Finite-time stability of genetic regulatory networks with impulsive effects

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Abstract

We study the finite-time stability of genetic regulatory networks with impulsive effects. Using the method of Lyapunov function, sufficient conditions of the finite-stability, in terms of linear matrix inequalities, are established. A numerical example is provided to further illustrate the significance of our results.

Keywords: impulsive genetic regulatory networks, finite-time stability, linear matrix inequalities approach, Lyapunov functions method

1. Introduction

It is well-known that the study of genetic regulatory networks (GRNs) has become a focus of interest in the past few years. It has been proved that many diseases are actually originated from the sicknesses of GRNs of the corresponding cell lineage, which mainly includes cancer, diabetes and AIDS. It is crucial to examine the dynamical behaviors of the GRNs. The stability of GRNs has been analyzed ([1, 2, 3, 4, 5, 6]) to some extent. GRNs illustrate the reciprocities among products of mRNA, genes and proteins, in terms of transcriptional and translational processes. It has been studied extensively over the last couple of decades through biological experiments (see ([7, 8]) for example). However, it is formidable to fully understand this kind of networks. One way is to set up mathematical models and perform rigorous mathematical analysis. Since time delays are ubiquitous in these processes and have influences on the stability of GRNs, it is indispensable in studying the effects from the time delays on the stability of GRNs. In 2002, L. Chen ([9]) presented a model for GRNs, which is described by delayed differential equations. Based on this model, some interesting results have been obtained ([10, 11, 12, 13, 14]). Considering the environment fluctuations between intracellular and extracellular, we focus on the stability of impulsive GRNs with time-varying delays. It is widely accepted that the study of finite-time stability for networks is significant ([15, 16, 17]). Meanwhile, the expression of gene and mRNA translated protein are accomplished in a much relatively short period of time. It is natural to consider the finite-time stability of GRNs. Some results have been obtained ([18, 19, 20]). In our paper, we will study the finite-time stability of GRNs with impulsive effects and time-varying delays.

The paper is organized as follows: In section 2, we give a brief account of the model, problem formulation and some preliminaries. In section 3, we design a lyapunov function that consists of three parts, and derive sufficient conditions represented by linear matrix inequalities for the finite-time stability of GRNs with impulsive effects. In section 4, we give a numerical example to demonstrate the significance of our result.

2. Problem Formulation and some Preliminaries

We study GRNs containing n mRNAs and n proteins represented by the differential equations as

$$\begin{cases} \dot{m}_i(t) = -k_{mi}m_i(t) + c_i(p(t - \tau_p(t))) \\ \dot{p}_i(t) = -k_{pi}p_i(t) + r_i m_i(t - \tau_{mi}(t)) \end{cases} \quad (1)$$

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where $i = 1, 2, \dots, n$, and $m_i(t) \in R_+^n$ denote the concentrations of mRNAs i , $p_i(t) \in R_+^n$ denote the concentrations of protein i . k_{mi} and k_{pi} are positive real numbers that describe the degradation rates for mRNAs i and protein i , respectively. r_i is a positive constant which indicate the rate of translating mRNAs i to protein i . $c_i(p(t - \tau_p(t)))$ is a non-linear function of $p_1(t - \tau_{p1}(t)), \dots, p_n(t - \tau_{pn}(t))$ which present the regulation function of gene i . $\tau_{mi}(t)$ and $\tau_{pi}(t)$ are time delays for mRNAs i and protein i , respectively.

In the translational process represented by the second equation in model (1), $r_i m_i(t)$ mean that one protein is produced by only one mRNA. In the transcriptional process described by the first equation in model (1), a gene or mRNA is usually influenced by multiple proteins. $c_i(p(t))$ is a function of the concentrations $p(t)$ of all possible proteins that represents the relative promoter or repressor activity of all possible proteins to gene i . In this paper, we take $c_i(p(t)) = \sum_{j=1}^n c_{ij}(p_j(t))$, where $c_{ij}(p_j(t))$ is a function of the Hill form below,

$$c_{ij}(p_j(t)) = a_{ij} \frac{1}{1 + (p_j(t)/b_j)^{h_j}},$$

assume that transcription factor j is a repressor of gene i , or

$$c_{ij}(p_j(t)) = a_{ij} \frac{(p_j(t)/b_j)^{h_j}}{1 + (p_j(t)/b_j)^{h_j}},$$

assume that transcription factor j is an activator of gene j . Meanwhile a_{ij} and b_j are non-negative constants, and h_j is the Hill coefficient that represent the degree of cooperativity. Suppose that $h_j \geq 1$ in this paper. So we have the following equation

$$\frac{1}{1 + (p_j(t)/b_j)^{h_j}} = 1 - \frac{(p_j(t)/b_j)^{h_j}}{1 + (p_j(t)/b_j)^{h_j}},$$

