Switched linear control for refrigerant superheat recovery in vapor compression systems

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Abstract—Extended durations of liquid refrigerant ingestion by the compressor of a vapor compression system (VCS) can lead to damage or failure of this component. While this can be prevented by inserting an accumulator between the evaporator and compressor, this addition of hardware may be undesirable for applications in which the weight or size of the thermal management system is critical. As an alternative, this paper proposes a switched Linear Quadratic Gaussian (LQG) design to quickly recover the presence of a superheated phase at the exit of the evaporator using feedback control. Stability analysis of the closed-loop switched system is presented, and application of the control approach in both simulation and on an experimental VCS testbed demonstrate the success of the control design.

Keywords: Vapor compression system; superheat recovery; switched linear control

1. Introduction

A basic vapor compression system (VCS) consists of four components: 1) An evaporator, 2) a compressor, 3) a condenser, and 4) an expansion valve, connected as shown in Fig. 1. In this thermodynamic system, refrigerant cycles through the four components, undergoing phase changes as a result of heat transfer and changes in pressure, transporting thermal energy from the evaporator secondary side to the condenser secondary side.

Fig. 1 Schematic of a basic vapor compression system

Operational objectives for a VCS can be characterized by three criteria. First, the system should achieve a given performance metric. For example, this metric might be to provide a desired cooling or heating capacity, or to regulate a conditioned space within temperature constraints. Second, the system should operate efficiently. This may be evaluated by the coefficient of performance (the

ratio of cooling or heating provided to the electrical energy consumed), by the electrical power consumption itself, or by the monetary cost of operation.

The third operational objective, and most pertinent to this work, is for the system to operate safely, in particular so as to minimize component wear. In many systems, the inclusion of an accumulator between the evaporator and compressor performs an important safety function by preventing liquid from being ingested by the compressor, which if unchecked can lead to compressor damage or failure [1]. However, in some applications the system configuration of Fig. 1 may be preferred in order to reduce the weight and/or size of the VCS by not including an accumulator. Such applications include next-generation aircraft, for which VCSs have been investigated as a lighter, less expensive, and more efficient alternative to air-cycle systems for thermal energy management [2].

In absence of an accumulator, minimizing compressor liquid ingestion must be achieved by feedback control—using measurements of system temperatures, pressures, etc. to ensure that the refrigerant exiting the evaporator is of superheated phase. Such control traditionally involves set-point regulation of evaporator superheat, T_{SH} , defined as:

$$T_{SH} = T_{evap,ref,out} - T_{sat}(P_{evap,ref,out}) \quad (1)$$

where $T_{sat}(P_{evap,ref,out})$ is the saturation temperature of the refrigerant as a function of evaporator pressure. While increasing the superheat set-point does improve robustness against the complete loss of the superheated phase, this comes at the expense of reduced efficiency. For example, in [3], it is reported that an increase in superheat set point from 5°C to 10°C resulted in a 10% increase in energy consumption of a VCS. In this light, selection of the superheat set point is seen as a tradeoff between safety and efficiency. As discussed in Section II, this selection is made even more critical by the fact that many VCS control designs are not functional at low values of superheat, nor are they designed to be capable of recovering a superheated phase at the exit of the evaporator in the event of its loss [4]. Such control is especially difficult because the superheat response in the transitional region between zero and low (typically under 5°C) values of superheat is known to exhibit highly nonlinear behavior [5].

This paper presents a switched linear control approach that is able to rapidly recover a superheated phase when it is lost in the evaporator in systems without an accumulator, demonstrating that the need for additional hardware to prevent extended durations of compressor liquid ingestion can be avoided by instead applying a more intelligent control design to the system. This switched approach allows powerful and reliable linear tools to be applied to an inherently highly nonlinear system. Stability analysis of the closed-loop system is presented, and application of the control approach in both simulation and on an experimental VCS testbed demonstrate the success of the control design.

The rest of this paper is organized as follows. Section 2 briefly reviews the relevant literature on VCS control design. Section 3 motivates the need for switched control and discusses the linear models used in control. Section 4 presents the control architecture and formulation, as well as stability results. Section 5 provides simulation and experimental results demonstrating the proposed approach. Lastly, Section 6 presents conclusions and future work. Portions of the work presented in this paper have been previously included in the thesis of [6], and in an abbreviated form without experimental results and detailed stability analysis in [7].

2. Relevant VCS control in the literature

The literature includes many control designs for VCSs, including decentralized single-input-single-output (SISO) loops [8], multi-input-multi-output (MIMO) control [3], and gain-scheduled MIMO control [9]. While the minimum superheat set point at which various control designs are found to robustly regulate a given system is not often reported, the high nonlinearity in superheat response at low values limits the performance of many designs, as discussed with regard to PID controllers in [10]. The work in [5] presents a nonlinear cascaded control architecture to produce a more linear input-to-output relationship between actuator commands and superheat response at low values, while [10] proposes a backstepping method, however controlled superheat recovery is not considered.

The authors in [4] demonstrate that for a particular experimental system and MIMO linear quadratic Gaussian (LQG) controller, superheat regulation is possible between 9.5°C and 22°C of superheat, but becomes unstable below 8.5°C of superheat due to changing input-to-output gains. This use of a linear plant model and controller allows well known results on optimal control of linear systems to be employed, but linearization about a single operating condition limits the range over which the control is valid. The operational range of linear controllers is extended in [11] and [12], in which a control design that switches among multiple linear matrix inequality (LMI) representations of linear quadratic regulation (LQR) is applied to switch among cooling capacity set points by driving the system between operating conditions which include different phase flow conditions in both the condenser and the evaporator.

This paper builds upon previous efforts in switched linear control by presenting an alternative formulation that includes feedforward control, zero-offset tracking, and other beneficial features, by focusing specifically on the case of superheat recovery (which was not previously treated), by experimentally validating the proposed control approach, and by providing further results on the stability of the switched system.

3. System modeling

3.1 Nonlinear models

The nonlinear VCS models used in this work are taken from the Thermosys™ toolbox for MATLAB/Simulink® developed at the University of Illinois at Urbana-Champaign [1]. The heat exchangers of the system (the evaporator and condenser) are modeled using a lumped-parameter approach known as the moving boundary (MB) method [1]. In this method, the heat exchanger is partitioned into control volumes (CVs) that track each refrigerant phase flow zone (subcooled, two-phase, or superheated). Conservation equations of refrigerant energy, refrigerant mass, and tube wall energy are applied to each CV to develop a dynamic model of the heat exchanger. As given in [13], the nonlinear heat exchanger models take the descriptor form:

$$\begin{aligned} Z(x)\dot{x} &= f(x, u) \\ y &= g(x, u) \end{aligned} \tag{2}$$

where x , u , and y are the states, inputs, and outputs, respectively.

In a “switched” MB (SMB) model, the governing equations are derived separately for each possible combination of fluid phase zones in the heat exchanger, which are treated as modes of the

plant [14]. The model then switches between the set of equations for each mode in order to accurately capture the component behavior across multiple phase flow combinations. Fig. 2 shows the two evaporator modes used in this work. These consist of the “nominal” mode, for which both two-phase and superheated refrigerant flow are present in the evaporator, and the “off-nominal” mode, for which the refrigerant is two-phase for its entire flow through the evaporator. These “nominal” and “off-nominal” labels follow from the assumption in this work that the control objectives include maintaining the superheated flow at the exit of the evaporator exhibited by the nominal mode.

