A combination of EFG-SBM and a temporally-piecewise adaptive algorithm to solve viscoelastic problems

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Abstract: This paper combines Element-Free Galerkin Scaled Boundary Method (EFG-SBM) with a temporally-piecewise adaptive algorithm to solve viscoelastic problems. By expanding variables at a discretized time interval, the variations of variables can be described more precisely, and a space-time domain coupled problem can be converted into a series of recurrent boundary value problems which are solved by EFG-SBM via an adaptive computing process. Numerical tests including creep and relaxation are given to verify the proposed algorithm.

Key Words: *Viscoelasticity; Element-free Galerkin scaled boundary method; temporally-piecewise adaptive algorithm.*

1. Introduction

The Scaled Boundary Method (SBM) is a semi-analytical method originally developed by Wolf and Song in 1996 [1-2]. By introducing a normalised radial coordinate system based on a scaling centre and a defining curve (usually taken as the boundary), the governing differential equations are weakened in the circumferential direction and then solved analytically in the normalised radial direction.

As its advantages in dealing with the problems involving unbounded domains and singularity, SBM has been applied in various fields [3-9]. However there seems no any direct report relevant to its exploration on the viscoelastic problem that relates to many engineering issues [10-17]. As matter of a fact, there is a pool containing a number of spatially discretized algorithms employed for the viscoelastic analysis, such as Finite Element Method (FEM) [18-23], Boundary Element Method (BEM) [24-26] and Element-Free Galerkin Method (EFGM) [27-29], but not yet popularly used with Scaled Boundary Method.

This paper attempts to develop a SBM based numerical algorithm that may be particularly beneficial to solve viscoelastic problems with unbounded domains or singularity. On the other hand, due to the viscoelastic constitutive relationship that is time dependent, usually a step-by-step based numerical technique is required temporally. Actually there is a long list including a variety of such

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techniques[23, 24, 27, 28, 30-33] among which a temporally-piecedwised algorithm [23, 27, 28] is one that is not only easy to combine with spatially discretized algorithms, but also able to adapt the change of the size of time step via a self-adaptive process. Therefore this paper combines advantages of Scaled Boundary Method (SBM) and a temporally-piecedwised adaptive algorithm, and proposes a new approach to solve viscoelastic problems.

The paper is organized as follows, Section 1 is the introduction; Sections 2 gives recurrent viscoelastic governing equations; Section 3 presents recursive equations of EFG-SBM; Section 4 provides four numerical examples, and Section 5 goes to the conclusion.

2. Recurrent governing equations

The governing equations of 2-D viscoelastic problems are described by [34]

$$\text{Equilibrium equation} \quad \mathbf{H}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad (1)$$

$$\text{Relationship of strain and displacement} \quad \boldsymbol{\varepsilon} = \mathbf{H}^T \mathbf{u} \quad (2)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote the vector of stress and strain, respectively, $\mathbf{b}$ is the vector of body force, $\mathbf{u}$ is the vector of displacement,

$$\mathbf{H} = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \quad (3)$$

The viscoelastic constitutive relationships is specified by a three-parameter solid viscoelasticity model (see Fig.1) in a differential form [10-12]

$$\alpha_1 \mathbf{D}\boldsymbol{\varepsilon}(t) + p_1 \mathbf{D} \frac{d\boldsymbol{\varepsilon}(t)}{dt} = (\alpha_1 + \alpha_2) \boldsymbol{\sigma}(t) + p_1 \frac{d\boldsymbol{\sigma}(t)}{dt} \quad (t > 0) \quad (4)$$

$$\boldsymbol{\sigma}(t) = \mathbf{D}\boldsymbol{\varepsilon}(t) \quad (t = 0) \quad (5)$$

where

$$\mathbf{D} = \frac{E_2}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \quad \text{for the plane stress problem} \quad (6)$$

$$p_1 = \frac{\eta_1}{E_1 + E_2} \quad \alpha_1 = \frac{E_1}{E_1 + E_2} \quad \alpha_2 = \frac{E_2}{E_1 + E_2} \quad (7)$$

where ν , E_1 , E_2 and η_1 are constitutive parameters.

For the plane strain problems, E_1 , E_2 , η_1 and ν need to be replaced by $\frac{E_1}{1-\nu^2}$, $\frac{E_2}{1-\nu^2}$, $\frac{\eta_1}{1-\nu^2}$, and $\frac{\nu}{1-\nu}$, respectively.

The boundary conditions are given by [34]

$$\mathbf{u} = \tilde{\mathbf{u}} \quad \text{on} \quad \Gamma_u \quad (8)$$

$$\mathbf{p} = \tilde{\mathbf{p}} \quad \text{on} \quad \Gamma_\sigma \quad (9)$$

where $\mathbf{p}$ denotes the vector of traction, $\tilde{\mathbf{u}}$, $\tilde{\mathbf{p}}$ are the prescribed functions on the boundary.

