A piecewise partitioning Scaled Boundary Finite Element algorithm to solve viscoelastic problems with cyclic symmetry

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Abstract

Scaled Boundary Finite Element Method (SBFEM) and a temporally adaptive algorithm are combined to solve viscoelastic problems. By expanding variables at a discretized time interval, a spatially and temporally coupled viscoelastic problem is decoupled into a series of recursive spatial problems, which are solved by SBFEM, the computing accuracy in the time domain is controlled via a self-adaptive process. For the cyclic symmetric structures, the cyclic symmetry is exploited to reduce the computational expense of SBFEM, both the eigenvalue and system equations of SBFEM are partitioned into a number of smaller independent problems, which are solved by a partitioning algorithm. Two numerical examples are given to verify and illustrate the proposed approach.

Key Words: Viscoelasticity, Cyclic symmetry, SBFEM, Computational expense.

1 Introduction

The numerical analysis for viscoelastic materials/structures is an important issue in many aspects of engineering [1], particularly because the analytical solution is limited due to the time dependent constitutive relationship, the complex geometry and boundary conditions. A numerical approach of viscoelastic analysis mainly consists of two parts. One is related to spatial discretized algorithm such as FEM [2] and [3], EFG [4] and [5], BEM [6], etc, and another is associated with the discretized algorithm in the time domain, such as various kinds of FDM [7] and [8].

The Scaled Boundary Finite Element Method (SBFEM) [9] and [10] is a spatial discretized algorithm that is semi-analytical, and exhibits good performance in dealing

with problems of stress singularities and unbounded domains [11], [12], [13], [14], [15], [16], [17] and [18]

On the other hand, there are numbers of discretized algorithms in the time domain employed to solve viscoelastic problems, among which a temporally-piecewise algorithm [19], [20] and [21] is one that is not only easy to combine with spatially discretized algorithms, but also able to adapt the change of the size of time step via a self-adaptive process.

This paper attempts to combine advantages of SBFEM and the temporally-piecewise adaptive algorithm, and proposes a new approach to solve viscoelastic problems.

It is noteworthy that SBFEM is more computationally expensive than FEM in an elastic analysis mainly because an eigenvalue problem needs to be solved in generating system equations [22]. If SBFEM is employed at every discretized time interval in the numerical analysis for the viscoelastic problem [23], the computational cost will be more expensive. Therefore, a piecewise partitioning algorithm [22] is proposed for a special kind of structure, i.e. structure with cyclic symmetry that can be observed in many engineering equipment and structures, such as aircraft rotor, turbine blades, satellite antenna dishes, and some large space structures etc [5] and [22]. A proof that the recursive eigenvalue matrices of SBFEM equations are block-circulant is presented, and on the basis of this proof, both the recursive eigenvalue and system equations of SBFEM are partitioned into a number of smaller independent problems, which are solved by a partitioning algorithm.

The benefits gained using the SBFEM based piecewise partitioning algorithm are two folds. (1) In the spacial domain, the implementation of SBFEM may further improve computing accuracy, particularly for the problems of stress singularities and unbounded boundary, the computational cost will be reduced by solving a series of independent subproblems instead of the whole SBFEM eigenvalue and system equations. (2) In the time domain, variables can be described more precisely, and computing accuracy can be controlled via an adaptive process.

The paper commences with the recursive governing equations for the viscoelastic problem in the section 2; section 3 derives recursive SBFEM equations; section 4 sets up recursive partitioning SBFEM equations; section 5 provides a numerical verification via two examples, and section 6 gives conclusions.

2 Recursive governing equations for viscoelastic problems

The governing equations of viscoelasticity can be described by [24]

$$[H]\{\sigma\} + \{b\} = \{0\} \quad (1)$$

$$\{\varepsilon\} = [H]^T \{u\} \quad (2)$$

where

$$[H] = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \quad (3)$$

The boundary conditions are specified by [24]

$$\{u\} = \{\tilde{u}\} \quad \text{on } \Gamma_u \quad (4)$$

$$\{p\} = \{\tilde{p}\} \quad \text{on } \Gamma_\sigma \quad (5)$$

where $\{\sigma\}$ and $\{\varepsilon\}$ denote the vector of stress and strain, respectively, $\{b\}$ is the vector of body force, $\{u\}$ is the vector of displacement, $\{p\}$ denotes the vector of traction, $\{\tilde{u}\}, \{\tilde{p}\}$ are the prescribed values of $\{u\}$ and $\{p\}$ on the boundary, respectively.

$\Gamma = \Gamma_u + \Gamma_\sigma$ represents the boundary of Ω , and the subscript u and σ refer to stress and displacement, respectively.

