Optimal Loading and Protection of Multi-state Systems Considering Performance Sharing Mechanism

Hui Xiao, Daimin Shi, Yi Ding and Rui Peng

Abstract—Engineering systems are designed to carry the load. The performance of the system largely depends on how much load it carries. On the other hand, the failure rate of the system is strongly affected by its load. Besides internal failures, such as fatigue and aging process, systems may also fail due to external impacts such as nature disasters and terrorism. In this paper, we integrate the effect of loading and protection of external impacts on multi-state systems with performance sharing mechanism. The objective of this research is to determine how to balance the load and protection on system elements. An availability evaluation algorithm of the proposed system is suggested and the corresponding optimization problem is solved utilizing genetic algorithms.

Index Terms—multi-state systems, performance sharing, load dependent failure, protection, universal generating function, genetic algorithms

NOMENCLATURE

e_i : multi-state element i
 E_j : the set of multi-state elements located at sub-system j
 L_i : the load on e_i
 $g(L_i)$: the performance of e_i when its load is L_i
 $L_{i,\min}$: the minimum allowable load on e_i
 $L_{i,\max}$: the maximum allowable load on e_i
 N : the number of sub-systems connected in series
 M : the total number of multi-state elements
 G_i : the random performance of e_i
 O_j : the random performance of sub-system j
 W_j : the random demand of sub-system j
 C : the random transmission capacity of the common bus

performance sharing system
 S_j : the random performance surplus of sub-system j
 D_j : the random performance deficiency of sub-system j
 Z : the random amount of redistributed performance
 $\hat{D}$: the system deficiency after redistribution
 x_i : the protection effort allocated to e_i
 f : constant frequency of external impacts
 d : expected impacts intensity
 c_{ri} : constant repair cost of e_i for internal failures
 c_{p_i} : constant repair cost of e_i for failure caused by external impacts
 c_{pi} : unit protection cost of e_i
 t_{ri} : constant repair time of e_i for internal failures
 t_{pi} : constant repair time of e_i for failure caused by external impacts
 T : system lifetime
 C_T : the total system cost
 A^* : the pre-specified system availability requirement
 $\theta(j)$: the number of states of the demand of sub-system j
 q_{jr} : the probability that the demand of sub-system j is in state r
 w_{jr} : the demand of sub-system j when it is in state r
 Q : The number of states of the common bus performance sharing system
 α_β : the probability that the common bus performance sharing system is in state β
 ς_β : the transmission capacity when the common bus performance sharing system is in state β

I. INTRODUCTION

A multi-state series-parallel system consists of N sub-systems connected in series and each sub-system contains several multi-state elements connected in parallel [1-6]. The system fails if and only if at least one of the sub-systems cannot meet its required demand. In the literature, most of the works assume that each sub-system satisfies its own demand individually. However, many examples in real situations such as power systems, communication systems and data processing systems indicate that the surplus performance from one

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sub-system can be transmitted to other sub-systems that are experiencing the performance deficiency. This type of performance sharing mechanism was first studied in [7], and extended in [8, 9].

Many research works have considered the element allocation and maintenance of the multi-state series-parallel system [5, 10, 11], but very limited work considered the effect of loading on the system elements. However, a majority of the engineering systems are designed to carry the load, such as coal conveyors, cargo trucks, and power generating units. The performance of such systems depends on the amount of load it is carrying. Besides, many studies have empirically shown that the failure rate of the system element is strongly affected by its working load [12, 13]. Therefore, it is important to consider the effect of loading when analyzing the availability of multi-state systems. As a result, several recent research works have studied the optimal loading of multi-state systems [14-19].

Besides the internal failures such as aging process and fatigue, the system element may also fail due to external impacts such as natural disasters and terrorism [20]. One approach for improving the survivability of system elements is to make defensive investment to protect the elements [21-23]. The probability that an element is destroyed by the external impact is usually modeled as a function of the external impacts intensity and the protection effort allocated on the element [24-27]. The survival probability of an element is higher if more protection effort is allocated onto it. The optimal trade-off between investment into the maintenance and protection of the elements in a simple parallel system that is subject to both internal failure and natural disasters was studied in [27].

In this paper, we consider a multi-state series-parallel system with common bus performance sharing as shown in Figure 1. The performance of each sub-system is the cumulative performance of the elements within the sub-system. If the performance of any sub-system is more than its demand, the surplus performance can be transmitted to any other deficient sub-system via the common bus. However, the total amount of the performance that can be transmitted among different sub-systems is subjected to the capacity of the common bus. In this research, the capacity is also assumed to be a random variable since the common bus may be concurrently used by several systems. The performance of any system element depends on the load it is carrying. In general, the working performance of each element increases with increasing load. However, increasing the load will also increase the failure rate of the element, and thus reduce the availability of the element. Therefore, the expected performance of the element is not a monotonic function of its load. Besides internal failures, the system elements are also subject to external impacts that occur with a constant frequency. Hence, it is important to consider providing protection to the system elements to increase their survivability against external impacts. For example, if the external impacts are from earthquakes, anti-seismic devices can be installed to protect system elements.