Divide time domain into a number of time intervals, the initial points and sizes of the time intervals are defined by $t_0, t_1, t_2, \dots, t_k \dots$ and $T_1, T_2, \dots, T_k \dots$, respectively. At the k -th discretized time interval, in order to describe the variation of variables more precisely, all variables are expanded in terms of s_t

$$\boldsymbol{\sigma} = \sum_{m=0} \boldsymbol{\sigma}^{km} s_t^m \quad (10)$$

$$\boldsymbol{\varepsilon} = \sum_{m=0} \boldsymbol{\varepsilon}^{km} s_t^m \quad (11)$$

$$\mathbf{b} = \sum_{m=0} \mathbf{b}^{km} s_t^m \quad (12)$$

$$\mathbf{u} = \sum_{m=0} \mathbf{u}^{km} s_t^m \quad (13)$$

$$\tilde{\mathbf{u}} = \sum_{m=0} \tilde{\mathbf{u}}^{km} s_t^m \quad (14)$$

$$\mathbf{p} = \sum_{m=0} \mathbf{p}^{km} s_t^m \quad (15)$$

$$\tilde{\mathbf{p}} = \sum_{m=0} \tilde{\mathbf{p}}^{km} s_t^m \quad (16)$$

$$s_t = \frac{t - t_{k-1}}{T_k} \quad (17)$$

where t_{k-1} and T_k represent the initial point and size of the k -th time interval, respectively, $\boldsymbol{\sigma}^{km}$ and $\boldsymbol{\varepsilon}^{km}$ represent the expanding coefficients of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ at k -th time interval, respectively, m stands for the power order of expansion, $\mathbf{b}^{km}$ denotes the expanding coefficient of $\mathbf{b}$, $\mathbf{u}^{km}$, $\mathbf{p}^{km}$, $\tilde{\mathbf{u}}^{km}$ and $\tilde{\mathbf{p}}^{km}$ are the expanding coefficients of $\mathbf{u}$, $\mathbf{p}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{p}}$, k -th time interval, respectively.

The conversion relationship between the differentiations respect to t and s_t is

$$\frac{d}{dt} = \frac{1}{T_k} \frac{d}{ds_t} \quad (18)$$

Substituting Eqns (10–16) into Eqns (1, 2, 5, 8, 9) then yields

$$\mathbf{H}\boldsymbol{\sigma}^{km} + \mathbf{b}^{km} = \mathbf{0} \quad (19)$$

$$\boldsymbol{\varepsilon}^{km} = \mathbf{H}^T \mathbf{u}^{km} \quad (20)$$

$$\mathbf{u}^{km} = \tilde{\mathbf{u}}^{km} \quad \text{on} \quad \Gamma_u \quad (21)$$

$$\mathbf{p}^{km} = \tilde{\mathbf{p}}^{km} \quad \text{on} \quad \Gamma_\sigma \quad (22)$$

At the first time interval ($k=1$), when $m=0$ i.e $t=0$,

$$\boldsymbol{\sigma}^{km} = \boldsymbol{\sigma}^{10} = \mathbf{D}\boldsymbol{\varepsilon}^{km} = \mathbf{D}\boldsymbol{\varepsilon}^{10} \quad (23)$$

σ^{10} , ϵ^{10} , and u^{10} can be obtained by solving Eqns (19-23).

At other time intervals ($k=2,3,\dots$)

$$\epsilon^{k0} = \sum_{m=0} \epsilon^{(k-1)m} \quad (24)$$

$$\sigma^{k0} = \sum_{m=0} \sigma^{(k-1)m} \quad (25)$$

$$u^{k0} = \sum_{m=0} u^{(k-1)m} \quad (26)$$

For $m \neq 0$, substitution Eqns (10-11) for Eqn (4) then yields

$$\sigma^{km} = D\epsilon^{km} + C^{km} \quad k=1,2,3,\dots \quad m=1,2,\dots \quad (27)$$

$$C^{km} = \frac{T_k E_1}{m\eta_1} D\epsilon^{k(m-1)} - \frac{T_k}{mp_1} \sigma^{k(m-1)} \quad (28)$$

Futhermore

$$\sigma^{km} = D\epsilon^{km} + \beta_m (\alpha_2 D \sum_{i=1}^{m-1} \gamma_i \epsilon^{ki} - \alpha_1 D\epsilon^{k0} + \sigma^{k0}) \quad (29)$$

where

$$\beta_m = \frac{1}{m!} \left(\frac{-T_k}{p_1} \right)^{(m)} \quad (30)$$

$$\gamma_i = i! \left(\frac{-p_1}{T_k} \right)^i \quad (31)$$

3. Recursive equations of EFG-SBM

The scaled boundary method introduces such a ξ - s coordinate system by scaling a defining curve relative to a scaling centre (x_0, y_0) selected within the domain, as shown in Fig. 2. The coordinate ξ runs from the scaling centre towards the curve, and has values of zero at the scaling centre and unity at the curve. The coordinate s specifies a distance around the boundary from an origin on the boundary [6-9].

The scaled and Cartesian coordinate systems are related by Deeks and Wolf [6-9]

$$x = x_0 + \xi x_s(s) \quad (32a)$$

$$y = y_0 + \xi y_s(s) \quad (32b)$$

For Eqns (15-18)

$$u^{km}(\xi, s) = N(s) u_h^{km}(\xi) \quad (33)$$

where $u^{km}(\xi, s)$ represents the expanding coefficient of $u = \sum_{m=0} u^{km} s_t^m$, $N(s)$ is a matrix of

shape function and $u_h^{km}(\xi)$ is a set of n functions analytical in ξ .

