Exponentially Stable Guaranteed Cost Control for Continuous and Discrete-time Takagi-Sugeno Fuzzy Systems

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Abstract

This paper investigates exponentially stable guaranteed cost control (GCC) for a class of nonlinear systems which is represented by Takagi-Sugeno (T-S) fuzzy systems. State feedback controllers of parallel distributed compensation (PDC) structure are designed by the means of GCC for continuous and discrete-time T-S fuzzy systems respectively. GCC methods in this paper adopt quadratic performance functions, which take effects of control efforts, regulation errors and convergence rates into consideration simultaneously, to provide desirable performance and fast response for closed-loop systems. Sufficient design conditions that guarantee exponential stabilities of resulting closed-loop systems with predefined convergence rates are presented. By setting different convergence rates, response speed of the closed-loop nonlinear system can be adjusted. The proposed design procedures are eventually

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converted into linear matrix inequalities (LMIs) problems of minimizing upper bounds of the guaranteed cost functions. Finally, a well-known nonlinear benchmark control example and a truck-trailer example demonstrate the effectiveness and feasibility of all the proposed methods.

Keywords: Exponential stability, Convergence rate, Guaranteed cost control (GCC), T-S fuzzy system, Linear matrix inequality (LMI)

1. Introduction

In real world, it is not an easy task to design satisfactory controllers for complex systems due to serious inherent **nonlinearities** and parameter **uncertainties** in these systems. Although approximate **nonlinear** mathematical models could be established to describe parts of the nonlinearities and uncertainties, sometimes these models are too complicated to be tackled directly for controller design with existing mathematic tools. Moreover, in some cases limited knowledge and measurement to the system's mechanism even prevent people from obtaining appropriate nonlinear mathematical models. At this point, fuzzy technique provides one of powerful ways to reduce the difficulties, which has been successful in modeling and control of a wide range of various nonlinear systems, **for example, non-adaptive fuzzy control [1, 2], adaptive fuzzy control [3, 4], etc.** Among them, **Takagi-Sugeno (T-S)** fuzzy model which is firstly introduced in [5] have found popularity both in literature and industry since many kinds of nonlinear systems can be approximated by the model succinctly and human's experience could be incorporated in. T-S fuzzy model uses linear combinations of system inputs to form linear subsystems as their inference consequences and then nonlinearly synthesizes these

subsystems to reconstruct the original system outputs. In this indirect way, T-S fuzzy model provides a powerful bridge connecting the actual nonlinear systems and classical linear systems so that the abundant theories and mature techniques in linear systems are possible to be applied, which mitigates the pain of designing controllers for nonlinear systems directly.

Many studies have been done in terms of applying linear control theories into T-S fuzzy models. However, most of existing works concern only the ultimate control performance, for example, stability, tracking, robustness and so on [2, 6, 7, 8, 9]. Little attention has been paid to transient control performance, such as response speed, settling time, overshoot, *etc.* In general, there are infinite controllers which guarantee stabilities of corresponding systems. It is necessary to select one controller which satisfies predefined transient performance from them. So it is interesting to figure out how to obtain a satisfactory controller under some specific transient performance for T-S fuzzy model.

One natural idea is to extend optimal control results of linear system to the T-S fuzzy model. By using the concept of PDC [1], locally optimal T-S fuzzy controllers were designed in [10] through solving standard LQR problem for each subsystems. If the synthesized controller satisfied the sufficient stability conditions for T-S models proposed in [2], a suboptimal controller was obtained, otherwise the design procedure repeated. Obviously this method is trial and error. By relaxing the Riccati equation constraint into inequality, *guaranteed cost control (GCC)* approach was achieved via LMIs through minimizing an upper bound of a quadratic performance measure in [11, 12]. Based on relaxed stability conditions [13], a *globally* suboptimal

fuzzy controller was designed by minimizing the upper bound of a given global quadratic performance function [14]. In [15] a globally optimal and stable fuzzy controller design method for finite and infinite horizons was claimed to be found. However the controller may only be considered as suboptimal according to comments [16, 17].

In reality, in addition to stability and optimality, fast response is also a desired property for control systems. The time optimal control problem is formulated for this objective. Although it is well known that for some classes of nonlinear systems the time optimal control is a bang-bang control [18], this control strategy is rarely implemented in real systems due to the difficulties of characterizing the switching surface. Another concept related to fast response control is the convergence rate [19], which could be explained in the definition of exponential stability, a special case of asymptotic stability [20]. If the system state variable $x(t)$ satisfies

$$\|x(t)\| \leq k\|x(0)\|e^{-\alpha t}, \quad \forall x(0) \in D_0, \quad \forall t > 0$$

where $D_0 = \|x\| < r_0$ is a domain containing $x = 0$, α , r_0 and k are positive constants, then the system is said to be exponentially stable and α is the convergence rate. Tanaka et.al proposed a fuzzy controller design method which can guarantee the predefined convergence rate of closed-loop system in [21]. By using LMI regions relevant to pole location constraints in [22], [23] presents a number of controller design methods for T-S fuzzy systems via H^∞ optimization with pole assignment specifications.

However, all the aforementioned studies aim at either minimizing the standard quadratic performance function or achieving fast response. To author's best knowledge, few results which consider both the optimality and re-

response speed at the same time are available for T-S fuzzy systems. Motivated by above concerns, in this paper, we try to figure out an effective approach of GCC design with desirable convergence rate for a class of nonlinear systems represented by linear T-S fuzzy model. Firstly, the continuous-time modified quadratic cost function and its discrete counterpart are borrowed from classic linear system theories. Secondly, by utilizing the PDC concept, sufficient conditions are presented respectively for continuous- and discrete-time systems such that the designed controllers are able to exponentially stabilize the closed-loop system with predefined convergence rate and simultaneously minimize the upper bound of the cost functions. Thirdly, we show that the controller design problems can be transformed into LMI problems solved efficiently by available convex optimization tools. Finally, simulation results of two numerical examples are presented to illustrate the effectiveness and advantages of the proposed design methods. Compared with the previous literature, the main contributions of this paper are as follows.

