Global Exponential Stability of Uncertain Neural Networks with Discontinuous Lurie-type Activation and Mixed Delays

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Abstract

This paper deals with the problem of the global exponential stability of a class of uncertain neural networks with discontinuous Lurie-type activation and mixed delays. By establishing a new sufficient condition, we first prove the existence of the equilibrium point by using the Leary-Schauder alternative theorem. Then, by employing a new Lyapunov functional, we obtain the global exponential stability of the equilibrium point of the uncertain neural network. In the end, some comparisons and numerical examples are given to show the improvement of the conclusions in this paper.

Keywords: Neural network with Lurie-type activation and mixed delays; global exponential stability; Leary-Schauder alternative theorem; Lyapunov functional

1. Introduction

In recent years, neural networks have been extensively investigated since their wide applications, such as pattern recognition, associative memories, automatic control, optimization, image processing and other areas (see [1, 2, 3, 4, 5, 6, 7, 8]). It is well known that the applications of neural networks heavily depend on their stability. Therefore, in order to successfully employ neural networks in applications, we should study the stability of the designed neural network in advance. However, due to the finite switching speed of the neuron amplifiers and transmissions of signals in a network, time delays are actually unavoidable in the electronic implementation. Meanwhile, in

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both biological and artificial neural networks, the delay may create the loss of stability, since it may originate the onset of nonvanishing oscillations (see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]). On the other hand, the estimation errors are unavoidable when we measure the vital data of deterministic neural networks such as the neuron fire rate and the synaptic interconnection weights. Moreover, in deterministic neural networks, the stability of neural networks can often be destroyed by its compulsory uncertainty issuing from the existence of modeling errors, external disturbance and parameter fluctuations. Hence, it is important and necessary to study the robust stability of neural networks with time delays in presence of uncertainties (see [13, 14, 15, 26, 27]).

In the past years, the stability of uncertain neural network has received considerable attentions, and varieties of interesting results have been presented in the literatures (see [10, 13, 14, 15, 26, 27, 28]). For example, in [10], authors investigated the problem of the existence, uniqueness and global asymptotic stability of the equilibrium point for the class of neural networks with multiple time delays and parameter uncertainties. By means of the homeomorphism theory and Lyapunov functional method, Arik in [13] studied the global asymptotic stability problem of dynamical neural networks with multiple time delays under parameter uncertainties. Guo et.al in [26] presented a systematic method for analyzing the robust stability of a class of interval neural networks with uncertain parameters and time delays. In [28], we studied the global robust exponential stability of the neural networks with possibly unbounded activation functions.

However, most of the results concerning the robust stability of neural networks are based on a common assumption that the activations are continuous or even Lipschitz continuous (see [9, 10, 13, 14, 15, 29, 30]). In practice, as mentioned by [31], discontinuous or non-Lipschitz neuron activations have been introduced into neural network systems due to their theoretical and practical significance in recent years. Nowadays, more and more scholars observe the importance of neural network with discontinuous activations (see [18, 32, 33, 34, 35, 35, 36, 37]). For example, Qin et. al. in [18] studied the global exponential stability and global convergence in finite time of neural networks with discontinuous activations. In [38], a novel class of Cohen-Grossberg neural networks with delays and inverse Hölder neuron activation functions are presented. In [37], authors integrated a class of delayed neural networks with discontinuous activations by means of the Leray-Schauder theorem and Viability theorem. In [23], authors investigated the global dynamics of equilibrium point for delayed competitive neural networks with different time scales and discontinuous activations.

Inspired by previous studies, in this paper, we will study the existence, global exponential stability of uncertain neural networks with discontinuous Lurie-type activation and mixed delays. The remainder of this paper is arranged as follows. In section 2, we state some preliminaries including some necessary definitions and lemmas. Our main results are contained in Sections 3 and 4, where the sufficient conditions are given to guarantee the existence and global exponential stability of equilibrium for neural networks in this paper. Section 5 present

some illustrative numerical examples to verify our results.

Notation Given the column vector $x = (x_1, x_2, \dots, x_n)^T$, where the superscript T is the transpose operator, and $\|x\| := (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and define $\|A\| = \sqrt{\lambda_M(A^T A)}$, where $\lambda_M(A)$ stands for the operation of taking the maximum eigenvalue of A . $I \in \mathbb{R}^{n \times n}$ is the $n \times n$ identity matrix. For a real symmetric matrix A , $A < 0$ means that A is negative definite.

2. Preliminaries

In this section, we will introduce some definitions and lemmas, which will be used in the remainder of this paper. The neural network we consider in this paper is described as follows,

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -d_i x_i(t) + \sum_{j=1}^n a_{ij} g_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t - \tau_j)) \\ & + \sum_{j=1}^n c_{ij} \int_{t-h_j}^t g_j(x_j(s)) ds + u_i, \end{aligned} \quad (1)$$

or equivalently,

$$\begin{aligned} \dot{x}(t) = & -Dx(t) + Ag(x(t)) + Bg(x(t - \tau)) \\ & + C \int_{t-h}^t g(x(s)) ds + U, \end{aligned} \quad (2)$$

where $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ denotes the state of the neurons; $D = \text{diag}(d_1, d_2, \dots, d_n)^T \in \mathbb{R}^n$ is a positive diagonal matrix; $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$, $C = (c_{ij})_{n \times n}$, are the $n \times n$ real connection weight matrices representing the weighting coefficients of the neurons; $g = (g_1(x_1), g_2(x_2), \dots, g_n(x_n))^T \in \mathbb{R}^n$ represents the neuron activations.