The expanding coefficient of the strain $\epsilon = \sum_{m=0} \epsilon^{km} s_t^m$ in the scaled coordinate system is described

by [6-9]

$$\boldsymbol{\varepsilon}^{km}(\xi, s) = \mathbf{B}_1(s) \mathbf{u}^{km}(\xi)_{,\xi} + \frac{1}{\xi} \mathbf{B}_2(s) \mathbf{u}^{km}(\xi) \quad k=1,2,\dots m=0,1,2,\dots \quad (34)$$

$$\mathbf{B}_1(s) = \mathbf{b}_1(s) \mathbf{N}(s) = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y_s(s)}{\partial s} & 0 & -\frac{\partial x_s(s)}{\partial s} \\ 0 & -\frac{\partial x_s(s)}{\partial s} & \frac{\partial y_s(s)}{\partial s} \end{bmatrix}^T \cdot \mathbf{N}(s) \quad (35)$$

$$\mathbf{B}_2(s) = \mathbf{b}_2(s) \mathbf{N}(s)_{,s} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} -y_s(s) & 0 & x_s(s) \\ 0 & x_s(s) & -y_s(s) \end{bmatrix}^T \cdot \mathbf{N}(s)_{,s} \quad (36)$$

$$|\mathbf{J}| = x_s(s) y_s(s)_{,s} - y_s(s) x_s(s)_{,s} \quad (37)$$

where $\mathbf{N}(s)_{,s}$ refers to the derivative of $\mathbf{N}(s)$ with respect to s .

Utilizing the virtual displacement principle then gives (in the absence of body force) [6-9]

$$\int_V \delta \boldsymbol{\varepsilon}(\xi, s)^T \boldsymbol{\sigma}^{km}(\xi, s) dV - \int_s \delta \mathbf{u}(s)^T \mathbf{p}^{km}(s) ds = 0 \quad (38)$$

$$m=0,1,2,3,\dots k=1,2,3,\dots$$

where $\delta \mathbf{u}(\xi, s) = \mathbf{N}(s) \delta \mathbf{u}(\xi)$ denotes the vectour of virtual displacement, and

$$\delta \boldsymbol{\varepsilon}(\xi, s) = \mathbf{B}_1(s) \delta \mathbf{u}(\xi)_{,\xi} + \frac{1}{\xi} \mathbf{B}_2(s) \delta \mathbf{u}(\xi) \quad (39)$$

The second term in Eqn (38) becomes

$$\int_s \delta \mathbf{u}(s)^T \mathbf{p}^{km}(s) ds = \delta \mathbf{u}^T \int_s \mathbf{N}(s)^T \mathbf{p}^{km}(s) ds = \delta \mathbf{u}^T \mathbf{P}^{km} \quad k=1,2,\dots m=0,1,2,\dots \quad (40)$$

At the first time interval, when $k=1$ and $m=0$, i.e. $t=0$, substituting Eqn (23) into Eqn (38) then gives

$$\int_V \delta \boldsymbol{\varepsilon}(\xi, s)^T \mathbf{D} \boldsymbol{\varepsilon}^{k0}(\xi, s) dV - \int_s \delta \mathbf{u}(s)^T \mathbf{p}^{k0}(s) ds = 0 \quad (41)$$

Using Eqns (34), (39-40) and Green's Theorem then gives [6-9]

$$\begin{aligned} & \delta \mathbf{u}^T \left\{ \mathbf{E}_0 \mathbf{u}_{h,\xi}^{k0} + \mathbf{E}_1^T \mathbf{u}_h^{k0} \right\} - \delta \mathbf{u}^T \int_s \mathbf{N}(s)^T \mathbf{p}^{k0}(s) ds \\ & - \int_0^1 \delta \mathbf{u}(\xi)^T \left\{ \mathbf{E}_0 \xi \mathbf{u}_h^{k0}(\xi)_{,\xi\xi} + \left[\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1 \right] \mathbf{u}_h^{k0}(\xi)_{,\xi} - \mathbf{E}_2 \frac{1}{\xi} \mathbf{u}_h^{k0}(\xi) \right\} d\xi = 0 \end{aligned} \quad (42)$$

where

$$\mathbf{E}_0 = \int_s \mathbf{B}_1(s)^T \mathbf{D} \mathbf{B}_1(s) |\mathbf{J}(s)| ds \quad (43a)$$

$$\mathbf{E}_1 = \int_s \mathbf{B}_2(s)^T \mathbf{D} \mathbf{B}_1(s) |\mathbf{J}(s)| ds \quad (43b)$$

$$\mathbf{E}_2 = \int_S \mathbf{B}_2(s)^T \mathbf{D} \mathbf{B}_2(s) |\mathbf{J}(s)| ds \quad (43c)$$

Furthermore

$$\mathbf{P}^{*k0} = \mathbf{E}_0 \mathbf{u}_h^{k0} + \mathbf{E}_1^T \mathbf{u}_h^{k0} \quad (44)$$

$$\mathbf{E}_0 \xi^2 \mathbf{u}_h^{k0}(\xi)_{,\xi\xi} + [\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1] \xi \mathbf{u}_h^{k0}(\xi)_{,\xi} - \mathbf{E}_2 \mathbf{u}_h^{k0}(\xi) = \mathbf{0} \quad (45)$$

where

$$\mathbf{P}^{*k0} = \mathbf{P}^{k0} = \int_S \mathbf{N}(s)^T \mathbf{p}^{k0}(s) ds \quad (46)$$

In Eqns (42), (44) and the following section, $\mathbf{u}_{h,\xi}^{km}$ and $\mathbf{u}_h^{km}$ are used to represent $\mathbf{u}_{h,\xi}^{km}(\xi=1)$ and $\mathbf{u}_h^{km}(\xi=1)$.