Then the system (1) change into the system as follows,

$$\begin{cases} \dot{m}_i(t) = -k_{mi}m_i(t) + \sum_{j=1}^n f_{ij}g_j(p_j(t - \tau_{pj}(t))) + l_i, \\ \dot{p}_i(t) = -k_{pi}p_i(t) + r_i m_i(t - \tau_{mi}(t)), \end{cases} \quad (2)$$

where $i = 1, 2, \dots, n$. $F = (f_{ij})$ is an $n \times n$ matrix which represent regulatory relations of the network. Its definition can be described as:

$$\begin{cases} f_{ij} = -a_{ij}, & j \text{ is a repressor of gene } i, \\ f_{ij} = 0, & j \text{ does not regulate gene } i, \\ f_{ij} = a_{ij}, & j \text{ is a activator of gene } i, \end{cases} \quad (3)$$

Assume that Re is the set of all represses of gene i . So we define as $l_i = \sum_{j \in Re} a_{ij}$, where l_i is a constant. Moreover, the function $g_j(\mu)$ can be written as the following form

$$g_j(\mu) = \frac{(\mu/b_j)^{h_j}}{1 + (\mu/b_j)^{h_j}}$$

For variable μ , it is clear that function $g_j(\mu)$ is monotonically increasing. When $h_j \geq 0$, it is obvious that the function $g_j(\mu)$ is continuously differential for $\mu \geq 0$. So, we get

$$\eta_j = \max_{\mu \geq 0} g'_j(\mu) = \frac{(h_j - 1)^{(h_j-1)/h_j} (h_j + 1)^{(h_j+1)/h_j}}{4b_j h_j} > 0 \quad (4)$$

Let us suppose that $(\bar{m}, \bar{p})$ is an equilibrium point of GRNs (2). Therefore, the following system are established

$$\begin{cases} 0 = -k_{mi}\bar{m}_i + \sum_{j=1}^n f_{ij}g_j(\bar{p}_j) + l_i, \\ 0 = -k_{pi}\bar{p}_i + r_i \bar{m}_i, \end{cases}$$

for $i = 1, 2, \dots, n$. Let $x(t) = m(t) - \bar{m}$ and $y(t) = p(t) - \bar{p}$, we translate the equilibrium $(\bar{m}, \bar{p})$ into the origin. It is not difficult to get

$$\begin{cases} \dot{x}_i(t) = -k_{mi}x_i(t) + \sum_{j=1}^n f_{ij}\tilde{g}_j(y_j(t - \tau_{pj}(t))) + l_i, \\ \dot{y}_i(t) = -k_{pi}y_i(t) + r_i x_i(t - \tau_{mi}(t)), \end{cases}$$

where $i = 1, 2, \dots, n$, and $\tilde{g}_j(y_j(t - \tau_{pj}(t))) = g_j(y_j(t - \tau_{pj}(t)) + \bar{p}_j) - g_j(\bar{p}_j)$. According to the inequation (4), we get

$$0 \leq \tilde{g}_j(y_j(t - \tau_{pj}(t))) \leq \eta_j y_j(t - \tau_{pj}(t)), \quad (5)$$

In this study, we discuss the impulsive GRNs as follows:

$$\begin{cases} \dot{x}(t) = -K_m x(t) + E \tilde{g}(y(t - \tau_p(t))), & t \neq t_k, \\ \dot{y}(t) = -K_p y(t) + D x(t - \tau_m(t)) & , \quad t \neq t_k, \\ \Delta x(t) = J_k(x(t_k^-)) & , \quad t = t_k, \\ \Delta y(t) = H_k(y(t_k^-)) & , \quad t = t_k, \end{cases} \quad (6)$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$, $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T$, $D = \text{diag}(r_1, r_2, \dots, r_n)$, $K_m = \text{diag}(k_{m1}, k_{m2}, \dots, k_{mn})$, $K_p = \text{diag}(k_{p1}, k_{p2}, \dots, k_{pn})$, $E = \text{diag}(\sum_{j=1}^n f_{ij}, \dots, \sum_{j=1}^n f_{nj})$, $\tilde{g}(\cdot) = [\tilde{g}_1(\cdot), \tilde{g}_2(\cdot), \dots, \tilde{g}_3(\cdot)]^T$, the time sequence t_k satisfies $0 < t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$, and $\lim_{k \rightarrow \infty} t_k = \infty$, $J_k, H_k \in R^n$. These sequence J_k, H_k are the effect of suddenly changing the state of system (6). Assume that $\Delta x = x(t_k^+) - x(t_k^-)$ at fixed points sequence t_k , where $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \rightarrow 0^+} x(t_k - h)$. The quadratic form $\|x(t)\|_Q^2$ is definite as: $\|x(t)\|_Q^2 = x^T(t)Q(t)x(t)$ for any state vector $x(t)$, and $Q(t)$ is non-negative definite matrix.