Fig. 2. Evaporator modes

The condenser is assumed here to operate at all times in the mode shown in Fig. 3, which includes superheated, two-phase, and subcooled flow phase zones. In this paper, refrigerant-to-air, cross-flow, tube-and-fin heat exchangers are used. The states, inputs, and outputs of the heat exchanger models are listed in Table 1 of Appendix A. When the evaporator enters the off-nominal mode, its states associated with the superheated zone are calculated with pseudo-state equations, as detailed in [15].

Fig. 3. Condenser mode

The expansion device and compressor of the VCS are modeled with static relationships using empirically derived maps and first principles equations. In this work, an electronic expansion valve (EEV) is used as the expansion device. The static expansion device and compressor models take the form:

$$y = h(u). \quad (3)$$

The inputs and outputs of the EEV and compressor are listed in Table 2 of Appendix A.

All but six of the various inputs and outputs listed in Table 1 and Table 2 for each component are provided as an output of either its upstream or downstream neighboring component. These six remaining signals, bolded in the tables, are:

1. Evaporator Air Mass Flow Rate
2. Condenser Air Mass Flow Rate
3. EEV Orifice Opening %
4. Compressor Rotational Speed
5. Evaporator Air Inlet Temperature
6. Condenser Air Inlet Temperature

The first four of these signals compose the actuator commands available to control, assumed to be continuously variable with some upper and lower constraints, with the air mass flow rates as empirically determined functions of the rotational speeds of the fans blowing air across the heat exchangers. The air inlet temperature to each heat exchanger will be treated as a disturbance signal in this work. A complete derivation of the nonlinear VCS models can be found in [15], and their validation with a 1kW experimental system can be found in [12] and [14].

3.2 Switched behavior with evaporator mode

As indicated in the previous section, an evaporator with a superheated phase at its outlet is defined as being in “nominal mode” and an evaporator without a superheated phase as being in “off-nominal mode,” as labeled in Fig. 2. It will next be shown that the need to classify these operating conditions as distinct modes can be understood from the differences in their dynamic behavior.

Fig. 5 shows results for simulations of a VCS with an evaporator in both the nominal and off-nominal modes. In both cases, the EEV orifice opening command undergoes a step input decrease followed by a step input increase, as shown in Fig. 4, while all other actuator commands to the system remain constant. As identified by the circled sections of Fig. 5, the responses of several evaporator-side outputs for each mode have opposite gains. This result suggests that control designs using the same feedback gains regardless of the mode of operation may not be tractable. This is exemplified by the benchmark decentralized controller demonstrated in Appendix B, where it is shown that a VCS controller that is effective under positive superheat operation can fail to perform as adequately when superheat is lost. These results motivate the need for a control design that switches among control laws in accordance with the current mode of operation.

Fig. 4. Orifice opening actuator commands to each mode

Fig. 5. Differences in evaporator response between modes

3.3 Model linearization

To take advantage of the well-established design infrastructure and stability analysis tools for linear models, the nonlinear VCS models for each of the four components shown in Fig. 1 are linearized via a first-order Taylor Series expansion. Following from the descriptor form of the nonlinear heat exchanger models given in Eq. (2), the linear heat exchanger models take the form:

$$\begin{aligned}\dot{x} &= A\Delta x + B\Delta u \\ \Delta y &= C\Delta x + D\Delta u\end{aligned}\tag{4}$$

where

$$\begin{aligned} A &= Z^{-1}(x_o) \frac{\partial f(x, u)}{\partial x} \Big|_{x_o, u_o}, \quad B = Z^{-1}(x_o) \frac{\partial f(x, u)}{\partial u} \Big|_{x_o, u_o}, \\ C &= \frac{\partial g(x, u)}{\partial x} \Big|_{x_o, u_o}, \quad D = \frac{\partial g(x, u)}{\partial u} \Big|_{x_o, u_o} \end{aligned} \quad (5)$$

and

$$\Delta x = x - x_o, \quad \Delta u = u - u_o, \quad \Delta y = y - y_o, \quad (6)$$

In the above equations, x_o , u_o , and y_o are the states, inputs, and outputs of an equilibrium point of the nonlinear model about which the linearization is taken, respectively. It is assumed that f and g in Eq. (5) are differentiable at the equilibrium point and that $Z(x_o)$ is invertible.

Following from the descriptor form of the static nonlinear expansion device and compressor models given in Eq. (3), the linear static component models take the form:

$$\Delta y = H \Delta u \quad (7)$$

where:

$$H = \frac{dh}{du} \Big|_{u_o} \quad (8)$$

and it is assumed that h is differentiable at the equilibrium point.

After the four VCS components are individually linearized, they are combined by feedback interconnection into a linear model for the entire system of Fig. 1, which takes the form:

$$\begin{aligned} \dot{x}_{sys} &= A_{sys} \Delta x_{sys} + B_{sys} \Delta u_{sys} \\ \Delta y_{sys} &= C \Delta x_{sys} + D_{sys} \Delta u_{sys} \end{aligned} \quad (9)$$

Two such linearization formulations are used in this work: One in which the evaporator is in its nominal mode, and one in which the evaporator is in its off-nominal mode. A more detailed description of the linearization procedure can be found in [6].

To verify the accuracy of the linearization, outputs of the nonlinear and linear models are compared across an identical input sequence consisting of both individual step inputs in each actuator and overlapping steps in multiple actuators. The latter is generated as a pseudorandom binary sequence (PRBS). Fig. 6 shows the input sequence used to validate the linear model with an evaporator in the nominal mode. The sequence includes a step in the four actuator commands of the system. Fig. 7 shows the resulting evaporator-side outputs of the nonlinear and linear models. As can be seen by comparison of these results to the experimental validation in [14], the differences between these two models falls within the error typical of experimental validation. The condenser-side outputs for this mode, as well as comparison results for the model with an evaporator in the off-nominal mode, can be found in [6].

Fig. 6. Inputs for linear and nonlinear model comparison

Fig. 7. Outputs for linear and nonlinear model comparison

4. Control architecture and formulation

4.1 Architecture overview

In this work, for simplicity of exposition it is assumed that the desired performance criteria (e.g. cooling capacity) are constant and that the heat exchanger air inlet temperature disturbances are small. Therefore, only two linear models are required, both of which are linearized about operating conditions near the true air inlet temperatures. The first model is of the evaporator in the nominal mode, operating with the desired performance criteria. The second model is of the evaporator in the off-nominal mode.

The procedure for the more general case, in which desired performance criteria and air inlet temperatures may vary greatly, is outlined as follows. First, additional formulations would be generated by discretizing across the expected range of disturbances (e.g. defining “cold,” “medium,” and “hot” air inlet temperatures), and across the possible performance demands on the system (e.g. defining a “low,” “medium,” and “high” desired cooling capacity). Optimal steady-state operating conditions under each combination of evaporator mode, disturbance level and performance demand can then be found as solutions to set point optimization problems [11]. After

developing an observer and controller for each formulation and its associated set points, the specific formulation to be used at any instant in time would be determined by a switching manager. This approach allows one to define a finite family of closed-loop models among which the system switches à priori.

Fig. 8 diagrams the architecture of the closed-loop system. The switching manager uses measurements from the plant and knowledge of the current performance demands to determine the controller and observer formulations to use. In addition to the selected formulation, the observer receives measurements from the plant in order to estimate the states of the system. These state estimates are input to the controller, which calculates the actuator commands.

Fig. 8. Closed-loop system architecture

4.2 Switching manager

For simplicity of exposition in this paper, performance demands are assumed to be constant and disturbances are assumed to be small. Under these assumptions, the only tasks of the switching manager are to determine whether the evaporator is operating in its nominal or off-nominal mode and to determine whether a formulation with or without zero-offset tracking is to be used.