When $t>0$, substituting Eqn (29) into Eqn (38) then gives

$$\begin{aligned} \int_V \delta \mathbf{\epsilon}(\xi, s)^T \mathbf{D} \mathbf{\epsilon}^{km} dV + \beta_m \alpha_2 \sum_{i=1}^{m-1} \gamma_i \int_V \delta \mathbf{\epsilon}(\xi, s)^T \mathbf{D} \mathbf{\epsilon}^{ki} dV - \beta_m \alpha_1 \int_V \delta \mathbf{\epsilon}(\xi, s)^T \mathbf{D} \mathbf{\epsilon}^{k0} dV \\ + \beta_m \int_V \delta \mathbf{\epsilon}(\xi, s)^T \boldsymbol{\sigma}^{k0} dV - \int_S \delta \mathbf{u}(s)^T \mathbf{p}^{km}(s) ds = 0 \end{aligned} \quad (47)$$

Using Eqns (34), (38-40) and Green's Theorem then yields [6-7]

$$\begin{aligned} \delta \mathbf{u}^T \{ \mathbf{E}_0 \mathbf{u}_{h,\xi}^{km} + \mathbf{E}_1^T \mathbf{u}_h^{km} \} \\ + \delta \mathbf{u}^T \left\{ \beta_m \alpha_2 \sum_{i=1}^{m-1} \gamma_i \{ \mathbf{E}_0 \mathbf{u}_{h,\xi}^{ki} + \mathbf{E}_1^T \mathbf{u}_h^{ki} \} - \beta_m \alpha_1 \{ \mathbf{E}_0 \mathbf{u}_{h,\xi}^{k0} + \mathbf{E}_1^T \mathbf{u}_h^{k0} \} + \beta_m \mathbf{P}^{k0} - \mathbf{P}^{km} \right\} \\ - \beta_m \alpha_2 \sum_{i=1}^{m-1} \gamma_i \int_0^1 \delta \mathbf{u}(\xi)^T \left\{ \mathbf{E}_0 \xi \mathbf{u}_h^{ki}(\xi)_{,\xi\xi} + [\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1] \mathbf{u}_h^{ki}(\xi)_{,\xi} - \mathbf{E}_2 \frac{1}{\xi} \mathbf{u}_h^{ki}(\xi) \right\} d\xi \\ + \beta_m \alpha_1 \int_0^1 \delta \mathbf{u}(\xi)^T \left\{ \mathbf{E}_0 \xi \mathbf{u}_h^{k0}(\xi)_{,\xi\xi} + [\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1] \mathbf{u}_h^{k0}(\xi)_{,\xi} - \mathbf{E}_2 \frac{1}{\xi} \mathbf{u}_h^{k0}(\xi) \right\} d\xi \\ - \int_0^1 \delta \mathbf{u}(\xi)^T \left\{ \mathbf{E}_0 \xi \mathbf{u}_h^{km}(\xi)_{,\xi\xi} + [\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1] \mathbf{u}_h^{km}(\xi)_{,\xi} - \mathbf{E}_2 \frac{1}{\xi} \mathbf{u}_h^{km}(\xi) \right\} d\xi = 0 \end{aligned} \quad (48)$$

where

$$\mathbf{P}^{k0} = \int_S \mathbf{N}(s)^T \mathbf{p}^{k0}(s) ds = \sum_{m=0} \mathbf{P}^{(k-1)m} \quad k=2,3,\dots \quad (49)$$

Furthermore

$$\mathbf{P}^{*km} = \mathbf{E}_0 \mathbf{u}_{h,\xi}^{km} + \mathbf{E}_1^T \mathbf{u}_h^{km} \quad (50)$$

$$\mathbf{E}_0 \xi^2 \mathbf{u}_h^{ki}(\xi)_{,\xi\xi} + [\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1] \xi \mathbf{u}_h^{ki}(\xi)_{,\xi} - \mathbf{E}_2 \mathbf{u}_h^{ki}(\xi) = \mathbf{0} \quad i=0,1,2,\dots,m \quad (51)$$

where

$$\mathbf{P}^{*km} = \mathbf{P}^{km} - \beta_m \alpha_2 \sum_{i=1}^{m-1} \gamma_i \mathbf{P}^{*ki} + \beta_m \alpha_1 \mathbf{P}^{*k0} - \beta_m \mathbf{P}^{k0} \quad m=1,2,\dots \quad (52)$$

$$\mathbf{P}^{*ki} = \mathbf{E}_0 \mathbf{u}_h^{ki} + \mathbf{E}_1^T \mathbf{u}_h^{ki} \quad i=0,1,2,\dots,m-1 \quad (53)$$

$\mathbf{u}^{km}(\xi)$ is approximated by

$$\mathbf{u}^{km}(\xi) = c_1^{km} \xi^{-\lambda_1} \phi_1 + c_2^{km} \xi^{-\lambda_2} \phi_2 + \dots = \sum_{i=1}^n c_i^{km} \xi^{-\lambda_i} \phi_i \quad k=1,2,\dots \quad m=0,1,2,\dots \quad (54)$$

