matrix, is thus
nonsingular for distinct z_{i_n} . Thus, this $A(\mathcal{M}, N)$ has full spark. A less obvious
result can be traced back to Chebotarëv in the early 20th century, (see [20]).

Theorem 2.1. *Suppose N is prime. Then for all $\mathcal{M} \subset \mathbb{Z}_N$, $A(\mathcal{M}, N)$ has full
spark.*

The most sophisticated results are in [17] and provide a springboard for the
results of this paper. The one motivating us is a set of sufficient conditions.

Theorem 2.2. *Suppose for some $\mathcal{M} \subset \mathbb{Z}_N$, $A(\mathcal{M}, N)$ has full spark. Then so
does:*

(i) $A((\mathcal{M} + i) \bmod N, N)$ for all $i \in \mathbb{Z}_N$, i.e. the full spark condition is
preserved under all translations.

(ii) $A(M\mathcal{M}, N)$ for all M that is coprime with N .

(iii) $A(\mathbb{Z}_N \setminus \mathcal{M}, N)$.

This theorem permits one to build entire suites of $\mathcal{M}_i$ for which $A(\mathcal{M}_i, N)$ is
full spark, from any $\mathcal{M}$ for which $A(\mathcal{M}, N)$ is full spark. Consider the following
example.

Example 2.1. *Consider $N = 10$, and $\mathcal{M} = \{0, 1, 2\}$. Then using Theorem 2.2,
matrices obtained by retaining just the following sets of rows of the DFT matrix
have full spark: $\{0, 3, 6\}$, $\{2, 5, 8\}$ and $\{0, 1, 3, 4, 6, 7, 9\}$.*

In fact this result has been strengthened in [23].

Theorem 2.3. [23] *Suppose for positive integers M, K , $\mathcal{M} = \{i, i + M, i + 2M,$
 $\dots, i + (K - 1)M\}$, with $i + (K - 1)M < N$. Then $A(\mathcal{M}, N)$ has full spark
iff M and N are coprime. Further if $A(\mathcal{M}, N)$ does not have full spark, then
 $\text{spark}(A) = 2$.*

As illustrated by Example 2.1 starting from the base set $\mathcal{M} = \mathbb{Z}_{K+1}$, the full
spark nature of $A(\mathcal{M}, N)$ permits the construction of a plethora of subsets of $\mathbb{Z}_N$
that yield the full spark property. The hallmark of $\mathcal{M} = \mathbb{Z}_{K+1}$ is that consecutive
rows of W_N comprise the observation matrix. The rest of the paper asks: What if a
single frame from $\{0, 1, \dots, K\}$ is missing? What conditions guarantee full spark
observation matrices? These sufficient conditions, together with Theorem 2.2 then
generate a rich class of further row indices that guarantee full spark observations.

174 To fix notations, the base sets we will consider will be for some $K \in \mathbb{Z}_N \setminus$
 175 $\{N-1\}$ and $m \in \mathbb{Z}_K \setminus \{0\}$,

$$\mathcal{S}(K, m) = \mathbb{Z}_{K+1} \setminus \{m\} \quad (2.3)$$

176 i.e. all integers from 0 to K excepting m . Theorem 2.2 then permits even the base
 177 set $\mathcal{S}(K, 1)$ to be converted into sets that may have more than just one element
 178 missing. For example, consider $N = 8$, $K = 4$, and the base set $\{0, 2, 3, 4\}$.
 179 Using (i), (iii) of Theorem 2.2, one can deduce whether the submatrices of W_{10} ,
 180 comprising rows indexed by the sets $\{1, 5, 6, 7\}$ and $\{0, 4, 5, 6\}$ have full spark.
 181 Effectively, the second of these is a base set missing *three instead of a solitary*
 182 *element*.

183 That this problem is nontrivial is gauged by the following example from [17].
 184 Our results directly explain this example.

185 **Example 2.2.** Consider $N = 10$, $\mathcal{M} = \{0, 1, 3, 4\}$. Then $A(\mathcal{M}, 10)$ does not
 186 have full spark as its columns indexed by the set $\{0, 1, 2, 6\}$ are linearly dependent.

187 3. A Necessary and Sufficient Condition: Role of Vanishing Sums

188 This section establishes an equivalence between the full spark nature of
 189 $A(\mathcal{S}(K, n), N)$ and the existence of certain vanishing sums of N -th roots of unity.
 190 Consider a set of N -th roots of unity

$$\mathcal{W} = \{w_1, w_2, \dots, w_K\}. \quad (3.1)$$

191 We say these form a vanishing sum if

$$\sum_{i=1}^K w_i = 0, \quad (3.2)$$

192 In the sequel we will refer to the sum of m -products of a set of complex numbers.
 193 This is the sum of the products of the elements belonging to all subsets of the
 194 set with cardinality m . For example, the sum of 2-products of $\{a_1, a_2, a_3\}$ is
 195 $a_1a_2 + a_1a_3 + a_2a_3$. We will denote $\Sigma(\mathcal{W}, m)$ to be the sum of the m -products of
 196 the elements of $\mathcal{W}$, with

$$\Sigma(\mathcal{W}, 0) = 1. \quad (3.3)$$

197 Of course the sum of 1-products of the elements of $\{a_1, a_2, a_3\}$ is $a_1 + a_2 + a_3$.

198 Thus

$$a_1 a_2 + a_1 a_3 + a_2 a_3 = \Sigma(\{a_1, a_2, a_3\}, 2).$$

199 Before presenting and proving the main theorem we illustrate the basic idea with
 200 Example 2.2. Consider four columns of $A(\mathcal{M}, 10)$, where as in Example 2.2,
 201 $\mathcal{M} = \mathcal{S}(4, 2)$. The matrix does not have full spark if and only if it has four
 202 linearly dependent columns. Consider the matrix formed by such a set of four
 203 columns, i.e. for suitable distinct z_i, z_k, z_l, z_n as in (2.2)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ z_i & z_k & z_l & z_n \\ z_i^3 & z_k^3 & z_l^3 & z_n^3 \\ z_i^4 & z_k^4 & z_l^4 & z_n^4 \end{bmatrix}.$$

