Hamiltonian model and dynamic analyses for a hydro-turbine governing system with fractional item and time-lag

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ABSTRACT: This paper focus on a Hamiltonian mathematical modeling for a hydro-turbine governing system including fractional item and time-lag. With regards to hydraulic pressure servo system, an universal dynamical model is proposed, taking into account the viscoelastic properties and low-temperature impact toughness of constitutive materials as well as the occurrence of time-lag in the signal transmissions. The Hamiltonian model of the hydro-turbine governing system is presented using the method of orthogonal decomposition. Furthermore, a novel Hamiltonian function that provides more detailed energy information is presented, since the choice of the Hamiltonian function is the key issue by putting the whole dynamical system to the theory framework of the generalized Hamiltonian system. From the numerical experiments based on a real large hydropower station, we prove that the Hamiltonian function can describe the energy variation

of the hydro-turbine suitably during operation. Moreover, the effect of the fractional α and the time-lag τ on the dynamic variables of the hydro-turbine governing system are explored and their change laws identified, respectively. The physical meaning between fractional calculus and time-lag are also discussed in nature. All of the above theories and numerical results are expected to provide a robust background for the safe operation and control of large hydropower stations.

KEY WORDS: Hamiltonian function, Hamiltonian model, hydro-turbine governing system, fractional calculus, time-lag

1. Introduction

By 2014, China's total installed capacity of hydropower has passed three hundred million kilowatts, which represents about 25% of the current installed capacity of hydropower worldwide. Moreover, according to current projections provided by international hydropower industries and referred to the next thirty years, a significant growth in the sector is expected to bring the worldwide installed capacity from ten hundred to twenty hundred million kilowatts. Undoubtedly, the growth of hydropower represents a promising prospect. There are still several outstanding problems in the operation of large hydropower stations [1-3], such as the vibration of hydro-turbine generator units. So far, some large hydropower stations, e.g. Yantan, Liujiaxia, Ertan, Gezhouba, Three Gorges, and Tianhuangping, have already appeared obvious vibration. A recent study reveals that there is a link between the vibration and the operation range of a hydro-turbine-generator unit. The ranges of operation are however inseparably linked to the regulation of the hydro-turbine governing system. Several studies available in the literature focused on the definition of models for the study of ranges of operation and dynamics [4-12]. For example, Nagode et al. [4] proposed a mathematical model of a Francis turbine with a surge tank and suggested a novel method for the control of a hydropower

plant; Li et al. [5] focused on the modeling of a dynamical hydraulic turbine governing systems with nonelastic water hammer effect.

The transmission of speed signal in the hydraulic pressure servo system requires a not-negligible interval of time which causes the delay of the wicket gates operation with respect to the change of the electricity load of the power grid. Unfortunately, the effect of time-lag is not considered in most available mathematical modeling. Thus, the qualities of complex nonlinear, time-variant and non-minimum phase for a hydro-turbine governing system, together with the time-lag, represent challenging issues for the modeling and control of hydropower systems.

The first studies on fractional calculus date back to 1695, with the introduction of derivative, integral and eventually differential equations. Nowadays, thanks to the rapid development of applied mathematics and computer science, fractional calculus is widely used in several fields of science and engineering [13-21], e.g. dynamics, image processing, control engineering etc. When compared with integer-order differential, it shows unique advantages in modeling viscoelastic substance, medical image processing [22], compared with integer-order differential. Due to the obvious viscoelastic properties and low-temperature impact toughness of constitutive material for the hydraulic pressure servo system, integer order calculus is a typical algorithm with limited nature and not suitable to describe it.

The generalized Hamiltonian system is a purely academic offshoot of nonlinear science. Recently, it has been applied to the theory of nonlinear systems [23-26]. Its structure matrix and damping matrix provide more energy information as the theory of the generalized Hamiltonian system develops. Moreover, the Hamiltonian function is a sort of generalized energy which can be treated as the Lyapunov function. It also represents potentially a novel strategy for exploring dynamic behaviors of complex systems [27-29]. Sgino et al. [27] provided a detailed discussion on

the use of stochastic Hamiltonians for noncritical string field on the basis of matrix models; Geng et al. [28] proposed a new hierarchy of integrable evolution equations which contains the generalized Harry Dym type equation. Considering the qualities of complex nonlinear, time-variant and non-minimum phase and time-lag for a hydro-turbine governing system, it is surely a novel way to study the dynamic behaviors and the energy flow that by putting the hydro-turbine governing system to the theory framework of the generalized Hamiltonian system.

In light of these considerations, the novelty of the present study can be summarized into 4 steps. First, a new general dynamical model for hydro-turbine governing system with fractional item and time-lag is proposed. This approach is integrated in the frame of generalized Hamiltonian system by utilizing the method of orthogonal decomposition. Second, we present a new Hamiltonian function that provides more detailed energy information from the numerical experiments that from a real large hydropower station. Third, the effect of time-lag on the dynamic variables and its change laws are investigated through numerical applications. Similarly, the fractional order on the influence of the dynamic variables and change laws are also defined. Finally, the essential differences between fractional calculus and time-lag are discussed in detail.