In this paper, we will study the neural network (1) with general activation functions. That is, the activation functions are only assumed to be non-increasing. For convenience, we define

$$\mathcal{M} \triangleq \{\phi : \mathbb{R} \rightarrow \mathbb{R} \mid s\phi(s) \geq 0, \text{ and } D^+\phi(s) \geq 0, s \in \mathbb{R}\}. \quad (3)$$

For any $\phi \in \mathcal{M}$, we define

$$K[\phi(t)] = [\phi(t^-), \phi(t^+)] \quad (4)$$

In this paper, we always assume the following assumptions hold,

(**A**₁) The activation function of neural network (1) satisfies the Lurie-type condition. That is, $g_i \in \mathcal{M}$, and there exists $k_i > 0$ such that

$$x_i g_i(x_i) \leq k_i x_i^2$$

for any $x_i \in \mathbb{R}$.

Here, the constants $k_i, i = 1, 2, \dots, n$, are generally called to be Lurie constants.

(A₂) The parameters $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$, $C = (c_{ij})_{n \times n}$, $D = \text{diag}(d_i)$ are assumed to be intervalised as follows,

$$\begin{aligned} D_I &:= \{D = \text{diag}(d_i) : 0 \leq \underline{D} \leq D \leq \overline{D}, \text{i.e., } \underline{d_i} \leq d_i \leq \overline{d_i}, \forall i\} \\ A_I &:= \{A = (a_{ij}) : \underline{A} \leq A \leq \overline{A}, \text{i.e., } \underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}, \forall i, j\} \\ B_I &:= \{B = (b_{ij}) : \underline{B} \leq B \leq \overline{B}, \text{i.e., } \underline{b_{ij}} \leq b_{ij} \leq \overline{b_{ij}}, \forall i, j\} \end{aligned} \quad (5)$$

Next lemma is Leary-Schauder alternative theorem, which plays an important role in prove the existence of the equilibrium point of the neural network (2).

Lemma 2.1. [39] *If X is a Banach space, $C \subseteq X$ is nonempty convex with $0 \in C$ and $G : C \rightarrow P_{k(c)}(C)$ is an USC multifunction which maps bounded sets into relatively compact sets, then one of the following statements is true,*

- (1) *the set $\Gamma = \{x \in C : x \in \lambda G(x), \lambda \in (0, 1)\}$ is unbounded;*
- (2) *the $G(\cdot)$ has a fixed point in C , i.e., there exists $x \in C$ such that $x \in G(x)$.*

Definition 2.1. x^* is said to be an equilibrium point of neural network (2) if there exists $\gamma^* \in K[g(x^*)]$ such that

$$-Dx^* + (A + B)\gamma^* + CH\gamma^* + U = 0, \quad (6)$$

where $H = \text{diag}(h_i), i = 1, 2, \dots, n$. And γ^* is said to be an output equilibrium point of system (2) corresponding to x^* .

Definition 2.2. [35] *For any continuous function $\phi : [-\tau, 0] \rightarrow \mathbb{R}^n$ and any measurable selection $\psi : [-\tau, 0] \rightarrow \mathbb{R}^n$, such that $\psi(s) \in K[g(\phi(s))]$ for a.a. $s \in [-\tau, 0]$, by an initial value problem (IVP) associated to (2) with initial condition (ϕ, ψ) , we mean the following problem: find a couple of functions $[x, \gamma] : [-\tau, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$, such that x is a solution of (2) on $[-\tau, T]$ for some $T > 0$, γ is an output associated to x , and*

$$\begin{cases} \dot{x}(t) = -Dx(t) + A\gamma(t) + B\gamma(t - \tau) + C \int_{t-h}^t \gamma(s)ds + U, & \text{for a.a. } t \in [0, T) \\ x(r) = \phi(r), & \forall r \in [-\tau, 0] \\ \gamma(r) = \psi(r), & \text{for a.a. } r \in [-\tau, 0] \end{cases} \quad (7)$$

Lemma 2.2. [40] *Let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, if*

$$A \in A_I := \{A = (a_{ij}) : 0 \leq \underline{A} \leq A \leq \overline{A}, \text{i.e., } 0 \leq \underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}, \forall i, j\}$$

then, for any positive diagonal matrix P , the following inequality holds:

$$x^T(PA + A^T P)x \leq x^T(PA^* + A^{*T}P + \|PA_* + A_*^T P\|_2 I)x,$$

where $A^* = \frac{1}{2}(\underline{A} + \overline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

Lemma 2.3. [28] Let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, if

$$B \in B_I := \{B = (b_{ij}) : 0 \leq \underline{B} \leq B \leq \overline{B}, \text{ i.e., } 0 \leq \underline{b}_{ij} \leq b_{ij} \leq \overline{b}_{ij}, \forall i, j\},$$

then, the following inequality holds:

$$\|B\|_2 \leq b,$$

where $b = \min\{\|B^*\|_2 + \|B_*\|_2, \|\hat{B}\|_2, \sqrt{\|B^*\|_2^2 + \|B_*\|_2^2 + 2\|B_*^T\|_2\|B^*\|_2}\}$, $B^* = \frac{1}{2}(\underline{B} + \overline{B})$, and $B_* = \frac{1}{2}(\overline{B} - \underline{B})$. Define $\hat{B} = \hat{b}_{ij} = \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$.

Lemma 2.4. Let $H < 0$. Given any $x, y \in \mathbb{R}^n$, and matrix $A \in \mathbb{R}^{n \times n}$, the following inequality holds:

$$x^T A y \leq \frac{1}{2\epsilon} x^T A A^T x + \frac{\epsilon}{2} y^T y.$$

Lemma 2.5. [41] For any constant matrix $\chi \in \mathbb{R}^{n \times n}$, $\chi = \chi^T$, scalar $v > 0$, vector function $F : [0, v] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well-defined, then,

$$v \int_v^0 F^T(s) \chi F(s) ds \geq \left(\int_v^0 F(s) ds \right)^T \chi \left(\int_v^0 F(s) ds \right).$$

3. Existence of the equilibrium point of neural network (2)

In this section, we will study the existence of an equilibrium for neural network (2) by the Leray-Schauder alternative theorem.