1. By using the modified quadratic cost functions which not only represent the control effort and the regulation error but also take response speed into account [24], the proposed control method can provide closed-loop systems with ability of fast response while makes the cost functions minimized.
2. The relationship between response speed and convergence rate of the proposed control method is more intuitive than that between response speed and weighted matrices of classical GCC. By setting different convergence rates, the response speeds of closed-loop systems can be adjusted efficiently.

Notation: Throughout this paper, if not explicitly stated, matrices are assumed to have compatible dimensions. “Continuous-time fuzzy systems” and ”discrete-time fuzzy systems” are abbreviated as CFS and DFS, respectively. $A > 0$ means that the matrix A is positive definite; the notation $A > B$ means that $A - B > 0$ is positive definite; I is the identity matrix of appropriate dimension; 0 is the zero matrix of appropriate dimension; In the square matrix, the symbol ‘*’ stands for the transposed elements in the symmetric positions. In addition, $\sum_{i<j}^r$ and $\sum_{i\neq j}^r$ mean, for instance,

$$\sum_{i<j}^3 x_{ij} = x_{12} + x_{23} + x_{13}$$

$$\sum_{i\neq j}^3 x_{ij} = x_{12} + x_{13} + x_{21} + x_{23} + x_{31} + x_{32}$$

2. Problem formulation and preliminaries

The T-S fuzzy models compose of a set of fuzzy “IF-THEN” rules with fuzzy sets in the antecedents and dynamic LTI systems in the consequents [5]. For a plant to be controlled, we can use the following sets of rules to describe it:

i^{th} Plant Rule: IF $z_1(t)$ is M_{i1} , $z_2(t)$ is M_{i2} , $\dots$, $z_p(t)$ is M_{ip} , THEN $\nabla x(t) = A_i x(t) + B_i u(t)$, $y(t) = C_i x(t)$. $i = 1, 2, \dots, r$.

where for expression simplicity, t is used to represent either continuous or discrete time without causing ambiguity, ∇ represents an operator. For CFS cases $\nabla x(t)$ means $\dot{x}(t)$ and for DFS cases $\nabla x(t)$ represents $x(t+1)$ [25]. $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input vector, $y(t) \in R^l$ is the output vector, $A_i \in R^{n \times n}$, $B_i \in R^{n \times m}$ and $C_i \in R^{l \times n}$. $z(t) \in R^p$ is the

premise variable vector, which is assumed to be independent of the input vector $u(t)$.

Each linear state equation in the consequent parts is called a “subsystem”. By aggregating all the subsystems, the synthesized model is obtained as:

$$\begin{aligned}\nabla x(t) &= \frac{\sum_{i=1}^r w_i(z(t))(A_i x(t) + B_i u(t))}{\sum_{i=1}^r w_i(z(t))} \\ &= \sum_{i=1}^r h_i(z(t))(A_i x(t) + B_i u(t))\end{aligned}\tag{1}$$

$$y(t) = \frac{\sum_{i=1}^r w_i(z(t))C_i x(t)}{\sum_{i=1}^r w_i(z(t))} = \sum_{i=1}^r h_i(z(t))C_i x(t)\tag{2}$$

where

$$\begin{aligned}w_i(z(t)) &= \prod_{j=1}^p M_{ij}(z_j(t)) \geq 0 \\ Z(t) &= [z_1(t) \quad z_2(t) \quad \cdots \quad z_p(t)] \\ h_i(z(t)) &= \frac{w_i(z(t))}{\prod_{i=1}^r w_i(z(t))}\end{aligned}\tag{3}$$

$M_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in the j^{th} fuzzy set M_{ij} for all t . From (1) to (3), there **exist**

$$\begin{cases} 1 \geq h_i(z(t)) \geq 0 \\ \sum_{i=1}^r h_i(z(t)) = 1 \end{cases} \quad i = 1, 2, \dots, r\tag{4}$$

The structure of the PDC controller for above fuzzy **model** is

i^{th} Controller Rule: IF $z_1(t)$ is M_{i1} , $z_2(t)$ is M_{i2} , $\dots$, $z_p(t)$ is M_{ip} , THEN $u_i(t) = -K_i x(t)$. $i = 1, 2, \dots, r$.

where the antecedents are exactly the same **with those of the fuzzy model** and the consequents are linear state feedback laws similar to those in linear

system theories. Aggregating all the controller rules, the synthesized fuzzy controller is represented by:

$$u(t) = -\frac{\sum_{i=1}^r w_i(z(t))K_i x(t)}{\sum_{i=1}^r w_i(z(t))} = -\sum_{i=1}^r h_i(z(t))K_i x(t) \quad (5)$$

Replacing (5) into (1), the closed-loop system is represented by:

$$\begin{aligned} \nabla x(t) &= \sum_{i=1}^r \sum_{j=1}^r h_i(z(t))h_j(z(t))[A_i - B_i K_j]x(t) \\ &= \sum_{i=1}^r h_i^2(z(t))G_{ii}x(t) + 2 \sum_{i < j}^r h_i(z(t))h_j(z(t))\frac{G_{ij} + G_{ji}}{2}x(t) \end{aligned} \quad (6)$$

where

$$G_{ij} = A_i - B_i K_j \quad (7)$$

The purpose of this paper is to design a fuzzy state feedback controller (5) to exponentially stabilize the closed-loop T-S fuzzy system (6) and minimize the upper bound of corresponding performance function.

