Finite-time H_∞ control for one-sided Lipschitz systems with auxiliary matrices

Yuehua Huang, Shiqi Fu, Yanjun Shen

College of Electrical Engineering and New Energy,

China Three Gorges University, Yichang, Hubei, China, 443002

Abstract

In this paper, finite-time H_∞ control is investigated for a class of one-sided Lipschitz systems by using the Finsler's lemma. Finite-time boundedness conditions are firstly provided for the one-sided Lipschitz system. The proposed conditions are less conservative since a new Lyapunov function with an additional matrix is constructed, and auxiliary matrices are introduced to make the Lyapunov matrix separate from the system matrix. Then, finite-time H_∞ controller is designed by dynamic output feedback. The observer gain and output feedback control gain can be designed in one step optimization. Further discussions and comparisons are also presented in two aspects. At last, numerical simulations are given to verify the validity of the proposed methods.

Index Terms

finite-time stability, H_∞ control, one-sided Lipschitz systems, LMI, time-varying exogenous disturbances

I. INTRODUCTION

In the past few decades, researchers have paid considerable attention on Lipschitz nonlinear systems [1], [2], [3], [4], [5]. Recently, some researchers have transferred interest from Lipschitz nonlinear systems to one-sided Lipschitz nonlinear systems [6], [7], [8], [9], [10], [11]. The reason is that Lipschitz conditions may fail to be utilized when the Lipschitz constant is large [6], [9]. Therefore, one-sided Lipschitz nonlinear systems have been extensively studied by researchers.

It should be noted that the results in the aforementioned papers related to stability and performance criteria are mostly defined over an infinite time interval. In some practical applications, the main concerns are the behavior of the considered system don't exceed a given bound in a finite time interval, that is, large values of the states are not acceptable in the presence of saturations [16]. In this sense, it is reasonable to define stability for a system as its states, with some initial conditions, remains within a given bound in the fixed finite time interval. For this purpose, definition of finite time stability (FTS) is presented in [12], [13]. Later, the definition of FTS is extended to definition of finite-time boundedness (FTB) by taking external disturbances into consideration [16]. By linear matrix inequality (LMI) theory, computationally appealing conditions are presented to ensure FTS and FTB for linear systems. The problem of FTB by dynamic output feedback is also considered in [14]. The proposed approach allowed to fit the FTB control problem in the general framework of the LMI approach to the multi-objective synthesis. Sub-optimal controller can be derived by a two-step optimization. There are some results on finite-time stabilization for discrete-time systems [15]. The disturbances considered in [15], [16] are assumed to be constant. The results on FTB for linear systems with time-varying disturbances and for linear time-varying systems can be referred to [17], [18] and [19], respectively. In order to reduce conservativeness, new sufficient conditions for FTB are derived by using the Finsler's lemma [21]. However, the synthesis of FTS and FTB controllers by the proposed conditions is intractable. More recently, definition of finite-time H_∞ control is presented for a class of linear systems [20]. A state feedback controller is designed to ensure that the closed-loop system is FTB, and at the same time to reduce the effect of the disturbance input on the output to a prescribed level. The results in [20] have been extended to one-sided Lipschitz nonlinear systems [23] and time-delayed nonlinear systems [24].

In this paper, motivated by the ideas in [21], [23], we consider robust finite-time H_∞ control for one-sided Lipschitz systems based on Finsler's lemma. Firstly, sufficient conditions for FTB are presented. The proposed conditions are less conservative since a new Lyapunov function is constructed, and auxiliary matrices are introduced to make the Lyapunov

matrix separate from the system matrix. Then, a dynamic output feedback controller is designed to ensure the closed-loop systems are FTB, and the effect of the disturbance input on the output to a prescribed level simultaneously. The observer gain and output feedback control gain can be designed in one step optimization. Further discussions and comparisons are also presented in two aspects: robust finite-time H_∞ control by state feedback, and finite-time stabilization by output feedback for a class of linear systems considered in [14]. Finally, numerical simulations are given to verify the validity of the proposed approach. The main contributions of this paper includes: a new Lyapunov function with an auxiliary matrix is constructed, and computationally tractable and less conservative sufficient conditions are presented based on the Finsler's lemma.

This paper is organized as follows. In Section II, some definitions and robust finite-time H_∞ control are presented. The problem of finite-time H_∞ control by dynamic output feedback is studied for a class of one-sided Lipschitz nonlinear systems by using the Finsler's lemma in Section III. Section IV gives further discussions and comparisons in two aspects. In Section V, numerical simulations are provided to illustrate the validity of the proposed design methods. Finally, the paper is concluded in Section VI.