We divide time domain into a number of intervals where the initial points and the sizes of time intervals are defined by $t_0, t_1, t_2, \dots, t_k, \dots$ and $\Delta t_0, \Delta t_1, \Delta t_2, \dots, \Delta t_k, \dots$, respectively. At a discretized time interval, in order to describe the variation of variables more precisely, all variables are expanded in the term of s

$$\{\sigma\} = \sum_{m=0} \{\sigma_m\} s^m \quad (6)$$

$$\{\varepsilon\} = \sum_{m=0} \{\varepsilon_m\} s^m \quad (7)$$

$$\{b\} = \sum_{m=0} \{b_m\} s^m \quad (8)$$

$$\{u\} = \sum_{m=0} \{u_m\} s^m \quad (9)$$

$$\{\tilde{u}\} = \sum_{m=0} \{\tilde{u}_m\} s^m \quad (10)$$

$$\{p\} = \sum_{m=0} \{p_m\} s^m \quad (11)$$

$$\{\tilde{p}\} = \sum_{m=0} \{\tilde{p}_m\} s^m \quad (12)$$

$$s = \frac{t - t_{k-1}}{\Delta t_k} \quad (13)$$

where t_{k-1} and Δt_k represent the initial point and size of the k -th time interval, respectively. $\{\sigma_m\}$ and $\{\varepsilon_m\}$ denote the expanding coefficients of $\{\sigma\}$ and $\{\varepsilon\}$, respectively, $\{b_m\}$, $\{u_m\}$, $\{p_m\}$, $\{\tilde{u}_m\}$ and $\{\tilde{p}_m\}$ denote the expanding coefficients of $\{b\}$, $\{u\}$, $\{p\}$, $\{\tilde{u}\}$ and $\{\tilde{p}\}$, respectively.

In this paper, we assume $\Delta t_k = \Delta t$, $k = 0, 1, 2, \dots$

The conversion relationship of differential from t to s is

$$\frac{d}{dt} = \frac{1}{\Delta t} \frac{d}{ds}; \quad \frac{d^2}{dt^2} = \frac{1}{\Delta t^2} \frac{d^2}{ds^2} \quad (14)$$

Substituting Eqs. (6–12) into Eqs. (1–5) then yields

$$[H]\{\sigma_m\} + \{b_m\} = \{0\} \quad (15)$$

$$\{\varepsilon_m\} = [H]^T \{u_m\} \quad (16)$$

$$\{u_m\} = \{\tilde{u}_m\} \quad \text{on } \Gamma_u \quad (17)$$

$$\{p_m\} = \{\tilde{p}_m\} \quad \text{on } \Gamma_\sigma \quad (18)$$

For the three-parameter solid viscoelastic model (Fig. 1) [25], the constitutive relationship can be written in a differential form

$$q_0 \{\varepsilon(t)\} + q_1 \frac{d\{\varepsilon(t)\}}{dt} = [G] \left(\{\sigma(t)\} + p_1 \frac{d\{\sigma(t)\}}{dt} \right) \quad (t > 0) \quad (19)$$

$$\{\varepsilon(t)\} = \frac{1}{E_2} [G] \{\sigma(t)\} \quad (t = 0) \quad (20)$$

where

$$p_1 = \frac{\eta_1}{E_1 + E_2}; \quad q_0 = \frac{E_1 E_2}{E_1 + E_2}; \quad q_1 = \frac{E_2 \eta_1}{E_1 + E_2} \quad (21)$$

$$[G] = \begin{bmatrix} G_{11} & G_{12} & 0 \\ & G_{22} & 0 \\ \text{symm} & & G_{33} \end{bmatrix} \quad (22)$$

For the plane stress problem

$$G_{11} = G_{22} = 1; \quad G_{33} = 2(1 + \nu); \quad G_{12} = -\nu \quad (23)$$

For the plane strain problem

$$G_{11} = G_{22} = 1 - \nu^2; \quad G_{33} = 2(1 + \nu); \quad G_{12} = -\nu(1 + \nu) \quad (24)$$

where ν , E_1 , E_2 and η_1 represent constitutive parameters.

Substituting Eqs. (6), (7) and Eq. (14) into Eq. (19) and equating the power of the two sides of the equation then gives

$$\{\sigma_m\} = [D]\{\varepsilon_m\} + \{C_m\}, \quad m = 1, 2, 3, \dots \quad (25)$$

where

$$\{C_m\} = \frac{\Delta t}{m} \left(\frac{1}{\eta_1} [D]\{\varepsilon_{m-1}\} - \frac{1}{p_1} \{\sigma_{m-1}\} \right) \quad (26)$$

$$[D] = \begin{pmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ \text{symm} & & D_{33} \end{pmatrix} \quad (27)$$

For the plane stress problem

$$D_{11} = s_1 + \frac{\nu}{s_2}; \quad D_{22} = \frac{1}{s_2}; \quad D_{33} = \frac{1}{s_3}; \quad D_{12} = \frac{\nu}{s_2} \quad (28)$$

where

$$s_1 = \frac{q_1}{p_1}; \quad s_2 = (1 - \nu^2) \frac{p_1}{q_1}; \quad s_3 = 2(1 + \nu) \frac{p_1}{q_1} \quad (29)$$

For the plane strain problem; E_1 , E_2 and ν need to be replaced by $\frac{E_1}{1 - \nu^2}$,

$\frac{E_2}{1 - \nu^2}$ and $\frac{\nu}{1 - \nu^2}$, respectively.

At the first time interval.

$$\{\sigma_0\} = [D]\{\varepsilon_0\} \quad (30)$$

At the $(k + 1)$ th time interval

$$\begin{cases} \{\varepsilon_0\}_{(k+1)} = \sum_{i=0} \{\varepsilon_i\}_k \\ \{\sigma_0\}_{(k+1)} = \sum_{i=0} \{\sigma_i\}_k \end{cases}, \quad k = 1, 2, 3, \dots, \quad (31)$$

where subscript $(k + 1)$ and k refer to $(k + 1)$ th and k th time intervals, respectively.