The initial value for the system (6) is : $x(t_0) = \varphi(t)$, $y(t_0) = \psi(t)$.

Definition 1. Given a non-negative definite matrix $Q(t)$, two positive real numbers c_1, c_2 , the system (6) is said to be finite-time stable with respect to $c_1, c_2, Q(t), T$ if and only if:

$$\|\varphi(t)\|_Q^2 + \|\psi(t)\|_Q^2 \leq c_1 \implies \|x(t)\|_Q^2 + \|y(t)\|_Q^2 \leq c_2, \quad t \in [0, T]$$

Assumption 1. $\tau_p(t)$ and $\tau_m(t)$ are time-varying delays satisfying

$$0 \leq \tau_{p1} \leq \tau_p(t) \leq \tau_{p2}, \quad \dot{\tau}_p(t) \leq \tau_{p3} < \infty$$

$$0 \leq \tau_{m1} \leq \tau_m(t) \leq \tau_{m2}, \quad \dot{\tau}_m(t) \leq \tau_{m3} < \infty$$

Assumption 2. For $t \in [t_0, t_0 + T]$, there exist positive-definite matrices Q , such that

$$\|J_k(x(t_k^-))\|_Q^2 \leq \|x(t_k^-)\|_Q^2,$$

$$\|H_k(x(t_k^-))\|_Q^2 \leq \|y(t_k^-)\|_Q^2,$$

Lemma 1. If $M \in R^{(n \times n)}$ is a positive definite matrix, $N \in R^{(n \times n)}$ is a symmetric matrix, then

$$\lambda_{\min}(M^{-1}N)x^T Mx \leq x^T Nx \leq \lambda_{\max}(M^{-1}N)x^T Mx$$

Lemma 2. For any positive definite matrix $P \in R^{n \times n}$, there exist a scalar $q > 0$ and a vector-valued function $\rho : [0, q] \rightarrow R^n$ such that

$$\left(\int_0^q \rho(s) ds \right)^T P \left(\int_0^q \rho(s) ds \right) \leq q \int_0^q \rho^T(s) P \rho(s) ds.$$

3. Main Results

Theorem 1. For impulsive networks (6) with Assumptions 1 and 2, given a scalar $\alpha > 0$, if there exist symmetric positive definite matrices $P_1, P_2, G_1, R_1, R_2, R_3, R_4$, the diagonal matrix $\Gamma_m = \text{diag}(\gamma_{m1}, \dots, \gamma_{mn}) \geq 0, m = 1, 2$, matrices G_2, G_3, M_1 , and scalars λ_1, λ_2 such that the following inequalities

$$\begin{bmatrix} G_2 & M_1 \\ * & G_3 \end{bmatrix} \geq 0$$

$$\Phi = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1,10} \\ & -R_1 - R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & * & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & * & * & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & * & * & * & a_{55} & 0 & 0 & 0 & M_1 + \Gamma_1 N & 0 \\ & * & * & * & * & -R_3 - R_4 & 0 & 0 & 0 & 0 \\ & * & * & * & * & * & a_{77} & 0 & 0 & a_{7,10} \\ & * & * & * & * & * & * & -R_4 & 0 & 0 \\ & * & * & * & * & * & * & * & G_3 - 2\Gamma_1 & 0 \\ & * & * & * & * & * & * & * & * & a_{10,10} \end{pmatrix} < 0 \quad (7)$$