To determine the current mode of operation, the switching manager calculates the evaporator superheat as defined in Eq. (1) from measurements of the evaporator refrigerant outlet temperature and pressure, as well as property lookup tables generated using the REFPROP database released by NIST [16]. A positive value of superheat indicates that the evaporator is in its nominal mode, while a superheat of zero indicates the off-nominal mode. To prevent erroneous switching due to measurement noise, hysteresis is introduced into the determination of the current mode as shown in Fig. 9, with T_{low} as the threshold to switch from the nominal to the off-nominal mode, and T_{high} as the threshold to switch from the off-nominal to the nominal mode.

Fig. 9. Hysteresis in switching manager

4.3 Observer

A linear Kalman filter is used as the observer. In accordance with the signal from the switching manager, the linear model for the current mode of operation is used by the Kalman filter at every

time step. Following the occurrence of a mode switch, one can treat the estimation that follows as a new process with initial conditions for the estimates given by the state and covariance estimates from just prior to mode switch. This design can exhibit switching dynamics in the estimates as they converge to values based on the current mode's model from the initial conditions based on the previous mode's model. It is left to future work to further study the effect of these switching dynamics in estimation on the closed-loop performance of the switched controller.

4.4 Error dynamics

To formulate the controller, it is important to first characterize the error dynamics of the switched system. It has already been discussed that the system is modeled in two modes, with an evaporator in both nominal and off-nominal phase flow conditions. In the equations that follow, the subscript 'a' denotes the off-nominal mode, and the subscript 'b' denotes the nominal mode. Whenever the plant is found to be operating in the off-nominal mode, the controller's primary objective is to drive the system from mode 'a' to mode 'b'. The subscript 'j' is used to designate a generic mode, which can be either 'a' or 'b' (i.e., $j \in \{a, b\}$). In the form of Eq. (9) (but excluding the 'sys' subscript for brevity of notation) the state equation of the system in mode 'j' can be written as:

$$\dot{x} = A_j \Delta x_j + B_j \Delta u_j \quad (10)$$

where

$$\Delta x = x - x_j, \quad \Delta u = u - u_j, \quad \Delta y = y - y_j. \quad (11)$$

Here, x_j , u_j , and y_j are states, inputs, and outputs at an equilibrium point of mode 'j' about which the linearization is taken for that mode, respectively.

Because the control objective involves regulation of the system to x_b , u_b , and y_b , it is beneficial to express the equations of all modes in terms of the errors from these values. Writing Eq. (10) in terms of $\Delta x_b = x - x_b$ and $\Delta u_b = u - u_b$ gives:

$$\dot{x} = A_j \Delta x_b + B_j \Delta u_b + A_j(x_b - x_j) + B_j(u_b - u_j). \quad (12)$$

Similarly, the output equation for mode 'j' can be written in terms of $\Delta y_b = y - y_b$, Δx_b , and Δu_b as:

$$\Delta y_b = C_j \Delta x_b + D_j \Delta u_b + C_j(x_b - x_j) + D_j(u_b - u_j) - y_b + y_j. \quad (13)$$

Eqs. (12) and (13) can be written compactly as:

$$\begin{aligned} \dot{x} &= A_j \Delta x_b + B_j \Delta u_b + V_j d_j \\ \Delta y_b &= C_j \Delta x_b + D_j \Delta u_b + W_j d_j \end{aligned} \quad (14)$$

where:

$$\begin{aligned}
V_j &= \begin{pmatrix} A_j & B_j & 0 \end{pmatrix} \\
W_j &= \begin{pmatrix} C_j & D_j & -I \end{pmatrix} . \\
d_j &= \begin{pmatrix} (x_b - x_j) \\ (u_b - u_j) \\ (y_b - y_j) \end{pmatrix}
\end{aligned} \tag{15}$$

From Eqs. (14)-(15), one can see that when $j = a$ (i.e., the system is in the off-nominal mode) the system equations in terms of the errors from the desired operating conditions have additive terms $V_a d_a$ and $W_a d_a$, where d_a is a non-zero constant associated with the difference in linearization point between the off-nominal and nominal modes. When $j = b$ (i.e., the system is in the nominal mode), one can see that $d_b = 0$, and so the system equations do not have these additive terms.

4.5 Controller formulation

The controller regulates the plant to the nominal values about which the linear model for the nominal mode is linearized, given by x_b , u_b , and y_b . To achieve the control objectives, a switched infinite-horizon LQR design is used. Because the observer is a Kalman filter, the overall architecture is therefore a switched LQG design. The use of a linear design for the controller of each mode allows one to take advantage of well-established design infrastructure and stability analysis tools in ensuring that each mode is stabilized and exhibits good closed-loop performance, however the general approach of this paper would be compatible with other feedback control design methodologies, including nonlinear approaches. Once stabilizing controllers for the individual modes have been developed, results on the stability of switched systems can be employed to ensure that the switched assembly of the family of closed-loop models is also stable, as will be demonstrated in Section 4.7.

The LQR design includes three features beyond those found in basic implementations, such as that derived in [17]. These are:

- 1) An output-weighted cost function
- 2) Weights against the rate of change of the outputs in the cost function
- 3) Feed-forward compensation

The cost function for each mode takes the form:

$$J_j = \int_{t_0}^{\infty} \left(\Delta y_b^T Q_j \Delta y_b + \Delta u_b^T R_j \Delta u_b + \dot{u}^T S_j \dot{u} \right) dt . \tag{16}$$

The first term of Eq. (16) weighs against the error in the outputs of the system from their desired values, driving the plant to the desired operating conditions. The second term weighs against the error in the actuator inputs from their nominal values. This helps to ensure that the system inputs remain close to those selected by set-point optimization to achieve high efficiency and maintain within actuator constraints. The third term weighs against the derivative of the actuator inputs, and can be used to ensure that actuator commands are smooth in time. This prevents against the

occurrence of noisy actuator commands if measurement noise is not completely rejected by the observer, which could contribute to component wear if unchecked. The third term also ensures that the controller will provide smooth actuator commands during switches despite any switching dynamics that occur in the plant and observer. In this work, it is assumed that the weighting matrices Q_j , R_j , and S_j are positive definite.

Regrouping terms in Eq. (16), the cost function can be written as:

$$J_j = \int_{t_0}^{\infty} \left(\begin{pmatrix} \Delta y_b^T & \Delta u_b^T \end{pmatrix} \begin{pmatrix} Q_j & 0 \\ 0 & R_j \end{pmatrix} \begin{pmatrix} \Delta y_b \\ \Delta u_b \end{pmatrix} + \dot{u}^T S_j \dot{u} \right) dt. \quad (17)$$

The next step of the control formulation is to augment the system equations of Eq. (14) such that the derivatives of the inputs are included in the state dynamics. The resulting equations are:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} &= \begin{pmatrix} A_j & B_j \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_b \\ \Delta u_b \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} \dot{u}_j + \begin{pmatrix} V_j \\ 0 \end{pmatrix} d_j \\ \begin{pmatrix} \Delta y_b \\ \Delta u_b \end{pmatrix} &= \begin{pmatrix} C_j & D_j \\ 0 & I \end{pmatrix} \begin{pmatrix} \Delta x_b \\ \Delta u_b \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \dot{u}_j + \begin{pmatrix} W_j \\ 0 \end{pmatrix} d_j \end{aligned} \quad (18)$$

Note that in Eq. (18), the inputs have been appended to the output equation in accordance with their grouping in Eq. (17).