Comparing with other works on common bus systems, we have considered both the effects of internal failure and external impacts on the system elements. To increase the system

availability, the system elements are protected and immediately repaired in case of failure. Besides, the effect of load on element failure rate is considered. Since increasing load can increase the element failure rate, but also increases the element performance, the relationship between load and the system availability is already non-monotonic. With the incorporation of the external impact, it is impossible to give an intuitive answer to how to load and protect the system element. Therefore, it is essential to propose a framework to solve the joint optimization problem of system loading and protection.

Fig.1. Series-parallel system with common bus performance sharing

The rest of the paper is organized as follows. Section 2 discusses the load dependent failure rate and performance. Section 3 provides the availability modeling of the multi-state series-parallel system with performance sharing. Section 4 formulates the joint optimization model of loading and protection. The availability evaluation algorithm of the proposed system based on universal generating function is suggested in Section 5. Section 6 proposes the optimization technique. Numerical experiments are provided in Section 7. The paper is concluded in Section 8.

II. LOAD DEPENDENT FAILURE RATE AND PERFORMANCE

Every multi-state element e_i considered in this paper is able to carry load of different values. However, element e_i can only be in two states: failure state with zero performance and functioning state with working performance $g_i(L_i)$, where L_i is the load carried by element e_i and $g_i(L_i)$ which denotes the performance of element e_i is a function of load L_i . The load on element e_i can vary from $L_{i,\min}$ to $L_{i,\max}$, where $L_{i,\min}$ and $L_{i,\max}$ denote the minimum and maximum allowable load on element e_i respectively. This is the reason why element e_i is considered to be multi-state. As discussed earlier, the expected performance of element e_i is not a monotonic function of the load L_i .

It can be difficult to distinguish the load and the performance in some situations since they can be considered the same. For example, the pressure on a pipe which carries fluid in a laminar flow mode is proportional to the volume of the flow, i.e., the throughput (performance) is proportional to the pressure (load). However, the load-performance relationship can be nonlinear. For instance, consider a pipe carrying fluid in a turbulent flow

mode. The pressure on the pipe is a non-linear function of volume of the flow.

The load dependent performance function $g_i(L_i)$ can take different expressions depending on the situations. Without loss of generality, the function measuring relationship between the performance and the load can be assumed as follows.

$$g_i(L_i) = a_i + c_i L_i \quad (1)$$

where a_i and c_i are the coefficients of the linear equation.

To analyze the availability of systems made up by elements with load dependent failure rates, the load failure relationship must be known beforehand since the availability of each element can be derived based on its failure rate. In the literature, the accelerated life test models play an important role in determining the load-failure rate relationships. The existing accelerated life test models are summarized in [28]. In this paper, the commonly used proportional hazard model (PHM) will be discussed and used in the numerical experiments.

PHM, which was first proposed in [29], has received popularity in the field of reliability engineering in recent years [30-32]. PHM states that the failure rate of an element is the product of the baseline hazard rate and factors based on the conditions. In general, PHM can be expressed as follows.

$$h(t | \mathbf{X}) = h_0(t) \cdot e^{(\beta_1 X_1 + \beta_2 X_2 + \dots + \beta_d X_d)} \quad (2)$$

where $h_0(t)$ denotes the baseline hazard rate as a function of time. $X_1, \dots, X_d$ are the factors that affect the hazard rate function and $\beta_1, \dots, \beta_d$ are the corresponding coefficients.

This paper assumes that load is the only factor that affects the failure rate of each element. Thus, we can simplify the PHM to be a single factor model.

$$h(t | L) = h_0(t) \cdot e^{L\beta} \quad (3)$$

In this research, we consider the scenario where the load on each element is static, i.e., the load on each element does not change during the entire system lifetime. In addition, we assume that the baseline failure rate is constant across time. Given the above conditions, equation (2) can be further simplified to be the following expression.

$$h(L) = h(t | L) = \lambda \cdot e^{L\beta} \quad (4)$$

where $\lambda = h_0(t)$ for all t is the baseline failure rate.

PHM is the most frequently used accelerated life test model in reliability, though there exists many other models in the literatures. Different models take different expressions, but they can be similarly applied to the considered problem like PHM.

