Quantifying model uncertainty in dynamical systems driven by non-Gaussian Lévy stable noise with observations on mean exit time or escape probability

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Abstract

Complex systems are sometimes subject to non-Gaussian α -stable Lévy fluctuations. A new method is devised to estimate the uncertain parameter α and other system parameters, using observations on mean exit time or escape probability for the system evolution. It is based on solving an inverse problem for a deterministic, nonlocal partial differential equation via numerical optimization. The existing methods for estimating parameters require observations on system state sample paths for long time periods or probability densities at large spatial ranges. The method proposed here, instead, requires observations on mean exit time or escape probability only for an arbitrarily small spatial domain. This new method is beneficial to systems for which mean exit time or escape probability is feasible to observe.

Key Words: Stochastic dynamical systems; parameter estimation; optimization; non-Gaussian Lévy motion; mean exit time; double-well system

Mathematics Subject Classifications (2010): 65N21, 60H15, 60F10

1 Introduction

Random fluctuations in complex systems are sometimes non-Gaussian α -stable Lévy motions [35, 33, 34]. We consider a system under such fluctuations

modeled by a scalar stochastic differential equation (SDE)

$$dX_t = f(\beta, X_t)dt + dL_t^\alpha, \quad X_0 = x, \quad (1)$$

where X_t is the system state process, f is a vector field (or drift), β is a system parameter, and L_t^α is a scalar symmetric α -stable Lévy motion ($0 < \alpha < 2$) defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For example, the calcium signal, as a proxy for climate state, in paleoclimatic data is approximately described [9] by a model like (1).

An α -stable Lévy motion is a non-Gaussian process, while the well-known Brownian motion is a Gaussian process. Non-Gaussian dynamical systems like (1) have attracted considerable attention recently [2], as they are appropriate models for various systems under heavy tail fluctuations [1, 30].

The process L_t^α has heavy tail or power law distribution in the sense that

$$\mathbb{P}(|L_t^\alpha| > x) \sim \frac{1}{x^\alpha},$$

for large x . The α is called the power parameter, or stability index, or non-Gaussianity index. (In fact, Brownian motion corresponds to the special case $\alpha = 2$. But, we will not consider this case in the present work.)

The α -stable fluctuations arise in various situations, including modeling for optimal foraging, human mobility and geographical spreading of emergent infectious disease. GPS data are used to track the wandering black bowed albatrosses around an island in the Southern Indian Ocean to study their movement patterns in searching for food. The movement patterns obey a power law distribution [18] with power parameter $\alpha \approx 1.25$. One way to examine the human mobility is to collect data by online bill trackers, which provide successive spatial-temporal trajectories. A bill traveling at certain distances within a short period of time (less than one week) follows a power law distribution [6] with power parameter $\alpha \approx 1.6$. Moreover, the spreading patterns of human influenza, as described by the classic susceptibility-infection-recovery (SIS) epidemiologic model, is also strikingly similar to an α -stable Lévy motion.

To make (1) a predictive model, it is essential to estimate the parameter α , using observations on the system evolution. Methods for estimating the other system parameter β , when α is known, have been considered in literature ([16, 17, 36, e.g.]) and thus it is not a focus here. There are a couple of attempts in estimating α . For example, assuming the drift ' f ' insignificant (which is an inappropriate assumption in many situations), it is suggested [9, 35] to roughly estimate this α value using data on probability density function for X_t . The tail of the probability density function $p(x)$ behaves like $1/x^\alpha$ for $x \gg 1$, after ignoring the drift ' f '. Thus the $\log p$ vs. $\log x$

plot is a straight line with slope ‘ $-\alpha$ ’. This provides an estimate α by data fitting. This method is not accurate as it assumes that the drift f does not alter the tail behavior of X_t . Another approach to estimate α is suggested in [23] and it requires observations on lots of system state sample paths or sample characteristic functions for long time periods.