Eqs. (17)-(18) can be written compactly as:

$$J_j = \int_{t_0}^{\infty} \left(\Delta \bar{y}_b^T \bar{Q}_j \Delta \bar{y}_b + \dot{u}^T S_j \dot{u} \right) dt \quad (19)$$

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}_j \Delta \bar{x}_b + \bar{B}_j \dot{u} + \bar{V}_j d_j \\ \Delta \bar{y}_b &= \bar{C}_j \Delta \bar{x}_b + \bar{D}_j \dot{u} + \bar{W}_j d_j \end{aligned} \quad (20)$$

where the definition of the barred variables ($\bar{\cdot}$) is as follows:

$$\begin{aligned} \bar{Q}_j &= \begin{pmatrix} Q_j & 0 \\ 0 & R_j \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \Delta \bar{x}_b = \begin{pmatrix} \Delta x_b \\ \Delta u_b \end{pmatrix}, \quad \Delta \bar{y}_b = \begin{pmatrix} \Delta y_b \\ \Delta u_b \end{pmatrix} \\ \bar{A} &= \begin{pmatrix} A_j & B_j \\ 0 & 0 \end{pmatrix}, \quad \bar{B}_j = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \bar{V}_j = \begin{pmatrix} V_j \\ 0 \end{pmatrix} \\ \bar{C}_j &= \begin{pmatrix} C_j & D_j \\ 0 & I \end{pmatrix}, \quad \bar{D}_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{W}_j = \begin{pmatrix} W_j \\ 0 \end{pmatrix} \end{aligned} \quad (21)$$

By following similar steps as given in [17] for deriving the standard LQR problem, the LQR problem posed by Eqs. (19)-(20) can be solved. While only the final result is presented here, a full derivation of the solution can be found in [6]. The derivative of the optimal actuator inputs for the j^{th} mode, $\dot{u}_j^*$, is given by:

$$\dot{u}_j^* = -\tilde{R}_j^{-1} \left(\bar{B}_j^T P_{1,j} + \tilde{N}_j^T \right) \Delta \bar{x}_b - \tilde{R}_j^{-1} \left(\bar{B}_j^T P_{2,j} + \tilde{M}_j \right) d_j \quad (22)$$

where

$$\begin{aligned} \tilde{Q}_j &= \bar{C}_j^T \bar{Q}_j \bar{C}_j, & \tilde{R}_j &= S_j + \bar{D}_j^T \bar{Q}_j \bar{D}_j \\ \tilde{M}_j &= \bar{D}_j^T \bar{Q}_j \bar{W}_j, & \tilde{N}_j &= \bar{C}_j^T \bar{Q}_j \bar{D}_j, & \tilde{H}_j &= \bar{C}_j^T \bar{Q}_j \bar{W}_j \end{aligned} \quad (23)$$

$P_{1,j}$ is the solution to the Algebraic Riccati Equation:

$$0 = P_{1,j} \bar{A}_j + \bar{A}_j^T P_{1,j} - \left(P_{1,j} \bar{B}_j + \tilde{N}_j \right) \tilde{R}_j^{-1} \left(\bar{B}_j^T P_{1,j} + \tilde{N}_j^T \right) + \tilde{Q}_j, \quad (24)$$

and $P_{2,j}$ is given by:

$$P_{2,j} = - \left(\bar{A}_j^T - \tilde{N}_j \tilde{R}_j^{-1} \bar{B}_j^T - P_{1,j} \bar{B}_j \tilde{R}_j^{-1} \bar{B}_j^T \right)^{-1} \left[P_{1,j} \left(\bar{V}_j - \bar{B}_j \tilde{R}_j^{-1} \tilde{M}_j \right) - \tilde{N}_j \tilde{R}_j^{-1} \tilde{M}_j + \tilde{H}_j \right]. \quad (25)$$

Eq. (22) can be seen as the addition of a static feedback gain on the augmented state and a static feedforward gain on d_j when written in the compact form:

$$\dot{u}_j^* = -k_{\bar{x},j} \Delta \bar{x}_b - k_{d,j} d_j \quad (26)$$

with

$$\begin{aligned} k_{\bar{x},j} &= \tilde{R}_j^{-1} \left(\bar{B}_j^T P_{1,j} + \tilde{N}_j^T \right) \\ k_{d,j} &= \tilde{R}_j^{-1} \left(\bar{B}_j^T P_{2,j} + \tilde{M}_j \right) \end{aligned} \quad (27)$$

In implementation, integration of Eq. (26) over time is performed to calculate the optimal actuator inputs to the VCS.

Note that d_j as defined in Eq. (15) is a constant for each j associated with the difference between the linearization point of the j^{th} mode of operation and that of nominal mode. As stated at the end of Section 4.4, d_a (active when operating in the off-nominal mode) is non-zero, while d_b (active when operating in the nominal mode) is zero. Therefore, there is no feedforward action in the control when the system operates in the nominal mode.

4.6 Special cases of controller formulation

The above LQR solution relates closely to formulations used frequently in the literature, as can be seen by applying simplifying assumptions to generate several special cases. While the special cases that follow are not implemented in this paper, they are included here to illustrate the relationships between the LQR design of this paper and results from the literature with which readers may be familiar.

First, in the case when $\bar{W}_j = 0$ and $\bar{V}_j = 0$, one can see that $P_{1,j}$ and $k_{\bar{x},j}$ are unchanged, but $k_{d,j} = 0$. This demonstrates that there should be no feedforward action when the $\bar{V}_j d_j$ and $\bar{W}_j d_j$ terms are absent from the system dynamics of Eq. (20). This special case is then seen to match the MATLAB function 'lqry', which computes an output weighted LQR state-feedback gain.

Second, in the case when $\bar{C}_j = I$, $\bar{D}_j = 0$, and $\bar{W}_j = 0$, the cost function of Eq. (19) reduces to:

$$J_j = \int_{t_0}^{\infty} (\Delta \bar{x}_b^T \bar{Q}_j \Delta \bar{x}_b + \dot{u}^T S_j \dot{u}) dt. \quad (28)$$

The Algebraic Riccati Equation of Eq. (24) then simplifies to:

$$0 = P_{1,j} \bar{A}_j + \bar{A}_j^T P_{1,j} - P_{1,j} \bar{B}_j S_j^{-1} \bar{B}_j^T P_{1,j} + \bar{Q}_j, \quad (29)$$

and $P_{2,j}$ is given by:

$$P_{2,j} = -(\bar{A}_j - P_{1,j} \bar{B}_j S_j^{-1} \bar{B}_j^T)^{-1} P_{1,j} \bar{V}_j. \quad (30)$$

The optimal control input derivatives are then given by Eq. (26) with

$$\begin{aligned} k_{\bar{x},j} &= S_j^{-1} \bar{B}_j^T P_{1,j} \\ k_{d,j} &= -S_j^{-1} \bar{B}_j^T (\bar{A}_j^T - P_{1,j} \bar{B}_j S_j^{-1} \bar{B}_j^T)^{-1} P_{1,j} \bar{V}_j. \end{aligned} \quad (31)$$

This special case corresponds to a LQR formulation with a state-based weighting and constant additive disturbances, which can also be found as a special case of the results in [18].

4.7 Closed-loop stability

This section presents an analysis of the closed-loop stability of the switched linear system. In particular, it shows that when subject to an average dwell-time switching signal as defined in [19], the switched system is input-to-state stable (ISS) with respect to d_j .