For positive definite matrix $Q \in R^{n \times n}$ and positive definite matrix $R \in R^{m \times m}$, the following quadratic performance functions are considered [24]

$\langle \text{CFS} \rangle$

$$J_c = \int_0^\infty e^{2\alpha t} [y^T(t)Qy(t) + u^T(t)Ru(t)]dt \quad (8)$$

$\langle \text{DFS} \rangle$

$$J_d = \sum_{t=0}^\infty e^{2\alpha t T_s} [y^T(t)Qy(t) + u^T(t)Ru(t)] \quad (9)$$

where the convergence rate of the exponential function $\alpha > 0$ reflects the requirement of response speed and T_s is the sample time of **DFS**. Then the problem of exponentially stable GCC design can be formulated as follows

Definition 1. Given a properly predefined real number $\alpha > 0$, if there *exist* a state feedback PDC control law $u(t)$ in the form of (5), such that: (a) the closed-loop system (6) is exponentially stable with the convergence rate *no less than* α ; (b) the cost *J_c or J_d* of quadratic performance function (8) (or (9)) *is no more than* J_0 . Then $u(t)$ is said to be exponentially stable GCC law for system (1) with (2) and J_0 is the guaranteed cost.

The stability conditions of fuzzy closed-loop control system (6) are derived using Lyapunov approach [13].

Lemma 1. (6) is asymptotically stable in the large if there *exist* a common matrix $P > 0$ such that

$\langle \text{CFS} \rangle$

$$G_{ii}^T P + P G_{ii} < 0 \quad (10)$$

$$\left(\frac{G_{ij} + G_{ji}}{2} \right)^T P + P \left(\frac{G_{ij} + G_{ji}}{2} \right) < 0 \quad (11)$$

$\langle \text{DFS} \rangle$

$$G_{ii}^T P G_{ii} - P < 0 \quad (12)$$

$$\left(\frac{G_{ij} + G_{ji}}{2} \right)^T P \left(\frac{G_{ij} + G_{ji}}{2} \right) - P < 0 \quad (13)$$

for all i and j excepting the pairs (i, j) such that $h_i(z(t))h_j(z(t)) = 0, \forall t > 0$.

Although Lemma 1 has guaranteed the stability of closed-loop systems, they may be quite conservative because no matter what the membership functions are, the conditions retain unchanged. Thus a relaxed stability condition is derived by considering the effect of membership functions as follows [13].

Lemma 2. Assume that the number of rules that fire for all t is *no more than* s , where $1 < s \leq r$. The fuzzy closed-loop control system (6) is asymptotically stable in the large if there *exist* a common matrix $P > 0$ and a common matrix $U_0 \geq 0$ such that

$\langle \text{CFS} \rangle$

$$G_{ii}^T P + P G_{ii} + (s - 1)U_0 < 0 \quad (14)$$

$$\left(\frac{G_{ij} + G_{ji}}{2} \right)^T P + P \left(\frac{G_{ij} + G_{ji}}{2} \right) - U_0 < 0 \quad (15)$$

$\langle \text{DFS} \rangle$

$$G_{ii}^T P G_{ii} - P + (s - 1)U_0 < 0 \quad (16)$$

$$\left(\frac{G_{ij} + G_{ji}}{2} \right)^T P \left(\frac{G_{ij} + G_{ji}}{2} \right) - P - U_0 < 0 \quad (17)$$

for all i and j excepting the pairs (i, j) such that $h_i(z(t))h_j(z(t)) = 0, \forall t > 0$.

The word 'fire' means that a rule's weight $h_i(z(t))$ for a given $z(t)$ in (3) is not zero for all t . Noticing that for any P satisfying Lemma 1, there must exist some U_0 satisfying Lemma 2, but it is not always the case vice versa. Thus the less conservative Lemma 2 contains more possible stable controller candidates and would give us more design choices. In the next section, main results of this paper are presented.

3. Exponentially stable GCC design for **CFS**

In this section, the sufficient conditions on the exponential stability with predefined convergence rate for **CFS** are provided by using the GCC approach and PDC concept. Then the design is transformed into a convex optimization represented by LMIs.

Theorem 1. *The system (1) with (2) for CFS case is exponentially stabilized with **convergence rate no less than** $\alpha > 0$ by the PDC fuzzy controller (5) if there **exist** a common matrix $P > 0$ and a common matrix $U_0 \geq 0$ satisfying*

$$G_{ii}^T P + P G_{ii} + N_{ii} + (s - 1)U_0 + 2\alpha P < 0 \quad \forall i \quad (18)$$

$$G_{ij}^T P + P G_{ji} + G_{ji}^T P + P G_{ij} + N_{ij} + N_{ji} - 2U_0 + 4\alpha P < 0 \quad i < j \quad (19)$$

excepting the pairs (i, j) such that $h_i(z(t))h_j(z(t)) = 0$, $s > 1$, $\forall t > 0$. G_{ij} is as in (7) and

$$N_{ij} = \begin{bmatrix} C_i^T & -K_j^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} C_i \\ -K_j \end{bmatrix} \quad (20)$$

Then the performance index (8) satisfies $\mathbf{J}_c < x^T(0)Px(0) = J_0$.

Proof. To begin with, we derive an equivalent problem of the original exponentially stable GCC problem stated in Definition 1 by defining the following new variables

$$\begin{aligned} x^\alpha(t) &= e^{\alpha t}x(t), \quad u^\alpha(t) = e^{\alpha t}u(t) \\ \hat{y}^\alpha(t) &= e^{\alpha t} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \sum_{i=1}^r h_i(z(t)) \begin{bmatrix} C_i \\ -K_i \end{bmatrix} x^\alpha(t) \end{aligned} \quad (21)$$

Then cost function (8) could be rewritten as

$$\begin{aligned} \mathbf{J}_c &= \int_0^\infty \left(\sum_{i=1}^r h_i(z(t)) \begin{bmatrix} C_i \\ -K_i \end{bmatrix} x^\alpha(t) \right)^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \sum_{i=1}^r h_i(z(t)) \begin{bmatrix} C_i \\ -K_i \end{bmatrix} x^\alpha(t) dt \\ &= \int_0^\infty (\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t) dt \end{aligned} \quad (22)$$

and the new closed-loop system in terms of new variables is derived by using (1) and (5)

$$\begin{aligned}\dot{x}^\alpha(t) &= \alpha e^{\alpha t} x(t) + e^{\alpha t} \dot{x}(t) \\ &= \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) [(A_i - B_i K_j + \alpha I)] x^\alpha(t)\end{aligned}\quad (23)$$

Therefore through the definition of (21), the original problem regarding (6) and (8) is transformed to the equivalent problem regarding (23) and (22).