III. AVAILABILITY MODELING

The multi-state series-parallel system considered in this paper consists of N sub-systems connected in series. In sub-system j , there are H_j multi-state elements connected in parallel. In total, there are M multi-state elements within the system, i.e., $\sum_{j=1}^N H_j = M$. Let W_j denote the demand of sub-system j . The demand is a random variable with probability mass function $q_{j,r} = P(W_j = w_{j,r})$ and $\sum_{r=1}^{\theta(j)} q_{j,r} = 1$, where $q_{j,r}$ denotes the probability that the demand of sub-system j is in state r with the rate of demand being $w_{j,r}$. In total, there are $\theta(j)$ possible states for the random demand W_j . The performance of sub-system j is the cumulative performance of all elements connected in parallel. Let G_i denote the performance of element e_i . Given that the load on element e_i is L_i , it can be known that $G_i = g_i(L_i)$ if element e_i is working and $G_i = 0$ if element e_i fails. Therefore, the total performance of sub-system j can be denoted as follows.

$$O_j = \sum_{e_i \in E_j} G_i \quad (5)$$

where E_j denotes the set of elements within sub-system j .

Hence, the performance surplus of the series-parallel system with common bus performance sharing can be evaluated by summing up performance surplus from each sub-system.

$$S = \sum_{j=1}^N S_j = \sum_{j=1}^N \max(O_j - W_j, 0) \quad (6)$$

Similarly, the performance deficiency of the system can be expressed as follows.

$$D = \sum_{j=1}^N D_j = \sum_{j=1}^N \max(W_j - O_j, 0) \quad (7)$$

The total amount of transmission among different sub-systems cannot be greater than the minimum of the surplus and deficiency. In addition, the total amount of transmission is further constrained by the capacity of the performance sharing system. The capacity of the performance sharing system is a random variable C with the probability mass function: $\alpha_\beta = P(C = \varsigma_\beta)$ and $\sum_{\beta=1}^Q \alpha_\beta = 1$. Therefore, the total amount of transmission is the minimum of S , D , C .

$$\begin{aligned} Z &= \min(S, D, C) \\ &= \min\left(\sum_{j=1}^N \max(O_j - W_j, 0), \sum_{j=1}^N \max(W_j - O_j, 0), C\right) \quad (8) \end{aligned}$$

The system performance deficiency after performance redistribution is,

$$\begin{aligned}\hat{D} &= D - Z = D - \min(S, D, C) = \max(0, D - \min(S, C)) \\ &= \max\left(0, \sum_{j=1}^N \max(W_j - O_j, 0) - \min\left(\sum_{j=1}^N \max(O_j - W_j, 0), C\right)\right)\end{aligned}\quad (9)$$

The availability of the system is the fraction of time when no performance deficiency exists in the system. Mathematically, it can be denoted as follows.

$$\begin{aligned}R &= P\{\hat{D} = 0\} \\ &= P\left\{\max\left(0, \sum_{j=1}^N \max(W_j - O_j, 0) - \min\left(\sum_{j=1}^N \max(O_j - W_j, 0), C\right)\right) = 0\right\}\end{aligned}\quad (10)$$

It can be verified that the proposed series-parallel system with performance sharing mechanism reduces to a general series-parallel system if $C = 0$ or a parallel system with all elements connected in parallel if $C = \infty$.

IV. OPTIMAL LOADING AND PROTECTION MODEL

Each element within the series-parallel system can be arbitrarily loaded subjected to the minimum and maximum allowable limit, i.e., the load L_i on element e_i must satisfy the condition $L_{i,\min} \leq L_i \leq L_{i,\max}$. Given that the load on element e_i is L_i , its failure rate can be derived based on PHM.

$$h_i(L_i) = \lambda_i \cdot e^{\beta_i L_i} \quad (11)$$

When element e_i fails, it will be repaired immediately and the repair is assumed to be perfect. The cost and time needed for repair are c_{ri} and t_{ri} respectively. It is reasonable to assume that the cost and time for repair is constant for the same element.

In addition, the element is also subjected to external impacts. Let x_i denote the protection effort allocated to element e_i and c_{pi} be the unit protection cost for element e_i . The constant frequency of the external impacts is f with the expected impact intensity being d . The element vulnerability, which is defined as the conditional probability of an element failure caused by an external impacts, is commonly evaluated using the contest function model as follows [33, 34].

$$v(x_i, d) = \frac{d^m}{x_i^m + d^m} \quad (12)$$

where m is the contest intensity parameter.

Let c_{pi} be the constant repair cost for element e_i if the failure is caused by external impacts and t_{pi} be the constant repair time for element e_i . Given that the system lifetime is T , the availability of the element e_i can be expressed as follows.

$$\begin{aligned}A_i(L_i, x_i) &= (T - t_{ri} \cdot \lambda_i e^{\beta_i L_i} \cdot T - f \cdot v(x_i, d) \cdot t_{pi} \cdot T) / T \\ &= 1 - t_{ri} \cdot \lambda_i e^{\beta_i L_i} - f \cdot v(x_i, d) \cdot t_{pi}\end{aligned}\quad (13)$$

The total cost includes the repair cost of both internal failures and external impacts and the cost spent on protection. Assuming that the repair costs c_{ri} and c_{pi} for element e_i are constant and independent of the system state, the total system cost can be expressed as follows.

$$C_T = \sum_{i=1}^M (c_{pi} \cdot x_i + c_{ri} \cdot \lambda_i e^{\beta_i L_i} \cdot T + c_{pi} \cdot f \cdot v(x_i, d) \cdot T) \quad (14)$$

The first term in the parentheses denotes the investment cost of protecting element e_i . The second term denotes the repair cost of failures caused by external impacts for element e_i . The last term represents the repair cost of internal failures for element e_i .