The present paper is organized as follows: Section 2 describes the proposed strategy for dynamical and Hamiltonian modeling of hydro-turbine governing system with fractional item and time-lag. Numerical experiments along with detailed analyses are presented in Section 3, where the essential difference between fractional calculus and time-lag are also discussed. Finally, the conclusions are presented in Section 4.

Table 1 Variables of the hydro-turbine governing system.

Symbol	Quantity
x_1	state variable of penstock system, p.u.
x_2	state variable of penstock system, p.u.
x_3	state variable of penstock system, p.u.

T_0	elastic time constant of the penstock, p.u.
h_q	Relative of head loss of the penstock
Z_0	Relative values of impedance of surges, p.u.
q_1	relative flow of hydro turbine, p.u.
h_0	gross head of hydropower station, p.u.
h_{qT}	Relative value of head of diversion tunnel p.u.
f_T	coefficient of head loss of diversion tunnel, p.u.
f_p	coefficient of head loss of the penstock, p.u.
y_r	deviation of incremental guide vane/wicket gate position, p.u.
y	the incremental deviation of the guide vane opening, p.u.
q_{nl}	relative flow of the operation of a system with no load,
D_t	damping coefficient of synchronous generator,
h_t	relative value of hydro-turbine head, p.u.
A_t	coefficient scale, p.u.
ω	deviation of relative angular speed, p.u.
k_d	differential adjustment coefficient,
k_p	proportional adjustment coefficient,
k_i	integral adjustment coefficient,
u	control signal of PID controller,
r	reference input, p.u.
T_y	major relay connector response time, s
P_m	output power of hydro-turbine, p.u.
m_t	the deviation of the incremental torque, p.u.
H_1	Hamiltonian function of unit 1, p.u.
H_2	Hamiltonian function of unit 1, p.u.
ω_B	rated angular speed of synchronous generator, rad/s
δ	rotor angle of synchronous generator, rad
D	damping factor of synchronous generator, p.u.
m_g	rate torque of synchronous generator, p.u.
E_q	internal transient voltage, p.u.
$T_{d0'}$	transient time constant of axis d , p.u.
U_s	voltage of infinite power system
E_f	output of excitation controller, p.u.

2. Hamiltonian model of a hydro-turbine governing system with fractional item and time-lag

2.1 Generalization

Most large Chinese hydropower stations are characterized by layouts of one unit with signal penstock, consisting of an upper reservoir, a diversion tunnel, a surge tank, a penstock, a hydro-turbine-generator unit and a draft tube. A general layout of such installations is shown in Fig.

1.

Fig. 1. A general layout of a large hydropower station with a surge tank.

As shown in Fig. 1, the hydro-turbine governing system of this type of large hydropower stations can be divided into two parts. The first includes the hydro-turbine system including the upstream reservoir, the diversion tunnel, the surge tank, the penstock and the hydro-turbine unit. The second consists of the generator system only including the synchronous generator unit.

2.2 Dynamical modeling of the hydro-turbine system with fractional item and time-lag

The dynamic behaviors of diversion tunnel are expressed by

$$\begin{cases} h_r = h_0 - h_{fT} - h_{qT} \\ h_{fT} = f_{pT} q_T^2 \\ h_{qT} = T_{wc} \frac{dq_T}{dt} \end{cases}, \quad (1)$$

where h_r is the head of the surge tank, h_0 the static head of the water column, h_{fT} the friction head loss in the diversion tunnel, the h_{qT} the head loss caused by water hammer in the diversion tunnel, f_{pT} the friction factor of the diversion tunnel, q_T the water flow in the diversion tunnel, and T_{wc} water inertia time. Water inertia time constant T_{wc} [31] is given by

$$T_{wc} = \frac{Q_r}{gH_r} L, \quad (2)$$

where Q_r is rated discharge, H_r rated head, g acceleration of gravity, and L the length of diversion tunnel. With reference to the surge tank, the equations are

$$\begin{cases} h_r = \frac{1}{C_s} \int q_s dt - f_s q_s^2, \\ q_l = q_T - q_s \end{cases}, \quad (3)$$

where q_s is the water flow in the surge tank, f_s the friction factor of the surge tank, and q_l the discharge of the hydro-turbine. According to second Newton's law, penstock dynamics are

$$\begin{cases} h_{t1} = h_r - h_{f1} - h_{q1} \\ h_{f1} = f_1 q_l^2 \\ h_{q1} = Z_{01} q_l \tanh(T_{01} s) \end{cases}, \quad (4)$$

where h_{t1} is the head of the hydro-turbine, h_{f1} is the friction head loss in the penstock, the h_{q1} is the head loss caused by water hammer in the penstock, and f_1 is the friction factor of the penstock.