Now we prove the stability. Assuming that there are a common **matrix** $P > 0$ and a common **matrix** $U_0 \geq 0$ satisfying (18) and (19), and noticing that $Q > 0$, $R > 0$, we immediately obtain

$$\begin{aligned}G_{ii}^T P + P G_{ii} + (s-1)U_0 + 2\alpha P &< 0 \quad \forall i \\ G_{ij}^T P + P G_{ij} + G_{ji}^T P + P G_{ji} - 2U_0 + 4\alpha P &< 0 \quad i < j\end{aligned}\quad (24)$$

which exactly shows that the fuzzy control system (23) is asymptotically stable in the large according to Lemma 2.

i.e.

$$\lim_{t \rightarrow \infty} x^\alpha(t) = \lim_{t \rightarrow \infty} e^{\alpha t} x(t) = 0$$

which means that the original state $x(t)$ would convergence to zero at an exponential speed no less than $k\|x(0)\|e^{-\alpha t}$, k is a positive constant. This implies that the original T-S fuzzy system (6) is exponentially stable.

Next, we prove that the quadratic performance function satisfies $J_c < x^T(0)Px(0)$.

Consider the **Lyapunov** function $(x^\alpha(t))^T P x^\alpha(t)$ and from (18), (19) and

Corollary 2 in [13]

$$\begin{aligned}
\frac{d}{dt}(x^\alpha(t))^T P x^\alpha(t) &= (\dot{x}^\alpha(t))^T P x^\alpha(t) + (x^\alpha(t))^T P \dot{x}^\alpha(t) \\
&= \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) (x^\alpha(t))^T [(A_i - B_i K_j + \alpha I)^T P \\
&\quad + P(A_i - B_i K_j + \alpha I)] x^\alpha(t) \\
&= \sum_{i=1}^r h_i^2(z(t)) (x^\alpha(t))^T [G_{ii}^T P + P G_{ii} + 2\alpha P] x^\alpha(t) \\
&\quad + \sum_{i < j}^r h_i(z(t)) h_j(z(t)) (x^\alpha(t))^T [G_{ij}^T P + P G_{ij} + G_{ji}^T P + P G_{ji} \\
&\quad + 4\alpha P] x^\alpha(t) \\
&< - \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) (x^\alpha(t))^T \begin{bmatrix} C_i^T & -K_i^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \\
&\quad \begin{bmatrix} C_j \\ -K_j \end{bmatrix} x^\alpha(t) - \left[(s-1) \sum_{i=1}^r h_i^2(z(t)) - 2 \sum_{i < j}^r h_i(z(t)) h_j(z(t)) \right] \\
&\quad x^\alpha(t)^T U_0 x^\alpha(t) \leq -(\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t)
\end{aligned}$$

Therefore

$$\frac{d}{dt}(x^\alpha(t))^T P x^\alpha(t) < -(\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t)$$

Integrating both sides from 0 to ∞ , we have

$$\mathbf{J}_c = \int_0^\infty (\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t) dt < -(x^\alpha(t))^T P x^\alpha(t) \big|_0^\infty$$

since the fuzzy system (23) is stable

$$\mathbf{J}_c < (x^\alpha(0))^T P x^\alpha(0) = x^T(0) P x(0)$$

□

However in Theorem 1, the cost function J_c depends on initial state conditions. To eliminate this matter, assume that the initial conditions are random variables with zero expectation and a covariance equal to identity matrix, i.e.

$$\Xi [x_0 x_0^T] = I \quad \Xi [x_0] = 0 \quad (25)$$

where Ξ is the expectation operator. Then the exponentially stable GCC design is transformed to be independent of initial conditions as stated in Corollary 1.

Corollary 1. *The system (1) with (2) for CFS case is exponentially stabilized with convergence rate no less than $\alpha > 0$ by the PDC fuzzy controller (5) if there exist a common matrix $P > 0$ and a common matrix $U_0 \geq 0$ satisfying (18) and (19). Then the expectation of the cost function (8) satisfies*

$$\Xi_{x_0} J_c = \Xi_{x_0} \int_0^\infty e^{2\alpha t} [y^T(t) Q y(t) + u^T(t) R u(t)] dt < tr(P) \quad (26)$$

Proof. Notice that from (25) we have

$$\Xi_{x_0} [x_0^T P x_0] = tr(P)$$

Then the procedure of proof is similar to Theorem 1. □

In order to achieve as good control performance as possible, the cost function (8) needs to be minimized. Based on Corollary 1, the exponentially stable GCC design is formulated into an optimization problem of minimizing the upper bound of cost function in (26) by using LMI techniques in [26].

Theorem 2. For a given proper convergence rate $\alpha > 0$, the PDC feedback control gain K_i in each rule's consequent of the fuzzy control system (6) can be calculated by solving the following *LMI problem*

$$\begin{aligned} & \min_{X,Y,M_1,\dots,M_r} \text{tr}(Z) \\ \text{s.t.} \quad & X > 0, \quad Y \geq 0 \\ & \begin{bmatrix} Z & I \\ I & X \end{bmatrix} > 0 \end{aligned} \quad (27)$$

$$U_{ii} < 0, \quad \forall i \quad (28)$$

$$V_{ij} < 0, \quad i < j \quad (29)$$

excepting all the pairs (i, j) such that $h_i(z(t))h_j(z(t)) = 0, \forall t > 0$ and $s > 1$, where

$$\begin{aligned} U_{ii} &= \begin{bmatrix} XA_i^T + A_iX - M_i^T B_i^T - B_i M_i + 2\alpha X + (s-1)Y & XC_i^T & -M_i^T \\ * & -Q^{-1} & 0 \\ * & * & -R^{-1} \end{bmatrix} \\ V_{ij} &= \begin{bmatrix} \begin{pmatrix} XA_i^T + A_iX - M_j^T B_i^T - B_i M_j \\ + XA_j^T + A_jX - M_i^T B_j^T - B_j M_i \\ + 4\alpha X - 2Y \end{pmatrix} & XC_i^T & -M_j^T & XC_j^T & -M_i^T \\ * & -Q^{-1} & 0 & 0 & 0 \\ * & * & -R^{-1} & 0 & 0 \\ * & * & * & -Q^{-1} & 0 \\ * & * & * & * & -R^{-1} \end{bmatrix} \\ & X = P^{-1}, \quad M_i = K_i X, \quad Y = XU_0 X \end{aligned}$$

If this problem is feasible, then the PDC feedback control gains can be obtained as $K_i = M_i X^{-1}$ for all i .