The objective of this research is to find the optimal loading and protection on each element in order to minimize the total system cost while meeting the pre-specified system availability requirement. Let $\mathbf{L} = (L_1, \dots, L_M)$ be the vector representing the load on each element and $\mathbf{X} = (x_1, \dots, x_M)$ be the vector denoting the protection effort allocated to each element. Let $A(\mathbf{L}, \mathbf{X})$ denote the availability of the system given the load on each element is $\mathbf{L}$ and protection effort for each element is $\mathbf{X}$. A^* is the pre-specified system availability requirement. The optimization model can be written as follows.

$$\begin{aligned}\min \quad & C_T = \sum_{i=1}^M (c_{pi} \cdot x_i + c_{ri} \cdot \lambda_i e^{\beta_i L_i} \cdot T + c_{pi} \cdot f \cdot v(x_i, d) \cdot T) \\ \text{s.t.} \quad & A(\mathbf{L}, \mathbf{X}) \geq A^* \\ & x_i \geq 0, \quad \forall i = 1, \dots, M \\ & L_{i,\min} \leq L_i \leq L_{i,\max}, \quad \forall i = 1, \dots, M\end{aligned}\quad (15)$$

In the optimization model above, the objective is to minimize the total system cost C_T . The decision variables are the load given to each element, i.e., $\mathbf{L} = (L_1, \dots, L_M)$, and the protection allocated to each element, i.e., $\mathbf{X} = (x_1, \dots, x_M)$. The first constraint requires that the availability of the system for any combination of $\mathbf{L} = (L_1, \dots, L_M)$ and $\mathbf{X} = (x_1, \dots, x_M)$ must be at least A^* . The second constraint forces the protection effort of any element to be nonnegative. The last constraint defined the maximum and minimum allowable load on each element.

V. AVAILABILITY EVALUATION ALGORITHM

To solve the optimization model (15), the availability of the considered system must be evaluated first. In other words, we must find a procedure which can compute the system availability given the the load on each element is $\mathbf{L}$ and the protection efforts on each element is $\mathbf{X}$.

In this section, we suggest an availability evaluation algorithm for the proposed system based on the universal generating function (UGF) which was first introduced by Ushakov[35]. The UGF technique has been proven to be effective in analyzing the system reliability of different structures such as the networks reliability model [36, 37], sliding window systems[38-40], voting systems [41], linear consecutively connected systems [42], series-parallel systems [5, 9]. The technique has also been applied to analyze the reliability of various real industrial and engineering systems such as distributed generation systems [43], the stress-strength interference model [44] and maritime transportation systems [45]. A comprehensive summary of the applications of the UGF technique to reliability analysis and optimization can be found in [46].

The UGF for a discrete random variable X is defined as the following polynomial.

$$u(z) = \sum_{k=1}^K \zeta_k \cdot z^{X_k} \quad (16)$$

where the variable X has K possible realizations and $\zeta_k = P(X = X_k)$.

In this paper, element e_i is able to support load from $L_{i,\min}$ to $L_{i,\max}$. When the load on element e_i is L_i , the performance of e_i is either $g(L_i)$ if it is working or 0 if it fails. Therefore, the UGF of element e_i can be written as follows.

$$u_i(z | L_i) = (1 - A_i(L_i)) \cdot z^0 + A_i(L_i) \cdot z^{g_i(L_i)} \quad (17)$$

where $A_i(L_i)$ is the probability that element e_i is working given the load on e_i is L_i . Equation (17) indicates that the state distribution of the element e_i depends on the load it is carrying.