The first step in demonstrating this property is to describe the closed-loop model for each mode of the system. The $k_{\bar{x},j} \Delta \bar{x}_b$ term of Eq. (26) can be expanded as:

$$k_{\bar{x},j} \Delta \bar{x}_b = k_{x,j} \Delta x_b + k_{u,j} \Delta u_b. \quad (32)$$

Substituting the control law into the system dynamics of Eq. (18) gives the linear closed-loop model for each mode as:

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} A_j & B_j \\ -k_{x,j} & -k_{u,j} \end{pmatrix} \begin{pmatrix} \Delta x_b \\ \Delta u_b \end{pmatrix} + \begin{pmatrix} V_j \\ -k_{d,j} \end{pmatrix} d_j. \quad (33)$$

Eq. (33) can be written compactly (and explicitly defining time arguments and initial conditions) as:

$$\dot{\bar{x}}(t) = \widehat{A}_j \Delta \bar{x}_b(t) + \widehat{B}_j d_j, \quad \Delta \bar{x}_b(t_0) = \Delta \bar{x}_{b,0} \quad (34)$$

where

$$\widehat{A}_j = \begin{pmatrix} A_j & B_j \\ -k_{x,j} & -k_{u,j} \end{pmatrix}, \quad \widehat{B}_j = \begin{pmatrix} V_j \\ -k_{d,j} \end{pmatrix}. \quad (35)$$

From the above closed-loop dynamics of each mode, one can describe the closed-loop dynamics of the switched system as:

$$\dot{\bar{x}}(t) = \widehat{A}_{\sigma(t)} \Delta \bar{x}_b(t) + \widehat{B}_{\sigma(t)} d_{\sigma(t)} \quad (36)$$

where $\sigma : [t_0, \infty) \rightarrow \{a, b\}$ is a piecewise constant time-based switching signal identifying the active mode at every instant in time. In keeping with the assumptions of [19], it is required that there are no jumps in the state at the switching instant, that each $\widehat{A}_j$ is Hurwitz (i.e., that the controller for each mode asymptotically stabilizes the plant in that mode), and that there are a finite number of switches on every bounded time interval. Note that the requirement of a finite number of switches is ensured by the hysteresis in the switching manager.

It is well known that under arbitrary switching conditions, stability of all the individual modes does not guarantee stability of the switched system [20]. In the previous work on switched linear VCS control of [11] and [12], it is discussed that common and multiple Lyapunov functions, respectively, are generated as part of the solution to a LMI-based LQR design. However, the approach does not account for the effects of the $\widehat{B}_{\sigma(t)} d_{\sigma(t)}$ term on closed-loop stability.

To rigorously describe the alternative stability result provided in this paper, two definitions from the literature are first required:

Definition 4.1: As defined in [19], a switching signal, $\sigma(t)$, is said to have average dwell-time, τ_a , if there are numbers $\eta_o, \tau_a > 0$ such that:

$$\eta_\sigma(T, t) \leq \eta_o + \frac{T - t}{\tau_a} \quad (37)$$

$\forall T \geq t \geq 0$, where $\eta_\sigma(T, t)$ is the number of switches on the interval $[t, T)$.

Definition 4.2: As defined in [21], a system is ISS if there exists a function $\beta \in \text{class-}KL$ and a function $\gamma \in \text{class-}K$ such that for any initial state $x(t_0)$ and any bounded $d(t)$, $x(t)$ exists $\forall t \geq t_0$ and satisfies:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|d(\tau)\|\right). \quad (38)$$

With the above definitions established, Theorem 3.1 in [19] (restated below as Theorem 4.3) can be used to find the average dwell time, τ_a , such that the switched system of Eq. (36) retains an ISS property with respect to d_σ . This is achieved using the fact that under average dwell-time switching signals, ISS of each mode necessitates that the switched system is ISS.

Theorem 4.3: The following is proved in [19]. Consider the switched system:

$$\dot{x}(t) = f_{\sigma(t)}(x(t), d(t)) \quad (39)$$

with $x \in \mathbb{R}^n$, $d \in \mathbb{R}^l$, and $\sigma: [0, \infty) \rightarrow p \in P$, where P is an index set. Assume that each f_p is locally Lipschitz, that $f_p(0, 0) = 0$, that there are no jumps in the state at the switching instant, and that there are a finite number of switches on every bounded time interval. Furthermore, assume that there exist continuously differentiable functions $V_p: \mathbb{R}^n \rightarrow [0, \infty)$, $\alpha_1, \alpha_2, \alpha_3 \in \text{class-}K_\infty$ and numbers $\nu > 0$, $\mu \geq 1$ such that $\forall \xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^l$ and $\forall p, q \in P$, the following hold:

$$\begin{aligned} \alpha_1(\|\xi\|) &\leq V_p(\xi) \leq \alpha_2(\|\xi\|) \\ \frac{\partial V_p}{\partial \xi} f_p(\xi, \eta) &\leq -\nu V_p(\xi) + \alpha_3(\|\eta\|). \\ V_p(\xi) &\leq \mu V_q(\xi) \end{aligned} \quad (40)$$

Then, if $\sigma(t)$ is a switching signal with average dwell time $\tau_a > \frac{\log \mu}{\nu}$, the switched system of Eq. (39) is ISS.

The constructions required of the above theorem are now derived for the closed-loop switched system of Eq. (36). The first step is to show that the closed-loop dynamics of each individual mode are ISS. To demonstrate ISS of each mode, note that for each mode in the family of Eq. (34) there exists a k_j and λ_j such that:

$$\|\Delta \bar{x}(t)\| \leq k_j e^{-\lambda_j(t-t_0)} \|\Delta \bar{x}(t_0)\| + \frac{k_j}{\lambda_j} \|\bar{B}_j\| \|d_j\|, \quad (41)$$

similarly to the linear system example in [21]. Therefore, each mode of the family of Eq. (34) is clearly ISS with β_j and γ_j defined as:

$$\begin{aligned} \beta_j(\|\Delta \bar{x}(t_0)\|, t - t_0) &= k_j e^{-\lambda_j(t-t_0)} \|\Delta \bar{x}(t_0)\| \\ \gamma_j\left(\sup_{t_0 \leq \tau \leq t} \|d(\tau)\|\right) &= \frac{k_j}{\lambda_j} \|\bar{B}_j\| \|d_j\|. \end{aligned} \quad (42)$$

As noted in [19], every ISS system admits an ISS-Lyapunov function. Therefore $\forall j$ there exists $V_j: \mathbb{R}^n \rightarrow [0, \infty)$, $\alpha_{1,j}, \alpha_{2,j}, \alpha_{3,j} \in \text{class-}K_\infty$, and $\nu_j > 0$ such that $\forall \xi, \eta$:

$$\alpha_{1,j}(\|\xi\|) \leq V_j(\xi) \leq \alpha_{2,j}(\|\xi\|) \quad (43)$$

and

$$\dot{V}_j(\xi) \leq -\nu_j V_j(\xi) + \alpha_{3,j}(\|\eta\|). \quad (44)$$

The ISS-Lyapunov function for each mode in the family of Eq. (34) can be found as follows. Let L_j be the symmetric positive semidefinite solution to the algebraic Riccati equation:

$$\widehat{A}_j^T L_j + L_j \widehat{A}_j + 2I = 0. \quad (45)$$

Furthermore, let:

$$V_j(\xi) = \xi^T L_j \xi. \quad (46)$$

Then $\forall \xi$,

$$\delta_{\min}(L_j) \|\xi\|^2 \leq V_j(\xi) \leq \delta_{\max}(L_j) \|\xi\|^2 \quad (47)$$

where $\delta_{\min}(L_j)$ and $\delta_{\max}(L_j)$ represent the minimum and maximum eigenvalues of L_j , respectively. Then, Eq. (43) clearly holds with:

$$\begin{aligned} \alpha_{1,j}(\|\xi\|) &= \delta_{\min}(L_j) \|\xi\|^2 \\ \alpha_{2,j}(\|\xi\|) &= \delta_{\max}(L_j) \|\xi\|^2. \end{aligned} \quad (48)$$