Proof. By using Schur complement and congruent transformation in [26], conditions (28) and (29) follow easily from Theorem 1. The original design objective is to minimize the trace of P . However after variable transformation, we need to minimize the trace of X^{-1} , which is unsolvable by LMI technique. So the new variable Z is introduced and let $Z - X^{-1} > 0$, if it holds then $\text{tr}(Z) > \text{tr}(X^{-1}) = \text{tr}(P) > J_c$. By using Schur complement, (27) is derived. $\square$

Remark 1. *Theorem 1 and Corollary 1 provide sufficient conditions for the exponentially stable GCC designs. This design approach only requires to adjust one parameter α , which is much more easier to guarantee the stability of the closed-loop system compared with the trial and error method in [10].*

Remark 2. *On one hand, it is interesting to find that, if all the terms regarding the quadratic performance parameter matrix Q and R in (18) and (19) are removed, then the stability sufficient conditions are exactly the Decay Rate Design in [21]. On the other hand, if convergence rate α is set to zero in (28) and (29), then the stability LMIs conditions are the same with those in [14]. Therefore, the approaches proposed in [21] and [14] are special cases of the method presented in this section, which synthesizes the respective advantages of two approaches and provides more design choices.*

4. Exponentially stable GCC design for DFS

In this section, the sufficient conditions on the exponentially stability with predefined convergence rate are extended to discrete-time T-S fuzzy systems.

Theorem 3. The system (1) with (2) for DFS case is exponentially stabilized with *convergence rate no less than $\alpha > 0$* by the PDC fuzzy controller (5) if there *exist* a common matrix $P > 0$ and a common matrix $U_0 \geq 0$ satisfying

$$(G_{ii}^\alpha)^T P G_{ii}^\alpha - P + N_{ii} + (s-1)U_0 < 0 \quad \forall i \quad (30)$$

$$\left(\frac{G_{ij}^\alpha + G_{ji}^\alpha}{2} \right)^T P \left(\frac{G_{ij}^\alpha + G_{ji}^\alpha}{2} \right) - P + \frac{1}{2}N_{ij} + \frac{1}{2}N_{ji} - U_0 < 0 \quad i < j \quad (31)$$

excepting the pairs (i, j) such that $h_i(z(t))h_j(z(t)) = 0$, $s > 1$, $\forall t > 0$. N_{ij} is as in (20) and

$$\begin{aligned} G_{ij}^\alpha &= A_i^\alpha - B_i^\alpha K_j \\ A_i^\alpha &= e^{\alpha T_s} A_i, \quad B_i^\alpha = e^{\alpha T_s} B_i \end{aligned} \quad (32)$$

Then the performance index (9) satisfies $J_d < x^T(0)Px(0) = J_0$.

Proof. Similar to the proof of Theorem 1, an equivalent problem of the original fuzzy exponentially stable GCC problem stated in Definition 1 is derived by defining the new variables as follows

$$\begin{aligned} x^\alpha(t) &= e^{\alpha t T_s} x(t), \quad u^\alpha(t) = e^{\alpha t T_s} u(t) = - \sum_{i=1}^r h_i(z(t)) K_i x^\alpha(t) \\ \hat{y}^\alpha(t) &= e^{\alpha t T_s} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \sum_{i=1}^r h_i(z(t)) \begin{bmatrix} C_i \\ -K_i \end{bmatrix} x^\alpha(t) \end{aligned} \quad (33)$$

Then cost function (9) could be rewritten as

$$\begin{aligned} J_d &= \sum_{t=0}^{\infty} \left(\sum_{i=1}^r h_i(z(t)) \begin{bmatrix} C_i \\ -K_i \end{bmatrix} x^\alpha(t) \right)^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \sum_{i=1}^r h_i(z(t)) \begin{bmatrix} C_i \\ -K_i \end{bmatrix} x^\alpha(t) \\ &= \sum_{t=0}^{\infty} (\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t) \end{aligned} \quad (34)$$

And the new closed-loop system in terms of new variables defined in (32) is derived by using (1) and (5)

$$\begin{aligned} x^\alpha(t+1) &= A_i^\alpha x^\alpha(t) + B_i^\alpha u^\alpha(t) \\ &= \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) [(A_i^\alpha - B_i^\alpha K_j)] x^\alpha(t) \end{aligned} \quad (35)$$

Therefore through the definition of (32) and (33), the original problem regarding (6) and (9) is transformed to the equivalent problem regarding (35) and (34).

Now we prove the stability. Assuming that there is a common **matrix** $P > 0$ and a common **matrix** $U_0 \geq 0$ satisfying (30) and (31), and noticing that $Q > 0, R > 0$, we immediately obtain

$$\begin{aligned} (G_{ii}^\alpha)^T P G_{ii}^\alpha - P + (s-1)U_0 &< 0 \quad \forall i \\ \left(\frac{G_{ij}^\alpha + G_{ji}^\alpha}{2} \right)^T P \left(\frac{G_{ij}^\alpha + G_{ji}^\alpha}{2} \right) - P - U_0 &< 0 \quad i < j \end{aligned} \quad (36)$$

which exactly shows that the fuzzy control system (35) is asymptotically stable in the large according to Lemma 2.

i.e.

$$\lim_{t \rightarrow \infty} x^\alpha(t) = \lim_{t \rightarrow \infty} e^{\alpha t T_s} x(t) = 0$$

which means that the original state $x(t)$ would convergence to zero at an exponential speed no less than $k\|x(0)\|e^{-\alpha t T_s}$, k is a positive constant. This implies that the original T-S fuzzy system (6) is exponentially stable.