When $\xi = 1$

$$\mathbf{u}_h^{km} = c_1^{km} \phi_1 + c_2^{km} \phi_2 + \dots = \sum_{i=1}^n c_i^{km} \phi_i = \Phi \mathbf{c}^{km} \quad (55)$$

where

$$\Phi = [\phi_1 \quad \phi_2 \quad \dots \quad \phi_n] \quad (56)$$

$$\mathbf{c}^{km} = [c_1^{km}, c_2^{km}, \dots, c_n^{km}]^T \quad (57)$$

Therefore

$$[\lambda_i^2 \mathbf{E}_0 - \lambda_i [\mathbf{E}_1^T - \mathbf{E}_1] - \mathbf{E}_2] \phi_i = \mathbf{0} \quad (58)$$

$$\mathbf{P}^{*km} = \mathbf{Q} \mathbf{c}^{km} \quad (59)$$

where

$$\mathbf{Q} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \dots \quad \mathbf{q}_n] \quad (60)$$

$$\mathbf{q}_i = [\mathbf{E}_1^T - \lambda_i \mathbf{E}_0] \phi_i \quad (61)$$

Eqns (58) and (61) can be further expressed by [6-9]

$$\begin{bmatrix} \mathbf{E}_0^{-1} \mathbf{E}_1^T & -\mathbf{E}_0^{-1} \\ \mathbf{E}_1 \mathbf{E}_0^{-1} \mathbf{E}_1^T - \mathbf{E}_2 & -\mathbf{E}_1 \mathbf{E}_0^{-1} \end{bmatrix} \begin{Bmatrix} \phi \\ \mathbf{q} \end{Bmatrix} = \lambda \begin{Bmatrix} \phi \\ \mathbf{q} \end{Bmatrix} \quad (62)$$

Left-multiplying both sides of Eqn (55) with Φ^{-1} then yields

$$\mathbf{c}^{km} = \Phi^{-1} \mathbf{u}_h^{km} \quad (63)$$

Thus

$$\mathbf{Q} \mathbf{c}^{km} = \mathbf{Q} \Phi^{-1} \mathbf{u}_h^{km} = \mathbf{K} \mathbf{u}_h^{km} = \mathbf{P}^{*km} \quad k=1,2,\dots \quad m=0,1,2,\dots \quad (64)$$

The boundary $\xi = 1$ is divided into several segments at each of which $N(s)$ is defined by EFGM [7-9, 35, 36].

$$[N_1(s), \dots, N_{n_i}(s)] = \boldsymbol{\varphi}(s)^T \boldsymbol{\Theta}^{-1} \quad (65)$$

where

$$\boldsymbol{\varphi}(s)^T = [\varphi_1(s) \quad \varphi_2(s) \quad \dots \quad \varphi_{n_i}(s)] \quad (66)$$

$$\Theta = \begin{bmatrix} \varphi_1(s_1) & \dots & \varphi_{n_l}(s_1) \\ \dots & \dots & \dots \\ \varphi_1(s_{n_l}) & \dots & \varphi_{n_l}(s_{n_l}) \end{bmatrix} \quad (67)$$

$$\varphi_l(s) = \mathbf{p}(s)^T \mathbf{A}(s)^{-1} \mathbf{H}_l(s) \quad (68)$$

$$\mathbf{A}(s) = \sum_{l=1}^{n_x} w_l(s) \mathbf{p}(s_l) \mathbf{p}(s_l)^T \quad (69)$$

$$\mathbf{H}_l(s) = w_l(s) \mathbf{p}(s_l) \quad (70)$$

$$\mathbf{p}(s)^T = [1, s] \quad (71)$$

n_l is the total number of nodes at the l th segment. n_x is the total number of nodes in the domain of definition of point s at the segment.

A spline weight function [7-9] is employed in this paper, i.e.

$$w_l(s) = \begin{cases} 1 - 6\left(\frac{|s - s_l|}{r_l}\right)^2 + 8\left(\frac{|s - s_l|}{r_l}\right)^3 - 3\left(\frac{|s - s_l|}{r_l}\right)^4 & 0 < |s - s_l| < r_l \\ 0 & |s - s_l| \geq r_l \end{cases} \quad (72)$$

where s_l is s coordinate of l th node, and r_l is the size of support domain.

Therefore

$$\begin{bmatrix} N_1(s)_{,s}, & \dots, & N_{n_l}(s)_{,s} \end{bmatrix} = \boldsymbol{\varphi}(s)_{,s}^T \Theta^{-1} \quad (73)$$

where

$$\boldsymbol{\varphi}(s)_{,s}^T = \mathbf{p}(s)_{,s}^T \mathbf{A}(s)^{-1} \mathbf{H}(s) + \mathbf{p}(s)^T \left[\mathbf{A}(s)_{,s}^{-1} \mathbf{H}(s) + \mathbf{A}(s)^{-1} \mathbf{H}(s)_{,s} \right] \quad (74)$$

$$\mathbf{A}(s)_{,s}^{-1} = -\mathbf{A}(s)^{-1} \mathbf{A}(s)_{,s} \mathbf{A}(s)^{-1} \quad (75)$$

Using Eqn (64) in which stiffness matrix $\mathbf{K}$ needs to be generated only one time, the solution of a viscoelastic problem can be achieved in a recursive process, at the beginning of each time interval, $\mathbf{u}^{k0}$ is determined by solving an elastic problem defined via Eqn (23) or by Eqn (26).

A self-adaptive computation is carried out at each of the time intervals with a convergence criterion

$$\text{Abs} \left(\left(u_{hj}^{kR} s_t^R / \sum_{m=0}^{R-1} u_{hj}^{km} s_t^m \right)_{s_t=1} \right) \leq \beta \quad (76)$$

where β is an error bound, u_{hj}^{km} denotes the j -th component of $\mathbf{u}_h^{km}$ ($m = 1, 2, \dots, R$). Every $\mathbf{u}_h^{kR}$ ($R = 1, 2, \dots$) is required to be checked with the above criterion, if the criterion is satisfied consecutively 3 times, computing will stop at the time interval considered, and step into the next one. If the criterion is not met, the next order ($R + 1$) computation will continue till reaching the convergence.