Consider a pair of elements e_e and e_f connected in parallel. The total performance of element e_e and e_f is the cumulative performance of the two elements. Define an operator $\oplus$ to derive the combined UGF of the two elements as follows.

$$\begin{aligned} & \oplus(u_e(z | L_e), u_f(z | L_f)) \\ &= \oplus \left(\begin{pmatrix} (1 - A_e(L_e)) \cdot z^0 + A_e(L_e) \cdot z^{g_e(L_e)}, \\ (1 - A_f(L_f)) \cdot z^0 + A_f(L_f) \cdot z^{g_f(L_f)} \end{pmatrix} \right) \\ &= \sum_{h_e=0}^1 \sum_{h_f=0}^1 \left(\begin{pmatrix} (A_e(L_e))^{h_e} \cdot (1 - A_e(L_e))^{1-h_e} \\ (A_f(L_f))^{h_f} \cdot (1 - A_f(L_f))^{1-h_f} \end{pmatrix} \cdot z^{h_e g_e(L_e) + h_f g_f(L_f)} \right) \end{aligned} \quad (18)$$

It can be verified that the operator $\oplus$ satisfies the communicative law.

$$\oplus(u_e(z | L_e), u_f(z | L_f)) = \oplus(u_f(z | L_f), u_e(z | L_e)) \quad (19)$$

and the associative law,

$$\begin{aligned} & \oplus(u_e(z | L_e), u_f(z | L_f), u_g(z | L_g)) \\ &= \oplus(\oplus(u_e(z | L_e), u_f(z | L_f)), u_g(z | L_g)) \quad (20) \\ &= \oplus(u_e(z | L_e), \oplus(u_f(z | L_f), u_g(z | L_g))) \end{aligned}$$

Let $E_j = \{e_{j_1}, \dots, e_{j_{|E_j|}}\}$ denote the set of elements connected in parallel within sub-system j . The performance of sub-system j is the cumulative performance of elements $e_{j_1}, \dots, e_{j_{|E_j|}}$. Therefore, the UGF denoting the performance of sub-system j can be derived by applying the operator $\oplus$ over all elements within sub-system j .

$$\begin{aligned} U_j(z | \mathbf{L}) &= \oplus(u_{j_1}(z | L_{j_1}), \dots, u_{j_{|E_j|}}(z | L_{j_{|E_j|}})) \\ &= \sum_{h_{j_1}=0}^1 \dots \sum_{h_{j_{|E_j|}}=0}^1 \left(\prod_{i=1}^{|E_j|} \left((A_{j_i}(L_{j_i}))^{h_{j_i}} (1 - A_{j_i}(L_{j_i}))^{1-h_{j_i}} \right) \right) \cdot z^{\sum_{i=1}^{|E_j|} h_{j_i} \cdot g_{j_i}(L_{j_i})} \end{aligned} \quad (21)$$

Equation (21) expresses the performance distribution of sub-system j using UGF. Similarly, the demand of sub-system j can be expressed as follows.

$$W_j(z) = \sum_{r=1}^{\theta(j)} q_{jr} \cdot z^{w_{jr}} \quad (22)$$

The combination of the performance rate and demand level of sub-system j constructs a state of sub-system j . The state of sub-system j is described by the performance surplus S_j and performance deficiency D_j . Since S_j and D_j are statistically dependent, a single UGF $\Delta_j(z | \mathbf{L})$ should be derived to represent the state distribution of the sub-system j .

considering both S_j and D_j . $\Delta_j(z | \mathbf{L})$ can be derived using the following operator $\oplus$ as follows.

$$\begin{aligned}
\Delta_j(z | \mathbf{L}) &= U_j(z | \mathbf{L}) \oplus W_j(z) \\
&= \left(\sum_{h_{j1}=0}^1 \cdots \sum_{h_{j|E_j|=0}^1 \prod_{i=1}^{|E_j|} \left((A_{j_i}(L_{j_i}))^{h_{ji}} (1-A_{j_i}(L_{j_i}))^{1-h_{ji}} \right) \cdot z^{\sum_{i=1}^{|E_j|} h_{ji} g_{ji}(L_{j_i})} \right) \\
&\quad \oplus \left(\sum_{r=1}^{\theta(j)} q_{jr} \cdot z^{w_{jr}} \right) \\
&= \sum_{b_j=1}^{B_j} \pi_{b_j}(\mathbf{L}) \cdot z^{\max\left\{\sum_{i=1}^{|E_j|} h_{ji} g_{ji}(L_{j_i}) - w_{jr}, 0\right\}, \max\left\{w_{jr} - \sum_{i=1}^{|E_j|} h_{ji} g_{ji}(L_{j_i}), 0\right\}} \\
&= \sum_{b_j=1}^{B_j} \pi_{b_j}(\mathbf{L}) \cdot z^{s_{b_j}(\mathbf{L}), d_{b_j}(\mathbf{L})}
\end{aligned} \tag{23}$$

where $\pi_{b_j}(\mathbf{L}) = P\left\{(S_j = s_{b_j}(\mathbf{L})) \cap (D_j = d_{b_j}(\mathbf{L}))\right\}$.