3 Recursive SBFEM equations

The Scaled Boundary Finite Element Method (SBFEM) [9], [10] and [11] introduces a normalised radial coordinate system by scaling a defining curve (usually the domain

boundary or a part of the domain boundary) relative to a scaling centre (x_0, y_0) selected within the domain or at the intersection of two straight sections of the boundary (Fig.2). The normalised radial coordinate ξ runs from the scaling centre towards the defining curve, and has values of zero at the scaling centre and unity at the defining curve. The other circumferential coordinate η specifies a distance around the defining curve from an origin on the curve. The scaled boundary and Cartesian coordinate systems are related by the scaling equations

$$x = x_0 + \xi x_\eta(\eta) \quad (32)$$

$$y = y_0 + \xi y_\eta(\eta) \quad (33)$$

Displacement and stress components are retained in the original Cartesian coordinate directions, while position is specified in terms of the scaled boundary coordinates.

For the problem defined by Eqs. (15-18), an approximate solution of $\{u_m(\xi, \eta)\}$ is sought in the form [26]

$$\{u_m(\xi, \eta)\} = [N(\eta)] \{u_m(\xi)\} \quad (34)$$

where $[N(\eta)]$ is a matrix of shape functions.

By the virtue of the weighted residual method [11], one has (in the absence of body force)

$$\int_V \{\delta \varepsilon(\xi, \eta)\}^T \{\sigma(\xi, \eta, t)\} dV = \int_\eta \{\delta u(\eta)\}^T \{p(\eta, t)\} d\eta \quad (35)$$

Substituting Eq. (6) into Eq. (35) then yields

$$\int_V \{\delta \varepsilon(\xi, \eta)\}^T \{\sigma_m(\xi, \eta)\} dV = \int_S \{\delta u(\eta)\}^T \{p_m(\eta)\} ds \quad (36)$$

Substituting Eq. (25) into Eq. (36) gives

$$\int_V \{\delta \varepsilon(\xi, \eta)\}^T [D] \{\varepsilon_m(\xi, \eta)\} dV = \int_\eta \{\delta u(\eta)\}^T \{p_m(\eta)\} ds - \int_V \{\delta \varepsilon(\xi, \eta)\}^T \{C_m\} dV \quad (37)$$

The left-side of Eq. (37) becomes

$$\begin{aligned} & \int_V \{\delta \varepsilon(\xi, \eta)\}^T [D] \{\varepsilon_m(\xi, \eta)\} dV \\ &= \{\delta u\}^T \left\{ [E^0] \{u_m\}_{,\xi} + [E^1] \{u_m\} \right\} \\ & \quad - \int_0^1 \{\delta u(\xi)\}^T \left\{ [E^0] \xi \{u_m(\xi)\}_{,\xi\xi} + [[E^0] + [E^1]^T - [E^1]] \{u_m(\xi)\}_{,\xi} - [E^2] \frac{1}{\xi} \{u_m(\xi)\} \right\} d\xi \end{aligned}$$

(38)

where

$$[E^0] = \int_{\eta} [B^1(\eta)]^T [D] [B^1(\eta)] |J| d\eta \quad (39)$$

$$[E^1] = \int_{\eta} [B^2(\eta)]^T [D] [B^1(\eta)] |J| d\eta \quad (40)$$

$$[E^2] = \int_{\eta} [B^2(\eta)]^T [D] [B^2(\eta)] |J| d\eta \quad (41)$$

The right-side of Eq. (37) is written as

$$\{\delta u\}^T \{P_m\} = \{\delta u\}^T (\{PS_m\} - \{PC_m\}) \quad (42)$$

where

$$\{PS_m\} = \int_{\eta} [N(\eta)] \{p_m(\eta)\} d\eta \quad (43)$$

$$\{\delta u\}^T \{PC_m\} = \int_V \{\delta \varepsilon(\xi, \eta)\}^T \{C_m\} dV \quad (44)$$

The arbitrariness of $\{\delta u(\xi)\}$ leads to

$$\{P_m\} = [E^0] \{u_m\}_{,\xi} + [E^1]^T \{u_m\} \quad (45)$$

$$[E^0] \xi^2 \{u_m(\xi)\}_{,\xi\xi} + [[E^0] + [E^1]^T - [E^1]] \xi \{u_m(\xi)\}_{,\xi} - [E^2] \{u_m(\xi)\} = \{0\} \quad (46)$$

By inspection, solutions to the homogeneous set of Euler-Cauchy differential equations represented by Eq. (46) must be of the form [\[11\]](#)

$$\{u_m(\xi)\} = c_1 \xi^{-\lambda_1} \{\phi_1\} + c_2 \xi^{-\lambda_2} \{\phi_2\} + \dots \quad (47)$$

$\{\phi_i\}$ and λ_i are obtained by solving the following eigenproblem [\[11\]](#)

$$\begin{bmatrix} [E^0]^{-1} [E^1]^T & -[E^0]^{-1} \\ [E^1] [E^0]^{-1} [E^1]^T - [E^2] & -[E^1] [E^0]^{-1} \end{bmatrix} \begin{Bmatrix} \{\phi\} \\ \{q\} \end{Bmatrix} = \lambda \begin{Bmatrix} \{\phi\} \\ \{q\} \end{Bmatrix} \quad (48)$$

At $\xi = 1$

$$\{c\} = [\Phi]^{-1} \{u_m\} \quad (49)$$

where

$$[\Phi] = [\{\phi_1\}, \{\phi_2\}, \dots, \{\phi_n\}] \quad (50)$$

Furthermore

$$\{u_m\} = [K]^{-1} \{P_m\} \quad (51)$$

where

$$[K]=[Q][\Phi]^{-1} \quad (52)$$

$$[Q]=[\{q_1\}, \{q_2\}, \dots, \{q_n\}] \quad (53)$$

$$\{P_m\} = \{PS_m\} - \{PC_m\} \quad (54)$$

$$\{PC_m\} = \frac{\Delta t}{m} \left(\left(\frac{1}{\eta_1} - \frac{1}{p_1} \right) [K] \{u_{m-1}\} - \frac{1}{p_1} \{PC_{m-1}\} \right) \quad (55)$$

$\{q_i\}$ refers to the dual variable of $\{\phi_i\}$ [11].