In the computation, mm and mim , the upper and lower bound of R , will be prescribed in advance.

If condition (76) is not satisfied when $R = mm$, a size decrement of time step is necessary to restart the recursive procedure at the current time interval; if condition (76) is satisfied when $R < mim$, a size increment of the time step can be considered at the next time interval.

It seems difficult to give a rigorous way to determine mm , mim , and sizes increment and decrement of time steps, in this paper, $mm=20$, $mim=5$, size increment and decrement are both half size of current time step for the convenience.

Eqn (64) can be rearranged as

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{ha}^{km} \\ \bar{\mathbf{u}}_{hb}^{km} \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{P}}_a^{*km} \\ \mathbf{P}_b^{*km} \end{Bmatrix} \quad k=1,2,\dots m=0,1,2,\dots \quad (77)$$

where $\mathbf{u}_{ha}^{km}$ and $\bar{\mathbf{u}}_{hb}^{km}$ represent the unknown and known expanding coefficients of nodal displacements, respectively, while $\bar{\mathbf{P}}_a^{*km}$ and $\mathbf{P}_b^{*km}$ represent the corresponding known and unknown expanding coefficients of nodal forces, respectively.

Furthermore, the following equations can be obtained

$$\mathbf{K}_{aa} \mathbf{u}_{ha}^{km} + \mathbf{K}_{ab} \bar{\mathbf{u}}_{hb}^{km} = \bar{\mathbf{P}}_a^{*km} \quad (78)$$

$$\mathbf{K}_{ba} \mathbf{u}_{ha}^{km} + \mathbf{K}_{bb} \bar{\mathbf{u}}_{hb}^{km} = \mathbf{P}_b^{*km} \quad (79)$$

Via Eqn (78), the expanding coefficient of unknown nodal displacements can be obtained as follow

$$\mathbf{u}_{ha}^{km} = \mathbf{K}_{aa}^{-1} (\bar{\mathbf{P}}_a^{*km} - \mathbf{K}_{ab} \bar{\mathbf{u}}_{hb}^{km}) \quad k=1,2,\dots m=0,1,2,\dots \quad (80)$$

For a special case where

$$\bar{\mathbf{u}}_{hb}^{k0} = \bar{\mathbf{u}}_{hb}^{(k-1)0} \quad k=2,3,\dots m=0 \quad (81)$$

$$\bar{\mathbf{u}}_{hb}^{km} = \mathbf{0} \quad k=1,2,\dots m=1,2,\dots \quad (82)$$

$$\bar{\mathbf{P}}_a^{km} = \mathbf{0} \quad k=1,2,\dots m=0,1,2,\dots \quad (83)$$

It can be proved

$$\bar{\mathbf{P}}_a^{*km} = \mathbf{0} \quad k=1,2,\dots m=0,1,2,\dots \quad (84)$$

$$\mathbf{u}_{ha}^{km} = \mathbf{0} \quad k=1,2,\dots m=1,2,\dots \quad (85)$$

Furthermore

$$\mathbf{u}_h^{km} = \begin{Bmatrix} \mathbf{u}_{ha}^{km} \\ \bar{\mathbf{u}}_{hb}^{km} \end{Bmatrix} = \mathbf{0} \quad k=1,2,\dots m=1,2,\dots \quad (86)$$

$$\mathbf{u}_h^{k0} = \begin{Bmatrix} \mathbf{u}_{ha}^{k0} \\ \bar{\mathbf{u}}_{hb}^{k0} \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_{ha}^{(k-1)0} \\ \bar{\mathbf{u}}_{hb}^{(k-1)0} \end{Bmatrix} = \mathbf{u}_h^{(k-1)0} \quad k=2,3,\dots m=0 \quad (87)$$

$$\boldsymbol{\varepsilon}^{km} = \mathbf{0} \quad k=1,2,\dots m=1,2,\dots \quad (88)$$

$$\boldsymbol{\varepsilon}^{k0} = \boldsymbol{\varepsilon}^{(k-1)0} \quad k=2,3,\dots m=0 \quad (89)$$

$$\boldsymbol{\sigma}^{km} = \beta_m (-\alpha_1 \boldsymbol{\sigma}^{j0} + \boldsymbol{\sigma}^{k0}) \quad k=1,2,\dots m=1,2,\dots j=1 \quad (90)$$

4. Numerical examples

Four numerical examples including creep and relaxation analysis are given to verify the proposed algorithm.

For the creep analysis, we assume $\mathbf{b}=\mathbf{0}$, $\tilde{\mathbf{u}}=\mathbf{0}$, $\tilde{\mathbf{p}}$ is time independent; for the relaxation analysis we assume $\mathbf{b}=\mathbf{0}$, $\tilde{\mathbf{p}}=\mathbf{0}$, $\tilde{\mathbf{u}}$ is time independent.

Under the above assumptions, it can be proved that the solution of displacement in a creep problem and the solution of stress in a relaxation problem can be presented using a product of creep compliance with an elastic displacement solution or a product of relaxation modulus with an elastic stress solution. The elastic solutions can be obtained by either analytical means or ANSYS [11]. In this way, the reference solutions can be gained, and used for the comparison with the solutions of proposed algorithm.

For all examples, $\beta=1\times 10^{-9}$.