Define an operator $\otimes$ to compute the UGF that denotes the joint states of sub-system i and sub-system j as follows.

$$\begin{aligned}
\Delta_j(z | \mathbf{L}) \otimes \Delta_i(z | \mathbf{L}) &= \left(\sum_{b_j=1}^{B_j} \pi_{b_j}(\mathbf{L}) \cdot z^{s_{b_j}(\mathbf{L}), d_{b_j}(\mathbf{L})} \right) \otimes \left(\sum_{b_i=1}^{B_i} \pi_{b_i}(\mathbf{L}) \cdot z^{s_{b_i}(\mathbf{L}), d_{b_i}(\mathbf{L})} \right) \\
&= \sum_{b_j=1}^{B_j} \sum_{b_i=1}^{B_i} \pi_{b_j}(\mathbf{L}) \cdot \pi_{b_i}(\mathbf{L}) \cdot z^{s_{b_j}(\mathbf{L}) + s_{b_i}(\mathbf{L}), d_{b_j}(\mathbf{L}) + d_{b_i}(\mathbf{L})}
\end{aligned} \tag{24}$$

It can be proven that $\otimes$ satisfies the communicative law and associative law like operator $\oplus$. Hence, the UGF of the total system performance surplus and deficiency can be obtained by applying the operator $\otimes$ over all sub-systems.

$$\begin{aligned}
\Gamma(z | \mathbf{L}) &= \otimes(\Delta_1(z | \mathbf{L}), \dots, \Delta_N(z | \mathbf{L})) \\
&= \otimes\left(\left(\sum_{b_1=1}^{B_1} \pi_{b_1}(\mathbf{L}) \cdot z^{s_{b_1}(\mathbf{L}), d_{b_1}(\mathbf{L})}\right), \dots, \left(\sum_{b_N=1}^{B_N} \pi_{b_N}(\mathbf{L}) \cdot z^{s_{b_N}(\mathbf{L}), d_{b_N}(\mathbf{L})}\right)\right) \\
&= \sum_b \pi_b(\mathbf{L}) \cdot z^{s_b(\mathbf{L}), d_b(\mathbf{L})}
\end{aligned} \tag{25}$$

where $\pi_b(\mathbf{L}) = P\left\{(S = s_b(\mathbf{L})) \cap (D = d_b(\mathbf{L}))\right\}$.

Equation (25) represents the UGF for the total system performance surplus and deficiency. However, the capacity of the performance sharing system has not been considered yet. Recall that the transmission capacity of the common bus performance sharing system is a random variable C with the probability mass function: $\alpha_\beta = P(C = \varsigma_\beta)$ and $\sum_{\beta=1}^Q \alpha_\beta = 1$. We can express the state distribution of the random variable C using UGF by the following polynomial.

$$\eta(z) = \sum_{\beta=1}^Q \alpha_\beta \cdot z^{\varsigma_\beta} \tag{26}$$

The system UGF $\Phi(z)$ after considering the transmission capacity can be derived by applying the following operator $\otimes_\varphi$ over the $\Gamma(z | \mathbf{L})$ and $\eta(z)$ such that,

$$\begin{aligned}
\Phi(z | \mathbf{L}) &= \Gamma(z | \mathbf{L}) \otimes_\varphi \eta(z) \\
&= \left(\sum_b \pi_b(\mathbf{L}) \cdot z^{s_b(\mathbf{L}), d_b(\mathbf{L})} \right) \otimes_\varphi \left(\sum_{\beta=1}^Q \alpha_\beta z^{\varsigma_\beta} \right) \\
&= \sum_b \sum_{\beta=1}^Q \alpha_\beta \cdot \pi_b(\mathbf{L}) \cdot z^{\max\{0, d_b(\mathbf{L}) - \min\{s_b(\mathbf{L}), \varsigma_\beta\}\}} \\
&= \sum_{\theta=1}^\Theta \gamma_\theta(\mathbf{L}) \cdot z^{\hat{d}_\theta(\mathbf{L})}
\end{aligned} \tag{27}$$

where $\gamma_\theta(\mathbf{L}) = P\left\{\hat{D} = \hat{d}_\theta(\mathbf{L})\right\}$ is for the entire system.

From equation (10), the availability of the system can be computed as follows.

$$A(\mathbf{L}, \mathbf{X}) = \sum_{\theta=1}^\Theta \gamma_\theta(\mathbf{L}) \cdot I(\hat{d}_\theta(\mathbf{L}) = 0) \tag{28}$$

where $I(x)$ is an indicator function such that $I(x) = 1$ if x is true and $I(x) = 0$ if x is not true.

VI. OPTIMIZATION TECHNIQUE

The optimization model formulated in (15) is a complex combinatorial deterministic optimization problem. It is not realistic to examine all possible solutions exhaustively. In the literature, the genetic algorithm (GA) has been proven efficient in solving large-scale reliability optimization problems [5, 9, 11, 42, 47-49]. The details of GA can be found in [50]. In this section, the GA is adapted in order to implement the specific optimization problem in this paper.