Considering the elastic water hammer effect in the penstock, the head loss h_{q1} can be rewritten as [23]

$$h_{q1}(s) = Z_{01} \frac{\pi^2 T_{01} s + T_{01}^3 s^3}{\pi^2 + 4 T_{01}^2 s^2} q_l(s). \quad (5)$$

Eq. (5) can be rewritten as the following state space equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\frac{\pi^2}{T_{01}^2} x_2 + \frac{1}{Z_{01} T_{01}^3} h_{q1} \\ \dot{q}_l = -3\pi^2 x_2 + \frac{4}{Z_{01} T_{01}} h_{q1} \end{cases}. \quad (6)$$

The hydraulic servo system provides forces to regulate the guide vane opening so that the output power of the hydro-turbine and the frequency of the speed of the generator can meet the requirements of a hydropower station. Its operational principle is shown in Fig. 2.

Fig. 2. The operational principle of the hydraulic pressure servo system.

The measuring module, shown in Fig. 2, measures the speed signal of the generator continuously and transmutes it to the voltage signal; the measuring module sends it then to the adder. Once received the signal, the adder sends out a regulate order to the main servomotor, comparing the voltage signal with a given signal from the setting module. Finally, when an imbalance between the torque of the hydro-turbine and the mechanical torque of the synchronous generator occurs, the main servomotor provides a powerful force to promote the change of the opening of the wicket vane. In view of the time cost associated with the above process, the opening of the wicket vane remains unchanged until the main servomotor provides a powerful force. Therefore, this results in an operation process time-lag of hydro-turbine governing system. In this work time-lag is considered in the model of hydraulic pressure servo system, and its equation can be written as

$$\dot{y}(t) = \frac{1}{T_y} (u(t) - y(t - \tau) + y_0) . \quad (7)$$

Then, considering the viscoelastic properties and low-temperature impact toughness of constitutive materials of the hydraulic pressure servo system, fractional calculus is also introduced into Eq. (7) which can be rewritten as

$$D^\alpha y(t) = \frac{1}{T_y} (u(t) - y(t - \tau) + y_0) . \quad (8)$$

According to the principles of orifice outflow, the head of the hydro-turbine h_{t1} can be expressed as

$$h_{t1}(t) = \frac{y_r^2}{y(t - \tau)^2} q_1(t)^2 . \quad (9)$$

Similarly, the head loss in the penstock can be rewritten as

$$h_{q1}(t) = h_0 - h_{fT}(t) - h_{qT}(t) - h_{f1}(t) - h_{t1}(t) = h_0 - f_{pT} q_T(t)^2 - T_{oc} \dot{q}_T(t) - f_1 q_1(t)^2 - \frac{y_r^2}{y(t - \tau)^2} q_1(t)^2 . \quad (10)$$

Nowadays, most hydropower stations in China adopt PID controller in the hydraulic pressure servo system. Under this assumption, the output signal u can be written as

$$u = k_p (r - \omega) + k_i \int (r - \omega) dt + k_d (r - \dot{\omega}) . \quad (11)$$

Combining Eq. (1) and (11) organically, the mathematical equations regulating the hydro-turbine system with fractional item and time-lag become

$$\left\{ \begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = -\frac{\pi^2}{T_{01}^2} x_2(t) + \frac{1}{Z_{01}T_{01}^3} (h_0 - f_{pT}q_T(t)^2 - T_{\omega c}\dot{q}_T(t) - f_1q_1(t)^2 - \frac{y_r^2}{y(t-\tau)^2} q_1(t)^2) \\ \dot{q}_1(t) = -3\pi^2 x_2(t) + \frac{4}{Z_{01}T_{01}} (h_0 - f_{pT}q_T(t)^2 - T_{\omega c}\dot{q}_T(t) - f_1q_1(t)^2 - \frac{y_r^2}{y(t-\tau)^2} q_1(t)^2) \\ \dot{q}_T(t) = \frac{1}{T_{\omega c}} (h_0 - h_r - f_{pT}q_T(t)^2) \\ \dot{h}_r(t) = \frac{1}{C_s} q_s(t) - 2f_s q_s(t) \\ D^\alpha y(t) = \frac{1}{T_y} (u(t) - y(t-\tau) + y_0) \end{array} \right. \quad (12)$$