In the present paper, we devise a method to estimate α (and other system parameters), using observations on mean exit time or escape probability. Recall the first exit time of X_t starting at x (or ‘a particle starting at x ’) from a bounded domain D is defined as

$$\tau(\omega) := \inf\{t \geq 0, X_t(\omega, x) \notin D\},$$

and the mean exit time (also called mean residence time) is denoted as $u(x) := \mathbb{E}\tau$. The likelihood of a particle, starting at a point x , first escapes from a domain D and lands in a subset E of D^c (the complement of D) is called escape probability and is denoted by $P_E(x)$.

Both the mean exit time $u(x)$ and escape probability $P_E(x)$ satisfy deterministic, nonlocal (i.e., integral) differential equations with exterior Dirichlet boundary conditions. For the scalar SDE (1), these are nonlocal ordinary differential equations, while for a SDE system, these become nonlocal partial differential equations. The non-Gaussianity of the noise manifests as nonlocality at the level of the mean exit time and escape probability.

When we have observations on the mean exit time $u(x)$ or escape probability $P_E(x)$, it is thus possible to estimate α and other system parameters, by solving an inverse problem for the nonlocal differential equations.

It is sometimes too costly to observe system state sample paths X_t over very long time periods [24], but is more feasible to observe (or to infer from collected data) other quantities about system evolution, such as mean exit time and escape probability. Mean residence time has been observed or measured in chemical, industrial and physiological systems [15, 26]. For example, the mean residence time for xenon in intact and surgically isolated muscles can be measured [27]. The mean residence time of xenon is longer than that predicted by a single-compartment model of gas exchange, and this leads to the understanding that a multiple-compartment model might be more accurate according to larger relative dispersion (the standard deviation of residence time divided by the mean). Escape probability has also been observed or measured in certain physical and electronic systems [11, 12, 13].

This paper is organized as follows. In section 2, we formulate our method, i.e., an inverse problem for nonlocal differential equations to estimate parameters. Numerical simulation results are presented in section 3. The paper ends with some discussions in section 4.

2 Methods

A scalar symmetric α -stable Lévy motion L_t^α is characterized [2, 21] by a shift coefficient which is often taken to be zero (for convenience) and a non-negative measure ν_α defined on the state space $\mathbb{R}^1$:

$$\nu_\alpha(dx) = C_\alpha |x|^{-(1+\alpha)} dx,$$

with $\alpha \in (0, 2)$ and $C_\alpha = \frac{\alpha}{2^{1-\alpha}\sqrt{\pi}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}$. For more information see [7, 32]. The generator for the solution process $\tilde{X}_t$ of (1) is ([2, p. 402]):

$$\begin{aligned} A\varphi &= f(\beta, x)\varphi'(x) \\ &+ \int_{\mathbb{R}^1 \setminus \{0\}} [\varphi(x+y) - \varphi(x) - I_{\{|y|<1\}} y\varphi'(x)] \nu_\alpha(dy), \end{aligned} \quad (2)$$

where I_S is the indicator function of the set S , i.e.,

$$I_S(y) = \begin{cases} 1, & \text{if } y \in S; \\ 0, & \text{if } y \notin S. \end{cases}$$

We consider the mean exit time, $u(x)$, for an orbit starting at x , from a bounded open interval D . By the Dynkin formula [28, 31] for general Markov processes, as in [25, 4, 5, 14], we know that $u(x)$ satisfies the following nonlocal differential equation:

$$Au(x) = -1, \quad x \in D \quad (3)$$

$$u = 0, \quad x \in D^c, \quad (4)$$

where $D^c = \mathbb{R}^1 \setminus D$ is the complement of D .

Suppose that we have observed the mean exit time $u(x)$, $x \in D = (a, b)$ (a small interval). We then solve the inverse problem for a nonlocal differential with exterior boundary condition (3)-(4), in order to estimate α and β . See [20, 3, 22] for discussions on inverse problems for partial differential equations. This is achieved by a numerical optimization

$$\min_{\alpha, \beta} G(\alpha, \beta), \quad (5)$$

where the objective function $G = \text{dist}(u(x), u_{ob})$, for an appropriate distance function ‘dist’ between u and its observation u_{ob} . Note that the observation u_{ob} , depending on spatial location x , is usually in data format. To evaluate the objective function G at initially guessed or approximated values of (α, β) ,

we need to numerically solve (3)-(4) by a finite difference scheme in [14]. See also [8] for discussions on related simulations.

