A Nonconvex Nonsmooth Regularization Method with Structure Tensor Total Variation [☆]

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Abstract

In this paper, a novel regularization method for image restoration and reconstruction is introduced which is accomplished by adopting a nonconvex nonsmooth penalty that depends on the eigenvalues of structure tensor of the underlying image. At first, an alternating minimization scheme is developed in which the problem can be decomposed into three subproblems, two of them are convex and the remaining one is smooth. Then, the convergence of the sequence which generate by the alternating minimization algorithm is proved. Finally, the efficient performance of the proposed method is demonstrated through experimental results for both grayscale and vector-value images.

Keywords: Nonconvex nonsmooth regularization, structure tensor, image restoration and reconstruction, alternating minimization methods.

1. Introduction

Image processing, such as Computerized Tomography, CCD sensors (see [14, 24, 23]) e.t.c., plays an important role in major of engineering and medicine. The problem only provides indirect information of the unknowns, our purpose is to determine the unknowns from the measure by constructing stable maps, while considering the uniqueness and stability of the inverse maps [16]. In this paper, we consider a basic forward image

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restoration model, which can be formulated as

$$g = Au + n,$$

where $u = [u_1(x), \dots, u_M(x)] : \Omega \in \mathbb{R}^2 \mapsto \mathbb{R}^M$ is the original vector-value image with M channels, $n = [n_1(x), \dots, n_M(x)] : \mathbb{R}^2 \mapsto \mathbb{R}^M$ accounts for the perturbations and A is a linear operator that represents for a blurring in optics or a transform operator in medical tomography. We would like to restore an image u from the measurement g where Au should be closed to g in the forward image restoration model and u need to satisfy some prior requirements. For most cases of practical interest, the operator A is either ill-conditioned or singular. And Regularization is the popular method to deal with this kind of ill-posedness problems [8, 30]. The restoration of u is cast as a minimization problem:

$$\mathcal{J}(u) = \|Au - g\|^2 + \tau\Phi(u). \quad (1.1)$$

Such a cost function consists of two terms, the first is a data fidelity term which measures how well a candidate solution explains the observed data, the second is a regularized term which encodes any available prior information about the underlying image and τ is a positive regularized parameter that controls the tradeoff between these two terms.

As state earlier, total variation (TV) semi-norm have been applied to grayscale images ($M=1$) processing [6, 5],

$$\Phi(u) := \sup \left\{ \int_{\Omega} u \nabla \cdot (v) : v \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^n), |v| \leq 1 \right\},$$

where u is defined on the bounded domain $\Omega \subset \mathbb{R}^n$ and $\mathcal{C}_0^\infty(\Omega; \mathbb{R}^n)$ denote the space of arbitrarily differentiable functions with compact support. TV regularizer is convex and rotation-invariant which preserve corners and edges well. Due to this favorable property, TV have been extended to vector-value images [2]. However, a known drawback of TV is that favors piecewise-constant solutions and, thus it can create strong staircase artifacts in smooth regions of the image. In addition, TV relies only on one of the points of Ω without taking into account the available information from its neighborhood.

In literature [28, 29], a more sophisticated descriptor of image variations has been proposed: structure tensor TV function. It generalizes several existing variational penalties, including TV and vectorial extension of TV, and has the ability to capture first-order

information around a local neighborhood. As a consequence, the structure tensor TV function can provide more robust measures of image variation. From another point of view, the discrete structure tensor TV function can be formulate as

$$\Phi(u) = \sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \left\| \left(\sqrt{\lambda_{i,k}^+}, \sqrt{\lambda_{i,k}^-} \right) \right\|_1, \quad (1.2)$$

where $\lambda_{i,k}^+$ and $\lambda_{i,k}^-$ denote the two nonnegative eigenvalues of the discrete structure tensor at pixel (i, k) (see Sect. 2 for detail).

structure tensor TV function is the summation of the $\left\| \left(\sqrt{\lambda_{i,k}^+}, \sqrt{\lambda_{i,k}^-} \right) \right\|_1$ of the underlying image over the domain. The combination leads to an ℓ_1 weighting that dose not overpenalize high variation of u , but the ℓ_1 weighting minimization problem may yield the all the $\left\| \left(\sqrt{\lambda_{i,k}^+}, \sqrt{\lambda_{i,k}^-} \right) \right\|_1$ with lower sparse rate than the real one. Another limitation of the ℓ_1 -norm minimization is its bias caused by shrinking all the $\left\| \left(\sqrt{\lambda_{i,k}^+}, \sqrt{\lambda_{i,k}^-} \right) \right\|_1$ toward zero simultaneously [15]. In order to overcome the limitations of ℓ_1 weighting minimization problem, many nonconvex penalty functions have been proposed, include ℓ_p -norm ($0 < p < 1$) [12], Smoothly Clipped Absolute Deviation (SCAD) [10], Minimax Concave Penalty (MCP) [36], Nonconvex Nonsmooth Potential Functions (PF) [20, 21, 31], and so on. What's more, In [31], we know that nonconvex regularization methods are effective for restoring images with neat edges. The work of Nikolova, Ng and Tam [20] showed that the nonconvex image restoration models enjoy better properties than the convex ones and the nonconvex nonsmooth PF model also performs well. Therefore, we are interested in studying the nonconvex nonsmooth PF penalty with structure tensor TV.

The main contribution of this work: (1) We proposed a nonconvex nonsmooth regularizer which penalizes the image variation at every point of the domain by taking into account the information in a local neighborhood. This is accomplished by adopting a nonconvex nonsmooth penalty that depends on the eigenvalues of structure tensor. (2) Due to the nonconvex nonsmooth property of our proposed regularizer, minimizing the cost function (1.1) become challengeable. Thus, we propose an alternating minimization scheme in which the problem can be decomposed into three subproblems, the two of them are convex and the remaining one is smooth. Therefore, some powerful optimization tools can be applied to deal with these three subproblems. (3) We prove that our

alternating minimization algorithm converges to a critical point.

The rest of this paper is organized as follows. In Section 2, we focus on the nonconvex nonsmooth regularization model and present an alternating minimization algorithm. In Section 3, we study the convergence of our proposed algorithm. In Section 4, we assess the reconstruction and restoration performance of our algorithm by designing some numerical examples. Conclusions are drawn in section 5, while some mathematical proofs are provided in Appendix.