Such system of equations can be expressed as an affine nonlinear mathematical model

$$\dot{X}(t) = f(x(t), y(t-\tau)) + g(x(t))u(t), \quad (13)$$

where

$$X(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad q_1(t) \quad q_T(t) \quad h_r(t) \quad y(t)], \quad g(x(t)) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{T_y}], \text{ and}$$

$$f_1(x(t)) = \begin{bmatrix} x_2(t) \\ x_3(t) \\ -\frac{\pi^2}{T_{01}^2} x_2(t) + \frac{1}{Z_{01}T_{01}^3} (h_0 - f_{pT}q_T(t)^2 - T_{\omega c}\dot{q}_T(t) - f_1q_1(t)^2 - \frac{y_r^2}{y(t-\tau)^2} q_1(t)^2) \\ -3\pi^2 x_2(t) + \frac{4}{Z_{01}T_{01}} (h_0 - f_{pT}q_T(t)^2 - T_{\omega c}\dot{q}_T(t) - f_1q_1(t)^2 - \frac{y_r^2}{y(t-\tau)^2} q_1(t)^2) \\ \frac{1}{T_{\omega c}} (h_0 - h_r - f_{pT}q_T(t)^2) \\ \frac{1}{C_s} q_s(t) - 2f_s q_s(t) \\ \frac{1}{T_y} (y_0 - y(t-\tau)) \end{bmatrix}.$$

The frame diagram of transfer functions of the hydro-turbine governing system with fractional item and time-lag is shown in Fig. 3.

Fig. 3. The transfer functions of the hydro-turbine governing system with fractional item and time-lag

2.3 Hamiltonian modeling of the hydro-turbine system with fractional item and time-lag

Assuming that $H_1(t)$ is the Hamiltonian function of the hydro-turbine system, its output can be expressed as

$$H_{output}(t) = g(X)^T \frac{\partial H_1(t)}{\partial X}. \quad (14)$$

The relationship between the output power $P_m(t)$ of the hydro-turbine and the torque of the generator $m_t(t)$ is

$$P_m(t) = m_t(t)\omega(t), \quad (15)$$

where $\omega(t)$ is the generator speed. Hence, from Eqs. (14) and (15), the Hamiltonian function $H_1(t)$ can be rewritten as

$$H_1(t) = \int g(X)H_{output}(t) = \int g(X)P_m(t) = \int g(X)m_t(t)\omega(t). \quad (16)$$

The mathematical equation of the output power for a hydro-turbine, proposed by IEEE Working Group in 1993, is often used in the mathematical modeling of a hydro-turbine system [31] in the form shown as

$$P_{m-IEEE}(t) = A_t h_t(t)(q_1(t) - q_{nl}) - D_t y(t)\omega(t). \quad (17)$$

To respect the condition $P_{m-IEEE}(t) > 0$, where $P_{m-IEEE}(t)$ is the output power of the hydro-turbine, the following condition must be satisfied

$$q_1(t) - q_{nl} > \frac{D_t y(t-\tau) \omega(t)}{A_t h_t(t)} > 0.$$

Eq. (17) is just a general formula that does not take into account the characteristics of the output power for a selected hydro-turbine in transient process, since the organization structures (the mounting height of the guide vane, the discharge angle of the guide vane etc.) of different types of hydro-turbines are largely different. To overcome this limitation, in this study, the output power dynamics derived from internal characteristics method is represented as

$$\begin{cases} h_{t1}(t) = \frac{\omega(t)}{g} \left[\left(\frac{\cot \gamma}{2\pi b_0} + r \frac{\cot \beta}{F} \right) q_1(t) - \omega(t) r^2 \right] \\ m_{t1}(t) = \rho \left[\left(\frac{\cot \gamma}{2\pi b_0} + r \frac{\cot \beta}{F} \right) q_1(t) - \omega(t) r^2 \right] \end{cases}, \quad (18)$$

where γ is the discharge angle of the guide vane, β the discharge angle of the middle area of the runner, b_0 the mounting height of the guide vane, r the radius of the middle area of the runner, and F the area of the exit of the runner. On the basis of Eqs. (16) and (18), the Hamiltonian function can be rewritten as

$$H_1(t) = \rho T_y q_1(t) y(t-\tau) \omega(t) \left[\frac{r}{F} \cot \beta + \frac{1}{2b_0\pi} \cot \gamma - r^2 \frac{\omega(t)}{q_1(t)} \right]. \quad (19)$$

Assuming that $H_1(t)$ is a positive definite function, the condition in Eq. (20) is obtained as

$$r^2 \frac{\omega(t)}{q_1(t)} - \frac{r}{F} \cot \beta - \frac{1}{2b_0\pi} \cot \gamma > 0. \quad (20)$$