where $a_{11} = -2P_1K_m + G_1 + \tau_{m1}^2 K_m^T R_1 K_m + (\tau_{m2} - \tau_{m1})^2 K_m^T R_2 K_m - R_1 - \alpha P_1$, $a_{33} = -G_1(1 - \tau_{m3}) + \tau_{p1}^2 D^T R_3 D + (\tau_{p2} - \tau_{p1})^2 D^T R_4 D - 2R_2$, $a_{55} = -2P_2K_p + G_2 + \tau_{p1}^2 K_p^T R_3 K_p + (\tau_{p2} - \tau_{p1})^2 K_p^T R_4 K_p - R_3 - \alpha P_2$, $a_{77} = -G_2(1 - \tau_{p3}) - 2R_4$, $a_{10,10} = -G_3(1 - \tau_{p3}) + \tau_{m1}^2 E^T R_1 E + (\tau_{m2} - \tau_{m1})^2 E^T R_2 E - 2\Gamma_2$, $a_{1,10} = P_1 E - \tau_{m1}^2 E^T R_1 K_m - (\tau_{m2} - \tau_{m1})^2 E^T R_2 K_m$, $a_{7,10} = M_1(1 - \tau_{p3}) + \Gamma_2 U$, $(\lambda_1/\lambda_2)e^{\alpha T} c_1 \leq c_2$, and $\lambda_1 = \max \left\{ \lambda_{\max}(Q^{-1}P_1) + \tau_{m2}\lambda_{\max}(Q^{-1}G_1) + 1/2[\tau_{m1}^3 \lambda_{\max}(Q^{-1}R_1) + (\tau_{m2} - \tau_{m1})^3 \lambda_{\max}(Q^{-1}R_2)] \right\}$, $\tau_{p2}(\lambda_{\max}(Q^{-1}G_2) + \eta^2 \lambda_{\max}(Q^{-1}G_3) + 2\eta \lambda_{\max}(Q^{-1}M_1)) + \lambda_{\max}(Q^{-1}P_2) + 1/2[\tau_{p1}^3 \lambda_{\max}(Q^{-1}R_3) + (\tau_{p2} - \tau_{p1})^3 \lambda_{\max}(Q^{-1}R_4)]$, $\eta = \max \eta_i, i = 1, 2, \dots, n$. $\lambda_2 = \min \left[(\lambda_{\min}(Q^{-1}P_1), \lambda_{\min}(Q^{-1}P_2)) \right]$, then, the system (6) is finite time stable with respect to positive real numbers (c_1, c_2, T) .

proof. According to the system (6), we construct the Lyapunov-krasovskii functional as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

where

$$\begin{aligned} V_1(t) &= x^T(t)P_1x(t) + y^T(t)P_2y(t), \\ V_2(t) &= \int_{t-\tau_m(t)}^t x^T(s)G_1x(s)ds + \int_{t-\tau_p(t)}^t \begin{bmatrix} y^T(s), \tilde{g}^T(y(s)) \end{bmatrix} \begin{bmatrix} G_2 & M_1 \\ * & G_3 \end{bmatrix} \times \begin{bmatrix} y(s) \\ \tilde{g}(y(s)) \end{bmatrix} ds, \\ V_3(t) &= \int_{-\tau_{m1}}^0 \int_{t+\theta}^t \tau_{m1} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta + \int_{-\tau_{m2}}^{-\tau_{m1}} \int_{t+\theta}^t (\tau_{m2} - \tau_{m1}) \dot{x}^T(s)R_2\dot{x}(s)dsd\theta \\ &\quad + \int_{-\tau_{p1}}^0 \int_{t+\theta}^t \tau_{p1} \dot{y}^T(s)R_3\dot{y}(s)dsd\theta + \int_{-\tau_{p2}}^{-\tau_{p1}} \int_{t+\theta}^t (\tau_{p2} - \tau_{p1}) \dot{y}^T(s)R_4\dot{y}(s)dsd\theta, \end{aligned}$$

take the derivatives of V_i on the interval $t \in (t_0, t_1)$, we have

$$\dot{V}_1(t) = 2x^T(t)P_1\dot{x}(t) + 2y^T(t)P_2\dot{y}(t), \quad (8)$$

$$\begin{aligned} \dot{V}_2(t) &= x^T(t)G_1x(t) - x^T(t - \tau_p(t))G_1x^T(t - \tau_m(t))(1 - \dot{\tau}_m(t)) + \begin{bmatrix} y^T(t), \tilde{g}^T(y(t)) \end{bmatrix} \begin{bmatrix} G_2 & M_1 \\ * & G_3 \end{bmatrix} \begin{bmatrix} y(t) \\ \tilde{g}(y(t)) \end{bmatrix} \\ &\quad - \begin{bmatrix} y^T(t - \tau_p(t)), \tilde{g}^T(y(t - \tau_p(t))) \end{bmatrix} \begin{bmatrix} G_2 & M_1 \\ * & G_3 \end{bmatrix} \begin{bmatrix} y(t - \tau_p(t)) \\ \tilde{g}(y(t - \tau_p(t))) \end{bmatrix} (1 - \dot{\tau}_p(t)), \end{aligned} \quad (9)$$