Notion: To simplify the presentation, we use $\|\cdot\|$ to denote L_2 -norm in continuous case and to denote ℓ_2 -norm in discrete case. For any $x \in \mathbb{R}^d$ and $p \geq 1$, $\|\cdot\|_p$ denote the standard ℓ_p -norm in $\mathbb{R}^d$, i.e., $\|x\|_p = (\sum_{i=1}^d |x_i|^p)^{\frac{1}{p}}$. For any $Y \in \mathbb{R}^{n_1 \times n_2}$ We denote $\|\cdot\|_*$ as nuclear-norm of Y , i.e., $\|Y\|_* = \sum_{i=1}^n \sigma_i(Y)$, where σ_i is the i largest singular value of Y and n is the number of singular values, and $\|\cdot\|_F$ as Frobenius norm of Y , i.e., $\|Y\|_F = (\sum_{i=1}^n \sigma_i^2(Y))^{\frac{1}{2}}$. We use δ to denote the Dirac distribution, i.e., $\delta(0) = \infty$, $\delta(t) = 0, \forall t \neq 0$ and $\int_{-\infty}^{+\infty} \delta(t)dt = 1$. We say that $u \in \mathcal{C}^l(\Omega)$ if it is l times continuously differentiable functions with compact support, particularly, $\mathcal{C}_0^\infty(\Omega; \mathbb{R}^n)$ denote the space of infinitely differentiable functions with compact support. Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces and $\mathcal{G} \subset \mathcal{X}$ be a dense linear subspace and $A : \mathcal{G} \rightarrow \mathcal{Y}$ a linear operator. There exists a unique linear operator $A^* : \text{Dom}(A^*) \subset \mathcal{Y} \rightarrow \mathcal{X}$ such that $\langle A^*y, x \rangle = \langle y, Ax \rangle$, for any $x \in \mathcal{G}$ and $y \in \text{Dom}(A^*)$, where $\text{Dom}(A^*) := \{y \in \mathcal{Y} : x \rightarrow \langle y, Ax \rangle \text{ is continuous on } \mathcal{G}\}$. For any convex function h and $x, y \in \mathbb{R}^d$, $\nabla h(x)$ ($\partial h(x)$) is the set of gradient (subgradient) at x , in particular,

$$\partial h(x) = \{\xi \in \mathbb{R}^d : h(y) \geq h(x) + \langle \xi, y - x \rangle\}.$$

For any function $g : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}$ and $Y, \Delta Y \in \mathbb{R}^{n_1 \times n_2}$, operator $Dg(Y) : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}$ denote the Gateaux derivatives at Y , in particular,

$$\lim_{t \rightarrow 0} \frac{1}{t} |g(Y + t\Delta Y) - g(Y) - t\langle Dg(Y), \Delta Y \rangle| = 0.$$

For a tensor $Z \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, we use $Z(i, :, :)$ to denote the matrix $Z(i, j, k) \in \mathbb{R}^{n_2 \times n_3}$ with fixed i . One important property which we will use later on is the Kurdyka-Łojasiewicz (KL) property.

Definition 1.1. (KL function). A proper function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ is said to have the Kurdyka-Łojasiewicz (KL) property at $\hat{x} \in \text{Dom}(\partial f)(\equiv \{z \in \mathbb{R}^p : \partial f(z) \neq \emptyset\})$ if there

exist $\eta \in [0, \infty)$, a neighborhood $\mathcal{V}$ of $\hat{x}$, and a continuous concave function $\theta : [0, \eta) \rightarrow \mathbb{R}_+$ such that

- $\theta(0) = 0$;
- θ is continuous differentiable on $(0, \eta)$;
- for all $z \in (0, \eta)$, $\theta'(z) > 0$;
- for all $z \in \mathcal{V} \cap [f(\hat{z}) < f < f(\hat{z}) + \eta]$, the Kurdyka-Lojasiewicz inequality and

$$\theta'(f(z) - f(\hat{z})) \text{dist}(0, \partial f(z)) \geq 1,$$

hold, where $\text{dist}(z, \mathcal{U}) = \inf \|y - z\|, y \in \mathcal{U}$, for any $\mathcal{U} \subset \mathbb{R}^p$. Moreover, proper semi-continuous functions which satisfy the Kurdyka-Lojasiewicz inequality at each point of $\text{Dom}(\partial f)$ are called KL functions.

2. Nonconvex Structure Tensor TV

2.1. The Structure Tensor and Problem Formulation

In this subsection, we begin with reviewing the theory behind the structure tensor [18, 19, 9] and then propose our nonconvex nonsmooth regularization problem. First, we assume that vector-valued image $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^M)$, i.e. Sobolev space, hence $u = (u_1, \dots, u_M)$, $u_h = (\frac{\partial u_1}{\partial x^h}, \dots, \frac{\partial u_M}{\partial x^h})$, $h = 1, 2$ is a basis for 2-dimensional vector space of tangent vector of $W^{1,2}(\mathbb{R}^2, \mathbb{R}^M)$ at $u = u(x)$ [35]. The variation of the vector-valued image u at any specific image point x is given by

$$du^2 = [dx^1, dx^2] Su(x) [dx^1, dx^2]^T$$

where Su and Ju are defined as

$$Su(x) = [Ju^T Ju](x),$$

$$Ju(x) = [\nabla u_1(x), \dots, \nabla u_M(x)]^T,$$

$[dx^1, dx^2]$ is the dual basis of u_h and Su is the so called structure tensor [11]. In order to use the information from the neighborhood, nonlocal structure tensor has been proposed

$$S_K u(x) = K * [Ju^T Ju](x), \quad (2.1)$$

where $*$ denotes the convolution operation, $K(x)$ is a nonnegative, rotationally symmetric convolution kernel.

In image applications, we are interested in finding the direction through point x along u which has the maximum and minimum rate of change. Let ν be an arbitrary unit two-dimensional direction vector, the nonlocal directional variation is given by

$$RMS_K(\|\partial u/\partial \nu\|_2) = \sqrt{K * \|\partial u/\partial \nu\|_2^2} = \sqrt{\nu^T (Su) \nu}. \quad (2.2)$$

Let $\lambda_+ = \lambda_+(S_K u(x))$, $\lambda_- = \lambda_-(S_K u(x))$ to be the ordered eigenvalues of $S_K u(x)$, namely, $\lambda_+ \geq \lambda_-$, and θ_+ , θ_- to be the corresponding unit eigenvectors. Therefore, (2.2) can be reformulated as

$$RMS_K(\|\partial u/\partial \nu\|_2) = \sqrt{\nu^T [\theta_+, \theta_-] \Lambda [\theta_+, \theta_-]^T \nu},$$

where $\Lambda = \text{diag}(\lambda_+, \lambda_-)$. If we choose $\nu = \theta_+$, then the maximum of RMS_K is equal to $\sqrt{\lambda_+}$, and if we choose $\nu = \theta_-$, then the minimum of RMS_K is $\sqrt{\lambda_-}$. It's concluded that the structure tensor could measure the geometry of image structures in each point, eigenvalues θ_+ and θ_- describe the orientation of maximum and minimum directional variation of u and $\sqrt{\lambda_+}$ and $\sqrt{\lambda_-}$ describe measures of these variations.