Utilizing Eq. (19), Eq. (13) can be transformed into the Hamiltonian mathematical model [32],

which is

$$\dot{X}(t) = [J_1(t, x(t), y(t-\tau)) + P_1(t, x(t), y(t-\tau))] \frac{\partial H_1(x(t), y(t-\tau))}{\partial x(t)} + g(x(t))u(t), \quad (21)$$

where

$$J_1(t, x(t), y(t-\tau)) = \begin{bmatrix} 0 & j_{12} & j_{13} & j_{14} & j_{15} & j_{16} & j_{17} \\ -j_{12} & 0 & j_{23} & j_{24} & j_{25} & j_{26} & j_{27} \\ -j_{13} & -j_{23} & 0 & j_{34} & j_{35} & j_{36} & j_{37} \\ -j_{14} & -j_{24} & -j_{34} & 0 & j_{45} & j_{46} & j_{47} \\ -j_{15} & -j_{25} & -j_{35} & -j_{45} & 0 & j_{56} & j_{57} \\ -j_{16} & -j_{26} & -j_{36} & -j_{46} & -j_{56} & 0 & j_{67} \\ -j_{17} & -j_{27} & -j_{37} & -j_{47} & -j_{57} & -j_{67} & 0 \end{bmatrix}, \quad P_1(t, x(t), y(t-\tau)) = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p \end{bmatrix}.$$

The variable p appearing in symmetric matrix $P_i^{(1)}(x)$ can be expressed as

$$p = \frac{\langle f(x), \nabla H \rangle}{\|\nabla H\|^2} = \frac{1}{\|\nabla H\|^2} \{f_1 \nabla H_{x1} + f_2 \nabla H_{x2} + f_3 \nabla H_{x3} + f_4 \nabla H_{x4} + f_5 \nabla H_{x5} + f_6 \nabla H_{x6} + f_7 \nabla H_{x7}\} = rp + sp. \quad (22)$$

In light of this, the symmetric matrix $P_i^{(1)}(x)$ can be further written as

$$P_1(x(t), y(t-\tau)) = S_1(x(t), y(t-\tau)) - R_1(x(t), y(t-\tau)), \quad (23)$$

where

$$R_1(x, y(t-\tau)) = \begin{bmatrix} rp & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & rp & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & rp & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & rp & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & rp & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & rp & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & rp \end{bmatrix}, \quad S_1(x(t), y(t-\tau)) = \begin{bmatrix} sp & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & sp & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sp & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & sp & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sp & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & sp & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & sp \end{bmatrix}.$$

Therefore, the Hamiltonian model of the hydro-turbine governing system with fractional item and time-lag can be described as

$$D^{\alpha_i} X(t, y(t-\tau)) = [J_1(x(t), y(t-\tau)) + S_1(t, y(t-\tau)) - R_1(t, y(t-\tau))] \frac{\partial H_1(t, y(t-\tau))}{\partial x(t)} + g(t)u(t). \quad (24)$$

Finally, according to Wang et al. [32], Eq. (24) can be further rewritten as

$$\dot{X}(t) = [J'(x(t), y(t-\tau)) - R'(x(t), y(t-\tau))] \frac{\partial H_1(x(t), y(t-\tau))}{\partial x(t)} + g(x(t))u'. \quad (25)$$

2.4 The Hamiltonian model of the generator system

In this paper, a third synchronous generator model is used, which expressed by the system of equations

$$\begin{cases} \dot{\delta} = \omega_B(\omega - 1) \\ \dot{\omega} = \frac{1}{T_j}(P_m - m_g - D(\omega - 1)) \\ \dot{E}_q' = -\frac{\omega_B}{T_{d0'}} \frac{X_{d\Sigma}}{X_{d\Sigma'}} E_q' + \frac{\omega_B}{T_{d0'}} \frac{X_{d\Sigma} - X_{d\Sigma'}}{X_{d\Sigma'}} U_s \cos \delta + \frac{\omega_B}{T_{d0'}} E_f \end{cases}. \quad (26)$$

The Hamiltonian mathematical model of the generator system is [33]

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \\ \dot{E}_q \end{bmatrix} = \begin{bmatrix} 0 & C_1 & 0 \\ -C_1 & -C_D & 0 \\ 0 & 0 & -C_G \end{bmatrix} \begin{bmatrix} \frac{\partial H_2}{\partial \delta} \\ \frac{\partial H_2}{\partial \omega} \\ \frac{\partial H_2}{\partial E_q} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A & 0 \\ 0 & \frac{\omega_B}{T_{d0}} \end{bmatrix} \begin{bmatrix} m_t \\ E_f \end{bmatrix}. \quad (27)$$

Its Hamiltonian function is

$$H_2(t) = \frac{T_j}{2} (\omega(t) - 1)^2 + \frac{U_s^2}{2} \frac{X_{q\Sigma} - X_{d\Sigma}}{X_{q\Sigma} X_{d\Sigma}} \cos^2 \delta(t) + \frac{U_s^2}{2X_{q\Sigma}} + \frac{1}{2X_{d\Sigma} X_{d\Sigma} X_f} (X_{ad} U_s \cos \delta(t) - X_{d\Sigma} \frac{X_f}{X_{ad}} E_q(t))^2. \quad (28)$$

Specific parameters are presented in Zeng et al [33].