The time derivative of Eq. (46) is given by:

$$\dot{V}_j(\xi) = \dot{\xi}^T L_j \xi + \xi^T L_j \dot{\xi}. \quad (49)$$

Substituting Eq. (45) and Eq. (34) with $\Delta \bar{x}_b \equiv \xi$, $d_j \equiv \eta$ into Eq. (49) and rearranging gives:

$$\dot{V}_j(\xi) = -2\xi^T \xi + \eta^T \widehat{B}_j^T L_j \xi + \xi^T L_j \widehat{B}_j \eta. \quad (50)$$

By square completion,

$$\dot{V}_j(\xi) = -\xi^T \xi - (\xi - L_j \widehat{B}_j \eta)^T (\xi - L_j \widehat{B}_j \eta) + \eta^T \widehat{B}_j^T L_j L_j \widehat{B}_j \eta. \quad (51)$$

The second term of Eq. (51) can be removed by taking an upper bound on the right hand side as:

$$\dot{V}_j(\xi) \leq -\xi^T \xi + \eta^T \widehat{B}_j^T L_j L_j \widehat{B}_j \eta. \quad (52)$$

Upper bounding the last term of Eq. (52) by the largest eigenvalue of $(\widehat{B}_j^T L_j L_j \widehat{B}_j)$, one finds that:

$$\dot{V}_j(\xi) \leq -\xi^T \xi + \delta_{\max}(\widehat{B}_j^T L_j L_j \widehat{B}_j) \|\eta\|^2. \quad (53)$$

Also, there exists a $\nu_j > 0$ such that the first term in Eq. (53) can be upper bounded as:

$$-\xi^T \xi \leq -\nu_j \xi^T L_j \xi = -\nu_j V_j(\xi). \quad (54)$$

Then Eq. (44) clearly holds with ν_j as defined above and:

$$\alpha_{3,j}(\|\eta\|) = \delta_{\max}(\hat{B}_j^T L_j L_j \hat{B}_j) \|\eta\|^2. \quad (55)$$

Having defined the ISS-Lyapunov function for each individual mode, the conditions in Eq. (40) required for Theorem 4.3 can now be ensured for the switched system at hand. From $\alpha_{1,j}$, $\alpha_{2,j}$, $\alpha_{3,j}$, and $\nu_j \forall j$, one can construct α_1 , α_2 , α_3 , and ν such that $\forall j$ and $\forall \xi, \eta$:

$$\alpha_1(\|\xi\|) \leq V_j(\xi) \leq \alpha_2(\|\xi\|) \quad (56)$$

and

$$\dot{V}_j(\xi) \leq -\nu V_j(\xi) + \alpha_3(\|\eta\|) \quad (57)$$

by taking:

$$\begin{aligned} \nu &= \min_j \nu_j \\ \alpha_1(\|\xi\|) &= \min_j (\delta_{\min}(L_j)) \|\xi\|^2 \\ \alpha_2(\|\xi\|) &= \max_j (\delta_{\max}(L_j)) \|\xi\|^2 \\ \alpha_3(\|\eta\|) &= \max_j (\delta_{\max}(\hat{B}_j^T L_j L_j \hat{B}_j)) \|\eta\|^2 \end{aligned} \quad (58)$$

The only remaining condition of Eq. (40) to be established for the switched system of Eq. (36) is that $\forall \xi$ and $\forall i, j$, there exists a $\mu \geq 1$ such that:

$$V_j(\xi) \leq \mu V_i(\xi). \quad (59)$$

The above condition is satisfied by:

$$\mu \geq \frac{\max_j (\delta_{\max}(L_j))}{\min_i (\delta_{\min}(L_i))}. \quad (60)$$

To prove this, first note that Eq (60) implies that:

$$\mu \min_i (\delta_{\min}(L_i)) \geq \max_j (\delta_{\max}(L_j)), \quad (61)$$

and that the bounds from Eq. (47) state that $\forall \xi$ and $\forall i, j$:

$$\begin{aligned} \max_j \left(\delta_{\max}(L_j) \right) \|\xi\|^2 &\geq V_j(\xi) \\ \min_i \left(\delta_{\min}(L_i) \right) \|\xi\|^2 &\leq V_i(\xi) \end{aligned} \quad (62)$$

Applying the bounds in Eq. (62) to Eq. (61), one finds that $\forall \xi$ and $\forall i, j$:

$$\mu V_i(\xi) \geq V_j(\xi). \quad (63)$$

Having satisfied all conditions of Eq. (40) for the switched system of Eq. (36), Theorem 4.3 can be applied. Using ν from Eq. (58) and selecting μ such that equality holds in Eq. (60), the average dwell time condition for ISS is given as:

$$\tau_a > \frac{\log \left(\frac{\max_j \left(\delta_{\max}(L_j) \right)}{\min_i \left(\delta_{\min}(L_i) \right)} \right)}{\nu}. \quad (64)$$

If the above average dwell time is achieved, then the switched system of Eq. (36) is ISS with respect to d_σ . Therefore, the norm of the states of the switched system is at all times bounded, and

as time approaches infinity is bounded by a class- K function $\gamma \left(\sup_{t_0 \leq \tau \leq t} \|d_{\sigma(\tau)}\| \right)$. Furthermore, if $d_{\sigma(\tau)} \rightarrow 0$ as $t \rightarrow \infty$, which occurs if the VCS is regulated to the nominal mode (since $d_b = 0$), then the switched system is asymptotically stable with respect to the origin [19].

4.8 Zero-offset tracking

In applications where exact tracking of select output references is critical, zero-offset tracking of a subset of the outputs can be easily added to the proposed control framework. To avoid wind-up of the integrator dynamics associated with zero-offset tracking that can occur during large transients, the switching manager is constructed so as to always use a formulation without zero-offset tracking at the start of a superheat recovery process. Once transients have sufficiently settled, a control formulation with zero-offset tracking can be introduced to bring the system to exactly achieve the desired values of these critical outputs.

To derive the cost function and augmented system equations for the case when zero-offset tracking is to be used, a vector of signals, $\dot{z}$, is first defined as consisting of the subset of the signals in Δy_b for which achievement of offset-free tracking is desired under the effects of model error and disturbances. It is assumed that the outputs comprising $\dot{z}$ are selected such that offset-free tracking of all outputs in $\dot{z}$ can be achieved simultaneously without actuator saturation. The model for $\dot{z}$ is given by:

$$\dot{z} = -C_{z,j} \Delta x_b - D_{z,j} \Delta u_b - W_{z,j} d_j \quad (65)$$

where $C_{z,j}$, $D_{z,j}$, and $W_{z,j}$ contain the rows of C_j , D_j , and W_j corresponding to each element of Δy_b that is in $\dot{z}$. The cost function for each mode then takes the form:

$$J_j = \int_{t_0}^{\infty} (\Delta y_b^T Q_j \Delta y_b + z^T T_j z + \Delta u_b^T R_j \Delta u_b + \dot{u}^T S_j \dot{u}) dt. \quad (66)$$