$$\begin{aligned} V_3(t) &= \tau_{m1}^2 \dot{x}^T(t)R_1\dot{x}(t) - \int_{t-\tau_{m1}}^t \tau_{m1} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta + (\tau_{m2} - \tau_{m1})^2 \dot{x}^T(t)R_2\dot{x}(t) \\ &\quad - \int_{t-\tau_{m2}}^{t-\tau_{m1}} (\tau_{m2} - \tau_{m1}) \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta + \tau_{p1}^2 \dot{y}^T(t)R_3\dot{y}(t) - \int_{t-\tau_{p1}}^t \tau_{p1} \dot{y}^T(\theta)R_3\dot{y}(\theta)d\theta \\ &\quad + (\tau_{p2} - \tau_{p1})^2 \dot{y}^T(t)R_4\dot{y}(t) - \int_{t-\tau_{p2}}^{t-\tau_{p1}} (\tau_{p2} - \tau_{p1}) \dot{y}^T(\theta)R_4\dot{y}(\theta)d\theta. \end{aligned} \quad (10)$$

By lemma2, we get

$$-\int_{t-\tau_{m1}}^t \tau_{m1} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \leq -[x(t) - x(t - \tau_{m1})]^T R_1 [x(t) - x(t - \tau_{m1})], \quad (11)$$

$$-\int_{t-\tau_{p1}}^t \tau_{p1} \dot{y}^T(\theta)R_3\dot{y}(\theta)d\theta \leq -[y(t) - y(t - \tau_{p1})]^T R_1 [y(t) - y(t - \tau_{p1})], \quad (12)$$

$$\begin{aligned} -\int_{t-\tau_{m2}}^{t-\tau_{m1}} (\tau_{m2} - \tau_{m1}) \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta &= -(\tau_{m2} - \tau_{m1}) \left[\int_{t-\tau_{m2}}^{t-\tau_m(t)} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta + \int_{t-\tau_m(t)}^{t-\tau_{m1}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \right] \\ &= -\int_{t-\tau_{m2}}^{t-\tau_m(t)} (\tau_{m2} - \tau_m(t)) \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta - \int_{t-\tau_m(t)}^{t-\tau_{m1}} (\tau_m(t) - \tau_{m1}) \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \\ &\quad - \int_{t-\tau_{m2}}^{t-\tau_m(t)} (\tau_m(t) - \tau_{m1}) \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta - \int_{t-\tau_m(t)}^{t-\tau_{m1}} (\tau_{m2} - \tau_m(t)) \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \end{aligned}$$

$$\begin{aligned}
&\leq -\left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right]^T R_2 \left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right] \\
&\quad -\left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right]^T R_2 \left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right] \\
&\quad -\frac{\tau_m(t) - \tau_{m1}}{\tau_{m2} - \tau_m(t)} \left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right]^T R_2 \left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right] \\
&\quad -\frac{\tau_{m2} - \tau_m(t)}{\tau_m(t) - \tau_{m1}} \left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right]^T R_2 \left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right] \\
&\leq -\left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right]^T R_2 \left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right] \\
&\quad -\left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right]^T R_2 \left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right]
\end{aligned} \tag{13}$$

Similar to (13), we obtain

$$\begin{aligned}
-\int_{t-\tau_{p2}}^{t-\tau_{p1}} (\tau_{p2} - \tau_{p1}) \dot{y}^T(\theta) R_4 \dot{y}(\theta) d\theta &\leq -\left[y(t-\tau_p(t)) - y(t-\tau_{p2})\right]^T R_4 \left[y(t-\tau_p(t)) - y(t-\tau_{p2})\right] \\
&\quad -\left[y(t-\tau_{p1}) - y(t-\tau_p(t))\right]^T R_4 \left[y(t-\tau_{p1}) - y(t-\tau_p(t))\right].
\end{aligned} \tag{14}$$

In addition, for any $\Gamma_m = \text{diag}(\Gamma_{m1}, \dots, \Gamma_{m2}) \geq 0, m = 1, 2$, the inequality below is established from (5)

$$-2\tilde{g}(y(t))\Gamma_1[\tilde{g}(y(t)) - Ny(t)] - 2\tilde{g}^T(y(t-\tau_p(t)))\Gamma_2[\tilde{g}(y(t-\tau_p(t))) - Ny(t-\tau_p(t))] \geq 0. \tag{15}$$