Theorem 3.1. Suppose the assumptions $(\mathbf{A}_1, \mathbf{A}_2)$ hold, and the network parameters satisfy (5). Then, neural network (2) has an equilibrium point, if there exists a positive diagonal matrix $P = \text{diag}\{p_i\}$ such that

$$\Phi_1 = (PA^* + A^{*T}P + \|PA_* + A_*^T P\|_2 I) + 2\|P\|_2 b + \|PC\|_2^2 I + H^2 < 0, \quad (8)$$

where b is defined in Lemma 2.3.

PROOF. Let $F(x) = D^{-1}(A + B)K[g(x)] + D^{-1}CHK[g(x)] + D^{-1}U$. Then, x^* is a fixed point of F if and only if x^* is an equilibrium point of the neural network (2). By the Leray-Schauder alternative Theorem, to solve the fixed point problem $x^* \in F(x^*)$, it is sufficient to show that the set

$$\Gamma = \{x \in \mathbb{R}^n : x \in \theta F(x), \theta \in (0, 1)\}$$

is bounded. Next, we will prove there exists $R > 0$, such that $\Gamma \subseteq B(0, R)$, which means that Γ is bounded. Let

$$\begin{aligned} F(x, \theta) &= D[-x + \theta F(x)] \\ &= -Dx + \theta(A + B)K[g(x)] + \theta CHK[g(x)] + \theta U. \end{aligned} \quad (9)$$

For any $y \in F(x, \theta)$, there must exist $\gamma \in K[g(x)]$ such that

$$y = -Dx + \theta(A + B)\gamma + CH\gamma + \theta U.$$

Multiplying both sides of the equation above by $2\gamma^T P$, we have

$$\begin{aligned} 2\gamma^T Py &= 2\gamma^T P(-Dx + \theta(A + B)\gamma + \theta CH\gamma + \theta U) \\ &= -2\gamma^T PDx + 2\theta\gamma^T P(A + B)\gamma + 2\theta\gamma^T PCH\gamma + 2\theta\gamma^T PU \\ &= -2\gamma^T PDx + 2\theta\gamma^T PA\gamma + 2\theta\gamma^T PB\gamma + 2\theta\gamma^T PCH\gamma + 2\theta\gamma^T PU \\ &= -2\gamma^T PDx + \theta\gamma^T (PA + A^T P)\gamma + 2\theta\gamma^T PB\gamma + 2\theta\gamma^T PCH\gamma \\ &\quad + 2\theta\gamma^T PU. \end{aligned} \quad (10)$$

According to the assumption $(\mathbf{A}_1)$, we have

$$-2\gamma^T PDx \leq -2\gamma^T P\underline{D}x = -2 \mid \gamma^T \mid P\underline{D} \mid x \mid \leq -2\theta \mid \gamma^T \mid P\underline{D} \mid x \mid. \quad (11)$$

From Lemma 2.2, we can get

$$\theta\gamma^T (PA + A^T P)\gamma \leq \theta\gamma^T (PA^* + A^{*T}P + \parallel PA_* + A_*^T P \parallel_2 I)\gamma, \quad (12)$$

where $K = \text{diag}\{k_i\}$. It is clear that,

$$2\theta\gamma^T PB\gamma \leq 2\theta\gamma^T PB\gamma \leq 2\theta\gamma^T \parallel PB \parallel_2 \gamma \leq 2\theta\gamma^T \parallel P \parallel_2 \parallel B \parallel_2 \gamma. \quad (13)$$

From Lemma 2.2 and (13),

$$2\theta\gamma^T PB\gamma \leq 2\theta\gamma^T \parallel P \parallel_2 b\gamma. \quad (14)$$

Then, by (12), (14), we can get

$$\begin{aligned} &2\theta\gamma^T P(A + B)\gamma + 2\theta\gamma^T PCH\gamma \\ &\leq \theta\gamma^T (PA^* + A^{*T}P + \parallel PA_* + A_*^T P \parallel_2 I + 2 \parallel P \parallel_2 b)\gamma + \theta\gamma^T \parallel PC \parallel_2^2 \gamma + \theta\gamma^T H^2 \gamma \\ &= \theta\gamma^T \Phi_1 \gamma \\ &\leq 0, \end{aligned} \quad (15)$$

Asserting (11) and (15) into (10) results in:

$$\begin{aligned} 2\gamma^T Py &\leq -2\gamma^T PDx + 2\theta\gamma^T PU \\ &= 2 \sum_{i=1}^n P_i \gamma_i (-D_i x_i + U_i) \end{aligned} \quad (16)$$

Then, similar to [28], we can construct an open convex bounded set Ω , such that $2\gamma^T P y < 0$ for all $x \in \Omega$. Hence, $\Gamma \subseteq \Omega$ is bounded. Invoking Lemma 2.1, there exists $x^* \in \mathbb{R}^n$ such that $x^* \in F(x^*)$. Of course, x^* is an equilibrium point of neural network (2).