204 As this matrix is singular there exists a nonzero $\theta = [\theta_0, \theta_1, \theta_3, \theta_4]^T$ such that

$$\theta^T \begin{bmatrix} 1 & 1 & 1 & 1 \\ z_i & z_k & z_l & z_n \\ z_i^3 & z_k^3 & z_l^3 & z_n^3 \\ z_i^4 & z_k^4 & z_l^4 & z_n^4 \end{bmatrix} = [\theta(z_i) \quad \theta(z_k) \quad \theta(z_l) \quad \theta(z_n)] = 0,$$

205 where

$$\theta(z) = \theta_0 + \theta_1 z + \theta_3 z^3 + \theta_4 z^4. \quad (3.4)$$

206 Thus lack of full spark is equivalent to the existence of a nonzero polynomial as
 207 in (3.4) that has the four N -th roots of unity z_i, z_k, z_l, z_n . Observe, $\theta(z)$ in (3.4)
 208 has the degree two coefficient zero. This coefficient is the sum of the 2-products
 209 of $\{z_i, z_k, z_l, z_n\}$. As the polynomial has degree four these are the only roots of
 210 this polynomial, there thus holds:

$$z_i z_k + z_i z_l + z_i z_n + z_k z_l + z_k z_n + z_l z_n = 0. \quad (3.5)$$

211 As each column corresponds to a distinct N -th root of unity, and all columns are
 212 present and between then account for all N -th roots of unity, we have effectively
 213 shown for this example that lack of full spark is equivalent to the existence of
 214 four N -th roots of unity whose 2-products form a vanishing sum. Observe that
 215 this argument critically relies on the fact that only one row of the first five are
 216 missing. Also observe that missing coefficient in $\theta(z)$ corresponds to the index of
 217 the missing row among the first five. Further each summand in (3.5) is itself an

218 N -th root of unity. Since four distinct columns of the DFT matrix are involved z_i ,
 219 z_k, z_l, z_n are distinct. Their 2-products though need not be.

220 Generalizing this result leads to the following promised equivalence.

221 **Theorem 3.1.** *For integers $1 \leq n < K < N$, $A(\mathcal{S}(K, n), N)$, see (2.3), does*
 222 *not have full spark iff there exist K distinct N -th roots of unity whose n -products*
 223 *form a vanishing sum.*

224 *Proof. Proof of Sufficiency:* Suppose there are K distinct N -th roots of unity,
 225 $z_{n_1}, \dots, z_{n_K}$ whose n -products form a vanishing sum. Consider the polynomial

$$\prod_{i=1}^K (z - z_{n_i}) = \sum_{l=0}^K g_l z^l = g(z) \quad (3.6)$$

226 Observe g_n is the sum of n -products of the roots of $g(z)$ and hence by definition
 227 of vanishing sums, is zero. Define the vector

$$G = [g_0 \quad \dots \quad g_{n-1} \quad g_{n+1} \quad \dots \quad g_K]^T \in \mathbb{C}^K.$$

228 With z_n defined in (2.2) define

$$\mathcal{Z}_k = [1 \quad z_{n_k} \quad \dots \quad z_{n_k}^{n-1} \quad z_{n_k}^{n+1} \quad \dots \quad z_{n_k}^K]^T. \quad (3.7)$$

229 As the row indexed by n is missing from the other contiguous rows that are re-
 230 tained in $A(\mathcal{S}(K, n), N)$, $\mathcal{Z}_1, \dots, \mathcal{Z}_K$ comprise K -columns of $A(\mathcal{S}(K, n), N)$.
 231 From (3.6)

$$G^T [\mathcal{Z}_1, \dots, \mathcal{Z}_K] = 0$$

232 i.e. this submatrix of $A(\mathcal{S}(K, n), N)$ has rank less than K , proving that $A(\mathcal{S}(K, n), N)$
 233 has spark less than $K + 1$.

234 *Proof of Necessity:* Since it has K rows, $A(\mathcal{S}(K, n), N)$ does not have full spark
 235 only if it has K distinct columns that are linearly dependent. Suppose these
 236 columns are indexed from the set $\{n_1, \dots, n_K\} \subset \mathbb{Z}_N$. Then there is a nonzero
 237 $G \in \mathbb{C}^K$ such that under (3.7)

$$G^T \mathcal{Z}_k = g(z_{n_k}) = 0$$

238 As $g(z)$ has degree K , $z_{n_1}, \dots, z_{n_K}$ are the only roots of this polynomial. Further
 239 by definition of $A(\mathcal{S}(K, n), N)$ the coefficient of z^n in $g(z)$ is zero. As this coeffi-

cient is the sum of the n products of the distinct roots $z_{n_1}, \dots, z_{n_K}$, the n -products of these roots form a vanishing sum.

□

In subsequent sections we refine this result to pinpoint conditions on just K , N and n under which the stated vanishing sums exist. But first we have an immediate consequence of Theorem 3.1.