From (9) to (15), we have

$$\begin{aligned}
\dot{V}(t) &\leq 2x^T(t)P_1 \left[-K_mx(t) + E\tilde{g}(y(t-\tau_p(t)))\right] + 2y^T(t)P_2 \left[-K_py(t) + Dx(t-\tau_m(t))\right] + x^T(t)G_1x(t) \\
&\quad -x^T(t-\tau_p(t))G_1x^T(t-\tau_m(t))(1-\tau_{m3}) + y^T(t)G_2y(t) + 2y^T(t)M_1\tilde{g}(y(t)) + \tilde{g}^T(y(t))G_3\tilde{g}(y(t)) \\
&\quad -y^T(t-\tau_p(t))G_2y(t-\tau_p(t))(1-\tau_{p3}) - 2y^T(t-\tau_p(t))M_1\tilde{g}(y(t-\tau_p(t)))(1-\tau_{p3}) \\
&\quad -\tilde{g}(y^T(t-\tau_p(t)))G_3\tilde{g}(y(t-\tau_p(t)))(1-\tau_{p3}) + \tau_{m1}^2 \left[-K_mx(t) + E\tilde{g}(y(t-\tau_p(t)))\right]^T R_1 \left[-K_mx(t) \right. \\
&\quad \left. + E\tilde{g}(y(t-\tau_p(t)))\right] + (\tau_{m2} - \tau_{m1})^2 \left[-K_mx(t) + E\tilde{g}(y(t-\tau_p(t)))\right]^T R_2 \left[-K_mx(t) + E\tilde{g}(y(t-\tau_p(t)))\right] \\
&\quad + \tau_{p1}^2 \left[-K_py(t) + Dx(t-\tau_m(t))\right]^T R_3 \left[-K_py(t) + Dx(t-\tau_m(t))\right] + (\tau_{p2} - \tau_{p1})^2 \left[-K_py(t) + Dx(t-\tau_m(t))\right]^T \\
&\quad \times R_4 \left[-K_py(t) + Dx(t-\tau_m(t))\right] - \left[x(t) - x(t-\tau_{m1})\right]^T R_1 \left[x(t) - x(t-\tau_{m1})\right] - \left[y(t) - y(t-\tau_{p1})\right]^T \\
&\quad \times R_1 \left[y(t) - y(t-\tau_{p1})\right] - \left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right]^T R_2 \left[x(t-\tau_m(t)) - x(t-\tau_{m2})\right] \\
&\quad -\left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right]^T R_2 \left[x(t-\tau_{m1}) - x(t-\tau_m(t))\right] - \left[y(t-\tau_p(t)) - y(t-\tau_{p2})\right]^T \\
&\quad \times R_4 \left[y(t-\tau_p(t)) - y(t-\tau_{p2})\right] - \left[y(t-\tau_{p1}) - y(t-\tau_p(t))\right]^T R_4 \left[y(t-\tau_{p1}) - y(t-\tau_p(t))\right] \\
&\quad -2\tilde{g}(y(t))\Gamma_1[\tilde{g}(y(t)) - Ny(t)] - 2\tilde{g}^T(y(t-\tau_p(t)))\Gamma_2[\tilde{g}(y(t-\tau_p(t))) - Ny(t-\tau_p(t))] \\
&\leq \zeta^T \Phi \zeta + \alpha \left(x^T(t)P_1x(t) + y^T(t)P_2y(t)\right) \\
&\leq \zeta^T \Phi \zeta + \alpha V(t)
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\zeta^T &= \left[x^T(t), x^T(t-\tau_{m1}), x^T(t-\tau_m(t)), x^T(t-\tau_{m2}), y^T(t), y^T(t-\tau_{p1}), \right. \\
&\quad \left. y^T(t-\tau_p(t)), y^T(t-\tau_{p2}), \tilde{g}^T(y(t)), \tilde{g}^T(y(t-\tau_p(t)))\right].
\end{aligned}$$

By condition (7), we have

$$\dot{V}(t) \leq \alpha V(t), \tag{17}$$

Integrating (17) from t_k to t , we get

$$V(t) \leq e^{\alpha(t-t_k)} V(t_k),$$

Since the Assumption2, we have

$$\begin{aligned} V(t) &\leq e^{\alpha(t-t_k)} V(t_k^-) \\ &\leq e^{\alpha(t-t_{k-1})} V(t_{k-1}) \\ &\leq e^{\alpha(t-t_0)} V(t_0) \\ &\leq e^{\alpha T} V(\varphi(t), \psi(t)) \end{aligned}$$