In this paper, we introduce a novel a nonconvex nonsmooth regularized method which base on the nonlocal structure tensor. And the new regularizer $\Phi(u)$ reads:

$$\Phi(u) := \int_{\Omega} [\phi(\sqrt{\lambda_+}) + \phi(\sqrt{\lambda_-})] dx, \quad (2.3)$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}^+$ is a potential function (PF). Particularly, when we choose $K = \delta$ (Dirac function), and consider the grayscale image ($M = 1$), then (2.3) could be expressed as follows:

$$\Phi(u) = \int_{\Omega} \phi(\|\nabla u\|_2) dx.$$

And if we choose PF as $\phi(t) = |t|$, then, the regularizer $\Phi(u)$ is equal to TV semi-norm. With the result that our proposed method could generalize TV. In this work, we are interested in choosing a nonconvex nonsmooth PF. Due to the nonconvex nonsmooth property of our proposed regularizer ϕ , minimizing the cost function (1.1) become challengeable. From [7], we known that PF can be decomposed into $\phi(\cdot) = \psi(\cdot) + \phi'(0^+)|\cdot|$ and $\psi \in \mathcal{C}^2$. Subsequently, we can decompose (2.3) into two term:

$$\Phi(u) = \Psi(u) + \alpha \int_{\Omega} \|J_K u\|_* dx,$$

where

$$\Psi(u) := \int_{\Omega} [\psi(\sqrt{\lambda_+}) + \psi(\sqrt{\lambda_-})] dx,$$

$\alpha = \phi'(0^+)$ and $\|\cdot\|_*$ denote the nuclear norm. $\int_{\Omega} \|J_K u\|_* dx$ is convex and $\Psi(u)$ is Gateaux differentiable which will be proved in the next subsection. Then, problem (1.1) can be reformulated as

$$\mathcal{J}(u) = \|Au - g\|^2 + \tau \Psi(u) + \tau \cdot \alpha \int_{\Omega} \|J_K u\|_* dx, \quad (2.4)$$

In the next subsection, we would use an alternating algorithm to solve problem (2.4).

2.2. Discrete Nonconvex Structure Tensor TV and Algorithm

In this subsection, we transform the above problem (2.4) into discrete form and design alternating minimization algorithm via introducing a auxiliary variable. We assume that the discrete vector-value image $u = [u_1, \dots, u_M]$, and $u_j \in \mathbb{R}^{N_1 \times N_2}$ ($j = 1, \dots, M$) is defined on a rectangular grid with size $N_1 \times N_2$. We assume in (2.1), K is a nonnegative, rotation-invariable and discretized convolution kernel with size $(2L_K + 1) \times (2L_K + 1)$ and $i = 1, \dots, N_1, k = 1, \dots, N_2$. The patch-based Jacobian is defined in [18], $J_K : \mathbb{R}^{N_1 \times N_2 \times M} \mapsto \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2}$ and $L = (2L_K + 1)^2$. The patch-based Jacobian reads:

$$[J_K u](i, k) = [\tilde{\nabla} u(i, k, 1), \dots, \tilde{\nabla} u(i, k, M)]^T,$$

where

$$\begin{aligned} [\tilde{\nabla} u] &= [T_{-L_K, -L_K} \circ \nabla u, \dots, T_{L_K, -L_K} \circ \nabla u, \dots, T_{L_K, L_K} \circ \nabla u], \\ T_{x', y'} \circ \nabla u &= \sqrt{K(x', y')} \cdot \nabla u(i - x', k - y', j), \quad x', y' = -L_K, \dots, L_K. \end{aligned}$$

We have

$$\|J_K u\|_* = \left\| \begin{pmatrix} \sqrt{\lambda_+} \\ \sqrt{\lambda_-} \end{pmatrix} \right\|_1,$$

where λ_+ and λ_- are the eigenvalues of $S_K u(x)$. Then, the discrete form of $\int_{\Omega} \|J_K u\|_* d\Omega$ in (2.4) is given by (1.2) and the discrete form of $\Psi(u)$ in (2.4) can be formulated as

$$\tilde{\Psi}(u) = \sum_{i=1}^{N_1} \sum_{k=1}^{N_2} [\psi(\sqrt{\lambda_{i,k}^+}) + \psi(\sqrt{\lambda_{i,k}^-})],$$

In order to develop an alternating scheme which have been extended into many fields [32, 33, 34], we introduce an auxiliary variable $W \in \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2}$. Then, the augmented Lagrangian function of discrete form of (2.4) can be reformulated as

$$L_\rho(u, W, P) = \sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \left(\tau \cdot \alpha \|W_{j,k}\|_* + \frac{\rho}{2} \|(J_K u)_{i,k} - W_{i,k}\|_F^2 - \langle P_{i,k}, (J_K u)_{i,k} - W_{i,k} \rangle \right) + \frac{1}{2} \sum_{j=1}^M \|Au_j - g_j\|_F^2 + \tau \tilde{\Psi}(u).$$

where $P \in \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2}$ is the Lagrangian multiplier, $W_{i,k} = W(i, k, :, :)$, $P_{i,k} = P(i, k, :, :)$, $(J_K u)_{i,k} = (J_K u)(i, k, :, :)$, and $\rho > 0$ is the penalty parameter for the violation of the linear constraint. To simplify the presentation, we use $\|Au - g\|_{F,1}^2$, $\|W\|_{*,1}$, $\|J_K u - W\|_{F,1}^2$ and $\langle P, J_K u - W \rangle$ to represent $\sum_{j=1}^M \|Au_j - g_j\|_F^2$, $\sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \|W_{j,k}\|_*$, $\sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \|(J_K u)_{i,k} - W_{i,k}\|_F^2$ and $\sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \langle P_{i,k}, (J_K u)_{i,k} - W_{i,k} \rangle$, respectively. Subsequently, the Lagrangian function can be reformulate as

$$L_\rho(u, W, P) = \frac{1}{2} \|Au - g\|_{F,1}^2 + \tau \tilde{\Psi}(u) + \tau \cdot \alpha \|W\|_{*,1} + \frac{\rho}{2} \|J_K u - W\|_{F,1}^2 - \langle P, J_K u - W \rangle, \quad (2.5)$$

Given $(u^{(s-1)}, W^{(s-1)}, P^{(s-1)})$, the iteration scheme of problem (2.5) can be described as follows:

$$\begin{cases} W^{(s)} \in \arg \min_{W \in \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2}} L_\rho(W, u^{(s-1)}, P^{(s-1)}); \\ u^{(s)} \in \arg \min_{u \in \mathbb{R}^{N_1 \times N_2}} L_\rho(W^{(s)}, u, P^{(s-1)}); \\ P^{(s)} = P^{(s-1)} - \rho(J_K u^{(s)} - W^{(s)}). \end{cases}$$

where $\arg \min$ denotes the minimal set to an optimization problem.

Similar to [27, 26, 25], W -subproblem can be formulated as matrix factorization problem:

$$\begin{aligned} W^{(s)} &\in \arg \min_{W \in \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2}} \frac{\rho}{2} \|W - J_K u^{(s-1)}\|_{F,1}^2 + \alpha \tau \|W\|_{*,1} - \langle P^{(s-1)}, W - J_K u^{(s-1)} \rangle \\ &= \arg \min_{W \in \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2}} \frac{1}{2} \left\| \left(J_K u^{(s-1)} + \frac{1}{\rho} P^{(s-1)} \right) - W \right\|_{F,1}^2 + \alpha \tau \|W\|_{*,1}, \end{aligned} \quad (2.6)$$

where $\iota = \frac{\tau \alpha}{\rho}$. According to [3], it's easy to show the solution of (2.6) as

$$W_{i,k}^{(s)} = \mathcal{D}_\iota((Y^{(s-1)})_{i,k}), \quad i = 1, \dots, N_1, \quad k = 1, \dots, N_2,$$

where $Y^{(s)} = J_K u^{(s-1)} + \frac{1}{\rho} P^{(s-1)}$, $\mathcal{D}_l$ is the soft-thresholding operator, $\mathcal{D}_l = U \mathcal{D}_l(\Sigma) V^*$,
 $\mathcal{D}_l(\Sigma) = \text{diag}(\{\sigma_i - \kappa\}_+)$, t_+ is the positive part of t , namely, $t_+ = \max(0, t)$ and
 $(J_K u^{(s-1)})_{i,k} = U \Sigma V^*$ is the singular value decomposition (SVD) of matrix $(Y^{(s)})_{i,k}$.