Theorem 3.2. *For integers $1 \leq n < K < N$ and $\mathcal{S}(K, n)$ defined in (2.3), $A(\mathcal{S}(K, n), N)$ has full spark iff $A(\mathcal{S}(K, K - n), N)$ has full spark.*

Proof. From Theorem 3.1, $A(\mathcal{S}(K, n), N)$ does not have full spark iff there are K distinct N -th roots of unity whose n products form a vanishing sum. Suppose these are $w_1, \dots, w_K$. In other words $\Sigma(\{w_1, \dots, w_K\}, n) = 0$. Consequently as the w_i are all nonzero

$$\frac{\Sigma(\{w_1, \dots, w_K\}, n)}{\prod_{i=1}^K w_i} = 0.$$

Consider the product of any n of these roots, without loss of generality

$$\prod_{i=1}^n w_i.$$

As w_i are on the unit circle $1/w_i = w_i^*$. Thus

$$\begin{aligned} \frac{\prod_{i=1}^n w_i}{\prod_{i=1}^K w_i} &= \frac{1}{\prod_{i=n+1}^K w_i} \\ &= \left(\prod_{i=n+1}^K w_i \right)^*, \end{aligned}$$

which is the conjugate of the product of the $K - n$ roots that are not in the set $\{w_1, \dots, w_n\}$. Thus the sum of n -products of the roots in $w_1, \dots, w_K$ is zero iff the conjugate of the sum of $(K - n)$ -products of these very roots is zero. Thus the n -products form a vanishing sum iff the $(K - n)$ -products do. The result follows. □

Thus ascertaining whether $A(\mathcal{S}(K, 1), N)$ has full spark, also ascertains whether $A(\mathcal{S}(K, K - 1), N)$ has full spark, amplifying the virtues of just considering the base set $\mathcal{S}(K, 1)$.

262 4. Characterization of Vanishing Sums

263 The previous section squarely tied the full spark property of the base set
 264 $\mathcal{S}(K, n)$ to whether there are K distinct N -th roots of unity whose n -products
 265 form a vanishing sum. We now recount the seminal result of Lam and Leung, [18]
 266 and its subsequent refinement, [19]. Lam and Leung ask and answer the question:
 267 Given a number r , when are there r , N -th roots of unity, not necessarily distinct,
 268 that form a vanishing sum? In a subsequent paper, [19] answers the questions
 269 when these root are mandated to be distinct. Both results tie the answer to the
 270 *prime factors* of N . In the sequel we define $\mathbb{N}$ to be the set of all nonnegative
 271 integers.

272 Before providing these characterizations let us examine an example. Suppose
 273 $N = 6$. Its prime factors are 2 and 3. Consider the set

$$S_2(x) = x + 2\mathbb{Z}_3 = \{x, x + 2, x + 4\}. \quad (4.8)$$

274 Clearly for any

$$\begin{aligned} \sum_{s_i \in S_2(x)} e^{j2\pi \frac{s_i}{6}} &= e^{j2\pi \frac{x}{6}} \left(1 + e^{j2\pi \frac{2}{6}} + e^{j2\pi \frac{4}{6}}\right) \\ &= e^{j2\pi \frac{x}{6}} \left(1 + e^{j\frac{2\pi}{3}} + e^{j\frac{4\pi}{3}}\right) \\ &= 0. \end{aligned}$$

275 Similarly, with

$$S_3(y) = y + 3\mathbb{Z}_2 = \{y, y + 3\}, \quad (4.9)$$

$$\sum_{s_i \in S_3(y)} e^{j2\pi \frac{s_i}{6}} = 0.$$

277 Equally

$$\sum_{s_i \in S_3(y) \cup S_2(x)} e^{j2\pi \frac{s_i}{6}} = 0.$$

278 In fact Lam and Leung show that a set of 6-th roots of unity $\left\{e^{j2\pi \frac{s_i}{6}}\right\}_{i=1}^r$ sum to
 279 zero iff they can be separated into *cosets* such as (4.8, 4.9). This is equivalent to
 280 saying that r their number is for nonnegative integers α and β , $2\alpha + 3\beta$. In other
 281 words $r \in 2\mathbb{N} + 3\mathbb{N}$; 2 and 3 are of course the prime factors of 6.

282 We now formally state the Lam and Leung theorem.

283 **Theorem 4.1. [18]** Suppose $N > 1$ is a positive integer, with prime factors
 284 $p_1, \dots, p_l$. Then there exist r possibly nondistinct N -th roots of unity $\{w_i\}_{i=1}^r$ that
 285 form a vanishing sum iff

$$r \in \sum_{i=1}^l p_i \mathbb{N}.$$

286 In other words r is a *nonnegative integer combination of the prime factors of*
 287 N . Sivek, [19], extends Theorem 4.1 to distinct N -th roots of unity. The intuition
 288 behind this result is as follows. As there are precisely N distinct N -th roots of
 289 unity, and these themselves add to zero, r distinct N -th roots of unity form a
 290 vanishing sum iff the remaining $N - r$ do so as well.

291 **Theorem 4.2. [19]** Suppose $N > 1$ is a positive integer, with prime factors
 292 $p_1, \dots, p_l$. Then there exist r distinct N -th roots of unity $\{w_i\}_{i=1}^r$ that form a van-
 293 ishing sum iff

$$r \in \sum_{i=1}^l p_i \mathbb{N} \text{ and } N - r \in \sum_{i=1}^l p_i \mathbb{N}$$

294 Though Lam and Leung provide a characterization of the N -th roots of unity
 295 that form a vanishing sum, this characterization is particularly clean when N is
 296 a product of two primes p and q , i.e. $N = pq$. To provide this characterization
 297 we introduce two types of cosets that generalize (4.8,4.9). For some nonnegative
 298 integers x and y

$$S_p(x) = x + p\mathbb{Z}_q, \quad x \in \mathbb{Z}_q \quad (4.10)$$

299 and

$$S_q(y) = y + q\mathbb{Z}_p, \quad y \in \mathbb{Z}_p. \quad (4.11)$$

300 In the sequel we will call (4.10) and (4.11) p and q cosets, respectively. The
 301 elements of p -cosets, for example have differences that are multiples of p . Then
 302 [18] shows that $r = \alpha p + \beta q$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$, N -th roots of unity

$$\left\{ e^{\frac{2\pi j s_i}{N}} \right\}_{i=1}^r, \quad s_i \in \mathbb{Z}_N, \quad (4.12)$$

303 form a vanishing sum iff the s_i can be distributed across complete α complete
 304 q -cosets and β complete p -cosets. Formally, we have the following theorem.