In another area, by the Assumption1, we get

$$\begin{aligned} V(\varphi(t), \psi(t)) &= \varphi^T(t) P_1 \varphi(t) + \psi^T(t) P_2 \psi(t) \\ &\quad + \int_{t-\tau_m(t)}^t \varphi^T(s) G_1 \varphi(s) ds + \int_{t-\tau_p(t)}^t \begin{bmatrix} \psi^T(s), g^T(\psi(s)) \end{bmatrix} \begin{bmatrix} G_2 & M_1 \\ * & G_3 \end{bmatrix} \times \begin{bmatrix} \psi(s) \\ g(\psi(s)) \end{bmatrix} ds \\ &\quad + \int_{-\tau_{m1}}^0 \int_{t+\theta}^t \tau_{m1} \dot{\varphi}^T(s) R_1 \dot{\varphi}(s) ds d\theta + \int_{-\tau_{m2}}^{-\tau_{m1}} \int_{t+\theta}^t (\tau_{m2} - \tau_{m1}) \dot{\varphi}^T(s) R_2 \dot{\varphi}(s) ds d\theta \\ &\quad + \int_{-\tau_{p1}}^0 \int_{t+\theta}^t \tau_{p1} \dot{\psi}^T(s) R_3 \dot{\psi}(s) ds d\theta + \int_{-\tau_{p2}}^{-\tau_{p1}} \int_{t+\theta}^t (\tau_{p2} - \tau_{p1}) \dot{\psi}^T(s) R_4 \dot{\psi}(s) ds d\theta, \\ &\leq \lambda_{\max}(Q^{-1} P_1) \|\varphi(t)\|^2 + \lambda_{\max}(Q^{-1} P_2) \|\psi(t)\|^2 + \tau_{m2} \lambda_{\max}(Q^{-1} G_1) \|\varphi(t)\|^2 \\ &\quad + \tau_{p2} \left(\lambda_{\max}(Q^{-1} G_2) \|\psi(t)\|^2 + \eta^2 \lambda_{\max}(Q^{-1} G_3) \|\psi(t)\|^2 + 2\eta \lambda_{\max}(Q^{-1} M_1) \right) \|\psi(t)\|^2 \\ &\quad + 1/2 \left[\tau_{m1}^3 \lambda_{\max}(Q^{-1} R_1) + (\tau_{m2} - \tau_{m1})^3 \lambda_{\max}(Q^{-1} R_2) \right] \|\varphi(t)\|^2 \\ &\quad + 1/2 \left[\tau_{p1}^3 \lambda_{\max}(Q^{-1} R_3) + (\tau_{p2} - \tau_{p1})^3 \lambda_{\max}(Q^{-1} R_4) \right] \|\psi(t)\|^2 \\ &\leq \left[\lambda_{\max}(Q^{-1} P_1) + \tau_{m2} \lambda_{\max}(Q^{-1} G_1) + 1/2 \left[\tau_{m1}^3 \lambda_{\max}(Q^{-1} R_1) + (\tau_{m2} - \tau_{m1})^3 \lambda_{\max}(Q^{-1} R_2) \right] \right] \|\varphi(t)\|^2 \\ &\quad + \left[\tau_{p2} \left(\lambda_{\max}(Q^{-1} G_2) + \eta^2 \lambda_{\max}(Q^{-1} G_3) + 2\eta \lambda_{\max}(Q^{-1} M_1) \right) + \lambda_{\max}(Q^{-1} P_2) \right. \\ &\quad \left. + 1/2 \left[\tau_{p1}^3 \lambda_{\max}(Q^{-1} R_3) + (\tau_{p2} - \tau_{p1})^3 \lambda_{\max}(Q^{-1} R_4) \right] \right] \|\psi(t)\|^2 \\ &\leq \lambda_1 \left(\|\varphi(t)\|^2 + \|\psi(t)\|^2 \right) \end{aligned} \tag{18}$$

In the meanwhile,

$$V(t) \geq \min \left[\lambda_{\min}(P_1) \|x(t)\|^2 + \lambda_{\min}(P_2) \|y(t)\|^2 \right] \geq \lambda_2 \left(\|x(t)\|^2 + \|y(t)\|^2 \right)$$

From (17) and (18) we obtain

$$\begin{aligned} \|x(t)\|^2 + \|y(t)\|^2 &\leq \frac{1}{\lambda_2} V(t) \leq \frac{1}{\lambda_2} e^{\alpha T} V(\varphi(t), \psi(t)) \\ &\leq \frac{1}{\lambda_2} e^{\alpha T} \lambda_1 c_1 \leq \frac{\lambda_1}{\lambda_2} e^{\alpha T} c_1 \leq c_2 \end{aligned}$$

Thus we complete the proof.