The u -subproblem can be formulated as follows

$$u^{(s)} = \arg \min_{u \in \mathbb{R}^{N_1 \times N_2 \times M}} \frac{1}{2} \|Au - g\|_{F,1}^2 + \tau \tilde{\Psi}(u) + \frac{\rho}{2} \|J_K u - W^{(s)}\|_{F,1}^2 - \langle P^{(s-1)}, W^{(s)} - J_K u \rangle, \quad (2.7)$$

we could use Quasi-Newton's method to solve this optimization problem (2.7):

$$(A^* A + \rho J_K^* J_K) \Delta u = A^*(g - Au^{(s-1)}) - \tau D_u \tilde{\Psi}(u^{(s-1)}) + \rho(W^{(s)} - J_K u^{(s-1)}) - J_K^* P^{(s-1)},$$

$$u^{(s)} = u^{(s-1)} + \Delta u,$$

where A^* is the adjoint of A , and J_K^* is the adjoint of J_K . In order to get Δu , in practice, it may be expensive to solve the above linear system directly, we can apply the conjugate method [17] for solving it. For the sake of simplicity, let $C = A^* A + \rho J_K^* J_K$, and $E_{s-1} = A^*(g - Au^{(s-1)}) - \tau D_u \tilde{\Psi}(u^{(s-1)}) + \rho(W^{(s)} - J_K u^{(s-1)}) - J_K^* P^{(s-1)}$. Let $\Delta u_0 = 0$, $\hat{R}_0 = C \Delta u_0 - E_{s-1}$ and $\hat{P}_0 = -\hat{R}_0$, and then the sequence $\{\Delta u_i\}$ can be computed iteratively as

$$\begin{cases} \alpha_i = -\frac{\langle \hat{R}_i, \hat{P}_i \rangle}{\langle \hat{P}_i, C \hat{P}_i \rangle}, \\ \Delta u_{i+1} = \Delta u_i + \alpha_i \hat{P}_i, \\ \hat{R}_{i+1} = C \Delta u_{i+1} - E_{s-1}, \\ \beta_{i+1} = \frac{\langle \hat{R}_{i+1}, C \hat{P}_i \rangle}{\langle \hat{P}_i, C \hat{P}_i \rangle}, \\ \hat{P}_{i+1} = -\hat{R}_{i+1} + \beta_{i+1} \hat{P}_i. \end{cases}$$

Next, we will compute the Gateaux derivatives of $\tilde{\Psi}(u)$. It has been show in [20] that $\psi \in \mathcal{C}^2$ on $(0, \infty)$. Thus, we have the following:

Proposition 2.1. *The Gateaux derivatives of $\tilde{\Psi}(u)$ is*

$$(D_u \tilde{\Psi}(u))_{i,k} = 2J_K^* P_{i,k}^* (\Theta_{i,k} P_{i,k} J_K u),$$

where $(D_u \tilde{\Psi}(u))_{i,k} = (D_u \tilde{\Psi}(u))(i, k, :)$, $\Theta_{i,k} = \frac{\partial \psi(\sqrt{\lambda_{i,k}^+})}{\partial \lambda_{i,k}^+} \cdot \theta_{i,k}^+ (\theta_{i,k}^+)^T + \frac{\partial \psi(\sqrt{\lambda_{i,k}^-})}{\partial \lambda_{i,k}^-} \cdot \theta_{i,k}^- (\theta_{i,k}^-)^T$,
 $P_{i,k} : \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2} \rightarrow \mathbb{R}^{L_K \cdot M \times 2}$, $P_{i,k} J_K u = (J_K u)_{i,k}$, $(J_K u)_{i,k} = (J_K u)(i, k, :, :)$,
and $P_{i,k}^*$ is the adjoint of $P_{i,k}$.

The proof of Proposition 2.1 is given in Appendix (A). Based on the analysis above, we give a basic framework of the alternating minimization scheme for solving our nonconvex nonsmooth augmented model as follows:

Algorithm 1 to Solve The Minimum Value of (2.5).

Input:

Initialize $u^{(0)}$, τ and $s = 1$;

Output:

- 1: **for** each $s = 1, 2, \dots$ **do**
 - 2: $W^{(s)} \in \arg \min_{W \in \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2}} L_\rho(u^{(s-1)}, W, P^{(s-1)});$
 - 3: $u^{(s)} = u^{(s-1)} + \Delta u$, where,
 - 4: $(A^*A + \rho J_K^* J_K) \Delta u = -D_u L_\rho(u, W^{(s)}, P^{(s-1)});$
 - 5: $P^{(s)} = P^{(s-1)} - \rho(J_K u^{(s)} - W^{(s)}).$
 - 6: **end for**
-

3. Convergence Analysis

The main aim of this section is to show that Algorithm 1 converges to a critical point of (2.5). Now we can establish the properties of $(u^{(s)}, W^{(s)}, P^{(s)})$, and use them to show the convergence of Algorithm 1.

Proposition 3.1. *Suppose that $\underline{\lambda} - \frac{L}{2} > 0$, where $\underline{\lambda} = \lambda_{\min}(A^*A + \rho J_K^* J_K)$ and L is the Lipschitz constant of $D_u L_\rho(u, W^{(s)}, P^{(s-1)})$. Then, the sequences $(u^{(s)}, W^{(s)}, P^{(s)})$ generated by Algorithm 1, satisfy the following inequalities and equality:*

$$\begin{aligned}
 (i) \quad & \frac{\rho}{2} \|J_K u^{(s-1)} - W^{(s)}\|_{F,1}^2 + \tau \alpha \|W^{(s)}\|_{*,1} - \langle P^{(s-1)}, W^{(s)} - J_K u^{(s-1)} \rangle + \mu_1 \|W^{(s-1)} - W^{(s)}\|_{F,1}^2 \\
 & \leq \frac{\rho}{2} \|J_K u^{(s-1)} - W^{(s-1)}\|_{F,1}^2 + \tau \alpha \|W^{(s-1)}\|_{*,1} - \langle P^{(s-1)}, W^{(s-1)} - J_K u^{(s-1)} \rangle.
 \end{aligned} \tag{3.1}$$

125 (ii)

$$\begin{aligned}
& \frac{1}{2} \|Au^{(s)} - g\|_{F,1}^2 + \tau \Psi(u^{(s)}) + \frac{\rho}{2} \|J_K u^{(s)} - W^{(s)}\|_{F,1}^2 - \langle P^{(s-1)}, W^{(s)} - J_K u^{(s)} \rangle \\
& + \mu_2 \|u^{(s-1)} - u^{(s)}\|_{F,1}^2 \\
& \leq \frac{1}{2} \|Au^{(s-1)} - g\|_{F,1}^2 + \tau \Psi(u^{(s-1)}) + \frac{\rho}{2} \|J_K u^{(s-1)} - W^{(s)}\|_{F,1}^2 - \langle P^{(s-1)}, W^{(s)} - J_K u^{(s-1)} \rangle.
\end{aligned} \tag{3.2}$$

(iii)

$$\langle P^{(s-1)} - P^{(s)}, J_K u^{(s)} - W^{(s)} \rangle = \rho \|P^{(s-1)} - P^{(s)}\|_{F,1}^2. \tag{3.3}$$

The proof of Proposition 3.1 is given in Appendix (B).