305 **Theorem 4.3. [18]** Consider the possibly, nondistinct r , N -th roots of unity in
 306 (4.12). Call the index set $\mathcal{I} = \{1, \dots, r\}$. Then the roots in (4.12) form a vanish-
 307 ing sum iff $\mathcal{I}$ admits a partitioning $\mathcal{I}_l$ that has the following property. For a given

308 *l* one of the following holds: either $\mathcal{I}_l$ has cardinality p and for some nonnegative
 309 integer x_l ,

$$\{s_i | i \in \mathcal{I}_l\} = S_p(x_l) \quad (4.13)$$

310 or $\mathcal{I}_l$ has cardinality q and for some nonnegative integer y_l ,

$$\{s_i | i \in \mathcal{I}_l\} = S_q(y_l). \quad (4.14)$$

311 For example, with $N = 6$ and $r = 5$, of the five indices, two of the s_i 's must
 312 form a 3-coset and the remaining three must form a 2-coset. Further for every x_l
 313 and y_l

$$\sum_{s \in S_p(x_l)} e^{j \frac{2\pi s}{N}} = \sum_{s \in S_q(y_l)} e^{j \frac{2\pi s}{N}} = 0. \quad (4.15)$$

314 5. Translation to full spark

315 We now apply the results on vanishing sum given in Section 4 to the full
 316 spark characterization of Section 3, beginning with two results for the base set is
 317 $\mathcal{S}(K, 1)$. The first is for general N . The second when N is the product of only
 318 two distinct primes. We will explain later why we are treating the $\mathcal{S}(K, 1)$ case
 319 differently from $\mathcal{S}(K, n)$ for a general n .

320 **Theorem 5.1.** *Suppose the prime factors of N are $p_1, \dots, p_l$. Consider for integer*
 321 *$1 < K < N$, and $\mathcal{S}(K, 1)$ and $\mathcal{S}(K, K - 1)$ defined in (2.3). Then $A(\mathcal{S}(K, 1), N)$*
 322 *and $A(\mathcal{S}(K, K - 1), N)$ do not have full spark iff*

$$K \in \sum_{i=1}^l p_i \mathbb{N} \text{ and } N - K \in \sum_{i=1}^l p_i \mathbb{N}.$$

323 *Proof.* From Theorem 3.1, $A(\mathcal{S}(K, 1), N)$ does not have full spark iff there are
 324 K distinct N -th roots of unity that form a vanishing sum. Further from Theorem
 325 3.2, $A(\mathcal{S}(K, 1), N)$ has full spark iff $A(\mathcal{S}(K, K - 1), N)$ has full spark. Then the
 326 result follows from Theorem 4.2. $\square$

327 Thus if $N = 5 \times 11 \times 13^2 = 9295$, $A(\mathcal{S}(8, 1), 9295)$ and $A(\mathcal{S}(14, 1), 9295)$
 328 have full spark. Similarly for any $n > 1$ $A(\mathcal{S}(K, 1), 2^n)$ has full spark iff K is
 329 odd.

330 It turns out when N is the product of just two prime numbers a much stronger
 331 result is possible.

332 **Theorem 5.2.** Suppose p and q are distinct prime numbers $N = pq$, $0 < K <$
333 N and $\mathcal{S}(K, 1)$ and $\mathcal{S}(K, K - 1)$ defined in (2.3). Then $A(\mathcal{S}(K, 1), N)$ and
334 $A(\mathcal{S}(K, K - 1), N)$ do not have full spark iff either

$$K \in p\mathbb{N}.$$

335 or

$$K \in q\mathbb{N}.$$

336 *Proof.* Because of Theorem 3.2 it suffices to work with $A(\mathcal{S}(K, 1), N)$. In view
337 of Theorem 5.1, there exist nonnegative integers a, b, c, d such that

$$K = ap + bq \tag{5.16}$$

338 and

$$N - K = cp + dq. \tag{5.17}$$

339 As $K \in \mathbb{Z}_N$, $N - K \in \mathbb{Z}_N$, and $N = pq$, a, c must be in $\mathbb{Z}_p$ and b, d in $\mathbb{Z}_q$. To
340 prove the “if part” suppose without loss of generality

$$K = ap, \quad a \in \mathbb{Z}_q.$$

341 Then

$$\begin{aligned} N - K &= pq - ap \\ &= (q - a)p, \end{aligned}$$

342 As $q - a \in \mathbb{Z}_q$, $N - K \in p\mathbb{N}$ and the result holds.

343 To prove the “only if” part, observe that one obtains from (5.16) and (5.17)
344 that

$$pq = N = (a + c)p + (b + d)q.$$

345 Further $a + c \in \mathbb{Z}_q$ and $b + d \in \mathbb{Z}_p$ as other wise the right hand side exceeds the
346 left hand side. Further

$$(q - (a + c))p = (b + d)q.$$

347 As p and q are prime and $(a + c) \in \mathbb{Z}_q$ and $b + d \in \mathbb{Z}_p$, either $a + c = q$ and
348 $b + d = 0$ or $b + d = p$ and $a + c = 0$. As a, b, c, d are all nonnegative, either
349 $b = d = 0$ or $a = c = 0$. □

Thus, for $N = 77$, $A(\mathcal{S}(K, 1), N)$ and $A(\mathcal{S}(K, K - 1), N)$ do not have full spark iff K is either a multiple of 11 or 7. However they have full spark for $K = 11 + 7 = 18$ as it is not a multiple of either 7 or 11.