At the initial points of k th time interval, $\{u_0\}$ is determined by

$$\begin{aligned} \{u_0\}^k &= [K]^{-1} \{PS_0\}; & \text{if } k=1 \\ \{u_0\}^k &= \sum_m \{u_m\}^{k-1} & \text{if } k>1 \end{aligned} \quad (56)$$

An adaptive computing process at a time interval can be realized with the increase of m until the following criteria is satisfied

$$\frac{\left\| \{u\}^m s^m \right\|_{s=1} \Big|_2}{\left\| \sum_{j=0}^{m-1} \{u\}^j s^j \right\|_{s=1} \Big|_2} \leq \beta \quad (57)$$

where β is a prescribed error tolerance, and $\|\bullet\|_2$ represents a L^2 -norm.

4 Recursive partitioning algorithm

A structure or a computational region Ω is said to possess cyclic symmetry of order N when its geometry and physical properties are invariant under the following N symmetry transformations [4]

$$\Psi_i = (i-1)\theta, \quad i=1, 2, \dots, N. \quad (58)$$

where Ψ_i represents a rotation of Ω about its axis of rotation with an angle of $\theta=2\pi/N$, and N is defined as the order of symmetry. For example, Fig.3 shows a [cyclic symmetric plate](#) with $N=6$.

To make full use of the symmetry in SBFEM analysis, nodes are required to be arranged in a symmetric way so that original symmetry of the system is retained.

Ω is naturally divided into N identical parts Ω_i , $i=1, 2, \dots, N$.

$$\Omega = \Omega_1 \cup \Omega_2 \cdots \cup \Omega_N, \quad \Omega_i = \Psi_i : \Omega_1 \quad (59)$$

Ω_i is arranged in an anti-clock sequence

Eq. (59) means that Ω_i can be obtained from Ω_1 , which is called the ‘basic region’ and can be arbitrarily chosen from those identical parts. By setting up nodal co-ordinates and integration points on Ω_1 only, one can then obtain the complete computational model by using Eq. (59). For any node or integration point in the basic region, there are another $N-1$ different nodes or integration points, which are located symmetrically on the other $N-1$ symmetry regions. These N nodes constitute a set of symmetric nodes, which is called a symmetric node orbit, and is designated as O_A

$$O_A = \{A_1, A_2, \dots, A_N\} \quad (60)$$

For the N nodes of O_A , the reference co-ordinate directions of node A_1 , which belongs to the basic region, are first defined, and then the reference co-ordinate directions for the other symmetric nodes can be defined by using Eq. (59) (see Fig.3, for example.) It is readily seen that the six interface nodes B_i ($i=1, 2, \dots, N$) constitute orbit O_B . Only those nodes that are located on the internal part and the ‘right’ interface of Ω_1 are regarded as belonging to the basic region. For example, of the two interface nodes of the basic region in Fig. 3, only B_1 is regarded as belonging to Ω_1 , while B_2 is regarded as belonging to Ω_2 . Assuming the number of nodes belonging to Ω_1 is d , the total number is $N \cdot d$.

For a viscoelastic cyclic symmetric system with order N , $\{u\}$ is designated by [3]

$$\{u\}^T = (\{u\}^{1T}, \{u\}^{2T}, \dots, \{u\}^{NT}) \quad (61)$$

where the nodal vector $\{u\}^i$ belonging to the i th symmetric region is a sub-vector of $\{u\}$ with dimension $2d$.

Therefore

$$\{u_m\}^T = (\{u_m\}^{1T}, \{u_m\}^{2T}, \dots, \{u_m\}^{NT}) \quad (62)$$

Using a symmetry-adapted reference coordinate system [22], the vector $\{u_m\}$, $\{P_m\}$, $\{\varphi\}$ and $\{q\}$ are expressed via

$$\{u_m\} = [T]^g \{\bar{u}_m\} \quad (63)$$

$$\{P_m\} = [T]^g \{\bar{P}_m\} = [T]^g (\{\bar{P}S_m\} - \{\bar{P}C_m\}) \quad (64)$$

$$\{\varphi\} = [T]^g \{\bar{\varphi}\} \quad (65)$$

$$\{q\} = [T]^g \{\bar{q}\} \quad (66)$$

$$\{q\} = [T]^g \{\bar{q}\} \quad (66)$$

where the [overbar](#) refers to vectors defined in the symmetry-adapted system [\[4\]](#)

$$[T]^g = \begin{bmatrix} [\bar{T}]_1^g & & 0 \\ & [\bar{T}]_2^g & \\ & & \dots \\ 0 & & & [\bar{T}]_N^g \end{bmatrix}_{n \times n}, \quad (67)$$