Next, we prove that the quadratic performance function satisfies $J_d < x^T(0)Px(0)$.

Consider the Lyapunov function $V(x(t)) = (x^\alpha(t))^T P x^\alpha(t)$ and from (30),

(31) and Corollary 2 in [13]

$$\begin{aligned}
\Delta V(x(t)) &= V(x(t+1)) - V(x(t)) \\
&= (x^\alpha(t))^T \frac{1}{4} \left[\sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r \sum_{l=1}^r h_i(z(t)) h_j(z(t)) h_k(z(t)) h_l(z(t)) \right. \\
&\quad \left. \left[(G_{ij}^\alpha + G_{ji}^\alpha)^T P (G_{kl}^\alpha + G_{lk}^\alpha) - 4P \right] x^\alpha(t) \right] \\
&\leq \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) (x^\alpha(t))^T \left[(G_{ij}^\alpha + G_{ji}^\alpha)^T P (G_{ij}^\alpha + G_{ji}^\alpha) - 4P \right] x^\alpha(t) \\
&= \sum_{i=1}^r h_i^2(z(t)) (x^\alpha(t))^T [(G_{ii}^\alpha)^T P G_{ii}^\alpha - P] x^\alpha(t) \\
&\quad + 2 \sum_{i < j}^r h_i(z(t)) h_j(z(t)) (x^\alpha(t))^T \left[\left(\frac{G_{ij}^\alpha + G_{ji}^\alpha}{2} \right)^T P \left(\frac{G_{ij}^\alpha + G_{ji}^\alpha}{2} \right) - P \right] x^\alpha(t) \\
&< -(x^\alpha(t))^T \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) \begin{bmatrix} C_i^T & -K_i^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} C_j \\ -K_j \end{bmatrix} x^\alpha(t) \\
&\quad - \left[(s-1) \sum_{i=1}^r h_i^2(z(t)) - 2 \sum_{i < j}^r h_i(z(t)) h_j(z(t)) \right] (x^\alpha(t))^T U_0 x^\alpha(t) \\
&\leq -(\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t)
\end{aligned}$$

Therefore

$$\Delta V(x(t)) = V(x(t+1)) - V(x(t)) < -(\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t)$$

Accumulate both sides from $t = 0$ to ∞ and noticing that the fuzzy system (35) is stable, we have

$$\mathbf{J}_d = \sum_{t=0}^{\infty} (\hat{y}^\alpha(t))^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \hat{y}^\alpha(t) < (x^\alpha(0))^T P x^\alpha(0) = x^T(0) P x(0)$$

□

Similarly, the Corollary 2 eliminates the effect of initial conditions on the design procedure.

Corollary 2. *The system (1) with (2) for DFS case is exponentially stabilized with convergence rate no less than $\alpha > 0$ by the PDC fuzzy controller (5) if there exist a common matrix $P > 0$ and a common matrix $U_0 \geq 0$ satisfying (30) and (31). Then the expectation of the cost function (9) satisfies*

$$\Xi_{x_0} \mathbf{J}_d = \Xi_{x_0} \sum_{t=0}^{\infty} e^{2\alpha t T_s} [y^T(t) Q y(t) + u^T(t) R u(t)] < \text{tr}(P) \quad (37)$$

Proof. The proof is trivial according to Corollary 1. □

Similar to last section, we have following LMI optimization problem to achieve better control performance.

Theorem 4. *For a given proper convergence rate $\alpha > 0$, the PDC feedback control gains K_i in each rule's consequent of the fuzzy control system (6) can be calculated by solving the following LMI problem*

$$\begin{aligned} & \min_{X, Y, M_1, \dots, M_r} \text{tr}(Z) \\ \text{s.t.} \quad & X > 0, \quad Y \geq 0 \\ & \begin{bmatrix} Z & I \\ I & X \end{bmatrix} > 0 \end{aligned} \quad (38)$$

$$U_{ii} < 0, \quad \forall i \quad (39)$$

$$V_{ij} < 0, \quad i < j \quad (40)$$

excepting all the pairs (i, j) such that $h_i(z(t))h_j(z(t)) = 0, \forall t > 0$ and $s > 1$, where

$$U_{ii} = \begin{bmatrix} (s-1)Y - X & (X(A_i^\alpha)^T - M_i^T(B_i^\alpha)^T) & XC_i^T & -M_i^T \\ * & -X & 0 & 0 \\ * & * & -Q^{-1} & 0 \\ * & * & * & -R^{-1} \end{bmatrix}$$

$$V_{ij} = \begin{bmatrix} -4X - 4Y & \begin{pmatrix} X(A_i^\alpha)^T - M_j^T(B_i^\alpha)^T \\ +X(A_j^\alpha)^T - M_i^T(B_j^\alpha)^T \end{pmatrix} & XC_j^T & -M_i^T & XC_i^T & -M_j^T \\ * & -X & 0 & 0 & 0 & 0 \\ * & * & -\frac{1}{2}Q^{-1} & 0 & 0 & 0 \\ * & * & * & -\frac{1}{2}R^{-1} & 0 & 0 \\ * & * & * & * & -\frac{1}{2}Q^{-1} & 0 \\ * & * & * & * & * & -\frac{1}{2}R^{-1} \end{bmatrix}$$

$$X = P^{-1}, \quad M_i = K_i X, \quad Y = XU_0 X$$

If this problem is feasible, then the PDC feedback control gains can be obtained as $K_i = M_i X^{-1}$ for all i .