We also consider estimation of parameters using observations on escape probability for the system (1). The escape probability of a particle, starting at a point x , first escapes from a bounded domain D and lands in a subset E of D^c , is denoted by $P_E(x)$, and it satisfies the following nonlocal differential equation [29]

$$A P_E(x) = 0, \quad x \in D, \quad (6)$$

$$P_E|_{x \in E} = 1, \quad P_E|_{x \in D^c \setminus E} = 0, \quad (7)$$

where A is the generator defined in (2). The subset E is usually taken to be an open set ([10, Ch. 7]). We again solve the inverse problem for a nonlocal differential with exterior boundary condition (6)-(7), in order to estimate α and β . This is also achieved by a numerical optimization

$$\min_{\alpha, \beta} G(\alpha, \beta), \quad (8)$$

where the objective function $G = \text{dist}(P_E(x), P_{Eob})$, for an appropriate distance ‘dist’ between P_E and its observation P_{Eob} . Note that the observation P_{Eob} , depending on spatial location x , is usually in data format. To evaluate the objective function G at initially guessed or approximated values of (α, β) , we need to numerically solve (6)-(7), also by a finite difference scheme in [14].

In both settings above, the domain D can be taken as small as we like (or arbitrarily small). This offers an advantage as it uses limited amount of observational resources.

In the present paper, we only consider scalar SDEs. For SDEs in higher dimensions, both mean exit time and escape probability satisfy nonlocal partial differential equations, and our method also applies.

3 Numerical experiments

We now consider three examples to illustrate our method for estimating parameters in stochastic dynamical systems with non-Gaussian Lévy noise. For numerical optimization, we use Matlab’s built-in function `fminbnd`, which is a hybrid scheme, using both successive parabolic interpolation and golden section search to find a minimizer for an objective function on a fixed interval. In the examples below, we use simulated data for u_{ob} and P_{Eob} , in order to illustrate our method. Our following numerical experiments are reproducible by Matlab coding the finite difference scheme in [14], together with the Matlab’s built-in function `fminbnd` for numerical optimization.

Example 1. Consider a scalar Ornstein-Uhlenbeck system

$$dX_t = -X_t dt + dL_t^\alpha, X_0 = x. \quad (9)$$

In this example, $f(x) = -x$. Suppose that we have observed the mean residence time $u_{ob}(x)$ for $x \in D = (-2, 2)$ and $(-0.1, 0.1)$. Let us find out estimation of α by solving the inverse problem of the following nonlocal differential equation:

$$\begin{aligned} Au(x) &= -1, \quad x \in D \\ u &= 0, \quad x \in D^c, \end{aligned} \quad (10)$$

where the generator A is

$$\begin{aligned} Au &= -xu'(x) \\ &+ \int_{\mathbb{R}^1 \setminus \{0\}} [u(x+y) - u(x) - I_{\{|y|<1\}} yu'(x)] \nu_\alpha(dy), \end{aligned} \quad (11)$$

and $D^c = \mathbb{R}^1 \setminus D$ is the complement set of D .

Using the L^2 norm, we define an objective function

$$G(\alpha) = \frac{\|u(\alpha, x) - u_{ob}(x)\|_2^2}{\|u_{ob}(x)\|_2^2},$$

and the estimation of $\alpha \in (0, 2)$ is taken to be the minimizer, i.e.,

$$\alpha_E = \arg \min_{\alpha} G(\alpha).$$

To test our method, we generate the observation $u_{ob}(x)$ by numerically solving (10) with $\alpha = 0.6$. The values of $u(\alpha, x)$ are from finite difference [14] discretization of (10) with α as a parameter to be determined. Figure 1 shows accurate estimation of α on a smaller domain $D = (-0.1, 0.1)$, as well as on a larger domain $D = (-2, 2)$.