4. Stability analysis

In this section, we will show that the equilibrium point of neural network (2) is globally robustly exponentially stable based on assumptions in Theorem 3.1. In order to simplify the proof, we need to shift the equilibrium point x^* of neural network (2) to the origin as follows,

$$z(\cdot) = x(\cdot) - x^*.$$

From Definition 2.1, there exists $\gamma^* \in K[g(x^*)]$ such that

$$-Dx^* + (A + B)\gamma^* + CHg(x^*) + U = 0, \quad (17)$$

Then the neural network (2) is equivalent to the following neural network,

$$\dot{z}(t) = -Dz(t) + A\tilde{G}(z(t)) + B\tilde{G}(z(t - \tau)) + C \int_{t-h}^t \tilde{G}(z(s))ds, \quad (18)$$

where $\tilde{G}_i(z)_i = g_i(z_i + x_i^*) - \gamma_i^*$. Obviously, $O(0, 0, \dots, 0)$ is an equilibrium point of neural network (18). By Definition 2.2, $z(t)$ is a solution of the neural network (18), if there exists $\tilde{\gamma}(z(t)) \in K[\tilde{G}(z(t))]$ such that

$$\dot{z}(t) = -Dz(t) + A\tilde{\gamma}(z(t)) + B\tilde{\gamma}(z(t - \tau)) + C \int_{t-h}^t \tilde{G}(z(s))ds. \quad (19)$$

Theorem 4.1. *Based on the hypotheses in Theorem 3.1, the equilibrium point of neural network (2) is globally exponentially stable.*

PROOF. Based on the hypotheses in Theorem 3.1, we choose a sufficiently small constant $\delta > 0$ such that

$$\begin{aligned} \Gamma = & (PA^* + A^{*T}P + \|PA_* + A_*^T P\|_2 I) + (1 + e^{\delta\tau}) \|P\|_2 b \\ & + \|PC\|_2^2 I + e^{\delta\tau} H^2 < 0, \end{aligned} \quad (20)$$

and $\delta < \min\{\frac{1}{2}d_i : 1 \leq i \leq n\}$. Consider the following Lyapunov functional

$$V(t, z(t)) = V_1(t, z(t)) + V_2(t, z(t)) + V_3(t, z(t)) + V_4(t, z(t)),$$

where

$$\begin{aligned}
V_1(t, z(t)) &= e^{\delta t} z^T(t) D^{-1} z(t) \\
V_2(t, z(t)) &= \alpha e^{\delta t} [2 \sum_{i=1}^n p_i \int_0^{z_i(t)} \tilde{G}_i(\theta) d\theta] \\
V_3(t, z(t)) &= (\alpha\gamma + \beta) \sum_{i=1}^n p_i \int_{t-\tau_i}^t \tilde{G}_i^2(z(s)) e^{\delta(s+\tau_i)} ds \\
V_4(t, z(t)) &= \sigma \sum_{i=1}^n p_i \int_{-h_i}^0 \int_{t+\theta}^t \tilde{G}_i^2(z_i(s)) e^{\delta(s+\tau_i)} ds d\theta.
\end{aligned} \tag{21}$$

Next, we will compute the derivation of V_i ($i = 1, 2, 3, 4$) separately. We first compute the derivation of $V_1(t, z(t))$ with respect to time along the solution of neural network (18) as follows,

$$\begin{aligned}
\dot{V}_1(t, z(t)) &= \delta e^{\delta t} z^T(t) D^{-1} z(t) + 2e^{\delta t} z^T(t) D^{-1} \dot{z}(t) \\
&= \delta e^{\delta t} z^T(t) D^{-1} z(t) - 2e^{\delta t} z^T(t) D^{-1} D z(t) + 2e^{\delta t} z^T(t) D^{-1} A(z) \tilde{\gamma}(z(t)) \\
&\quad + 2e^{\delta t} z^T(t) D^{-1} B(z) \tilde{\gamma}(z(t-\tau)) + 2e^{\delta t} z^T(t) D^{-1} C \int_{t-h}^t g(z(s)) ds \\
&\leq e^{\delta t} z^T(t) (\delta D^{-1} - \frac{I}{2}) z(t) + 2e^{\delta t} \tilde{\gamma}^T(z(t)) \|D^{-1} A\|_2^2 \tilde{\gamma}(z(t)) \\
&\quad + 2e^{\delta t} \tilde{\gamma}^T(z(t-\tau(t))) \|D^{-1} B\|_2^2 \tilde{\gamma}(z(t-\tau)) \\
&\quad + 2e^{\delta t} \int_{t-h}^t g(z(s)) ds \|D^{-1} C\|_2^2 \int_{t-h}^t g(z(s)) ds \\
&\leq 2e^{\delta t} \tilde{\gamma}^T(z(t)) \|D^{-1} A\|_2^2 \tilde{\gamma}(z(t)) \\
&\quad + 2e^{\delta t} \tilde{\gamma}^T(z(t-\tau(t))) \|D^{-1} B\|_2^2 \tilde{\gamma}(z(t-\tau)) \\
&\quad + 2e^{\delta t} \int_{t-h}^t g(z(s)) ds \|D^{-1} C\|_2^2 \int_{t-h}^t g(z(s)) ds.
\end{aligned} \tag{22}$$

Next, calculate the derivation of $V_2(t, z(t))$ along the neural network (18),

$$\begin{aligned}
\dot{V}_2(t, z(t)) &= \delta \alpha e^{\delta t} \left[2 \sum_{i=1}^n p_i \int_0^{z_i(t)} \tilde{\gamma}_i(\theta) d\theta \right] + 2e^{\alpha \delta t} \tilde{\gamma}^T(z(t)) P \dot{z}(t) \\
&= 2\delta \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P z(t) - 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P D z(t) \\
&\quad + 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P A \tilde{\gamma}(z(t)) + 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P B \tilde{\gamma}(z(t - \tau)) \\
&\quad + 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P C \int_{t-h}^t g(z(s)) ds \\
&\leq 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P (\delta I - D) z(t) \\
&\quad + 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P A \tilde{\gamma}(z(t)) + 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P B \tilde{\gamma}(z(t - \tau)) \\
&\quad + \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) \| P C \|^2_2 \tilde{\gamma}(z(t)) + \alpha e^{\delta t} \int_{t-h}^t g(z(s)) ds \int_{t-h}^t g(z(s)) ds \\
&\leq \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) (P A + A^T P) \tilde{\gamma}(z(t)) + 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P B \tilde{\gamma}(z(t - \tau)) \\
&\quad + \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) \| P C \|^2_2 \tilde{\gamma}(z(t)) + \alpha e^{\delta t} \int_{t-h}^t g(z(s)) ds \int_{t-h}^t g(z(s)) ds.
\end{aligned} \tag{23}$$

By Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned}
(1) \quad & \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) (P A + A^T P) \tilde{\gamma}(z(t)) \\
& \leq \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) (P A^* + A^{*T} P + \| P A_* + A_*^T P \|_2 I) \tilde{\gamma}(z(t)), \\
(2) \quad & 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P B \tilde{\gamma}(z(t - \tau)) \\
& \leq 2\alpha e^{\delta t} \tilde{\gamma}^T(z(t)) \| P \|_2 b \tilde{\gamma}(z(t - \tau)) \\
& \leq \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) \| P \|_2 b \tilde{\gamma}(z(t)) + \alpha e^{\delta t} \tilde{\gamma}(z(t - \tau))^T \| P \|_2 b \tilde{\gamma}(z(t - \tau)).
\end{aligned} \tag{24}$$

Substituting (24) into (23), we get

$$\begin{aligned}
\dot{V}_2(t, z(t)) &\leq \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) (P A^* + P A^{*T} + \| P A_* + A_*^T P \|_2 I) \tilde{\gamma}(z(t)) \\
&\quad + \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) \| P \|_2 b \tilde{\gamma}(z(t)) \\
&\quad + \alpha e^{\delta t} \tilde{\gamma}(z(t - \tau))^T \| P \|_2 b \tilde{\gamma}(z(t - \tau)) \\
&\quad + \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) P C \tilde{\gamma}(z(t)) + \alpha e^{\delta t} \int_{t-h}^t g(z(s)) ds P C \int_{t-h}^t g(z(s)) ds.
\end{aligned} \tag{25}$$

The time derivative of $V_3(t, z(t))$ along the solution of neural network (18) is obtained as follows

$$\begin{aligned}
\dot{V}_3(t, z(t)) &= (\alpha \gamma + \beta) \sum_{i=1}^n p_i \int_{t-\tau_i}^t \tilde{\gamma}_i(z(s)) e^{\delta(s+\tau_i)} ds \\
&= (\alpha \gamma + \beta) e^{t+\tau} \| \tilde{\gamma}(z(t)) \|_2^2 - (\alpha \gamma + \beta) e^t \| \tilde{\gamma}(z(t - \tau)) \|_2^2.
\end{aligned} \tag{26}$$

The derivation of $V_4(t, z(t))$ is shown as follows,

$$\begin{aligned}\dot{V}_4(t, z(t)) &= \sigma \sum_{i=1}^n e^{\delta(t+\tau_i)} p_i h_i^2 \tilde{\gamma}_i^2(z_i(t)) - \sigma \sum_{i=1}^n e^{\delta t} p_i h_i^2 \int_{t-h_i}^t \tilde{\gamma}_i^2(z_i(s)) ds \\ &\leq \sigma e^{\delta(t+\tau)} \tilde{\gamma}^T(z(t)) P H^2 \tilde{\gamma}(z(t)) - \sigma e^{\delta t} \int_{t-h}^t \tilde{\gamma}(z(s)) ds P \int_{t-h}^t \tilde{\gamma}(z(s)) ds.\end{aligned}\quad (27)$$

where $H^2 = \text{diag}(h_1^2, h_2^2, \dots, h_n^2)$. Hence, according to the derivation of $\dot{V}_i$ ($i = 1, 2, 3, 4$) above, we have

$$\begin{aligned}\dot{V}(t, z(t)) &= \dot{V}_1(t, z(t)) + \dot{V}_2(t, z(t)) + \dot{V}_3(t, z(t)) + \dot{V}_4(t, z(t)) \\ &\leq 2e^{\delta t} \tilde{\gamma}^T(z(t)) \|D^{-1}A\|_2^2 \tilde{\gamma}(z(t)) + 2e^{\delta t} \tilde{\gamma}^T(z(t-\tau)) \|D^{-1}B\|_2^2 \tilde{\gamma}(z(t-\tau)) \\ &\quad + 2e^{\delta t} \int_{t-h}^t g(z(s)) ds \|D^{-1}C\|_2^2 \int_{t-h}^t g(z(s)) ds \\ &\quad + \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) (PA^* + A^{*T}P + \|PA_* + A_*^T P\|_2 I) \tilde{\gamma}(z(t)) \\ &\quad + \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) \|P\|_2 b \tilde{\gamma}(z(t)) + \alpha e^{\delta t} \tilde{\gamma}(z(t-\tau))^T \|P\|_2 b \tilde{\gamma}(z(t-\tau)) \\ &\quad + \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) \|PC\|_2^2 \tilde{\gamma}(z(t)) + \alpha e^{\delta t} \int_{t-h}^t g(z(s)) ds \int_{t-h}^t g(z(s)) ds \\ &\quad + (\alpha\gamma + \beta) e^{\delta(t+\tau)} \|\tilde{\gamma}(z(t))\|_2^2 - (\alpha\gamma + \beta) e^t \|\tilde{\gamma}(z(t-\tau))\|_2^2 \\ &\quad + \sigma e^{\delta(t+\tau)} \tilde{\gamma}^T(z(t)) P H^2 \tilde{\gamma}(z(t)) - \sigma e^{\delta t} \int_{t-h}^t \tilde{\gamma}(z(s)) ds P \int_{t-h}^t \tilde{\gamma}(z(s)) ds \\ &= e^{\delta t} \tilde{\gamma}^T(z(t)) [2 \|D^{-1}A\|_2^2 I + \alpha(PA^* + A^{*T}P + \|PA_* + A_*^T P\|_2 I) \\ &\quad + \alpha \|P\|_2 b I + (\alpha\gamma + \beta) e^\tau + \sigma e^{\delta\tau} P H^2 + \|PC\|_2^2 I] \tilde{\gamma}(z(t)) \\ &\quad + \tilde{\gamma}(z(t-\tau))^T [2 \|D^{-1}B\|_2^2 I + \alpha \|P\|_2 b I - (\alpha\gamma + \beta) I] \tilde{\gamma}(z(t-\tau)) \\ &\quad + \int_{t-h}^t \tilde{\gamma}(z(s)) ds [2 \|D^{-1}C\|_2^2 I - \sigma P + \alpha I] \int_{t-h}^t \tilde{\gamma}(z(s)) ds.\end{aligned}\quad (28)$$