Proof. By using Schur complement and congruent transformation in [26], conditions (39) and (40) follow easily from Theorem 3. (38) is derived in the same way with (27). $\square$

5. Numerical Examples

In this section, two numerical examples are simulated to demonstrate the effectiveness and feasibility of the proposed methods in section 3 and section 4 respectively. All simulations are performed on MATLAB 2012a with the

convex optimization tools MOSEK [27] and YALMIP [28]. Some comparisons with other methods are presented to show the merits of our methods.

Example 1: Figure 1 shows a transitional oscillator with an eccentric rotational proof mass actuator (TORA) system, which is a well-known nonlinear benchmark control problem [29] and has been studied and reported in [11, 14, 30, 31, 32]. As Figure 1 illustrates, the nonlinear coupling between the rotational motion of the actuator and the translational motion of the oscillator provides the mechanism for control. After simplification, the system is described by the non-dimensional equations

Figure 1: TORA system

$$\dot{x}(t) = f(x) + g(x)u(t) + d(x)w(t)$$

$$f(x) = \begin{bmatrix} x_2 \\ \frac{\epsilon x_4^2 \sin(x_3) - x_1}{1 - \epsilon^2 \cos^2(x_3)} \\ x_4 \\ \frac{\epsilon \cos(x_3)(x_1 - \epsilon x_4^2 \sin(x_3))}{1 - \epsilon^2 \cos^2(x_3)} \end{bmatrix} \quad (41)$$

$$g(x) = \begin{bmatrix} 0 \\ \frac{-\epsilon \cos(x_3)}{1 - \epsilon^2 \cos^2(x_3)} \\ 0 \\ \frac{1}{1 - \epsilon^2 \cos^2(x_3)} \end{bmatrix}, \quad d(x) = \begin{bmatrix} 0 \\ \frac{1}{1 - \epsilon^2 \cos^2(x_3)} \\ 0 \\ \frac{-\epsilon \cos(x_3)}{1 - \epsilon^2 \cos^2(x_3)} \end{bmatrix}$$

where $u(t)$ is the control input, $w(t)$ is the disturbance input. x_1 and x_2 denote the transitional position and velocity of the cart with $\dot{x}_1 = x_2$. $x_3 = \theta$ and $x_4 = \dot{x}_3$ denote the angular position and velocity of the rotational proof mass.

To obtain the T-S fuzzy model of TORA for control design from (41), it is a simple and convenient way to linearize the system at local working points and then blend the local subsystems with properly designed membership functions. Commonly the more local working points the original system is linearized at, the more precise the resulting T-S fuzzy model is. However, simulation results here will demonstrate that even if only two working points are linearized for control design, the original TORA is still well controlled. By calculating the Jacobian of system (41) at points $\theta = 0^\circ$ and 80° , resulting T-S fuzzy model is defined by the rules

Plant Rule 1: IF $x_3(t)$ is M_1 , THEN $\dot{x}(t) = A_1x(t) + B_1u(t), y(t) = C_1x(t)$

Plant Rule 2: IF $x_3(t)$ is M_2 , THEN $\dot{x}(t) = A_2x(t) + B_2u(t), y(t) = C_2x(t)$

where M_1 and M_2 are input fuzzy sets defined by membership functions

$$\mu_1(x_3(t)) = 1 - \frac{2}{\pi}|x_3(t)|, \quad \mu_2(x_3(t)) = \frac{2}{\pi}|x_3(t)|$$

respectively, other parameters are given as follows

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-\epsilon^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon}{1-\epsilon^2} & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -\frac{\epsilon}{1-\epsilon^2} \\ 0 \\ \frac{1}{1-\epsilon^2} \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-\epsilon^2\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon\beta}{1-\epsilon^2\beta^2} & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -\frac{\epsilon\beta}{1-\epsilon^2\beta^2} \\ 0 \\ \frac{1}{1-\epsilon^2\beta^2} \end{bmatrix}, \quad C_2 = C_1$$

where $\beta = \cos(80^\circ)$.

Let $\epsilon = 0.5$ and $w(t) = 0$. The weighting matrices in quadratic performance (8) are chosen as

$$Q = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times 10^{-4}, R = 10^{-8}$$

By using Theorem 2 for fuzzy GCC design in section 3 and choosing the convergence rate $\alpha = 0.1$, resulting state feedback controller gains are

$$K_1 = [-4.7826, 0.0061, 0.5379, 1.8669]$$

$$K_2 = [-4.7272, -0.9135, 0.4816, 1.7448]$$

The matrices X and Y are

$$X = \begin{bmatrix} 0.0507 & -0.0168 & -0.0037 & -0.0133 \\ -0.0168 & 0.0292 & 0.0030 & 0.0082 \\ -0.0037 & 0.0030 & 0.0008 & 0.0018 \\ -0.0133 & 0.0082 & 0.0018 & 0.0054 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.0571 & 0.0008 & -0.1333 & 0.0187 \\ 0.0008 & 0.0005 & 0.0041 & -0.0044 \\ -0.1333 & 0.0041 & 0.4768 & -0.1401 \\ 0.0187 & -0.0044 & -0.1401 & 0.0723 \end{bmatrix}$$

with $J_0 = 0.0861$.

For comparison, the controller based on standard quadratic performance function (QPF) in [14], the model-based GCC (MB-GCC) controller in [11] and the decay rate design (DRD) in [21] are also applied to the system with the same settings of Q , R and α (if needed). Figure 2 to Figure 7 show the **performance comparisons** and control effort **comparisons** of the resulting simulation with initial condition $x_0 = [0.3 \ 0 \ -40^\circ \ 0]^T$.

Figure 2: Performance comparison of QPF controller and the proposed

Figure 3: Control effort comparison of QPF controller and the proposed

Figure 4: Performance comparison of MB-GCC controller and the proposed

Figure 5: Control effort comparison of MB-GCC controller and the proposed

Figure 6: Performance comparison of DRD controller and the proposed

Figure 7: Control effort comparison of DRD controller and the proposed

From Figure 2 and Figure 3, one can find that the responses with proposed controller is faster and milder than those with QPF controller while the control effort with proposed controller is larger than that with QPF controller. As stated in the Remark 2, the method in [14] is equivalent to the proposed method when $\alpha = 0$. Thus it is reasonable that the faster response the control task requires, the much control effort the system needs. The utilization of quadratic performance function (8) affords a trade-off between the convergence rate and the control effort such that the control effort would not become too large. This kind of trade-off is the very advantage of our

method.