To solve the optimization problem using GA, the solution must be represented in an integer string. Although the load on each element is generally assumed continuous, we can discretize the load based on the required accuracy. For example, the load L_i on element e_i can be represented by an integer k_i such that,

$$L_i = L_{i,\min} + \frac{k_i}{K} (L_{i,\max} - L_{i,\min}) \tag{29}$$

where $k_i = 0, 1, \dots, K$, i.e., the load on element e_i can be selected from the $K+1$ possible choices. The value of K determines the required level of accuracy. Another decision variable in the optimization model (15) is the allocation of the

protection effort $\mathbf{X}=(x_1, \dots, x_M)$. Let Λ be the maximum protection effort that can be allocated to an element. The protection effort allocated to element e_i can be represented by an integer between zero and Λ , i.e., $x_i \in \{0, 1, \dots, \Lambda\}$.

Taking consideration of both loading and protection, we define an integer string $\mathbf{Y}=(y_1, \dots, y_M)$ to represent a solution of the optimization model (15), where y_i refers to the solution corresponding to element e_i and the range of y_i is,

$$0 \leq y_i < (K+1)(\Lambda+1) \quad (30)$$

The integer y_i can be decoded into the load and the protection effort in the following manner.

$$k_i = \lceil y_i / (\Lambda+1) \rceil \quad (31)$$

$$x_i = \text{mod}_{(\Lambda+1)}(y_i) \quad (32)$$

where $\lceil x \rceil$ denotes the greatest integer less than x and $\text{mod}_x y = y - \lceil y/x \rceil \cdot x$ denotes the remainder of y/x .

For a given k_i and x_i , the corresponding y_i is composed as follows.

$$y_i = k_i \cdot (\Lambda+1) + x_i \quad (33)$$

After decoding the solutions, the system availability can be computed using the algorithm suggested in Section 5. Given that the resulted system availability is A , the fitness value in GA can be evaluated as follows.

$$C_{GA} = \sum_{i=1}^M (c_{p1i} \cdot x_i + c_{p2i} \cdot f \cdot v(x_i, d) \cdot T + c_{ri} \cdot \lambda_i \cdot e^{\beta_i L_i} \cdot T) + \omega \cdot I(A < A^*) \quad (34)$$

where ω is a sufficient large penalty imposed if the pre-specified system availability is not met.

For given parental strings $\mathbf{Y}_1$ and $\mathbf{Y}_2$, crossover operator produces offspring string $\mathbf{Y}_3$ by combining the first i elements of $\mathbf{Y}_1$ and last $M-i$ elements of $\mathbf{Y}_2$, where i is a randomly generated integer from 1 to M . The mutation operator swaps elements of the string initially located at two randomly chosen positions.

VII. NUMERICAL EXPERIMENTS

An example of the multi-state system with common bus performance sharing can be found in the collaborative computing system as illustrated in the literature [51]. In this section, we will consider a series-parallel collaborative

computing system with 7 multi-state elements and a common bus line for transmitting performance. The structure of the system is shown below in Figure 2. Each element has only one processor which can work at different performance levels depending on its working load. Elements connected in parallel are working together to satisfy the demand of the sub-system. The demand of each sub-system is determined by the complexity of computation problem in the system.

Fig. 2. Series-parallel system with 3 sub-systems and 7 elements

Each element is different and has different working capacities and failure rates. The working capacity can vary in the range $[L_{i,\min}, L_{i,\max}]$ for each $i=1, \dots, 7$. The parameters of the elements are summarized in Table 1. The demand of each sub-system is summarized in Table 2.

TABLE I
PARAMETERS OF MULTI-STATE ELEMENTS

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
λ_i	0.0105	0.0116	0.0113	0.0109	0.0108	0.0112	0.0095
β_i	0.14	0.17	0.11	0.13	0.14	0.13	0.11
$L_{i,\min}$	20	20	30	10	20	30	20
$L_{i,\max}$	50	40	60	50	50	50	60
a_i	1	2	2	1	3	2	1
c_i	2	2	2	2	2	2	2
t_{ri}	0.056	0.063	0.065	0.058	0.055	0.062	0.061
t_{pi}	0.12	0.09	0.11	0.13	0.09	0.08	0.11
c_{p1i}	5	6	4	3	5	6	7
c_{p2i}	15	16	15	14	16	17	13
c_{ri}	15	14	13	12	16	12	14

TABLE II
SUBSYSTEM DEMAND DISTRIBUTION

Sub-system 1		Sub-system 2		Sub-system 3	
W	p	W	p	W	p
50	0.6	120	0.7	80	0.5
60	0.4	140	0.3	110	0.5

In this experiment, the constant frequency of the external impact is assumed to be 0.5, The expected impact intensity is 30 and contest intensity parameter is 1, i.e., $f = 0.5$, $d = 30$, $m = 2$. The system lifetime is assumed to be 60. Two sets of numerical experiments are conducted. In this first set of

experiments, the capacity of the performance sharing system is defined to be $C=[60 \ 0.5; 30 \ 0.5]$, and the capacity is $C=[20 \ 0.5; 10 \ 0.5]$ for the second set of experiments. The protection investment on each element is assumed to be from zero to 50.