If we choose

$$\begin{aligned}\beta &= 2 \|D^{-1}B\|_2^2, \\ \gamma &= \alpha \|P\|_2 b, \\ \sigma &= \alpha p_m^{-1} + 2p_m^{-1} \|D^{-1}C\|_2^2,\end{aligned}\quad (29)$$

where $p_m = \min_{i=1, \dots, n} p_i$. Then, $\dot{V}(t, z(t))$ can be written as

$$\begin{aligned}
\dot{V}(t, z(t)) &\leq \alpha e^{\delta t} \tilde{\gamma}^T(z(t)) [(PA^* + A^{*T}P + \|PA_* + A_*^T P\|_2 I) + (1 + e^{\delta \tau}) \|P\|_2 I \\
&\quad + \|PC\|_2^2 I + e^{\delta \tau} H^2] \tilde{\gamma}(z(t)) \\
&\quad + e^{\delta t} \tilde{\gamma}^T(z(t)) [2 \|D^{-1}A\|_2^2 I + 2e^{\delta \tau} \|D^{-1}B\|_2^2 I \\
&\quad + 2e^{\delta \tau} \|D^{-1}C\|_2^2 H^2] \tilde{\gamma}(z(t)) \\
&= e^{\delta t} \tilde{\gamma}^T(z(t)) [\alpha \Gamma + 2 \|D^{-1}A\|_2^2 I + 2e^{\delta \tau} \|D^{-1}B\|_2^2 I \\
&\quad + 2e^{\delta \tau} \|D^{-1}C\|_2^2 H^2] \tilde{\gamma}(z(t))
\end{aligned} \tag{30}$$

Since Γ from (20) is a negative definite matrix, we can choose a sufficient large α such that

$$\alpha \Gamma + 2 \|D^{-1}A\|_2^2 I + 2e^{\delta \tau} \|D^{-1}B\|_2^2 I + 2e^{\delta \tau} \|D^{-1}C\|_2^2 H^2 < 0, \tag{31}$$

which implies,

$$\dot{V}(t, z(t)) \leq 0.$$

Hence, $V(t, z(t)) \leq V(0, z(0))$, i.e.,

$$V_1(t, z(t)) \leq V(t, z(t)) \leq V(0, z(0)),$$

which means

$$e^{\delta t} z^T(t) D^{-1} z(t) \leq V(0, z(0)). \tag{32}$$

By (32), we obtain

$$z^T(t) z(t) \leq V(0, z(0)) e^{-\delta t} D_M,$$

where $D_M = \max\{d_i\}$.

$$\|x(t) - x^*\| \leq D_M [V(0, z(0))]^{\frac{1}{2}} e^{-\frac{\delta t}{2}}.$$

This means that neural network (2) is globally robustly exponentially stable. Moreover, it also means that the equilibrium point of neural network (2) is unique.

5. Comparisons and examples

In this section, by making further analysis and giving some numerical examples, we compare our results with the previous ones derived in the previous literatures. We first compare our results with the results in [32]. The neural network considered in [32] is

$$\dot{x}(t) = -Dx(t) + Ag(x(t)) + Bg(x(t - \tau)) + U. \tag{33}$$

Obviously, the neural network (33) considered in [32] is a special case of the neural network (2) in this paper. Moreover, to some extent, the results in this paper improve the results in [32]. The following example explains it.

Example 1. we consider a delayed neural network model (2) with the following network parameters:

$$\begin{aligned} \bar{A} &= \begin{pmatrix} -107 & 0 \\ 0 & -103 \end{pmatrix}, \underline{A} = \begin{pmatrix} -111 & 0 \\ 0 & -107 \end{pmatrix}, \bar{B} = \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix} \\ \underline{B} &= \begin{pmatrix} 85 & 0 \\ 0 & 70 \end{pmatrix}, \bar{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0.9 \end{pmatrix}, \underline{C} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.4 \end{pmatrix}, \bar{D} = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix} \\ \underline{D} &= \begin{pmatrix} 9 & 0 \\ 0 & 11 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, g(x) = \begin{pmatrix} \frac{1}{3}x \\ \frac{1}{4}x \end{pmatrix}, U = (1, 1) \\ \tau &= 1 \end{aligned} \quad (34)$$