All but the second term of Eq. (66) are identical to terms in Eq. (16). The second term ensures that the outputs in $\dot{z}$ are tracked with zero offset at steady-state. As with the other weighting matrices, it is assumed that T_j is positive definite. Regrouping terms in Eq. (66), the cost function can be written as:

$$J_j = \int_{t_0}^{\infty} \left(\begin{pmatrix} \Delta y_b^T & z^T & \Delta u_b^T \end{pmatrix} \begin{pmatrix} Q_j & 0 & 0 \\ 0 & T_j & 0 \\ 0 & 0 & R_j \end{pmatrix} \begin{pmatrix} \Delta y_b \\ z \\ \Delta u_b \end{pmatrix} + \dot{u}^T S_j \dot{u} \right) dt. \quad (67)$$

Augmenting the system equations of Eq. (14) to include the derivatives of the inputs and the offset free tracking signals yields:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \\ \dot{u} \end{pmatrix} &= \begin{pmatrix} A_j & 0 & B_j \\ -C_{z,j} & 0 & -D_{z,j} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_b \\ z \\ \Delta u_b \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \dot{u}_j + \begin{pmatrix} V_j \\ -W_{j,z} \\ 0 \end{pmatrix} d_j \\ \begin{pmatrix} \Delta y_b \\ z \\ \Delta u_b \end{pmatrix} &= \begin{pmatrix} C_j & 0 & D_j \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \Delta x_b \\ z \\ \Delta u_b \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \dot{u}_j + \begin{pmatrix} W_j \\ 0 \\ 0 \end{pmatrix} d_j \end{aligned} \quad (68)$$

In Eq. (68), both the inputs and the zero-offset outputs have been appended to the output equation in accordance with their grouping in Eq. (67).

Similarly to the formulation without zero-offset tracking, Eq. (67)-(68) can be written in the compact form of Eq.(19)-(20) by a different definition of the barred variables and matrices. The derivative of the optimal actuator inputs for the j^{th} mode are again given by Eq. (22)-(25). In implementation, the z component of $\Delta \bar{x}_b$ to be used in Eq. (26) when computing the optimal control is calculated by integration of:

$$\dot{z} = y_{z,meas} - y_{z,des,j} \quad (69)$$

where $y_{z,meas}$ are measurements of the subset of outputs to be tracked with zero-offset and $y_{z,des,j}$ are the corresponding reference values to be tracked.

5. Experimental Validation and Comparison to Simulation

The proposed switched LQR control approach has been previously demonstrated on a simulated plant in [7]. In this section, its experimental application is presented. Experiments are conducted on the VCS test stand described in [1]. This stand has been used extensively for model

and control validation in other work, including in [14], [22], and [23]. In this work, the stand is configured as the 4-component system of Fig. 1, with an EEV as the expansion device.

To demonstrate the similarity in control performance between simulated and experimental results, simulation results with the VCS models of the Thermosys™ toolbox [1] using identical observer and controller tuning parameters and similar air inlet temperatures are also provided. To simulate experimental conditions, white Gaussian noise is added to the measurement signals provided to the observer in simulations. This noise is of similar standard deviation as that which occurs in the experimental system, at $\sqrt{3}$ kPa on pressure signals and 0.1 °C on temperature signals.

The design process for the proposed control approach is summarized as follows:

- 1) Generate equilibrium points for each mode by simulating the nonlinear models of each mode at steady-state. For the nominal mode, these steady-state conditions should represent the desired operating conditions to be tracked by the controller.
- 2) Run the linearization routine to generate linear models in the form of Eq. (9) about the equilibrium point of each mode.
- 3) Select weighting parameters for the cost function and solve the for the optimal controller gains of Eq. (26) for each mode.
- 4) Augment the system model and cost function of the nominal mode with any outputs for which zero-offset tracking is desired, as shown in Eq. (67)-(68). With these augmented equations, again solve the for the optimal controller gains of Eq. (26) for the nominal mode.

In both the simulations and experimental results that follow, the following plant measurements are available to the observer and switching manager, sampled at 10 Hz:

- Evaporator Pressure
- Evaporator Air Inlet and Outlet Temperature
- Evaporator Refrigerant Outlet Temperature
- Condenser Pressure
- Condenser Air Inlet and Outlet Temperature
- Condenser Refrigerant Outlet Temperature

Zero-offset tracking is implemented on the output of evaporator cooling capacity. Therefore, the zero-offset tracking dynamics are calculated by:

$$\dot{z} = \dot{Q}_{evap,meas} - \dot{Q}_{evap,o} \quad (70)$$

where $\dot{Q}_{evap,o}$ is the nominal cooling capacity to be tracked and $\dot{Q}_{evap,meas}$ is the measured cooling capacity, calculated from plant measurements and an empirical map between fan speed and mass flow rate as:

$$\dot{Q}_{evap,meas} = \dot{m}_{evap,air} c_p (T_{evap,air,out} - T_{evap,air,in}). \quad (71)$$

The specific heat of air, c_p , is approximated as $1.005 \frac{kJ}{kg \cdot K}$. Note that to prevent wind-up of the offset-free tracking integrator dynamics, a controller formulation without zero-offset tracking is used until transients associated with the superheat recovery process have largely settled.

To prevent erroneous switching due to measurement noise, T_{low} and T_{high} of Fig. 9 are set to 0.5 °C and 1.5 °C, respectively. Using the stability analysis of Section 4.7, the closed-loop switched linear system is found to be ISS under an average dwell-time switching signal as defined in Eq. (37) with $\tau_a > 14.6s$. Because superheat loss would generally be expected to occur on a relatively infrequent basis (on the order of minutes or hours), this constraint on the switching to ensure stability is easily achievable.

Fig. 11 shows simulation results under the proposed control approach, while Fig. 12 shows experimental results. Fig. 13 and Fig. 14 provide a “zoomed in” view of the first 250 seconds of Fig. 11 and Fig. 12, respectively. Fig. 14 also shows the result when a non-switched controller using only the formulation for the nominal mode is experimentally applied. The times at which closed-loop control begins, the mode switch in the observer and controller is triggered (for the switched case only), and zero-offset tracking of evaporator cooling capacity is turned on are indicated by dashed vertical lines in accordance with the legend of Fig. 10.

Fig. 10. Legend for simulated and experimental results

Fig. 11. Simulated results (Left: Control inputs, Right: Plant outputs)

Fig. 12. Experimental results (Left: Control inputs, Right: Plant outputs)

Fig. 13. First 250s of simulated results (Left: Control inputs, Right: Plant outputs)

Fig. 14. First 250s of experimental results (Left: Control inputs, Right: Plant outputs)

As can be seen by examination of Fig. 11 and Fig. 12, in both the simulation and experimental results, the switched controller first brings the system to recover superheat and then regulates the system outputs to near the nominal conditions. Following the onset of zero-offset tracking of evaporator cooling capacity, this output is seen to approach and eventually track its nominal value exactly, modulo the presence of noise in the experimental system.

The actuator commands leading up to and following the recovery of superheat are most easily studied in the left subplots of Fig. 13 and Fig. 14. One can see that following the onset of closed-loop control the actuator commands remain smooth in time. This comes as a result of both the rejection of measurement noise from state estimates by the observer and the weighting against large derivatives of inputs in the cost function of the controller. Furthermore, actuator saturation is avoided by the weighting against the deviation of the inputs from their nominal values in the cost function.

The most significant difference in performance between the switched and non-switched approach is the time to superheat recovery under each. As circled in Fig. 14, the switched controller recovers superheat in almost half the time of the non-switched controller. This occurs because the switched controller uses a model that better captures the dynamics of the system under loss of superheat (the off-nominal mode), and therefore is able to select actuator commands that better promote the control objective of superheat recovery.