For Figure 4 and Figure 5, it is observed that though response speeds are similar in both cases, the transient responses with proposed controller are generally superior than those with MB-GCC Controller while the control effort with proposed controller is smaller than that of MB-GCC controller.

Figure 6 and Figure 7 show that the responses with proposed controller are generally milder than those with DRD controller although the response with DRD controller is a little faster than that with proposed controller. Therefore, performance of two controller can be evaluated as almost same. However, the control effort with DRD controller is much larger than that with proposed controller. Thus, the overall performance of DRD controller is inferior to that of proposed controller.

To sum up, the proposed controller generally provides better performance than the other three ones do.

Remark 3. *In above simulation, the range of convergence rate α in which our method has feasible solutions is approximately $[0, 0.224]$. It should be pointed out that the approaches in [11] and [14] can provide only one controller (if the problem is feasible) for a given T-S fuzzy system and a given pair of Q and R while our method has the potential to provide more possible controllers by setting different $\alpha > 0$.*

Figure 8 show responses of state $x_1(t)$ under proposed control with different values of α . One can find that as α increases, the response speed of the original TORA system increases efficiently.

Figure 8: Responses comparison of the translational position $x_1(t)$ of the cart for TORA system with different α under proposed control

Remark 4. *In fact, the weighting matrices Q and R can also affect the response speed of closed-loop systems. However, compared with the much more intuitive relationship between α and resulting response speed in our method, the relationship between *response speed* and weighting matrices is too obscure to be utilized for adjusting *response speed* effectively, .*

In order to test the robustness of our proposed method, disturbance is added to the system with $w(t) = \sin(20t)$. Figure 9 shows the response of each state. The proposed methods can stabilize the original TORA system

in the presence of disturbances.

Figure 9: Responses of TORA system with sinusoidal disturbance

Example 2: This example will demonstrate the feasibility and efficiency of exponentially stable GCC design for **DFS** in section 4. Consider the backing-up control of a computer-simulated truck-trailer which is borrowed from [33, 34]. The truck-trailer model is formulated as

$$\begin{aligned} x_1(t+1) &= \left(1 - \frac{vt}{L}\right) x_1(t) + \frac{vt}{l} u(t) \\ x_2(t+1) &= x_2(t) + \frac{vt}{L} x_1(t) \\ x_3(t+1) &= x_3(t) + vt \sin \left[x_2(t) + \frac{vt}{2L} x_1(t) \right] \end{aligned}$$

The following T-S fuzzy models are used to design the fuzzy controller

Plant Rule 1: IF $\theta(t) = x_2(t) + \frac{vt}{2L}\mathbf{x}_1(t)$ is about 0,

THEN $x(t+1) = A_1x(t) + B_1u(t), y(t) = C_1x(t)$

Plant Rule 2: IF $\theta(t) = x_2(t) + \frac{vt}{2L}\mathbf{x}_1(t)$ is about π or $-\pi$,

THEN $x(t+1) = A_2x(t) + B_2u(t), y(t) = C_2x(t)$

$$A_1 = \begin{bmatrix} 1 - \frac{v\bar{t}}{L} & 0 & 0 \\ \frac{v\bar{t}}{L} & 1 & 0 \\ \frac{v^2\bar{t}^2}{2L} & v\bar{t} & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{v\bar{t}}{l} \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 - \frac{v\bar{t}}{L} & 0 & 0 \\ \frac{v\bar{t}}{L} & 1 & 0 \\ \frac{dv^2\bar{t}^2}{2L} & dv\bar{t} & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{v\bar{t}}{l} \\ 0 \\ 0 \end{bmatrix}, \quad C_2 = C_1$$

where the length of truck is $l = 2.8$, the length of trailer is $L = 5.5$, the constant speed of backing up is $v = -1.0$, the sampling time is $\bar{t} = 2.0$, $d = (0.01/\pi)$ and the membership functions are

$$\mu_1(\theta(t)) = \frac{\sin(\theta(t))}{\theta(t)}, \quad \mu_2(\theta(t)) = 1 - \mu_1(\theta(t))$$

Applying Theorem 4 and set $\alpha = 0.3$, the state feedback controller gains are designed as

$$K_1 = [-3.472, 5.005, -0.7264]$$

$$K_2 = [-3.215, 3.596, -0.6993]$$

Figure 10 shows the state trajectories of the closed-loop truck-trailer system. Figure 11 shows the corresponding control input. These figures demonstrated the effectiveness and feasibility of our proposed methods.

Figure 10: States responses of backing-up truck-trailer

Figure 11: Control input of backing-up truck-trailer

6. Conclusion

In this paper, the method of designing state feedback controllers which exponentially stabilize the T-S fuzzy systems was presented. Using guaranteed cost control, PDC controller was designed by minimizing the upper bound of a variation of quadratic performance function which could provide desirable control performance and fast responses simultaneously. Sufficient conditions which guarantee exponential stabilities of the resulting closed-loop systems with predefined convergence rates were proved for continuous- and discrete-time T-S fuzzy systems, respectively. The simulation results verified that by

appropriately setting convergence rate and weighting matrices in the design stage of our method, the response speed of the closed-loop system could be accelerated efficiently and the overall control performance was superior than those of other existing methods, while the control effort was still kept at an acceptable level.

Based on the main results, further works may be carried out, for instance:

1. extensions to deal with more complex nonlinear systems by introducing observers when state-variables are not measurable, as proposed in [35] and [36].
2. taking parameter uncertainty and actuator saturation into considerations, such as non-fragile guaranteed cost control [37] and robust control [38].

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