Obviously, for any positive diagonal matrix P ,

$$\begin{pmatrix} -P\tilde{A} - \tilde{A}^T P - (\|B^*\| + \|B_*\|)^2 I_n & -P \\ -P & I_n \end{pmatrix} \quad (35)$$

is not a positive definite matrix. Hence, The results in ([32]) cannot be used to verify the robust stability of the defined neural network. Here we use Theorem 4.1 to verify the global robust exponential stability of the defined neural network in the example. Firstly, it is obvious that g satisfy hypotheses $(\mathbf{A}_1, \mathbf{A}_2)$. After simple calculation, $b = 4$. Then, if we choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\Phi_1 = \begin{pmatrix} -12.95 & 0 \\ 0 & -4.95 \end{pmatrix} < 0$$

Hence, by Theorem 3.1 and Theorem 4.1, the defined neural network in the example has an equilibrium point, which is globally robustly exponentially stable. To test and verify the theoretical result, we give some simulations by the following cases:

Case 1. $A = \bar{A}, B = \bar{B}, C = \bar{C}, D = \bar{D}$, see Figure 1.

Case 2. $A = \begin{pmatrix} 109 & 0.1 \\ 0.1 & 105 \end{pmatrix} \in A_I, B = \begin{pmatrix} 90 & 0 \\ 0 & 85 \end{pmatrix} \in B_I, C = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.6 \end{pmatrix} \in C_I, D = \begin{pmatrix} 9.5 & 0 \\ 0 & 11.5 \end{pmatrix} \in D_I$, see Figure 1.

The following example is used to show the advantage of this paper compared with Theorem 3 in [40].

Figure 1: Trajectories of neural network in case 1 and 2 of Example 1 with five different initial points.

Example 2. We consider a delayed neural network model (2) with the following network parameters:

$$\begin{aligned} \bar{A} &= \begin{pmatrix} -6 & 0 \\ 0 & -5 \end{pmatrix}, \underline{A} = \begin{pmatrix} -8 & 0 \\ 0 & -7 \end{pmatrix}, \bar{B} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}, \underline{B} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \\ \bar{C} &= \begin{pmatrix} -1 & 0 \\ 0 & -0.5 \end{pmatrix}, \underline{C} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \bar{D} = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}, \underline{D} = \begin{pmatrix} 9 & 0 \\ 0 & 11 \end{pmatrix} \\ H &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, g(x) = \begin{pmatrix} \frac{1}{3}x \\ \frac{1}{4}x \end{pmatrix}, U = (1, 1), \tau = 1 \end{aligned} \quad (36)$$

Obviously, for any positive diagonal matrix P ,

$$\begin{aligned} \Phi_4 &= 2\underline{C}PK^{-1} - (PA^* + A^{*T}P + \|PA^* + A^{*T}P\| I) - 2\|P\|_2\|Q\|_2 I \\ &= \begin{pmatrix} 6p_1 - 10.809p - 2 & 0 \\ 0 & 10p_2 - 10.809p - 2 \end{pmatrix} \end{aligned} \quad (37)$$

is not a positive definite matrix, where $p = \max\{p_1, p_2\}$. Hence, Theorem 3 in [40] cannot be used to verify the robust stability of the defined neural network. Here we use Theorem 4.1 to verify the global robust exponential stability of the defined neural network in the example. Firstly, it is obvious that g satisfy hypotheses $(\mathbf{A}_1, \mathbf{A}_2)$. After simple calculation, $b = 4$. Then, if we choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $\Phi_1 = \begin{pmatrix} -2.188 & 0 \\ 0 & -0.188 \end{pmatrix} < 0$.

Hence, by Theorem 3.1 and Theorem 4.1, the defined neural network in the example has an equilibrium point, which is globally robustly exponentially stable. To test and verify the theoretical result, we give some simulations by the following cases:

Case 3. $A = \bar{A}, B = \bar{B}, C = \bar{C}, D = \bar{D}$, see Figure 2.

Case 4. $A = \begin{pmatrix} -7 & 0 \\ 0 & -6 \end{pmatrix} \in A_I, B = \begin{pmatrix} 4.5 & 0 \\ 0 & 2.5 \end{pmatrix} \in B_I, C = \begin{pmatrix} -1.5 & 0 \\ 0 & -0.7 \end{pmatrix} \in C_I, D = \begin{pmatrix} 9.5 & 0 \\ 0 & 11.5 \end{pmatrix} \in D_I$, see Figure 2.

Figure 2: Trajectories of neural network in case 3 and 4 of Example 2 with five different initial points.

Example 3. We consider a delayed neural network model (2) with the following network parameters:

$$\begin{aligned} \bar{A} &= \begin{pmatrix} -26 & 0 \\ 0 & -28 \end{pmatrix}, \underline{A} = \begin{pmatrix} -30 & 0 \\ 0 & -32 \end{pmatrix}, \bar{B} = \begin{pmatrix} 20 & 0 \\ 0 & 19 \end{pmatrix}, \underline{B} = \begin{pmatrix} 16 & 0 \\ 0 & 11 \end{pmatrix} \\ \bar{C} &= \begin{pmatrix} 3 & 0.01 \\ 0.01 & 2 \end{pmatrix}, \underline{C} = \begin{pmatrix} 2 & 0.01 \\ 0.01 & 1 \end{pmatrix}, \bar{D} = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}, \underline{D} = \begin{pmatrix} 9 & 0 \\ 0 & 11 \end{pmatrix} \\ H &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, g(x) = \begin{pmatrix} \frac{1}{3}x \\ \frac{1}{4}x \end{pmatrix}, U = (1, 1), \tau = 1 \end{aligned} \quad (38)$$

After simple calculation, for any positive diagonal matrix P ,

$$2PCK^{-1} - (PA^* + A^{*T}P + \|PA^* + A^{*T}P\|I) - L - \frac{1}{1-h}b_1\|P\|_2\|L\|_2I \quad (39)$$

is not a positive definite matrix. Hence, Theorem 3 in [42] cannot be used to verify the robust stability of the defined neural network. On the other hand, if we choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $\Phi_1 = \begin{pmatrix} -10.9973 & 0 \\ 0 & -14.9973 \end{pmatrix} < 0$.