We now turn to more general n .

Theorem 5.3. *Suppose the prime factors of N are $p_1, \dots, p_l$. Consider for integer $1 < n < K - 1 < N - 1$, $\mathcal{S}(K, n)$ defined in (2.3). Then $A(\mathcal{S}(K, n), N)$ has full spark if*

$$\binom{K}{n} \notin \sum_{i=1}^k p_i \mathbb{N}. \quad (5.18)$$

Proof. From Theorem 3.1, $A(\mathcal{S}(K, n), N)$ does not have full spark iff there are K distinct N -th roots of unity whose n -products form a vanishing sum. Each n -product of N -th roots of unity is itself an N -th root of unity. Further there are $\binom{K}{n}$ such n -products. Then the result follows from Theorem 4.1. $\square$

Unlike Theorem 5.1, this theorem does not provide a necessary and sufficient condition. All that the violation of (5.18) guarantees is that there are $\binom{K}{n}$, N -th roots of unity that form a vanishing sum. This does not necessarily imply that these can be expressed as the n -products of K distinct N -th roots of unity. Indeed in Section 5.1, we show that $A(\mathcal{S}(5, 2), 21)$ has full spark despite the fact that $\binom{5}{2} = 10 = 7 + 3$ where 7 and 3 are prime factors of 21.

It is also instructive that even though certain numbers may be distinct, their n -products need not be. Thus the additional condition imposed by Theorem 4.2 does not apply to $\binom{K}{n}$.

On the other hand, return to Example 2.2 in which $N = 10$, $K = 4$, and $n = 2$. Clearly $\binom{4}{2} = 6$ is even and hence a multiple of one of the prime factors of 10. Thus this example violates the sufficient condition of Theorem 5.3. However, as Theorem 5.3 does not provide a sufficient condition it *fails to provide a definitive explanation* of the lack of full spark evinced by Example 2.2. That explanation is postponed to Section 6.

5.1. The case of $\mathcal{S}(5, 2)$ when $N = 21$

Consider the case flagged earlier where $N = pq$, $p = 3$, $q = 7$ and $K = 5$. We will argue that no five distinct N -th roots of unity have 2-products that form a vanishing sum. Thus even though $\binom{K}{2} = 10 = 3 + 7$, $A(\mathcal{S}(5, 2), 21)$ has full spark.

To establish a contradiction, suppose there are 5 distinct s_i such that the 2-products of

$$e^{\frac{2\pi j s_i}{21}}$$

382 vanish. If these 2-products form a vanishing sum then so do the 2-products formed
383 by

$$e^{\frac{2\pi j(s_i-t)}{21}}$$

384 for any t . Thus without loss of generality assume one of the s_i is zero and the set
385 of s_i can be chosen to be

$$S = \{0, s_1, s_2, s_3, s_4\}. \quad (5.19)$$

386 As $\binom{5}{2} = 10 = p + q$, arguing as in the analysis of Example 2.2, from Theorem
387 4.3, the 2-sums of the s_i must break into two cosets: For some $z_i \in \mathbb{Z}_{21}$

$$S_7(z_1) = z_1 + \{0, 7, 14\} \quad (5.20)$$

388 and

$$S_3(z_2) = z_2 + \{0, 3, 6, 9, 12, 15, 18\}. \quad (5.21)$$

389 In the sequel, we will drop the arguments z_i and refer to these cosets as just S_3 and
390 S_7 . Now the 2-sums of the elements of S include $T = \{s_1, s_2, s_3, s_4\}$. Consider
391 the following cases.

392 **Case I: All elements of T are in the same coset:** As $|S_7| = 3$, $T \subset S_3$. Then,
393 without loss of generality for distinct $\alpha_i \in \mathbb{Z}_7$

$$s_i = s_1 + 3\alpha_i, i \in \{2, 3, 4\}. \quad (5.22)$$

394 Then every 2-sum takes the form for some nonnegative integers β_l

$$t_l = 2s_1 + 3\beta_l. \quad (5.23)$$

395 As three of these t_l form S_7 , without loss of generality $l \in \{1, 2, 3\}$, there holds:

$$\begin{aligned} 7 &= t_2 - t_1 \\ &= 3(\beta_2 - \beta_1). \end{aligned}$$

396 This cannot hold as 7 is a prime.

397 **Case II:** $\{s_1, s_2, s_3\} \subset S_r$, $r \in \{3, 7\}$ but $s_4 \notin S_r$: We argue that $r = 3$. To
398 establish a contradiction suppose $\{s_1, s_2, s_3\} \subset S_7$. Arguing as above, without

399 loss of generality for distinct $\alpha_i \in \mathbb{Z}_3$

$$s_i = s_1 + 7\alpha_i, i \in \{2, 3\}. \quad (5.24)$$

400 Further as $|S_7| = 3$, $s_4 + s_l \in S_3$ for all $l \in \{1, 2, 3\}$. Then $|s_2 - s_3|$ is divisible by
 401 3. From (5.24), $|s_2 - s_3|$ is also divisible by 7, i.e. is divisible by 21. This cannot
 402 hold as $|s_2 - s_3| \pmod{21} \neq 0$.