Numerical Example 1

Consider a square viscoelastic plate. Regarding to the creep and relaxation cases, two boundary conditions are given in Fig. 3 (a) and Fig. 3 (b) where $l=1\text{m}$, $q=1\text{N/m}$, and $u_x=5\text{mm}$. Plane stress conditions are assumed with $E_1=0.5\text{MPa}$, $E_2=1\text{MPa}$, $\eta_1=1\times 10^7 \text{ N}\cdot\text{s/m}^2$ and $\nu = 0.25$. Nodes are uniformly arranged along the boundary with a uniform $\Delta s = 0.25\text{m}$.

The solution of u_x at node A given by the proposed algorithm with uniform sizes of time steps (10 seconds) is presented in Table 1 and Fig.4, and is compared with a reference solution.

The solution of σ_{xx} at node B given by the proposed algorithm with uniform sizes of time steps (10 seconds) is presented in Table 2 and Fig.5, and is compared with a reference solution [37].

Numerical Example 2

Consider a square plate with a square hole. Regarding to creep and relaxation cases, two boundary conditions are given in Fig. 6 (a) and Fig. 6 (b) where $a=1\text{m}$, $q=1\text{N/m}$ and $u_x=5\text{cm}$. Due to the biaxial symmetry, only one quarter of the plate is taken into account as shown in Fig. 6 (c) and Fig. 6 (d). Plane stress conditions are assumed with $E_1=2000\text{Pa}$, $E_2=2000\text{Pa}$, $\eta_1=8\times 10^5 \text{ N}\cdot\text{s/m}^2$ and $\nu = 0.3$.

Nodes are arranged along the boundary with a uniform $\Delta s = 0.25\text{m}$, except side faces.

The results obtained by the proposed approach is compared with a reference solution whose elastic solution is given by ANSYS with 15309 nodes and 7362 6-node-triangular finite elements.

The solution of u_y at point O given by the proposed algorithm with adaptive sizes of time steps is presented in Table 3 where 1 check, 2 checks and 3 checks mean that the criterion (76) is satisfied only 1 time, consecutively 2 and 3 times respectively. The maximum relative error between the proposed algorithm and reference solution is about 0.03%. Fig. 8 and Fig. 9 exhibit the adaptive procedures of size of time interval and power order, respectively.

The solution of σ_{xx} at node H given by the proposed algorithm with adaptive sizes of time steps is presented in Table 4 and Fig. 10, and The maximum relative error between the proposed algorithm

and reference solution is about 0.41%.

Numerical Example 3

Consider an infinite viscoelastic plate containing a hole of unit radius, subjected to an internal uniform pressure of unity. Plane strain conditions are assumed with Young's Modulus $E_1=0.1\text{MPa}$, $E_2=0.1\text{MPa}$, $\eta_1=1\times 10^7 \text{ N}\cdot\text{s}/\text{m}^2$ and Poisson ratio $\nu = 0.3$.

Due to the biaxial symmetry of the problem, only one quarter of the plate is considered as shown in Fig. 11. If the centre of the hole is defined as the scaling centre, nodes only need to set along the boundary of the hole, and are arranged with a uniform $\Delta s = l/6$ where l is the length of the boundary.

The solution of u_x at node C given by the proposed algorithm with adaptive sizes of time steps is presented in Table 5 and Fig. 12, and is compared with a reference solution [37]. The maximum relative error between the proposed algorithm and reference solution is about 0.035%.

Numerical Example 4

Consider an infinite plate with a through crack subjected uniform tension, as shown in Fig. 13 (a) where $a=1\text{m}$ and $\sigma_0=1\text{N/m}$. Plane stress conditions are assumed with $E_1=2000\text{Pa}$, $E_2=2000\text{Pa}$, $\eta_1=8\times 10^5 \text{ N}\cdot\text{s}/\text{m}^2$, $\nu = 0.25$.

Due to the biaxial symmetry, only one quarter of the plate is considered as shown in Fig. 13 (b). where the computing model of EFG-SBM consists of a bounded domain with the scaling centre at the crack tip, and an unbounded domain with the scaling centre at the middle of the crack.

The solution of u_x at node A given by the proposed algorithm with adaptive sizes of time steps is presented in Table 6 and Fig. 14, and compared with a reference solution given by [11, 38]. The maximum relative error between the proposed algorithm and reference solution is about 0.9%.

Computing remarks

- (1) In the section 3, a consecutive 3 checks based principle is presented to secure the computing accuracy. Actually numerical test of example 2 indicates that only one check is sufficient.
- (2) For examples 1, 3, and 4, the elastic solutions related to the reference solutions can be obtained analytically, it means that these reference solutions are exact solutions. However example 2 needs a FEM based convergent elastic solution that requires more DOF of unknowns in comparison with EFG-SBM. In this example, the proposed algorithm needs only 50 DOF, the convergent elastic solution given by ANSYS needs 30618 DOF. An ANSYS based viscoelastic analysis for this problem may lead to more DOF.
- (3) Fig 8 exhibits the adaptive procedure of size of time interval. When creep tends to a stable state, the variation of displacement becomes less and less, and the size of time step gradually becomes relative bigger in the adaptive process. When the size of time step is enlarged, more recursive computing may be required to satisfy Eqn (76) as shown in Figs 9.

5. Conclusions

This paper proposes an EFG-SBM based numerical method via a combination with a temporally adaptive algorithm for the viscoelastic analysis, which may be particularly beneficial to solve viscoelastic problems with unbounded domains or singularity. The work presented extends the

application of SBM since there seems no any direct report relevant to SBM exploration on the viscoelastic problems. The major merits include

- a) A spatially and temporally coupled viscoelastic problem is converted into a series of recurrent boundary value problems which are solved by EFG-SBM that is semi-analytical, and advantageous in dealing with the problems involving unbounded domains and singularity.
- b) A self-adaptive computation can be realized in the time domain to adapt the change of the size of time step, and results in a more precise description for the variation of variables.
- c) In the whole recursive process, the eigenvalue equations and the stiffness matrix need to be solved only one time.
- d) Both creep and relaxation analyses can easily be carried out using the proposed algorithm.