$$[\bar{T}]_i^g = \begin{bmatrix} [T]_i & & 0 \\ & [T]_i & \\ & & \dots \\ 0 & & & [T]_i \end{bmatrix}_{2d \times 2d}, \quad (68)$$

$$[T]_i = \begin{bmatrix} \cos(i-1)\theta & -\sin(i-1)\theta \\ \sin(i-1)\theta & \cos(i-1)\theta \end{bmatrix}, \quad \theta = 2\pi / N \quad (69)$$

$$[T]_i^T [T]_i = [I] \quad (70)$$

$$[T]_{i+k-1} [T]_i = [T]_i [T]_k \quad (71)$$

Substituting Eqs. (65) and (66) into Eq (48) then yields

$$\begin{bmatrix} [E]_0^{-1} [E]_1^T & -[E]_0^{-1} \\ [E]_1 [E]_0^{-1} [E]_1^T - [E]_2 & -[E]_1 [E]_0^{-1} \end{bmatrix} \begin{bmatrix} [T]^g & 0 \\ 0 & [T]^g \end{bmatrix} \begin{Bmatrix} \{\bar{\varphi}\} \\ \{\bar{q}\} \end{Bmatrix} = \lambda \begin{bmatrix} [T]^g & 0 \\ 0 & [T]^g \end{bmatrix} \begin{Bmatrix} \{\bar{\varphi}\} \\ \{\bar{q}\} \end{Bmatrix} \quad (72)$$

Multiplying each side of Eq. (72) by $\begin{bmatrix} [T]^{gT} & 0 \\ 0 & [T]^{gT} \end{bmatrix}$ and applying the identity

stated in Eq. (70) to the right hand side then gives

$$\begin{bmatrix} [T]^{gT} & 0 \\ 0 & [T]^{gT} \end{bmatrix} \begin{bmatrix} [E]_0^{-1} [E]_1^T & -[E]_0^{-1} \\ [E]_1 [E]_0^{-1} [E]_1^T - [E]_2 & -[E]_1 [E]_0^{-1} \end{bmatrix} \begin{bmatrix} [T]^g & 0 \\ 0 & [T]^g \end{bmatrix} \begin{Bmatrix} \{\bar{\varphi}\} \\ \{\bar{q}\} \end{Bmatrix} = \lambda \begin{Bmatrix} \{\bar{\varphi}\} \\ \{\bar{q}\} \end{Bmatrix} \quad (73)$$

Eq. (73) can further be written as

$$\begin{bmatrix} [\bar{Z}]_{11} & [\bar{Z}]_{12} \\ [\bar{Z}]_{21} & [\bar{Z}]_{22} \end{bmatrix} \begin{Bmatrix} \{\bar{\varphi}\} \\ \{\bar{q}\} \end{Bmatrix} = \lambda \begin{Bmatrix} \{\bar{\varphi}\} \\ \{\bar{q}\} \end{Bmatrix} \quad (74)$$

where

$$[\bar{Z}]_{11} = [T]^{gT} [E]_0^{-1} [E]_1^T [T]^g \quad (75)$$

$$[\bar{Z}]_{12} = [T]^{gT} [E]_0^{-1} [T]^g \quad (76)$$

$$[\bar{Z}]_{21} = [T]^{gT} ([E]_1 [E]_0^{-1} [E]_1^T - [E]_2) [T]^g \quad (77)$$

$$[\bar{Z}]_{22} = -[T]^{gT} [E]_1 [E]_0^{-1} [T]^g \quad (78)$$

$[\bar{Z}]_{11}$, $[\bar{Z}]_{12}$, $[\bar{Z}]_{21}$, $[\bar{Z}]_{22}$ can be proved block-circulant [22], having the form

$$[\bar{Z}]_{ij} = \begin{bmatrix} [\bar{Z}]_{ij}^1 & [\bar{Z}]_{ij}^2 & \cdots & [\bar{Z}]_{ij}^N \\ [\bar{Z}]_{ij}^N & [\bar{Z}]_{ij}^1 & & \vdots \\ \vdots & & \ddots & [\bar{Z}]_{ij}^2 \\ [\bar{Z}]_{ij}^2 & \cdots & [\bar{Z}]_{ij}^N & [\bar{Z}]_{ij}^1 \end{bmatrix}, \quad i, j = 1, 2 \quad (79)$$

$\{\bar{u}_m\}$, $\{\bar{P}_m\}$, $\{\bar{\varphi}\}$ and $\{\bar{q}\}$ can further be expressed by [5] and [22]

$$\{\bar{u}_m\} = [E]^g \{\underline{u}_m\} \quad (80)$$

$$\{\bar{P}_m\} = [E]^g \{\underline{P}_m\} = [E]^g (\underline{PS}_m - \underline{PC}_m) \quad (81)$$

$$\{\bar{\varphi}\} = [E]^g \{\underline{\varphi}\} \quad (82)$$

$$\{\bar{q}\} = [E]^g \{\underline{q}\} \quad (83)$$

where

$$[E]^g = [[e]_{rss} \cdot [I]^g]^T \quad (84)$$

$[I]^g$ is a unit matrix of $2d$ dimensions. $[e]_{rss}$ is the ss th element of the basis $[e]_r$ defined by