403 Thus $r = 3$. Now $s_4 \in S_7$. We assert first that for $\{i, k\} \subset \{1, 2, 3\}$, $S_7 \neq$
 404 $\{s_4, s_4 + s_i, s_4 + s_k\}$. This is so as in this case $|s_i - s_k|$ is divisible by 7, while as
 405 s_i and s_k are both in S_3 , $|s_i - s_k|$ is also divisible by 3, and $|s_i - s_k| \pmod{21} = 0$.
 406 Likewise, for $\{i, k, l\} = \{1, 2, 3\}$, $S_7 \neq \{s_4, s_k + s_i, s_l + s_k\}$ as then $|s_i - s_l|$
 407 $\pmod{21} = 0$. Thus, without loss of generality, there remain two possibilities.
 408 Either

$$S_7 = \{s_4, s_4 + s_1, s_1 + s_2\} \quad (5.25)$$

409 or

$$S_7 = \{s_4, s_4 + s_1, s_3 + s_2\}. \quad (5.26)$$

410 We will now eliminate both. By definition of S_7 , (5.25) implies that s_1 is a nonzero
 411 multiple of 7. In particular,

$$s_1 = 7\alpha, \alpha \in \{1, 2\}. \quad (5.27)$$

412 Further, for some $z \in \mathbb{Z}_7$,

$$\{s_1, s_2, s_3, s_1 + s_3, s_2 + s_3, s_2 + s_4, s_3 + s_4\} = z + 3\mathbb{Z}_7. \quad (5.28)$$

413 Thus for distinct $\beta_i \in \mathbb{Z}_7$

$$s_2 = 7\alpha + 3\beta_2 \text{ and } s_3 = 7\alpha + 3\beta_3 \Rightarrow s_2 + s_3 = 14\alpha + 3(\beta_2 + \beta_3).$$

414 Thus $s_2 + s_3 - s_1 = 7\alpha + 3(\beta_2 + \beta_3)$ is a multiple of 3. This cannot happen unless
 415 α is a multiple of 3, violating the fact that $\alpha \in \{1, 2\}$.

416 Turn now to (5.26). Again (5.27) holds. In this case for some $z \in \mathbb{Z}_7$

$$\{s_1, s_2, s_3, s_1 + s_3, s_1 + s_2, s_2 + s_4, s_3 + s_4\} = z + 3\mathbb{Z}_7, \quad (5.29)$$

417 and for some $\beta \in \mathbb{Z}_7 \setminus \{0\}$, $s_2 = 3\beta + 7\alpha$. Thus $s_1 + s_2 = 14\alpha + 3\beta$. As both

418 $s_1 + s_2$ and s_1 are in the set on the left hand side of (5.29)

$$\begin{aligned} s_1 + s_2 - s_1 &= 14\alpha + 3\beta - 7\alpha \\ &= 7\alpha + 3\beta \end{aligned}$$

419 is a multiple of 3. Again this cannot happen as $\alpha \in \{1, 2\}$.

420 **Case III:** $\{s_1, s_2\} \subset S_7$ and $\{s_3, s_4\} \subset S_3$: Then for $\alpha \in \{1, 2\}$,

$$s_2 = s_1 + 7\alpha \quad (5.30)$$

421 As $|S_7| = 3$ at least three among $s_1 + s_3$, $s_1 + s_4$, $s_2 + s_3$ and $s_2 + s_4$ are in S_3 .
 422 without loss of generality assume $s_1 + s_3$ and $s_2 + s_3$ are in S_3 . Then $|s_2 - s_1|$ is a
 423 multiple of 3. From (5.30) $|s_2 - s_1|$ is also a multiple of 7. Again a contradiction
 424 obtains.

425 6. Some conditions for lack of full spark

426 Theorem 5.3 provides only a sufficient condition for $A(\mathcal{S}(K, n), N)$ to have
 427 full spark, when $1 < n < K - 1$. Section 5.1 proves by analyzing an example
 428 that this condition is not necessary. For the special case of $N = pq$, in this section
 429 we provide sufficient conditions under which $A(\mathcal{S}(K, n), N)$ *does not* have full
 430 spark. In view of Theorem 3.1 these sufficient conditions for lack of full spark
 431 will involve proving that m -products of certain distinct N -th roots of unity form
 432 vanishing sums.

433 Consider a set of numbers as in (3.1), $\Sigma(\mathcal{W}, m)$ the sum of its m -products.
 434 Observe that the following recursion obtains:

$$m\Sigma(\mathcal{W}, m) = \sum_{i=1}^K w_i \Sigma(\mathcal{W} \setminus \{w_i\}, m - 1) \quad (6.1)$$

435 This is so as each m -product appears m times in the right hand side of (6.1).
 436 Additionally, under (3.3), for any $n \in \{1, \dots, K\}$

$$\Sigma(\mathcal{W}, m) = w_n \Sigma(\mathcal{W} \setminus \{w_n\}, m - 1) + \Sigma(\mathcal{W} \setminus \{w_n\}, m) \quad (6.2)$$

437 Now suppose for some l

$$\Sigma(\mathcal{W}, l) = 0. \quad (6.3)$$

438 Then from (6.2), for any $n \in \{1, \dots, K\}$, there holds

$$\begin{aligned}\Sigma(\mathcal{W} \setminus \{w_n\}, l) &= \Sigma(\mathcal{W}, l) - w_n \Sigma(\mathcal{W} \setminus \{w_n\}, l-1) \\ &= -w_n \Sigma(\mathcal{W} \setminus \{w_n\}, l-1).\end{aligned}\tag{6.4}$$

439 Thus, in view of (3.3),

$$\Sigma(\mathcal{W}, i) = 0, \forall i \in \{1, \dots, l\}\tag{6.5}$$

440 implies

$$\Sigma(\mathcal{W} \setminus \{w_n\}, l) = (-w_n)^l\tag{6.6}$$

441 Then we have the following Theorem.

442 **Theorem 6.1.** *Suppose $N = pq$ with p and q prime numbers and K , $0 < K < N$*
 443 *is a multiple of p . Then there are K distinct N -th roots of unity w_i , such that for*
 444 *each $m \in \{1, \dots, p-1\}$, the m -products of w_i form a vanishing sum.*