Superheat recovery under the switched approach occurs within approximately 10s of the start of closed-loop control in the simulation, while taking several times longer in the experimental

results. Similarly, while the general trends in the transient behavior are similar, the settling times of the simulated plant are faster than those of the experimental system. These differences can be attributed to model mismatch, such as coupling effects in air temperatures due to air circulation in the experimental facility that are not captured by the model.

Modulo the differences in timescale, there is similitude between the simulation and experimental dynamics. This is exemplified by Fig. 15, which overlays simulation and experimental results of two signals under the switched control approach with time normalized by the time interval between the beginning of closed-loop control and the switch from the nominal to the off-nominal mode in each case.

Fig. 15. Similitude between simulation and experiment under switched control approach

6. Conclusions

This paper demonstrates the ability to apply a switched linear control formulation to a nonlinear VCS by switching among controllers matching distinct modes in the plant's operational range. To the authors' knowledge, this is the first control design demonstrating the ability to recover the superheated phase of an evaporator in the event of its loss. This provides a key capability to applications in which extended durations of compressor liquid ingestion are undesirable, but preventing against liquid ingestion by adding an accumulator to the system is not preferred due to the weight and/or size of this additional hardware. Such applications may include next generation aircraft, for which performance and power density requirements on the thermal management systems have resulted in recent investigations into replacing traditional air-cycle systems with VCSs [2].

Success of the proposed switched control approach for both rapid superheat recovery and regulation about nominal operating conditions is demonstrated both in simulation and in experimental implementation. Supporting these results, stability analysis is presented to demonstrate that the closed-loop linear switched system is input-to-state stable with respect to a piecewise constant term associated with the difference between the linearization point of each mode of operation and the linearization point of the nominal mode.

Future work will include extended stability results, such as investigation of robustness to model error, estimation error, and delay between the time of mode switches in the plant and the time at which these switches are detected by the switching manager.

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Appendix A: Component model inputs and outputs

Table 1: Heat exchanger states, inputs, and outputs

	Evaporator	Condenser
States	1. Refrigerant Pressure 2. Two-Phase Zone Normalized Length 3. Two-Phase Zone Wall Temperature 4. Two-Phase Zone Mean Void Fraction 5. Superheated Zone Wall Temperature 6. Superheated Zone Enthalpy	1. Refrigerant Pressure 2. Superheated Zone Normalized Length 3. Superheated Zone Wall Temperature 4. Superheated Zone Enthalpy 5. Two-Phase Zone Normalized Length 6. Two-Phase Zone Wall Temperature 7. Two-Phase Zone Mean Void Fraction 8. Subcooled Zone Wall Temperature 9. Subcooled Zone Enthalpy
Inputs	1. Air Inlet Temperature 2. Air Mass Flow Rate 3. Refrigerant Inlet Enthalpy 4. Refrigerant Inlet Mass Flow Rate 5. Refrigerant Outlet Mass Flow Rate	1. Air Inlet Temperature 2. Air Mass Flow Rate 3. Refrigerant Inlet Enthalpy 4. Refrigerant Inlet Mass Flow Rate 5. Refrigerant Outlet Mass Flow Rate
Outputs	1. Air Outlet Temperature 2. Air Cooling Capacity 3. Refrigerant Pressure 4. Refrigerant Outlet Enthalpy 5. Refrigerant Outlet Temperature 6. Refrigerant Superheat	1. Air Outlet Temperature 2. Air Heating Capacity 3. Refrigerant Pressure 4. Refrigerant Outlet Enthalpy 5. Refrigerant Outlet Temperature 6. Refrigerant Subcool

Table 2: EEV and compressor inputs and outputs

	EEV	Compressor
Inputs	1. Orifice Opening % 2. Refrigerant Inlet Enthalpy 3. Refrigerant Inlet Pressure 4. Refrigerant Outlet Enthalpy	1. Rotational Speed 2. Refrigerant Inlet Enthalpy 3. Refrigerant Inlet Pressure 4. Refrigerant Outlet Enthalpy
Outputs	1. Refrigerant Mass Flow Rate 2. Refrigerant Outlet Enthalpy	1. Refrigerant Mass Flow Rate 2. Refrigerant Outlet Enthalpy

Appendix B: Decentralized PI control approach

To demonstrate that feedback controllers which are effective under positive superheat operation can fail to perform as adequately when superheat is lost, this section presents experimental results for a decentralized SISO control approach from the literature as a benchmark. These experiments are conducted on the VCS test stand described in [1]. This stand has been used extensively for model and control validation in other work, including in [14], [22], and [23]. The stand is configured as the 4-component system of Fig. 1, with an EEV as the expansion device.

Industry applications of VCSs have typically favored the use of multiple decentralized SISO control loops over the use of a single MIMO control loop [8]. Therefore, the former approach is an appropriate benchmark with which to compare the approach proposed in this paper. The decentralized SISO design used here is identical to that in [12], wherein four decentralized proportional-integral (PI) controllers govern the four actuator inputs. As shown in Fig. 16, superheat is used as the feedback signal for the expansion valve orifice opening, the evaporator air outlet temperature is used as the feedback signal for the evaporator air mass flow rate, the condenser pressure is used as the feedback signal for the condenser air mass flow rate, and the difference between the condenser and evaporator pressures is used as the feedback signal for the compressor rotational speed.

Fig. 16. Legend for simulated and experimental results

Each of the four controllers takes the form:

$$K_i(s) = \left(k_{i,1} + \frac{1}{s} k_{i,2} \right) e_i, \quad i \in \{1, 2, 3, 4\} \quad (72)$$

$$u_i(s) = K_i(s) e_i + u_{i,o}$$

where e_i is the error between the reference value and the measured value, u_i is the actuator input, and $u_{i,o}$ is the nominal actuator input. The reference values are taken from steady-state operating conditions of the experimental system when operating with the nominal inputs. The controller gains, references, and nominal actuator inputs are given in Table 3.

Table 3: PI parameters

Controller Gains	References	Nominal Actuator Inputs
$k_{1,1} = -1$ $k_{1,2} = -0.02$	18 °C	40 %
$k_{2,1} = 0.001$ $k_{2,2} = 0.0009$	10.6 °C	0.12 kg/s
$k_{3,1} = 3$ $k_{3,2} = 0.4$	617 kPa	1400 RPM
$k_{4,1} = -0.0005$ $k_{4,2} = -0.0001$	854 kPa	0.29 kg/s

Fig. 17 demonstrates the efficacy of the decentralized PI control approach for regulating to the reference outputs from initial conditions which are relatively close to the references. The time at which closed-loop control begins is indicated with a dashed vertical line. As circled in the right subplot of Fig. 17, operating conditions when closed-loop control begins include approximately 8 °C of superheat in the evaporator, significantly away from the 0 °C that would indicate loss of superheat. The controller is seen to regulate the plant outputs to their references with sufficiently damped oscillations so as to settle within several minutes.

Fig. 17. PI control with positive initial superheat (Left: Control inputs, Right: Plant outputs)

Despite the ability of the decentralized PI approach to exhibit good performance in regulating the system from positive initial values of superheat to the references, the same cannot be said of its ability to regulate the system from an initial value of zero superheat to the references, as shown in Fig. 18. As circled in the right subplot of Fig. 18, in this case the operating conditions when closed-loop control begins include 0 °C of superheat in the evaporator. The resulting system response is highly oscillatory in both system outputs and actuator commands. Therefore, this benchmark case demonstrates the extent to which VCS controllers that are effective under positive superheat operation can fail to perform as adequately when superheat is lost.

Fig. 18. PI control with zero initial superheat (Left: Control inputs, Right: Plant outputs)

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