$$\begin{aligned} \{e\}_1 &= \{1, 1, \dots, 1\}^T / \sqrt{N} \\ \{e\}_{2i} &= \sqrt{2/N} \{\cos i\theta_1, \cos i\theta_2, \dots, \cos i\theta_N\}^T \\ \{e\}_{2i+1} &= \sqrt{2/N} \{\sin i\theta_1, \sin i\theta_2, \dots, \sin i\theta_N\}^T \\ i &= 1, \dots, \lceil (N-1)/2 \rceil, \quad \theta_k = (k-1)\theta \quad (k=1, 2, \dots, N) \\ \{e\}_N &= \{1, -1, 1, \dots, -1\}^T / \sqrt{N} \quad (\text{when } N \text{ is even}) \end{aligned} \quad (85)$$

where $\{\{e\}_1, \{e\}_2, \dots, \{e\}_N\}^T$ represents a group of complete symmetrized orthogonal unit vectors, $\lceil (N-1)/2 \rceil$ is the largest integer which does not exceed $(N-1)/2$ [4].

Utilizing Eq. (82) and Eq. (83), Eq. (74) becomes

$$\begin{bmatrix} [\underline{Z}]_{11} & [\underline{Z}]_{12} \\ [\underline{Z}]_{21} & [\underline{Z}]_{22} \end{bmatrix} \begin{Bmatrix} \{\underline{\varphi}\} \\ \{\underline{q}\} \end{Bmatrix} = \lambda \begin{Bmatrix} \{\underline{\varphi}\} \\ \{\underline{q}\} \end{Bmatrix} \quad (86)$$

where

$$[\underline{Z}]_{ij} = [E]^g{}^T [\bar{Z}]_{ij} [E]^g, \quad i, j = 1, 2 \quad (87)$$

$$[\underline{Z}]_{ij} = \sum_{r=0}^{\lfloor N/2 \rfloor} \oplus [\underline{Z}]_{ij}^{rr}, \quad i, j = 1, 2 \quad (88)$$

where $\oplus$ represents the direct sum of matrices, i.e., the matrices $[\underline{Z}]_{ij}$ are block-diagonal

$$\begin{aligned}
[\underline{Z}]_{ij}^{00} &= ([\bar{Z}]_{ij}^1 + [\bar{Z}]_{ij}^2 + \dots + [\bar{Z}]_{ij}^N) \\
[\underline{Z}]_{ij}^{rr} &= \begin{bmatrix} [\underline{Z}]_{ij}^{rr11} & [\underline{Z}]_{ij}^{rr12} \\ [\underline{Z}]_{ij}^{rr21} & [\underline{Z}]_{ij}^{rr22} \end{bmatrix}, \quad r=1,2,\dots,[(N-1)/2] \\
[\underline{Z}]_{ij}^{rr11} &= [\underline{Z}]_{ij}^{rr22} = \sum_{s=1}^N [\bar{Z}]_{ij}^s \cos(ss-1)i\theta \\
[\underline{Z}]_{ij}^{rr12} &= -[\underline{Z}]_{ij}^{rr21} = \sum_{s=1}^N [\bar{Z}]_{ij}^s \sin(ss-1)i\theta \\
[\underline{Z}]_{ij}^{N/2,N/2} &= ([\bar{Z}]_{ij}^1 - [\bar{Z}]_{ij}^2 + [\bar{Z}]_{ij}^3 - [\bar{Z}]_{ij}^4 \dots - [\bar{Z}]_{ij}^N) \text{ when } N \text{ is even.} \quad (89)
\end{aligned}$$

Eq. (86) is naturally partitioned into $[(N+2)/2]$ decoupled sub-eigenproblems

$$\begin{bmatrix} [\underline{Z}]_{11}^{rr} & [\underline{Z}]_{12}^{rr} \\ [\underline{Z}]_{21}^{rr} & [\underline{Z}]_{22}^{rr} \end{bmatrix} \begin{Bmatrix} \{\underline{\varphi}\}^r \\ \{\underline{q}\}^r \end{Bmatrix} = \lambda \begin{Bmatrix} \{\underline{\varphi}\}^r \\ \{\underline{q}\}^r \end{Bmatrix}, r=0,1,2,\dots,N/2 \quad (90)$$

Therefore, instead of solving the original eigenproblem Eq. (48), now one only needs to solve a series of smaller independent sub-eigenproblems of Eq. (90).

Substituting Eqs. (63), (64), (80) and (81) into Eq. (52) yields

$$\{\underline{u}_m\} = [\underline{K}]^{-1} \{\underline{P}_m\} \quad (91)$$

where

$$\begin{aligned}
\{\underline{P}_m\} &= [E]^{gT} [T]^{gT} \{\underline{P}_m\} \\
&= [E]^{gT} [T]^{gT} \{PS_m\} + \quad (92)
\end{aligned}$$

$$\frac{\Delta t}{m} \left(\left(\frac{1}{\eta_1} - \frac{1}{p_1} \right) [E]^{gT} [T]^{gT} [\underline{K}] \{u_{m-1}\} - \frac{1}{p_1} [E]^{gT} [T]^{gT} \{PC_{m-1}\} \right)$$

$$[\underline{K}] = [\underline{Q}][\underline{\Phi}]^{-1} \quad (93)$$

where $[\underline{\Phi}]$, $[\underline{Q}]$, $\{\underline{u}_m\}$, $\{\underline{P}_m\}$ and $[\underline{K}]$ have the form

$$[\underline{\Phi}] = \begin{bmatrix} [\underline{\Phi}]_0 & & & \\ & [\underline{\Phi}]_1 & & \\ & & \dots & \\ & & & [\underline{\Phi}]_{N/2} \end{bmatrix} \quad (94)$$