4. Numerical Examples

In this section, we put forward an example to explain the significance of our results. Let us consider a GRNs model of this system.

$$\begin{cases} \dot{x}(t) = -K_m x(t) + E \tilde{g}(y(t - \tau_p(t))) \\ \dot{y}(t) = -K_p y(t) + D x(t - \tau_m(t)) \\ \Delta x(t) = J_k(x(t_k^-)) \\ \Delta y(t) = H_k(y(t_k^-)) \end{cases} \tag{19}$$

where $K_m = \text{diag}[4, 4, 4]$, $K_p = \text{diag}[4, 4, 4]$, $J_k = \text{diag}[-0.75, -0.75, -0.75]$, $H_k = \text{diag}[-0.95, -0.95, -0.95]$
 $D = \text{diag}[0.5, 0.4, 0.6]$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The gene regulation function is taken as $g(\tilde{x}) = x^2/(1 + x^2)$, $N = \text{diag}(0.65, 0.65, 0.65)$, $t_k = 2.5k$, $T = 5$, $c_1 = 2$, $c_2 = 1.5$, $\alpha = 0.02$, the time delay $\tau_p(t)$, $\tau_m(t)$ are assumed to be:

$$\tau_p(t) = 0.6 + 0.4 \sin(t), \tau_m(t) = 0.3 + 0.1 \cos(t)$$

we obtain the parameters below:

$$\tau_{p1} = 0.2, \tau_{p2} = 1, \tau_{p3} = 0.4, \tau_{m1} = 0.2, \tau_{m2} = 0.4, \tau_{m3} = 0.1$$

We calculate that the following feasible solutions for LMI (7) :

$$P_1 = \begin{pmatrix} 1.7556 & 0.0199 & 0.0199 \\ 0.0199 & 1.7529 & 0.0199 \\ 0.0199 & 0.0199 & 1.7585 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1.0342 & -0.0006 & -0.0007 \\ -0.0006 & 0.9145 & -0.0006 \\ -0.0007 & -0.0006 & 1.1411 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 5.7466 & 0.0419 & 0.0418 \\ 0.0419 & 5.7240 & 0.0418 \\ 0.0418 & 0.0418 & 5.7720 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 8.2762 & -0.0003 & -0.0003 \\ -0.0003 & 8.1893 & -0.0003 \\ -0.0003 & -0.0003 & 8.3543 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 5.2570 & 0.0003 & 0.0003 \\ 0.0003 & 5.2570 & 0.0003 \\ 0.0003 & 0.0003 & 5.2570 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 3.0654 & 0.0739 & 0.0736 \\ 0.0739 & 3.0675 & 0.0737 \\ 0.0736 & 0.0737 & 3.0631 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 2.0856 & 0.0119 & 0.0119 \\ 0.0119 & 2.0778 & 0.0119 \\ 0.0119 & 0.0119 & 2.0943 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 3.4685 & 0.0000 & 0.0000 \\ 0.0000 & 3.4540 & 0.0000 \\ 0.0000 & 0.0000 & 3.4817 \end{pmatrix}$$

$$R_4 = \begin{pmatrix} 1.0090 & 0.0004 & 0.0006 \\ 0.0004 & 1.0915 & 0.0005 \\ 0.0006 & 0.0005 & 0.9355 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -1.6161 & 0.0000 & 0.0000 \\ 0.0000 & -1.6161 & 0.0000 \\ 0.0000 & 0.0000 & -1.6161 \end{pmatrix}$$

$$\Lambda_1 = \begin{pmatrix} 5.8675 & 0.0000 & 0.0000 \\ 0.0000 & 5.8675 & 0.0000 \\ 0.0000 & 0.0000 & 5.8675 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 3.1570 & 0.0000 & 0.0000 \\ 0.0000 & 3.1569 & 0.0000 \\ 0.0000 & 0.0000 & 3.1568 \end{pmatrix}$$

The initial condition is $x(0) = [0.2, 0.2, 0.3]^T$, $y(0) = [0.3, 0.2, 0.5]^T$. The simulation results of the mRNA concentrations are shown in Figures 1. The simulation results of the protein concentrations are shown in Figures 2. So, the system of GRNs with impulsive effects is finite-time stable.

5. Conclusions

We considered the problem of finite-time stability for GRNs with impulsive effects in this paper. The sufficient conditions were provided for the system to be finite-time stable based on lyapunov function method. We represent a numerical example to shown the proposed theoretical results. In the future, we will further study the finite-time stability for the genetic regulatory with noise perturbations([21, 22, 23]), and the dynamics of stochastic GRNS with mixed time-delays([24, 25]).

Figure 1: mRNA concentrations $x(t)$.

Figure 2: Protein concentrations $y(t)$.

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