Example 2. Consider

$$dX_t = (X_t - X_t^3)dt + dL_t^\alpha, X_0 = x. \quad (12)$$

$$f(x) = x - x^3.$$

We estimate α , using observation on escape probability P_{Eob} . Namely, we solve an inverse problem for the following nonlocal differential equation

$$A P_E(x) = 0, \quad x \in D, \quad (13)$$

$$P_E|_{x \in E} = 1, \quad P_E|_{x \in D^c \setminus E} = 0, \quad (14)$$

Figure 1: Estimation of α on domains $D = (-0.1, 0.1)$ (left) and $D = (-2, 2)$ (right) with observation on mean exit time: True value $\alpha = 0.6$.

Figure 2: Estimation of α on domains $D = (-0.1, 0.1)$ (left) and $D = (-2, 2)$ (right) using observation on escape probability: True value $\alpha = 1.5$.

where A is the generator defined in (2). Defining an objective function

$$G(\alpha) = \frac{\|P_E(\alpha, x) - P_{Eob}(x)\|_2^2}{\|P_{Eob}(x)\|_2^2},$$

the estimation of α is $\alpha_E = \arg \min_{\alpha} G(\alpha)$. To test our method, we generate the observation $P_{Eob}(x)$ by numerically solving (13)-(14) with $\alpha = 1.5$. The values of $P_E(\alpha, x)$ are from finite difference [14] discretization of (13)-(14) with α as a parameter to be determined. Figure 2 shows the estimation of $\alpha = 1.5$ on two different domains.

Example 3. Consider

$$dX_t = (X_t - \beta X_t^3)dt + dL_t^\alpha, X_0 = x. \quad (15)$$

Figure 3: Estimation of α and β by observing mean exit time with true value of $\alpha = 0.6$ and true value of $\beta = 1.5$. The estimated α is 0.59858 and the estimated β is 1.51.

$f(x) = x - \beta x^3$ where β is a positive parameter.

In this example, we use observations of either mean exit time or escape probability to estimate unknown parameters. Let the observation of mean exit time be u_{ob} and the observation of escape probability be P_{Eob} . Defining an objective function

$$G_1(\alpha, \beta) = \frac{\|u(x, \alpha, \beta) - u_{ob}(x)\|_2^2}{\|u_{ob}(x)\|_2^2},$$

and

$$G_2(\alpha, \beta) = \frac{\|P_E(x, \alpha, \beta) - P_{Eob}(x)\|_2^2}{\|P_{Eob}(x)\|_2^2},$$

respectively, we obtain estimations of parameters by minimizing these functions separately. Results are shown in Figure 3 for using observation of mean exit time and Figure 4 for using observation of escape probability.

4 Discussions

In summary, we have devised a method to estimate the non-Gaussianity parameter α , and other system parameters, for non-Gaussian stochastic dynamical systems, using observations on either mean exit time or escape probability. It is based on solving an inverse problem for a deterministic, nonlocal partial differential equation via numerical optimization.

When the noise has a Gaussian component modeled by a Brownian motion B_t , the generator A in nonlocal partial differential equations (3) and (6) contains an extra Laplacian term Δu and our method still works. Especially,

Figure 4: Estimation of α and β by observing escape probability with true value of $\alpha = 1.5$ and true value of $\beta = 0.4$. The estimated α is 1.5288 and the estimated β is 0.401.

if the noise has only Gaussian component, the generator A is Δu (and the nonlocal term is absent) and our method remains valid.

The existing methods for estimating the non-Gaussianity parameter α require observations on system state sample paths for long time periods or probability densities on very large spatial domain. The method proposed here, instead, requires observations on either mean exit time or escape probability only for an arbitrarily small spatial domain. This new method is especially beneficial for systems where either mean exit time or escape probability is relatively easy to observe.

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