$$[\underline{Q}] = \begin{bmatrix} [\underline{Q}]_0 & & & \\ & [\underline{Q}]_1 & & \\ & & \dots & \\ & & & [\underline{Q}]_{N/2} \end{bmatrix} \quad (95)$$

$$\{\underline{u}_m\} = (\{\underline{u}_m\}_0^T, \{\underline{u}_m\}_1^T, \{\underline{u}_m\}_2^T, \dots, \{\underline{u}_m\}_{N/2}^T)^T \quad (96)$$

$$\{\underline{P}_m\} = (\{\underline{P}_m\}_0^T, \{\underline{P}_m\}_1^T, \{\underline{P}_m\}_2^T, \dots, \{\underline{P}_m\}_{N/2}^T)^T \quad (97)$$

$$[\underline{K}] = \begin{bmatrix} [\underline{K}]_0 & & & \\ & [\underline{K}]_1 & & \\ & & \dots & \\ & & & [\underline{K}]_{N/2} \end{bmatrix} \quad (98)$$

where

$$\{\underline{u}_m\}_{ii} = [\underline{K}]_{ii}^{-1} \{\underline{P}_m\}_{ii}, \quad ii = 0, 1, 2, \dots, N/2 \quad (99)$$

$$[\underline{K}]_{ii} = [\underline{\Phi}]_{ii} [\underline{Q}]_{ii}^{-1}, \quad ii = 0, 1, 2, \dots, N/2 \quad (100)$$

Therefore, instead of solving Eq. (51), only a series of smaller independent system equations of Eq. (99) needs to be solved.

If eigenvalue problems of Eq. (48) and Eq. (90) are solved using the QR algorithm, the operations of multiplication and addition can be estimated by [27] and [28]

$$EC_E = \frac{7}{3}M^3 + 4rM^2 \quad (\text{for Eq (48)}) \quad (101)$$

$$EP_E = \frac{28N-42}{3N^3}M^3 + \frac{8N-8}{N^2}rM^2 \quad (\text{for Eq (90)}) \quad (102)$$

$$R_E = \frac{EC_E}{EP_E} = \frac{7N^3M + 12N^3r}{(28N-42)M + 24N^2r - 24Nr} \quad (103)$$

where M denotes the dimension of matrix, r denotes the number of iteration.

If inverse matrixes of $[\underline{K}]$ and $[\underline{K}]_{ii}$ ($ii = 0, 1, 2, \dots, N/2$) are generated using the Gauss elimination technique, the operations of multiplication and addition can be estimated by [27] and [28]

$$EC_K = \frac{5}{6}M^3 + M^2 - \frac{5}{6}M \quad (\text{for } [\underline{K}]) \quad (104)$$

$$EP_K = \frac{10N-15}{3N^3}M^3 + \frac{2N-2}{N^2}M^2 - \frac{5}{6}M \quad (\text{for } [\underline{K}]_{ii}) \quad (105)$$

$$R_K = \frac{EC_K}{EP_K} = \frac{N^3(5M^2 + 6M - 5)}{(10N - 15)M^2 + (12N^2 - 12N)M - 5N^3} \quad (106)$$

Fig.4 and Fig.5 give a variation of R_E and R_K with M and N

Actually, the real speed-up gained by partition [also](#) depends on the solver, computer used, and art of programming etc. In this paper, computing program is coded in MATLAB with its own solvers, and run on a PC computer with processor: Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz (4 CPUs).

5 Numerical examples

Example 1 - A hexagonal viscoelastic plate

Consider a hexagonal viscoelastic plate subjected to uniform tension as shown in Fig.6 where the length of hexagonal side is 2 m, $P = 1\text{N/m}^2$, $N = 6$, $E_1 = 1 \times 10^3\text{N/m}^2$, $\nu = 0.25$, $E_2 = 2 \times 10^3\text{N/m}^2$, $\eta = 1000\text{N}\cdot\text{s/m}^2$, plane stress conditions were assumed. The distribution of nodes on the basic region is shown in Fig.7.

Fig.8 and Fig.9 exhibit comparisons of displacement at point A and B given by the proposed algorithm and the ABAQUS.

Tab.1 shows the comparisons of t_E , t_K and t_T with and without partitioning. t_E is the CPU time to solve eigenvalue equation one time; t_K is the CPU time spending on two parts : 1. generating $[K]^{-1}$ or $[\underline{K}]_{ii}^{-1}$ one time, 2. calculating all $\{u_m\} = [K]^{-1}\{P_m\}$ or $\{\underline{u}_m\}_{ii} = [\underline{K}]_{ii}^{-1}\{\underline{P}_m\}_{ii}$ for all time [intervals](#); t_T refers to the total CPU time consuming on the [whole solution process](#).

Fig.10 exhibits the variation of R_{CP} with time t , [and](#) R_{CP} is defined by

$$R_{CP} = \frac{t_T \text{ without partitioning}}{t_T \text{ with partitioning}} \quad (107)$$

5.2 Example 2 - A dodecagon viscoelastic plate

Consider a dodecagon viscoelastic plate subjected to uniform tension as shown in Fig.11 where $R = 1\text{m}$, $P = 1\text{N/m}^2$, $N = 12$, $E_1 = 1000\text{N/m}^2$, $E_2 = 2000\text{N/m}^2$, $\eta = 1000\text{N}\cdot\text{s/m}^2$, $\nu = 0.25$. The distribution of nodes on the basic region is shown in Fig.12.