Theorem 3.1. *For any $T > 0$, and the sequences $(u^{(s)}, W^{(s)}, P^{(s)})$ generated by Algorithm 1, there exist a $\kappa > 0$ satisfy the following inequality*

$$\sum_{s=1}^T \text{dist}(\partial L_\rho(u^{(s)}, W^{(s)}, Z^{(s)}), 0)^2 \leq \frac{1}{\kappa} \left[L_\rho(u^{(0)}, W^{(0)}, P^{(0)}) - L_\rho(u^{(T)}, W^{(T)}, P^{(T)}) \right]. \tag{3.4}$$

130 The proof of theorem 3.1 is given in Appendix (C).

Remark 1. According to literature [13], we can use $\text{dist}(\partial L_\rho(u^{(s)}, W^{(s)}, P^{(s)}), 0)^2 \leq \epsilon$ to judge approximate stationarity of $(u^{(s)}, W^{(s)}, P^{(s)})$, which contrasts to convex optimization, where one uses $|L_\rho(u^{(s)}, W^{(s)}, P^{(s)}) - L_\rho(u^*, W^*, P^*)|$ or $\|u - u^*\|_{F,1}^2 + \|W - W^*\|_{F,1}^2 + \|P - P^*\|_{F,1}^2$ as criteria for convergence analysis.

135 **Remark 2.** From (3.4), it is easy to obtain a ϵ -accurate solution in time $o(\frac{1}{\epsilon^2})$ for our algorithm.

From the definition of Ψ we know that it is a KL function. Our objective is to show that the sequence which is generated by Algorithm 1 converges to a critical point of (2.5). For this purpose we use the KL property to analyze the convergence of the sequence.

Theorem 3.2. *Let $(u^{(s)}, W^{(s)}, P^{(s)})$ be a sequence generated by Algorithm 1 with an initial point $(u^{(0)}, W^{(0)}, P^{(0)})$. If L_ρ has the Kurdyka-Łojasiewicz (KL) property at a critical point (u^*, W^*, P^*) with*

$$L_\rho(u^*, W^*, P^*) \leq L_\rho(u^{(s)}, W^{(s)}, P^{(s)}) \leq L_\rho(u^*, W^*, P^*) + \eta,$$

and

$$C + 2\sqrt{\frac{L_\rho(u^{(0)}, W^{(0)}, P^{(0)}) - L_\rho(u^*, W^*, P^*)}{a}} + \frac{b}{a}\theta(L_\rho(u^{(0)}, W^{(0)}, P^{(0)}) - L_\rho(u^*, W^*, P^*)) < \varrho.$$

where $C = \|u^{(0)} - u^*\|_{F,1}^2 + \|W^{(0)} - W^*\|_{F,1}^2 + \|P^{(0)} - P^*\|_{F,1}^2$, $a = \min\{\mu_1, \mu_2, \rho\}$, $b = \tau L' + \rho + \frac{1}{\rho} + 1 + (1 + \rho)\bar{\lambda}$, $\bar{\lambda} = \lambda_{\max}\sqrt{(J_K^* J_K)}$ and $D\Psi$ is L' -Lipschitz function, then we have

$$\sum_{s=1}^{+\infty} \left[\|u^{(s)} - u^{(s-1)}\|_{F,1}^2 + \|W^{(s)} - W^{(s-1)}\|_{F,1}^2 + \|P^{(s)} - P^{(s-1)}\|_{F,1}^2 \right]^{\frac{1}{2}} \leq +\infty, \quad (3.5)$$

and

$$\lim_{s \rightarrow +\infty} (u^{(s)}, W^{(s)}, P^{(s)}) = (u^*, W^*, P^*).$$

The proof of theorem 3.2 is given in Appendix (D).

4. Numerical Experiments

4.1. Experimental settings

In this section, we present the experimental result to demonstrate the efficiency of our method. Peak-signal-to-noise ratio (PSNR) is used to measure the quality of the restored images, which are defined as

$$PSNR = 10 \cdot \log \frac{255^2}{MSE} [dB] \quad (4.1)$$

where MSE is the Mean-Squared-Error per pixel.

To show the effectiveness of our proposed methods we compare them with other related methods on several inverse image processing problems. We consider the problems of image reconstruction from a limited number of Fourier measurement and image restoration (image denosing and image deblurring). For the first problem, experiments are conducted on images show in Fig. 1. For the image restoration experiments, we present results on both grayscale images and vector-value images. In all the tests, we initialize the image intensities so that they lie in the rang $[0,1]$.

In our experiments, we compare our proposed method with TV [22], STV_δ , STV_K [18]. And we denote our proposed methods as $NSTV_\delta$ with convolution kernel δ and $NSTV_K$ with convolution kernel K . We choose K as a Gaussian kernel with standard

deviation $\sigma = 0.5$ pixels, truncated in a 5×5 pixel window, and choose the potential function (PF) as

$$\phi(t) = \frac{\alpha|t|}{1 + \alpha|t|}, \psi(t) = -\frac{\alpha^2 t^2}{1 + \alpha|t|}.$$

In general, we choose $\alpha = 0.5$.

155 4.2. Reconstruction from Sparse Fourier Measurement

In this subsection, we consider the problem of image reconstruction from a limited number of Fourier measurements, i.e., The signals have relatively few nonzero terms or whose coefficients in Fourier basis have relatively few nonzero entries. In this setting, the operator of (1.1) corresponds to $A = MF$, where F denotes the discrete Fourier transform and M is a masking operator that retains only a subset of the available Fourier coefficients [18], and the discrete Fourier transform is given by

$$(Ff)(j, k) := \frac{1}{\sqrt{N_1 N_2}} \sum_{t \in \mathbb{Z}_{N_1}, z \in \mathbb{Z}_{N_2}} f(t, z) e^{-i2\pi(\frac{kt}{N_1} + \frac{jz}{N_2})}$$

where $f \in \mathbb{R}^{N_1 \times N_2}$, and N_1, N_2 represent the discrete grayscale images matrix dimension, the frequency index k, z ranges over the sets $\mathbb{Z}_{N_1} := \{0, 1, \dots, N_1 - 1\}$, $\mathbb{Z}_{N_2} := \{0, 1, \dots, N_2 - 1\}$.