2.5 The Hamiltonian model of the whole system

Set $D^\alpha X(t) = [\dot{x}_1(t) \ \dot{x}_2(t) \ \dot{x}_3(t) \ \dot{q}_1(t) \ \dot{q}_r(t) \ \dot{h}_r(t) \ D^\alpha y(t) \ \dot{\delta}(t) \ \dot{\omega}(t) \ \dot{E}_q(t)]^T$, $H(t) = H_1(t) + H_2(t)$. From Eqs.

(25) and (27), the Hamiltonian mathematical model of the hydro-turbine governing system with fractional item and time-lag is

$$D^\alpha X(t) = [J(x(t), y(t-\tau)) - R(x(t), y(t-\tau))] \frac{\partial H(x(t), y(t-\tau))}{\partial x(t)} + g(x(t))u(t), \quad (29)$$

where

$$\begin{aligned} H(t) = & \frac{T_j}{2} (\omega(t) - 1)^2 + \frac{U_s^2}{2} \frac{X_{q\Sigma} - X_{d\Sigma}}{X_{q\Sigma} X_{d\Sigma}} \cos^2 \delta(t) + \frac{1}{2X_{d\Sigma} X_{d\Sigma} X_f} (X_{ad} U_s \cos \delta(t) - X_{d\Sigma} \frac{X_f}{X_{ad}} E_q(t))^2 \\ & + \rho T_y q_1(t) y(t-\tau) \omega(t) \left[\frac{r}{F} \cot \beta + \frac{1}{2b_0 \pi} \cot \gamma - r^2 \frac{\omega(t)}{q_1(t)} \right] + \frac{U_s^2}{2X_{q\Sigma}}. \end{aligned}$$

3. Numerical experiments

In order to investigate the performances of the proposed model, numerical examples have been implemented. Basic data were elected from an existent large hydropower station and the model applied to the hydro-turbine governing system with a penstock [23]. The parameters adopted for the hydropower station are

Hydro-turbine type HL99.5-LJ-344.2, nominal head $H_r=312$ m, nominal flow $Q_r=53.5$ m³/s, installed capacity $N_r=150$ MW, and nominal speed $n_r=333.3$ r/min. Other parameters are presented in Table 1.

3.1 Numerical experiments for the Hamiltonian function

As discussed in Section 2, the Hamiltonian function defines the structure matrix and the damping matrix of the whole Hamiltonian system. Therefore, the choice of the Hamiltonian function becomes the key issue by putting the hydro-turbine governing system to the theory framework of the generalized Hamiltonian system. The Hamiltonian function of the whole system in this paper includes two parts which are the Hamiltonian function of the hydro-turbine system and the determined Hamiltonian function of the generator system. To select a proper Hamiltonian function of the hydro-turbine system, which are derived from Eq. (17) and Eq. (18) respectively, are compared in this subsection in order to identify the most valid.

According to Eq. (17), the Hamiltonian function can be rewritten as

$$H_{IEEE} = \frac{1}{2} D_r T_y \omega(t) y(t - \tau)^2 + \frac{\omega(t)}{y(t - \tau)} A_r T_y q_1(t)^2 y_r^2 (q_1(t) - q_{ml}) . \quad (30)$$

Numerical results are computed for the following assumptions.

Assumption 1: The discharge angle of the guide vane remains unchanged in the transient process.

Assumption 2: The discharge angle of the middle area of the runner also remains unchanged in the transient process.

Fig. 4 shows the numerical results for $t=0.3$ second.

(a) The Hamiltonian function H_{IEEE} derived from Eq. (17) and the output power P_{m-IEEE} . (b) The Hamiltonian function H derived from Eq. (18) and the output power P_m .

Fig. 4. The dynamic behaviors of the Hamiltonian function and the output power with $\tau=0.3$ second.

As shown in Fig. 4(a) or Fig. 4(b), the output power function has the same trends of the Hamiltonian function. The values of the output power P_m and the Hamiltonian function H_1 grow significantly when the time t increases from 0 to about 1.5 seconds. When time t is over 1.5 second, their values synchronously decrease reaching stability. This trend demonstrate that the Hamiltonian function can adequately describe the dynamic behavior of the hydro-turbine in the transient process.

The obvious dissimilarities are highlighted from the comparison of Fig. 4(a) with Fig. 4(b). First, the output power and Hamiltonian function in Fig. 4(b) appear a palpable reverse regulation phenomenon at about 0.31 second, which is more close to practical engineering. Second, the values of their overshoot in Fig. 4(b) are much larger than that of Fig. 4(a). Third, the stable values of the output power are very different when the system enters into stable operation. The value of the output power derived from Eq. (17) is about 0.93, while the value derived from Eq. (18) becomes about 0.875. This result is due to the two assumptions initially introduced. More specifically, the discharge angle of the guide vane and the discharge angle of the middle area of the runner always remain unchanged with the increase of the discharge q_1 , which causes the increase of the friction head loss directly.