Tab.2 and Tab.3 show the comparisons of t_E , t_K and t_T with and without partitioning.

Fig.13 and Fig.14 exhibit the variation of R_{CP} with n and Δt , respectively.

Fig.15 exhibits the variation of the number of recursion with t using different Δt . With the increase of the size of time step, more recursive computing is required to satisfy Eq. (57).

Computing Remarks

(1) A good agreement can be observed in the comparison of solutions given by the proposed algorithm and ABAQUS. By contrast with the FEM based ABAQUS, fairly less number of DOF of unknowns are used in SBFEM analysis. For instance, in the Example 1, 768 DOF is used in the proposed algorithm, while 20624 DOF is used in ABAQUS.

(2) Exploiting cyclic symmetry, a substantial speed up can be achieved in solving the eigenvalue and system equations of SBFEM, as shown in Tab.2. Since the coefficient matrices of eigenvalue and system equations of SBFEM keep invariant during the whole recursive process, the Eq. (90) needs to be solved only one time, and the inverse matrix of $[K]_{ii}$ in Eq. (99) need to be generated only one time, the speed up gained from partitioning eigenvalue and system equations will be gradually diluted with time increase. A higher speed up can be expected when Eq. (90) and Eq. (99) are solved using parallel computing skills.

6 Conclusions

This paper attempts to develop a SBFEM based numerical method via a combination with a temporally adaptive algorithm for the viscoelastic analysis, and exploit the cyclic symmetry to reduce its computational expense. The proposed approach may be particularly beneficial to solve viscoelastic problems with cyclic symmetry, unbounded domains and stress singularity. The major merits include

- a) In virtue of a combination of SBFEM with a temporally adaptive algorithm, a spatially and temporally coupled viscoelastic problem is converted into a series of recurrent spatial problems which are solved by SBFEM that is semi-analytical, and advantageous in dealing with the problems involving unbounded domains and singularity. A self-adaptive computation can be realized in the time domain to adapt the change of the size of time step, and results in a more precise description for the

variation of variables.

- b) For cyclically symmetric structures, the recursive coefficient matrices of eigenvalue and system equations of SBFEM are proved block-circulant, and can be partitioned into a number of smaller independent problems to solve. Thus the whole computing process is speeded up via the proposed recursive partitioning algorithm. Two numerical examples are provided to verify the proposed approach, and illustrate the advantages of proposed approach.
- c) A higher computing efficiency can be expected since the proposed partitioning algorithm facilitates parallel processing.

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Fig.1 Three-parameter solid viscoelastic model

Fig.2 Scaled boundary coordinate system

Fig.3 A rotationally periodic symmetric plane with $N = 6$

Fig.4.The variation of R_k with M and N

Fig.5.The variation of R_E with M and N ($r=1$)

Fig.6.A hexagonal plate

Fig.7. Nodes distribution on the basic region

Fig.8. A comparison of u_y with t at point A $(0, \sqrt{3})$

Fig.9. A comparison of u_y with t at point B $(-0.5, \sqrt{3})$

Fig.10.The variation of R_{CP} with t

Fig.11.A dodecagon plate

Fig.12. Nodes distribution on the basic region

Fig.13.The variation of R_{CP} with n

Fig.14.The variation of R_{CP} with Δt

Fig.15.The variation of the number of recursion with Δt and t ($\beta = 1 \times 10^{-10}$)

Table 1 Comparison of CPU time($\Delta t = 0.01\text{s}$, $n = 768$)

		SBFEM without partitioning Time t_1 (s)	SBFEM with partitioning Time t_2 (s)	Ratio t_1/t_2
t_E		18.52	3.32	5.58
$t = 1\text{s}$	t_K	12.62	8.78	1.44
	t_T	36.21	18.27	1.98
$t = 3\text{s}$	t_K	36.43	26.01	1.40
	t_T	59.92	35.49	1.69
$t = 5\text{s}$	t_K	56.33	41.87	1.35
	t_T	80.89	51.68	1.57

Table 2 Comparison of CPU time($t = 4\text{ s}$, $\Delta t = 0.05\text{ s}$)

		SBFEM without partitioning Time t_1 (s)	SBFEM with partitioning Time t_2 (s)	Ratio t_1/t_2
$n = 480$	t_E	4.74	0.55	8.62
	t_K	4.52	2.92	1.55
	t_T	13.28	7.04	1.89
$n = 960$	t_E	33.16	2.14	15.50
	t_K	22.34	20.17	1.11
	t_T	64.69	31.25	2.07
$n = 1440$	t_E	116.02	5.43	21.37
	t_K	44.59	39.11	1.14
	t_T	185.78	63.26	2.94

Table 3 Comparison of CPU time($n = 960$, $t = 4$ s)

		SBFEM without partitioning Time t_1 (s)	SBFEM with partitioning Time t_2 (s)	Ratio t_1/t_2
t_E		32.50	2.04	15.93
$\Delta t = 0.02$ s	t_K	44.29	38.06	1.16
	t_T	87.39	48.78	1.79
$\Delta t = 0.06$ s	t_K	17.64	15.55	1.13
	t_T	61.02	26.22	2.33
$\Delta t = 0.1$ s	t_K	12.96	10.75	1.21
	t_T	55.44	21.57	2.57