Hence, by Theorem 3.1 and Theorem 4.1, the defined neural network in the example has an equilibrium point, which is globally robustly exponentially stable. For details, see case 4 and Figure 3 below. To test and verify the theoretical result, we give some simulations by the following cases:

Case 5. $A = \bar{A}, B = \bar{B}, C = \bar{C}, D = \bar{D}$, see Figure 3.

Case 6. $A = \begin{pmatrix} -28 & 0 \\ 0 & -30 \end{pmatrix} \in A_I, B = \begin{pmatrix} 18 & 0 \\ 0 & 15 \end{pmatrix} \in B_I, C = \begin{pmatrix} 2.5 & 0.01 \\ 0.01 & 1.5 \end{pmatrix} \in C_I, D = \begin{pmatrix} 9.5 & 0 \\ 0 & 11.5 \end{pmatrix} \in D_I$, see Figure 3.

Figure 3: Trajectories of neural network in case 5 and 6 of Example 3 with five different initial points.

Example 4. We consider the delayed neural network (2) with the following network parameters:

$$\begin{aligned} \bar{A} &= \begin{pmatrix} -26 & 0 \\ 0 & -28 \end{pmatrix}, \underline{A} = \begin{pmatrix} -30 & 0 \\ 0 & -32 \end{pmatrix}, \bar{B} = \begin{pmatrix} 20 & 0 \\ 0 & 19 \end{pmatrix}, \underline{B} = \begin{pmatrix} 16 & 0 \\ 0 & 11 \end{pmatrix} \\ \bar{C} &= \begin{pmatrix} 3 & 0.01 \\ 0.01 & 2 \end{pmatrix}, \underline{C} = \begin{pmatrix} 2 & 0.01 \\ 0.01 & 1 \end{pmatrix}, \bar{D} = \begin{pmatrix} 10 & 0 \\ 0 & 12 \end{pmatrix}, \underline{D} = \begin{pmatrix} 9 & 0 \\ 0 & 11 \end{pmatrix} \\ H &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, U = (1, 1), \tau = 1 \end{aligned} \quad (40)$$

where

$$g_1(x) = g_2(x) = \begin{cases} x + 1 & x > 1 \\ x & 1 \geq x \end{cases} \quad (41)$$

Obviously, for any positive diagonal matrix P ,

$$\begin{aligned} \Phi_5 &= 2PCK^{-1} - (PA^* + A^{*T}P + \|PA^* + A^{*T}P\| I) - L - \frac{1}{1-h}b_1 \|P\|_2 \|L\|_2 I \\ &= \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \end{aligned} \quad (42)$$

where $\Delta_1 = 68p_1 - \begin{vmatrix} 4p_1 & 0 \\ 0 & 4p_2 \end{vmatrix} \|_2 - l_1 - 400.004 * \frac{1}{1-h} \| P \|_2^2 \| L^{-1} \|_2$,

$\Delta_2 = 66p_2 - \begin{vmatrix} 4p_1 & 0 \\ 0 & 4p_2 \end{vmatrix} \|_2 - l_2 - 400.004 * \frac{1}{1-h} \| P \|_2^2 \| L^{-1} \|_2$.

Since $0 \leq l_i \leq 1, 0 \leq h \leq 1$,

$$\Phi_5 \leq \begin{pmatrix} 68p_1 - 4 \| P \|_2^2 - 400.004 \| P \|_2^2 & 0 \\ 0 & 66p_2 - 4 \| P \|_2^2 - 400.004 \| P \|_2^2 \end{pmatrix} \quad (43)$$

is not a positive definite matrix, where $p = \max\{p_1, p_2\}$. Hence, Theorem 3 in [43] cannot be used to verify the robust stability of the defined neural network. Here we use Theorem 4.1 to verify the global robust exponential stability of the defined neural network in the example. Firstly, it is obvious that g satisfy hypotheses $(\mathbf{A}_1, \mathbf{A}_2)$. After simple calculation, $b = 4$. Then, if we choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\Phi_1 = \begin{pmatrix} -10.9973 & 0 \\ 0 & -14.9973 \end{pmatrix} < 0$$

Hence, by Theorem 3.1 and Theorem 4.1, the defined neural network in the example has an equilibrium point, which is globally robustly exponentially stable. To test and verify the theoretical result, we give some simulations by the following cases:

Case 7. $A = \bar{A}, B = \bar{B}, C = \bar{C}, D = \bar{D}$, see Figure 4.

Case 8. $A = \begin{pmatrix} -28 & 0 \\ 0 & -30 \end{pmatrix} \in A_I, B = \begin{pmatrix} 18 & 0 \\ 0 & 15 \end{pmatrix} \in B_I, C = \begin{pmatrix} 2.5 & 0.01 \\ 0.01 & 1.5 \end{pmatrix} \in C_I, D = \begin{pmatrix} 9.5 & 0 \\ 0 & 11.5 \end{pmatrix} \in D_I$, see Figure 4.

6. Conclusions

In this paper, we study the global exponential stability of a class of neural networks with discontinuous Lurie-type activation and mixed delays. We propose a new sufficient condition for the existence and global exponential stability of the equilibrium point under parameter uncertainties of the neural network. We make a detailed comparison between our result and the previous corresponding results derived in the previous literature. These comparisons show that our result is new and improves some of the previously reported stability results.

Figure 4: Trajectories of neural network in case 7 and 8 of Example 4 with five different initial points.

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