445 *Proof.* Suppose for some $\alpha > 0$, $K = \alpha p$. As $K < N$, $\alpha \in \mathbb{Z}_q \setminus \{0\}$. Choose for
 446 distinct $x_i \in \{1, \dots, \alpha\} \subset \mathbb{Z}_q$,

$$S_p(x_i) = x_i + q\mathbb{Z}_p.\tag{6.7}$$

447 Choose

$$\mathcal{W} = \left\{ w_i = e^{j2\pi \frac{s_i}{N}} \left| s_i \in \bigcup_{i=1}^{\alpha} S_p(x_i) \right. \right\}\tag{6.8}$$

448 As each $S_p(x_i)$ has cardinality p ,

$$\left| \bigcup_{i=1}^{\alpha} S_p(x_i) \right| = \alpha p.$$

449 Further as the $x_i \in \mathbb{Z}_q$ are distinct so are the elements of this set. Finally, as all
 450 elements of this set are in $\mathbb{Z}_N$, the elements of $\mathcal{W}$ in (6.8) are distinct.

451 We now proceed to prove by induction that such a $\mathcal{W}$ obeys (6.3) for all $l \in$

452 $\{1, \dots, p-1\}$. First observe that for all $l \in \{1, \dots, p-1\}$

$$\begin{aligned}
\sum_{i=1}^{\alpha p} w_i^l &= \sum_{k=1}^{\alpha} \sum_{n=0}^{p-1} e^{j2\pi l \frac{x_k + nq}{pq}} \\
&= \sum_{k=1}^{\alpha} e^{j2\pi l \frac{x_k}{pq}} \sum_{n=0}^{p-1} e^{j2\pi \frac{nl}{p}} \\
&= \sum_{k=1}^{\alpha} e^{j2\pi l \frac{x_k}{pq}} \frac{1 - e^{j2\pi l}}{1 - e^{j2\pi \frac{l}{p}}} \\
&= 0,
\end{aligned} \tag{6.9}$$

453 where the penultimate equation critically relies on the facts that p is prime and
454 $l \in \{1, \dots, p-1\}$.

455 Substituting $l = 1$ in (6.9) proves the result for $m = 1$. Now suppose the
456 result holds for all $m \in \{1, \dots, p-2\}$. Then from (6.6) holds for $l = m$ and all
457 $w_n \in \mathcal{W}$. Thus from (6.1) there holds:

$$\begin{aligned}
(m+1)\Sigma(\mathcal{W}, m+1) &= \sum_{i=1}^{\alpha p} w_i \Sigma(\mathcal{W} \setminus \{w_i\}, m) \\
&= \sum_{i=1}^{\alpha p} (-w_i)^{m+1} \\
&= (-1)^{m+1} \sum_{i=1}^{\alpha p} w_i^{m+1} \\
&= 0,
\end{aligned}$$

458 where we have used (6.9) and the fact that $m+1 \in \{1, \dots, p-1\}$. $\square$

459 In view of Theorem 3.1 this thus shows that if K is a multiple of p then for all
460 $1 < m < p$, $A(\mathcal{S}(K, m), N)$ lacks full spark. A corresponding result also holds
461 for multiple of q . The proof does not work for $m = p$ for example. This is so as
462 in such a case (6.9) no longer holds. The next theorem provides a much broader
463 class of K for which $A(\mathcal{S}(K, m), N)$ lacks full spark.

464 **Theorem 6.2.** *Suppose $N = pq$ with p and q prime numbers and $N > K > 0$.
465 Suppose $m \in \{1, \dots, p-1\}$, and at least one member of $\{K-m+1, \dots, K\}$,
466 is αp , $\alpha > 0$. Then there are K distinct N -th roots of unity w_i , such that the*

467 *m*-products of w_i form a vanishing sum.

468 *Proof.* The result is clearly true for $m = 1$. Thus consider $m > 1$. Suppose for
 469 some $l \in \{1, \dots, m-1\}$, $K-l$ is αp , $\alpha > 0$. From Theorem 6.1, there are
 470 $K-l$ distinct N -roots of unity all whose n -products form a vanishing sum for
 471 all $n \in \{1, \dots, p-1\}$. In other words the set $\mathcal{W}_{K-l} = \{w_1, \dots, w_{K-l}\}$ has the
 472 property that

$$\Sigma(\mathcal{W}_{K-l}, n) = 0, \forall n \in \{1, \dots, p-1\}. \quad (6.10)$$

473 As $K < N$, there are clearly l , N -th roots of unity that do not belong to $\mathcal{W}_{K-l}$.
 474 Call these w_i , $i \in \{K-l+1, \dots, K\}$. Define $\mathcal{W}_{K-l+i} = \{w_1, \dots, w_{K-l+i}\}$.
 475 We will use induction to show that for all $i \in \{0, \dots, l\}$

$$\Sigma(\mathcal{W}_{K-l+i}, n) = 0, \forall n \in \{i+1, \dots, p-1\}. \quad (6.11)$$

476 As $l < m$, this will prove the result. From (6.10), (6.11) holds for $i = 0$. Now
 477 suppose (6.11) holds for some $i < l$. Then from (6.2), and all $n \in \{i+2, \dots, p-$
 478 $1\}$ and (6.11)

$$\begin{aligned} \Sigma(\mathcal{W}_{K-l+i+1}, n) &= w_{K-l+i+1} \Sigma(\mathcal{W}_{K-l+i}, n-1) + \Sigma(\mathcal{W}_{K-l+i}, n) \\ &= 0. \end{aligned}$$

479

□

480 Theorem 3.1, Theorem 6.1 and Theorem 6.2 effectively prove the following.

481 **Theorem 6.3.** Suppose $N = pq$ with p and q prime numbers and $N > K > 0$.
 482 Suppose $n \in \{1, \dots, p-1\}$ (respectively $n \in \{1, \dots, q-1\}$), and at least
 483 one member of $\{K-n+1, \dots, K\}$, is αp (respectively αq), $\alpha > 0$. Then
 484 $A(\mathcal{S}(K, n), N)$ does not have full spark.