3.2 The validity of the fractional model with time-lag

In 2014, Zeng et al [34] proposed a differential algebraic model for the *BuLuge* hydropower station, which is named Zeng's model for convenience in the following. To prove the validity and advantages of the fractional model with time-lag, a detail comparison of the dynamic behaviors of the hydro-turbine discharge q_1 , the head loss h_{q1} , the generator speed ω and the guide vane opening y for the fractional model with time-lag and Zeng's model are presented (Fig. 5).

Fig. 5. The dynamic behaviors of the hydro-turbine discharge q_1 , the head loss h_{q1} , the generator speed ω , and the guide vane opening y from the different models of the hydro-turbine governing system.

As shown from the graph, compared with the responses of Zeng's model, the dynamic behaviors of the four variables from the integer order model ($\alpha=1$, $\tau=0$) decrease obviously and show reverse regulation. The main difference between Zeng's model and the integer order model is that Zeng's model only addresses the elastic water column effect in the mathematical modeling, without considering the effect of surge tank. Nevertheless, the effect of surge tank on the dynamic behavior of the hydro-turbine governing system is apparent. The fractional order α affects the variation of the variables in the transient process, although it does not affect the stable value of the variables. Comparing Zeng's model and the integer order model ($\alpha=1$, $\tau=0$) with the integer order model with time-lag ($\alpha=1$, $\tau \neq 0$) and the fractional model with time-lag ($\alpha \neq 1$, $\tau \neq 0$), the similarity which is the time-lag τ has no impact on the stable values of the whole system in the

stable operation. The difference is that the responses of the variables from the models with time-lag show palpable overshoot signs in the transient process. From the above analysis, the integer order model ($\alpha=1$, $\tau=0$) is more exact and reliable in simulating the dynamic behaviors of the hydro-turbine governing system. Moreover, the fractional model with time-lag ($\alpha \neq 1$, $\tau \neq 0$) provides more information for the transient process of the hydro-turbine governing system.

3.3 The influence of time-lag τ on the dynamic behavior of the hydro-turbine

To investigate the influence of time-lag τ on the hydro-turbine dynamics, time-lag τ is assumed to change from 0.03 second to 0.75 seconds with the increment of 0.03 seconds. The numerical results representing the main dynamic variables of the governing system in function of τ are shown in Fig. 6. The value of the fractional order q results 1 when the time-lag τ is being varied.

(a) The output power P_m

(b) The hydro-turbine discharge q_1

(c) The guide vane opening y

(d) The generator speed ω

Fig. 6. The dynamic behavior of the main four variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω) of hydro-turbine governing system when τ changes from 0.03 second to 0.75 second with the increment of 0.03.

As shown in Fig. 6, the overshoot of the output power, the discharge of the hydro-turbine and the guide vane opening increase along with τ . In contrast with the above three variables, the overshoot of the generator speed decreases gradually. Moreover, a critical value τ_k can be identified, so that when $\tau > \tau_k$, the stability of the whole system results influenced by the time-lag.

Such trend is captured by the projection of the dynamic variables into the YZ plant plotted in Fig. 7.

(a) The output power P_m , $0.03 < \tau < 0.75$

(b) The hydro-turbine discharge q_1 , $0.03 < \tau < 0.75$

(c) The guide vane opening y , $0.03 < \tau < 0.75$

(d) The generator speed ω , $0.03 < \tau < 0.75$

Fig. 7. The change laws of the overshoot of the above main variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω) when τ changes from 0.03 second to 0.75 second with incremental steps of 0.03.

The time-lag τ affects the overshoots of the above four variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω), and the effect laws

are presented in Tab. 2. It also affects the stability of the four variables as represented by the green lines: the change laws of the stable ranges of the four variables are shown in Table 3.

Table 2 The change laws of the overshoot for the four variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω) of the hydro-turbine governing system with fractional item and time-lag.

Variable	Ranges	Laws	Ranges	Laws
P_m	(0.03, 0.20)	$P_m = 0.93$	(0.20, 0.75)	$P_m = 2.81(\tau - 0.20) + 0.93$
q_1	(0.03, 0.24)	$q_1 = 0.9866$	(0.24, 0.75)	$q_1 = 0.962(\tau - 0.24) + 0.9866$
y	(0.03, 0.21)	$y = 0.95$	(0.21, 0.75)	$y = 0.88(\tau - 0.21) + 0.95$
ω	(0.03, 0.35)	$\omega = 1.02$	(0.35, 0.75)	$\omega = 0.355(\tau - 0.35) + 1.02$

Table 3 The change laws of the four variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω) for the hydro-turbine governing system with fractional item and time-lag.