TABLE III
OPTIMAL LOADING AND PROTECTION WITH HIGH TRANSMISSION CAPACITY

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$A^* = 0.85, A = 0.8500, C_T = 3928, C_{rep} = 2476, C_{inv} = 1452$							
L	20	20	30	10	20	30	20
X	47	37	47	47	37	37	37
$A^* = 0.90, A = 0.9000, C_T = 4363, C_{rep} = 3409, C_{inv} = 954$							
L	20	20	33	18	20	30	28
X	22	22	35	33	22	22	33
$A^* = 0.92, A = 0.9201, C_T = 4671, C_{rep} = 3924, C_{inv} = 747$							
L	20	20	30	22	26	30	28
X	22	22	22	23	18	22	18
$A^* = 0.95, A = 0.9504, C_T = 6125, C_{rep} = 5423, C_{inv} = 702$							
L	23	24	36	30	26	32	40
X	2	36	36	14	36	2	14

TABLE IV
OPTIMAL LOADING AND PROTECTION WITH LOW TRANSMISSION CAPACITY

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$A^* = 0.85, A = 0.8501, C_T = 4925, C_{rep} = 3677, C_{inv} = 1248$							
L	20	22	33	30	26	32	20
X	35	34	34	23	43	34	35
$A^* = 0.88, A = 0.8803, C_T = 5371, C_{rep} = 4003, C_{inv} = 1368$							
L	26	24	36	30	26	30	28
X	26	44	44	34	44	49	26
$A^* = 0.90, A = 0.9001, C_T = 6786, C_{rep} = 5296, C_{inv} = 1490$							
L	29	24	39	30	29	36	36
X	46	33	46	12	46	46	48
$A^* = 0.92, A = 0.9201, C_T = 7096, C_{rep} = 5569, C_{inv} = 1527$							
L	29	26	39	38	29	30	36
X	41	42	42	44	44	45	40

The numerical results are summarized in Table 3 and Table 4 for high transmission capacity and low transmission capacity respectively. They are obtained by running GA for 1000 iterations. In the tables, A^* refers to the pre-specified system availability requirement and A is the actual system availability achieved. C_T denotes the total system cost. C_{rep} denotes all repair costs which include both repair cost of internal failures and repair cost caused by external impacts. C_{inv} denotes the investment on protection. Consistent with our intuition, the results indicate that the total system cost increases with more stringent system availability requirement. Higher transmission

capacity reduces the system cost given the same system availability requirement. However, the optimal trade-off between repair cost and protection investment can be different under different settings. In general, no trend can be found if we observe the repair cost and protection cost separately.

It is also interesting to test how the variation of the impact intensity d and the frequency f affect the system cost. To analyze the effect of d , we use the parameter values in Table 1 and 2. The low transmission capacity for the performance sharing system is assumed and the pre-specified system availability requirement is set to be 0.85. By varying d from 10 to 100, the minimal system cost is found by running GA when the frequency is kept to be 0.5. The results are shown in Table 5. Similarly, the minimal system cost is found for different values of frequency while the impact intensity is kept to be 30 as shown in Table 6.

TABLE V
EFFECT OF IMPACT INTENSITY

d	10	20	30	40	50
C_T	3283	4047	4925	5344	6065
d	60	70	80	90	100
C_T	6568	7010	7585	7956	8456

TABLE VI
EFFECT OF FREQUENCY

f	0.1	0.2	0.3	0.4	0.5
C_T	2689	3680	4038	4346	4925
f	0.6	0.7	0.8	0.9	1
C_T	5362	5857	6058	6155	6428

The numerical results in both tables clearly indicate that the system cost is an increasing function of f and d . The results are consistent with the contest function model as well as the system cost function in (15). The sensitivity analysis of d and f implies that reducing the potential impact frequency and the expect intensity can save the future system operation cost when designing an engineering system.

VIII. CONCLUSION

In this paper, we integrate the effect of loading on element failure rate and the protection of system element against external impacts into a multi-state series-parallel system with common bus performance sharing. Different from previous studies, the performance distribution of each multi-state element is unknown. The performance of each element depends on the load it is carrying. In addition, protection effort is able to reduce the system failure caused by external impacts such as nature disasters and terrorism. Although higher load on each element results in higher performance, the element also tends to fail more frequently. Hence, the repair cost increases and element availability reduces. This paper formulates an

optimization model to balance the repair cost and protection investment. An availability evaluation algorithm based on universal generation function is suggested for the proposed system. A collaborative computing system is analyzed as an example to illustrate the proposed model and suggest algorithms.

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