485 This theorem precludes a vast swathe of cases from having full spark and plays
 486 the important role of excluding many sampling patterns. For example with $p = 5$
 487 and $q = 11$, i.e. $N = 55$. Then $A(\mathcal{S}(K, 8), N)$ does not have full spark for
 488 $K \in \{22, 23, 24, 25, 26, 27\}$. Similarly, in Example 2.2 $N = 2 \times 5$, $K = 4$ and
 489 $n = 2$. While the example fails the sufficient condition of Theorem 5.3 as $\binom{K}{n} = 6$
 490 is a nonnegative multiple of the prime factor 2, the explanation of lack of spark is
 491 not conclusive from Theorem 5.3. However, Theorem 6.3 definitively provides the
 492 explanation as K is a multiple of 2.

7. Simulation Results

We now provide several simulations. First we illustrate the negative results of Section 6 with nontrivial matrices whose spark would be computationally prohibitive to check. First consider $N = 35 = 5 \times 7$, $n = 2$ and $K = 8$. As $K - 1 = 7$ by Theorem 6.3 $A(\mathcal{S}(K, n), N)$ does not have full spark. Indeed one can verify that rank of the 7×7 submatrix obtained by retaining columns indexed by multiples of 5 has rank 6. Checking the spark of a 7×35 matrix requires checking $\binom{35}{7} = 6724520$ submatrices.

More dramatically, consider the 38×2183 , $A(\mathcal{S}(K, n), N)$ obtained by $N = 2183 = 37 \times 59$, $K = 38$ and $n = 2$. Theorem 6.3 correctly predicts that this cannot have full spark as $K - 1$ is a multiple of a prime factor of 37. Indeed the 38×38 matrix obtained by retaining columns that are multiples of 59 has rank 36. Observe that to check the spark of this $A(\mathcal{S}(K, n), N)$ would require checking the rank of $\binom{2183}{38} > 10^{82}$ submatrices of $A(\mathcal{S}(K, n), N)$.

Next we turn to implications of satisfying the condition in Theorem 5.3. To this end we examine the performance of ℓ_1 -recovery of a k -sparse vectors x of length N with Gaussian random entries from observations $y = Ax$, where A is the submatrix of the N -point DFT matrix obtained by selecting the first K consecutive rows, with the row indexed by n missing. For all cases we choose $N = 37 \times 59$, $n = 2$ and $K = 36$. One can verify that $\binom{36}{2} = 630$ is not a nonnegative integer combination of the two prime factors 37 and 59.

We assume that the observation vector y is corrupted by noise to get $y_n = y + \sigma \xi \|y\|_2$, where $\sigma \|y\|_2$ is the noise standard deviation and ξ is a random vector of the same length as y with elements drawn from $\sim \mathcal{CN}(0, 1)$.

The recovery of such a x from a corrupted y_n is shown for various sparsity levels in Figure 1. Specifically, recovery was effected as the constrained optimization

$$\min \|x\|_1, \text{ subject to } \|y_n - Ax\|_2 \leq u \|y_n\|_2,$$

using the Matlab's Disciplined Convex Programming, CVX routine. The recovery error in x , $\frac{\|x - x_{\text{estimated}}\|_2}{\|x\|_2}$ is drawn on the y-axis, against the SNR on the x-axis.

SNR is defined as $10 \log \frac{\|y\|_2^2}{\|y_n - y\|_2^2}$. Each data point on the plot is obtained after averaging 50 realizations of x .

As by Theorem 5.3, the observation matrix has full spark 37, theoretically ℓ_0 recovery with $k \leq 17$ is possible. However ℓ_1 recovery should be possible only with a much smaller sparsity level. Yet at high SNRs even vectors with sparsity level of $k \in \{15, 16, 17\}$ are effectively recovered.

(a) $K = 36$, sparsity level $k = 6$

(b) $K = 36$, sparsity level $k = 15$

(c) $K = 36$, sparsity level $k = 16$

(d) $K = 36$, sparsity level $k = 17$

Figure 1: l_1 recovery error vs SNR for various sparsity levels.

8. Conclusion

We have addressed the following question: When is the submatrix of the $N \times N$, DFT matrix obtained by removing certain rows has full spark, when the rows that are retained are contiguous with precisely one missing. We have argued that because of Theorem 2.2, the full spark nature of such a matrix yields several others with the same property. We have established an equivalence between the full spark property of such matrices and the existence of roots of unity whose products, or themselves, form a vanishing sum. Some, necessary, some sufficient and some necessary and sufficient conditions on the prime factors of N , the number of rows that are retained and the index of the row whose absence prevents the set of retained rows from being completely contiguous.

536 9. Acknowledgments

537 This work is in part supported by US NSF grants EPS-1101284, ECCS-1150801,
538 CNS-1329657, CCF-1302456, CCF-0844812, CCF-1116067, NIH 1R21HL109710-
539 01A1, ACS RSG-11-267-01-CCE, ONR grant N00014-13-1-0202 and under the
540 Thousand Talents Program of the State and Shandong Province, administered by
541 the Shandong Academy of Sciences, China.

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