In literature [4], the strategy based on TV model has been proposed

$$\min_{f \in \mathbb{R}^{n_1 \times n_2}} \frac{1}{2} \|Af - g\|_2^2 + \tau \|f\|_{TV},$$

where $\|f\|_{TV}$ is the discrete total variation semi-norm for discrete $f(t_1, t_2)$,

$$\|f\|_{TV} = \sum_{t_1, t_2} \sqrt{|D_1 f(t_1, t_2)|^2 + |D_2 f(t_1, t_2)|^2},$$

where D_1 is the finite difference $D_1 f(t_1, t_2) = f(t_1, t_2) - f(t_1 - 1, t_2)$ and D_2 is the finite difference $D_2 f(t_1, t_2) = f(t_1, t_2) - f(t_1, t_2 - 1)$, $0 \leq t_1 \leq N_1, 0 \leq t_2 \leq N_2$.

The reported experiments are conducted on images shows in Fig. 1. The recovery is performed via the algorithms in this experiment. And we use

$$\sum_{j=1}^M \|f_j^{(s)} - f_j^{(s-1)}\|_F / \sum_{j=1}^M \|f_j^{(s-1)}\|_F < 10^{-4},$$

and the maximum iteration step $maxit = 500$ as stopping criterion.

Figure 1: (a)-(c) Images with size 205×205 ; (d) Radial sampling mask with 64 lines; (e) low-frequency sampling with 40% portion.

Table 1: PSNR comparisons on Fourier image reconstruction for several sampling patterned and noise conditions ($NSTV_\delta$ represent our proposed nonconvex regularization method).

Sampling	PSNR	Radial-32 lines				Radial-48 lines				Radial-64 lines				Low-frequency portion 20				Low-frequency portion 25				Low-frequency portion 30			
		10dB	20dB	30dB	∞ dB	10dB	20dB	30dB	∞ dB	10dB	20dB	30dB	∞ dB	10dB	20dB	30dB	∞ dB	10dB	20dB	30dB	∞ dB	10dB	20dB	30dB	∞ dB
Brain1	$NSTV_\delta$	24.798	24.780	24.734	24.736	27.868	27.745	27.806	27.812	30.449	30.735	30.761	30.778	25.252	25.309	25.310	25.308	27.614	27.784	27.811	27.815	29.079	29.386	29.421	29.426
	TV	23.956	23.907	23.862	23.866	26.865	26.568	26.499	26.506	30.239	29.614	29.581	29.603	24.988	25.035	25.031	25.031	27.459	27.631	27.659	27.668	29.107	29.551	29.608	29.631
Brain2	$NSTV_\delta$	24.362	24.256	24.213	24.215	27.351	27.021	27.063	26.960	29.614	29.570	29.559	29.566	23.408	23.438	23.447	23.449	24.992	25.050	25.066	25.052	26.701	26.813	26.826	26.825
	TV	23.538	23.405	23.440	23.356	26.262	25.755	25.796	25.800	29.112	28.338	28.256	28.271	23.247	23.302	23.310	23.313	24.787	24.825	24.858	24.858	26.550	26.612	26.641	26.641
Brain3	$NSTV_\delta$	23.840	23.813	23.835	23.838	26.966	26.748	26.762	26.708	29.304	29.205	29.260	29.265	23.387	23.405	23.411	23.407	24.907	24.991	24.998	25.000	26.147	26.299	26.321	26.321
	TV	23.176	23.111	23.058	23.062	26.090	25.519	25.580	25.585	28.870	28.063	27.980	27.985	23.287	23.303	23.309	23.308	24.772	24.864	24.894	24.895	26.063	26.229	26.267	26.269

To create the measured data we use two different Fourier sampling patterns, namely, a radial mask with 32, 48 and 64 radial lines and a low-frequency sampling with 20%, 25% and 30% portion. As an additional degradation factor we consider the presence of complex Gaussian noise in Fourier domain of four different levels. These correspond to a signal-to-noise-ratio (SNR) of $\{10, 20, 30, \infty\}$ dB, and the last SNR value indicates the absence of noise in the measurements. Peak-signal-to-noise ratio (PSNRs) is used to measure the quality of the restored images. In Table. 1 and Fig. 2 we report the reconstruction results we obtained using TV and our proposed method $NSTV_\delta$. From these results we clearly observe that our proposed regularizer consistently better preforms than TV .

4.3. Test on Image Deblurring

In this test, we test the performance of our proposed algorithm on image deblurring problems whose objective is to reconstruct the unknown image $f^* \in \mathbb{R}^{N_1 \times N_2 \times M}$ ($f^* \in \mathbb{R}^{N_1 \times N_2}$) from an observed image $f = Af^* + n$ degraded by a linear blurring operator A and a noise n , where we consider additive i.i.d. Gaussian noise n with the noise level

Figure 2: Reconstruct the image from Fourier data sampled with 48 radial lines and 20-dB SNR. (a) Original image; (b) Back-project image; (c) Our proposed nonconvex regularization method $NSTV_{\delta}$; (d) TV reconstruction.

$\sigma = 0.0025$. Setting the stopping criterion are the same with stopping criterion in section 4.2. And the relative error defines as

$$RelErr = \frac{\sum_{j=1}^M \|f_j - f_j^*\|_F^2}{\sum_{j=1}^M \|f_j^*\|_F^2}.$$

To simplify the presentation, we also use $\|f - f^*\|^2 / \|f^*\|^2$ to denote relative error. In Fig. 3, for the vector-value image test, we show the experimental results with the well-known "Lena" image which contains a variety of meaningful parts in it, such as the large flat area, the areas with gradual changes of the image intensity and the areas with different shapes. During this experiment, the first image is corrupted by "Gaussian" blur (PSNR=25.558) with standard deviation $\sigma = 1.5$ pixels, truncated in a 5×5 pixel window. The second image "House" blurred by "motion" kernel (PSNR=22.152) with standard deviation $\sigma = 1.5$ pixels, truncated in a 15×15 pixel window.

In Fig. 4, we give the relative error values of these methods, and in table 2, we give the PSNRs of these methods. In Fig. 5, for grayscale image deblur test, we show the experimental results with the well-known "Cameraman" and "Lenna". During this experiment, the first image is corrupted by "Gaussian" blur (PSNR=22.593) with standard deviation $\sigma = 1.5$ pixels, truncated in a 5×5 pixel window. The second image blurred by "motion" kernel (PSNR=18.481) with standard deviation $\sigma = 1.5$ pixels, truncated in a 15×15 pixel window.

Figure 3: (a) Original vector-value images; (b) Blurred images; (c) Recovered images by TV ; (d) Recovered images by STV_{δ} ; (e) Recovered images by $NSTV_{\delta}$.

Figure 4: Relative error values of (a) recovered image (Lenna) by TV ; (b) recovered image (Lenna) by STV_{δ} ; (c) recovered image (Lenna) by $NSTV_{\delta}$; (d) recovered image (House) by TV ; (e) recovered image (House) by STV_{δ} ; (f) recovered image (House) by $NSTV_{\delta}$.

Figure 5: (a) Original grayscale images; (b) Blurred images [psnr=22.593 (first image), 18.481 (second image)]; (c) Recovered images by TV [psnr=26.691 (first image), 27.197 (second image)]; (d) Recovered images by STV_δ [psnr=27.153 (first image), 27.208 (second image)]; (e) Recovered images by $NSTV_\delta$ [psnr=27.617 (first image), 27.345 (second image)].