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Fig.1. Three-parameter solid.

Fig. 2. Definition of the scaled boundary coordinate system.

Fig. 3. A square plate subjected to uniform tension or prescribed displacement along two opposite sides ($x_A=0.5\text{m}$, $y_A=0$ and $x_B=0.25\text{m}$, $y_B=0.5\text{m}$).

Fig. 4. Numerical comparison of u_x at node A.

Fig. 5. Numerical comparison of σ_{xx} at node B .

Fig. 6. A square plate with a square hole with uniform tension or prescribed displacement at two sides.

Fig. 7. Numerical comparison of u_y at node O .

Fig. 8. Size of time interval.

Fig. 9. The max power order of expansionl at each time interval.

Fig. 10. Numerical comparison of σ_{xx} at point H .

Fig. 11. Model for plane strain plate of infinite extent with circular hole subjected to an internal uniform pressure ($x_C= 0.9659\text{m}$, $y_C= 0.2588\text{m}$).

Fig. 12. Numerical comparison of u_x at node C.

Fig. 13. An infinte plate with a through crack subjected uniform tension.

Fig. 14. Numerical comparison of u_y at point A.

Table 1 Numerical comparison of u_x at node A ($x_A=0.5\text{m}$, $y_A=0$)

t (s)	Proposed algorithm(10^{-6} m)	Analytical solution(10^{-6} m)
10	0.8934693	0.8934693
20	1.1321206	1.1321206
30	1.2768698	1.2768698
40	1.3646647	1.3646647
50	1.4179150	1.4179150
60	1.4502129	1.4502129
70	1.4698026	1.4698026
80	1.4816844	1.4816844
90	1.4888910	1.4888910
100	1.4932621	1.4932621
110	1.4959132	1.4959132

Table 2 Numerical comparison of σ_{xx} at node B ($x_B=0.25\text{m}$, $y_B=0.5\text{m}$)

t (s)	Proposed algorithm(kPa)	Analytical solution(kPa)
10	4.8208677	4.8208677
20	3.6652471	3.6652471
30	3.4073933	3.4073933
40	3.3498583	3.3498583
50	3.3370206	3.3370206
60	3.3341561	3.3341561
70	3.3335169	3.3335169
80	3.3333743	3.3333743
90	3.3333425	3.3333425
100	3.3333354	3.3333354
110	3.3333338	3.3333338

Table 3 Numerical comparison of u_y at point O ($x_O= 1\text{m}$, $y_O= 1\text{m}$) after 1, 2 and 3 checks

t (s)	Proposed algorithm (mm)			Reference solution (mm)
	1 check	2 checks	3 checks	
100.00	1.8747711	1.8747711	1.8747711	1.8741744
200.00	2.1392382	2.1392382	2.1392382	2.1385574
300.00	2.3452054	2.3452054	2.3452054	2.3444591
400.00	2.5056128	2.5056128	2.5056128	2.5048154
500.00	2.6305382	2.6305382	2.6305382	2.6297011
600.00	2.7278303	2.7278303	2.7278303	2.7269621
700.00	2.8036014	2.8036014	2.8036014	2.8027091
800.00	2.8626119	2.8626119	2.8626119	2.8617009
900.00	2.9085694	2.9085694	2.9085694	2.9076438
1000.00	2.9443612	2.9443612	2.9443612	2.9434242

Table 4 Numerical comparison of σ_{xx} at point H ($x_H= 1.1\text{m}$, $y_H= 1\text{m}$)

t (s)	Proposed algorithm(Pa)	Reference solution(Pa)
100.00	59.1925326	59.4352083
200.00	50.3994418	50.6060678
300.00	45.0661626	45.2509234
400.00	41.8313652	42.0028641
500.00	39.8693615	40.0328166
600.00	38.6793460	38.8379224
700.00	37.9575652	38.1131824
800.00	37.5197830	37.6736054
900.00	37.2542546	37.4069884
1000.00	37.0932035	37.2452771

Table 5 Numerical comparison of u_x at node C ($x_C= 0.9659\text{m}$, $y_C= 0.2588\text{m}$)

t (s)	Proposed algorithm(10^{-6} m)	Analytical solution(10^{-6} m)
50.00	17.504106	17.497844
100.00	20.501931	20.494596
150.00	22.320203	22.312218
200.00	23.423041	23.414661
250.00	24.091946	24.083327
300.00	24.497657	24.488893
350.00	24.743734	24.734882
400.00	24.892987	24.884081
450.00	24.983513	24.974575
500.00	25.038420	25.029463

Table 6 Numerical comparison of u_y at point A ($x_A=0.9999\text{m}$, $y_A=0$)

t (s)	Proposed algorithm(10^{-6} m)	Analytical solution(10^{-6} m)
100.00	17.107717	17.270365
200.00	19.521041	19.706632
300.00	21.400539	21.603999
400.00	22.864293	23.081670
500.00	24.004266	24.232482
600.00	24.892079	25.128734
700.00	25.583507	25.826737
800.00	26.121992	26.370341
900.00	26.541365	26.793701
1000.00	26.867973	27.123414