Variable	Ranges	Laws	Ranges	Laws	Ranges	Laws
P_m	(0.03, 0.69)	$P_m = 0.87$	(0.69, 0.75)	$P_m = 10.65(\tau - 0.69) + 0.87$		
q_1	(0.03, 0.69)	$q_1 = 0.985$	(0.69, 0.75)	$q_1 = 2.67(\tau - 0.69) + 0.985$		
y	(0.03, 0.66)	$y = 0.95$	(0.66, 0.72)	$y = -0.88(\tau - 0.66) + 0.95$	(0.72, 0.75)	$y = 5.29(\tau - 0.72) + 0.8972$
ω	(0.03, 0.66)	$\omega = 1.00$	(0.66, 0.72)	$\omega = 0.467(\tau - 0.66) + 1.00$	(0.72, 0.75)	$\omega = -1.12(\tau - 0.72) + 1.028$

3.4 The effect of fractional α on the dynamic behaviors of the hydro-turbine

In this subsection, the effect of the fractional α on the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y and the generator speed ω are explored, respectively. Let the fractional α increase from 0.4 second to 2 second with the incremental steps of 0.08 seconds. The numerical results of the four dynamic variables are shown in Fig. 8. In addition, the value of the time-lag τ is zero when the fractional order α is being varied.

Fig. 8. The dynamic behaviors of the main four variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω) of hydro-turbine governing system when α changes from 0.4 to 2 with the incremental steps of 0.08.

From Fig. 8, we can get the change laws of the above four dynamic variables. The change laws of the overshoots for the above four variables are shown in Table 4. The change laws of the stable ranges of the four variables with the fractional order α varying are presented in Table 5. In addition, as α is more than around 2, the system is out of control (see Fig. 8).

Table 4 The change laws of the overshoot for the four variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω) of the hydro-turbine governing system with fractional item and time-lag.

Variable	Ranges	Laws	Ranges	Laws
P_m	(0.40, 0.88)	$P_m = 0.357(\alpha - 0.4) + 0.5387$	(0.88, 2.00)	$P_m = 1.27(\alpha - 0.88) + 0.7099$
q_1	(0.40, 1.04)	$q_1 = 0.07(\alpha - 0.4) + 0.9027$	(1.04, 2.00)	$q_1 = 0.471(\alpha - 1.04) + 0.9479$
y	(0.40, 1.04)	$y = 0.072(\alpha - 0.4) + 0.9045$	(1.04, 2.00)	$y = 0.494(\alpha - 1.04) + 0.9504$
ω	(0.40, 1.04)	$\omega = 1.004$	(1.04, 2.00)	$\omega = 0.1677(\alpha - 1.04) + 1.004$

Table 5 The stable ranges of the four variables (the output power P_m , the hydro-turbine discharge q_1 , the guide vane opening y , the generator speed ω) as the fractional order α is varying from 0.4 to 2.0.

Variable	Ranges	Laws	Ranges	Laws
P_m	(0.4, 1.0)	$P_m = 0.308(\alpha - 0.4) + 0.539$	(1.0, 1.84)	$P_m = 0.724$
	(1.84, 1.92)	$P_m = 2.538(\alpha - 1.84) + 0.724$	(1.92, 2.0)	$P_m = 14.51(\alpha - 1.92) + 0.927$
q_1	(0.4, 1.0)	$q_1 = 0.075(\alpha - 0.4) + 0.903$	(1.0, 1.84)	$q_1 = 0.948$
	(1.84, 1.92)	$q_1 = 0.913(\alpha - 1.84) + 0.948$	(1.92, 2.0)	$q_1 = 4.675(\alpha - 1.92) + 1.021$
y	(0.4, 1.0)	$y = 0.075(\alpha - 0.4) + 0.905$	(1, 1.76)	$y = 0.95$
	(1.76, 1.92)	$y = 0.44(\alpha - 1.76) + 0.95$	(1.92, 2)	$y = 5.05(\alpha - 1.92) + 1.021$
ω	(0.4, 1.84)	$\omega = 1.00$	(1.84, 1.92)	$\omega = 0.35(\alpha - 1.84) + 1$
	(1.92, 2.0)	$\omega = 1.22(\alpha - 1.92) + 1.028$		

4. Conclusions

In this paper, a fractional mathematical model with time-lag of a hydro-turbine governing system is proposed. Such model has been applied to the theory framework of the generalized Hamiltonian system. The Hamiltonian function, which is derived from the interval function, results into a reverse regulation function that allows to adequately describe the energy change of the hydro-turbine. Furthermore, some numerical experiments are presented to explore the effect of fractional order and time-lag on the characteristics of system variables. some key values of fractional order and time-lag where the state of the hydro-turbine governing system changes are highlighted. Finally, the difference between the function $y(t_f)$ which is solved from the fractional function $D^\alpha y(t_1)$ and the function $y(t_d - \tau)$ in nature has been discussed.

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