4.4. Test on Image Denoise

In this test, we test the performance of our proposed algorithm on image denoising problems whose objective is to reconstruct the unknown image $f^* \in \mathbb{R}^{N_1 \times N_2 \times M}$ ($\hat{f} \in \mathbb{R}^{N_1 \times N_2}$) from an observed image $f = f^* + n$, where we consider additive i.i.d. Gaussian noise n with the noise level $\sigma = 0.1$. Regarding the stopping criterion for the optimization algorithm are the same with stopping criterion in section 3.2. In Fig. 6, for the vector-value image test, we show the experimental results with the well-known "Flower" image. During this experiment, the image is corrupted by "Gaussian" noise (PSNR=23.027) with standard deviation $\sigma = 0.1$, (f) and (g) images are recovered by STV_K and our proposed $NSTV_K$ regularization methods, where K is the convolution kernel, we choose it to be a Gaussian with standard deviation $\sigma = 0.5$ pixels, truncated in a 5×5 pixel window.

For the grayscale image test, we show the experimental results with the well-known "Baboon" image which contains great texture and structure information. The noise added in it is also the white Gaussian noise with standard variance $\sigma = 0.1$ (PSNR=21.491) and the restored images by these five approaches are shown in Fig. 8. (f) and (g) images are recovered by STV_K and our proposed $NSTV_K$ regularization methods, where K is

Figure 6: (a) Original vector-value image; (b) Noising image; (c) Recovered image by TV ; (d) Recovered image by STV_{δ} ; (e) Recovered image by $NSTV_{\delta}$ (f) Recovered image by STV_K ; (g) Recovered image by $NSTV_K$.

Figure 7: Relative error values of (a) recovered image by TV ; (b) recovered image by STV_{δ} ; (c) recovered image by $NSTV_{\delta}$; (d) recovered image by STV_K ; (e) recovered image by $NSTV_K$.

Table 2: Result of vector-value images.

Methods	Lenna			House			Flower		
	PSNR(dB)	#Iter	Time(s)	PSNR(dB)	#Iter	Time(s)	PSNR(dB)	#Iter	Time(s)
TV	27.419	180	16.32	27.254	191	28.66	27.317	26	3.17
STV_{δ}	27.496	99	265.28	27.348	97	197.59	27.488	19	54.61
$NSTV_{\delta}$	27.592	59	212.62	27.445	79	160.41	27.854	15	70.91
STV_K	--	--	--	--	--	--	27.442	22	109.63
$NSTV_K$	--	--	--	--	--	--	27.501	17	108.37

the convolution kernel, we choose it to be a Gaussian with standard deviation $\sigma = 0.5$ pixels, truncated in a 5×5 pixel window.

Figure 8: (a) Original grayscale image; (b) Noising image (psnr=21.491); (c) Recovered image by TV (psnr=21.536); (d) Recovered image by STV_{δ} (psnr=21.543); (e) Recovered image by $NSTV_{\delta}$ (psnr=21.590); (f) Recovered image by STV_K (psnr=21.564); (g) Recovered image by $NSTV_K$ (psnr=21.574).

As we can see in the above experiment results, the performance (relative error and PSNR) of our proposed methods are better than other methods in this paper.

5. conclusions

In this paper, we have proposed a novel regularization method for image reconstruction and restoration, which is accomplished by adopting a nonconvex nonsmooth penalty that depends on the eigenvalues of structure tensor of the underlying image. And our theoretical result indicates that there exist a sequence which is generated by our algorithm converges to a critical point. The efficiency of the method are performed via various numerical simulations. In addition, we plan to construct a high order structure tensor in the future work, which can generalize the mean curvature or total generalized variation in grayscale images and vector-value images.

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Appendix

(A) PROOF OF PROPOSITION 2.1. We have $\tilde{\Psi}(u) = \sum_{i=1}^{N_1} \sum_{k=1}^{N_2} [\psi(\sqrt{\lambda_{i,k}^+}) + \psi(\sqrt{\lambda_{i,k}^-})]$, $\tilde{\Psi} : \mathbb{R}^{N_1 \times N_2 \times M} \rightarrow \mathbb{R}$. From [20] we know that $\psi \in \mathcal{C}^2(\mathbb{R})$, then $D_u \tilde{\Psi}(u) \in L(\mathbb{R}^{N_1 \times N_2 \times M}, \mathbb{R})$, and

$$(D_u \tilde{\Psi}(u))_{i,k} = \frac{\partial \psi}{\partial \lambda_{i,k}^+} \cdot D_u \lambda_{i,k}^+(u) + \frac{\partial \psi}{\partial \lambda_{i,k}^-} \cdot D_u \lambda_{i,k}^-(u),$$

where $i = 1, \dots, N_1$ and $k = 1, \dots, N_2$. On the other hand, we have

$$D_u \lambda_{i,k}^+ = \langle D_{(S_K u)_{i,k}} \lambda_{i,k}^+, D_u (S_K u)_{i,k} \rangle.$$

And another form of $(S_K u)_{i,k}$ is

$$(S_K u)_{i,k} = \lambda_{i,k}^+ \theta_{i,k}^+ (\theta_{i,k}^+)^T + \lambda_{i,k}^- \theta_{i,k}^- (\theta_{i,k}^-)^T. \quad (5.1)$$

Multiplying (5.1) by $(\theta_{i,k}^+)^T$ on the left and by $\theta_{i,k}^+$ on the right, we obtain

$$\lambda_{i,k}^+ = \langle \theta_{i,k}^+ (\theta_{i,k}^+)^T, (S_K u)_{i,k} \rangle,$$

where $\langle \theta_{i,k}^+(\theta_{i,k}^+)^T, (S_K u)_{i,k} \rangle = \text{Tr}(\theta_{i,k}^+(\theta_{i,k}^+)^T (S_K u)_{i,k})$, so we have

$$D_{(S_K u)_{i,k}} \lambda_{i,k}^+ = \theta_{i,k}^+(\theta_{i,k}^+)^T.$$

What's more, we know that $(S_K u)_{i,k} = (J_K u)_{i,k}^T (J_K u)_{i,k}$, $(J_K u)_{i,k} = (J_K u)(i, k, :, :)$, and we set the operator $P_{i,k} : \mathbb{R}^{N_1 \times N_2 \times L_K \cdot M \times 2} \rightarrow \mathbb{R}^{L_K \cdot M \times 2}$ to be $P_{i,k} J_K u = (J_K u)_{i,k}$. Thus, we have

$$D_u(S_K u)_{i,k} = (P_{i,k} J_K u)^T P_{i,k} J_K + (P_{i,k} J_K)^T P_{i,k} J_K u.$$

From above analysis, we know that

$$D_u \lambda_{i,k}^+ = 2J_K^* P_{i,k}^* (\theta_{i,k}^+(\theta_{i,k}^+)^T P_{i,k} J_K u),$$

where $P_{i,k}^*$ is the adual of $P_{i,k}$, and we have

$$(D_u \tilde{\Psi}(u))_{i,k} = 2J_K^* P_{i,k}^* (\Theta_{i,k} P_{i,k} J_K u),$$

$$\text{where } \Theta_{i,k} = \frac{\partial \psi(\sqrt{\lambda_{i,k}^+})}{\partial \lambda_{i,k}^+} \cdot \theta_{i,k}^+(\theta_{i,k}^+)^T + \frac{\partial \psi(\sqrt{\lambda_{i,k}^-})}{\partial \lambda_{i,k}^-} \cdot \theta_{i,k}^-(\theta_{i,k}^-)^T.$$

(B) PROOF OF PROPOSITION 3.1. The inequalities (3.1) and (3.3) can be obtained by direct computation. The proof for the inequality of (3.2) can be considered as follows.

$$\begin{aligned} L_\rho(u^{(s)}, W^{(s)}, P^{(s-1)}) &\leq L_\rho(u^{(s-1)}, W^{(s)}, Z^{(s-1)}) \\ &\quad + \langle D_u L_\rho(u^{(s-1)}, W^{(s)}, Z^{(s-1)}), u^{(s)} - u^{(s-1)} \rangle + \frac{L}{2} \|u^{(s)} - u^{(s-1)}\|_{F,1}^2, \end{aligned}$$

where L is a Lipschitz constant of $D_X L_\rho(X^{(s)}, W^{(s+1)}, Z^{(s)})$. By (2.5), we have

$$\begin{aligned} L_\rho(u^{(s)}, W^{(s)}, P^{(s-1)}) &\leq L_\rho(u^{(s-1)}, W^{(s)}, P^{(s-1)}) \\ &\quad - \langle (A^* A + \rho J_K^* J_K)(u^{(s)} - u^{(s-1)}), u^{(s)} - u^{(s-1)} \rangle + \frac{L}{2} \|u^{(s)} - u^{(s-1)}\|_{F,1}^2. \end{aligned}$$

We have

$$L_\rho(u^{(s)}, W^{(s)}, P^{(s-1)}) + (\underline{\lambda} - \frac{L}{2}) \|u^{(s)} - u^{(s-1)}\|_{F,1}^2 \leq L_\rho(u^{(s-1)}, W^{(s)}, Z^{(s-1)}).$$

From the assumption of the proposition, i.e. $\underline{\lambda} - \frac{L}{2} > 0$, the inequality (3.2) hold.

(C) PROOF OF THEOREM 3.1. According to (2.6) and (2.7), we have

$$\begin{aligned} 0 &\in \partial_W L_\rho(u^{(s-1)}, W^{(s)}, P^{(s-1)}) \\ &= \tau \alpha \partial \|W^{(s)}\|_{*,1} + P^{(s-1)} - \rho(W^{(s)} - J_K u^{(s-1)}), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} -(A^*A + \rho J_K^* J_K)(u^{(s)} - u^{(s-1)}) &= DL_\rho(u^{(s-1)}, W^{(s)}, P^{(s-1)}) \\ &= A^*(Au^{(s-1)} - g) + \tau D\Psi(u^{(s-1)}) - J_K^* P^{(s-1)} + \rho J_K^*(W^{(s)} - J_K u^{(s-1)}). \end{aligned} \quad (5.3)$$

From the definition of the augmented Lagrangian function (2.5), we have

$$\begin{aligned} \partial_W L_\rho(X^{(s)}, W^{(s)}, Z^{(s)}) \\ = \lambda \alpha \partial \|W^{(s)}\|_{*,1} + P^{(s)} - \rho(W^{(s)} - J_K u^{(s)}), \end{aligned} \quad (5.4)$$

$$\begin{aligned} D_u L_\rho(u^{(s)}, W^{(s)}, Z^{(s)}) \\ = A^*(Au^{(s)} - g) + \tau D\Psi(u^{(s)}) - J_K^* P^{(s)} + \rho J_K^*(W^{(s)} - J_K u^{(s)}), \end{aligned} \quad (5.5)$$

With (5.2) and (5.4), we obtain

$$\partial_W L_\rho(u^{(s)}, W^{(s)}, P^{(s)}) = P^{(s)} - P^{(s-1)} + \rho(J_K u^{(s)} - J_K u^{(s-1)}).$$

With (5.3) and (5.5), we obtain

$$D_u L_\rho(u^{(s)}, W^{(s)}, P^{(s)}) = \tau D\Psi(u^{(s)}) - \tau D\Psi(u^{(s-1)}) + J_K^* P^{(s-1)} - J_K^* P^{(s)}.$$

And via (2.5), we have

$$D_P L_\rho(u^{(s)}, W^{(s)}, P^{(s)}) = \frac{1}{\rho}(P^{(s)} - P^{(s-1)}).$$

From [1], we know that there exists a $\kappa > 0$, and

$$\kappa \cdot \text{dist}(\partial L_\rho(u^{(s)}, W^{(s)}, P^{(s)}), 0)^2 \leq L_\rho(u^{(s-1)}, W^{(s-1)}, P^{(s-1)}) - L_\rho(u^{(s)}, W^{(s)}, P^{(s)}).$$

Then, (3.4) hold.

(D) PROOF OF THEOREM 3.2. From (3.1), (3.2) and (3.3), we know that the sufficient decrease condition [1] is satisfied. Therefore we only need to prove

(1) relative error condition, i.e., there exists $(\widehat{u}^{(s)}, \widehat{W}^{(s)}, \widehat{P}^{(s)}) \in \partial L_\rho(u^{(s)}, W^{(s)}, P^{(s)})$

such that

$$\begin{aligned} \|\widehat{u}^{(s)}\|_{F,1} + \|\widehat{W}^{(s)}\|_{F,1} + \|\widehat{P}^{(s)}\|_{F,1} \\ \leq b \left(\|u^{(s)} - u^{(s-1)}\|_{F,1} + \|W^{(s)} - W^{(s-1)}\|_{F,1} + \|P^{(s)} - P^{(s-1)}\|_{F,1} \right). \end{aligned} \quad (5.6)$$

(2) continuity condition, i.e., there exists a subsequence $(u^{(s_j)}, W^{(s_j)}, P^{(s_j)})$ such that

$$\begin{aligned} (u^{(s)}, W^{(s)}, P^{(s)}) &\rightarrow (u^*, W^*, P^*), \\ L_\rho(u^{(s)}, W^{(s)}, P^{(s)}) &\rightarrow L_\rho(u^*, W^*, P^*), \text{ as } j \rightarrow +\infty. \end{aligned} \quad (5.7)$$

First, Define

$$\begin{cases} \widehat{u}^{(s)} = P^{(s)} - P^{(s-1)} + \rho(J_K u^{(s)} - J_K u^{(s-1)}); \\ \widehat{W}^{(s)} = \tau D\Psi(u^{(s)}) - \tau D\Psi(u^{(s-1)}) + J_K^* P^{(s-1)} - J_K^* P^{(s)}; \\ \widehat{P}^{(s)} = \frac{1}{\rho}(P^{(s)} - P^{(s-1)}). \end{cases}$$

Let $b = \tau L' + \rho + \frac{1}{\rho} + 1 + (1 + \rho)\bar{\lambda}$ and $\bar{\lambda} = \lambda_{\max} \sqrt{(J_K^* J_K)}$, then, (5.6) holds, where $D\hat{\Psi}$ is L' -Lipschitz function. By the semi-continuity of $L_\rho(u, W, P)$, it is easy to know that (5.7) also holds.

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