II. DEFINITIONS AND PROBLEM FORMULATION

In this section, we introduce some useful definitions and lemma and state our main aim of the paper.

Definition 1 ([23]): Consider the following nonlinear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Gw(t) + f(x(t)), & x(t_0) = x_0, \\ y(t) = Cx(t), \end{cases} \quad (1)$$

where the matrices $A \in \mathcal{R}^{n \times n}$, $G \in \mathcal{R}^{n \times l}$, and $C \in \mathcal{R}^{p \times n}$, the state $x(t) \in \mathcal{R}^n$, the output $y(t) \in \mathcal{R}^p$, $f(x(t)) : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is a real nonlinear vector field, the exogenous disturbance $w(t) \in \mathcal{W} \subset \mathcal{R}^l$, and the disturbance set $\mathcal{W}$ is given as

$$\mathcal{W} = \left\{ w(t) : \int_0^T w^T(t)w(t)dt < d^2, \quad d > 0 \right\}. \quad (2)$$

The system (1) is FTB with respect to $(c_1, c_2, \mathcal{W}, T, R)$, if

$$x_0^T R x_0 \leq c_1^2 \Rightarrow x^T(t) R x(t) \leq c_2^2, \forall t \in [0, T], w(t) \in \mathcal{W},$$

where $c_2 > c_1 > 0$, $T > 0$ and $0 < R \in \mathcal{R}^{n \times n}$.

Definition 2: The system (1) satisfies the following H_∞ performance index, if the following inequality

$$\int_0^T y^T(t)y(t)dt \leq \gamma^2 \int_0^T w^T(t)w(t)dt, \quad (3)$$

holds under the zero initial condition, where γ is a positive real number.

Now, consider the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Gw(t) + f(x(t)), \\ y(t) = Cx(t), \quad x(t_0) = x_0, \end{cases} \quad (4)$$

where the matrices $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $G \in \mathcal{R}^{n \times l}$, and $C \in \mathcal{R}^{p \times n}$, and the state $x(t) \in \mathcal{R}^n$, the input $u(t) \in \mathcal{R}^m$, the output $y(t) \in \mathcal{R}^p$, the exogenous disturbance $w(t) \in \mathcal{W} \subset \mathcal{R}^l$. We make the following assumptions, the matrix B has full column rank, and the nonlinear function $f(x(t))$ satisfies $f(0) = 0$ and the following conditions

a) one-sided Lipschitz condition

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \leq \rho_1 (x_1 - x_2)^T (x_1 - x_2), \quad (5)$$

where ρ_1 is a real number;

b) quadratic inner-boundedness in the region $\mathcal{D}$

$$(f(x_1) - f(x_2))^T (f(x_1) - f(x_2)) \leq \rho_2 (x_1 - x_2)^T (x_1 - x_2) + \rho_3 \langle f(x_1) - f(x_2), x_1 - x_2 \rangle, \quad (6)$$

for any $x_1, x_2 \in \mathcal{D}$, where ρ_2 and ρ_3 are real numbers.

In this paper, our aim is to design an output feedback controller in the form of

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + LC(\hat{x}(t) - x(t)) + f(\hat{x}(t)), & \hat{x}(0) = 0, \\ u(t) = K\hat{x}(t), \end{cases} \quad (7)$$

such that the closed-loop (4)-(7) is FTB, and at the same time the performance index (3) is satisfied. Then, the dynamic output feedback controller (7) are called robust output feedback finite-time H_∞ controller of the one-sided Lipschitz nonlinear system.

Remark 1: The matrix B has full column rank, which can be taken nonstrict at all. Since if $\text{rank}(B) = r < m$, then, there exist invertible matrices S and T such that $SBT = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where I_r denotes r -order unite matrix. Let $y(t) = Sx(t)$, then,

$$\dot{y}(t) = SAS^{-1}y(t) + SBT T^{-1}u(t) + Sw(t) + Sf(S^{-1}y(t)).$$

Let $v(t) = [v_1^T(t), v_2^T(t)]^T = T^{-1}u(t)$. We have

$$\dot{y}(t) = SAS^{-1}y(t) + \begin{bmatrix} I_r \\ 0 \end{bmatrix} v_1(t) + Sw(t) + Sf(S^{-1}y(t)).$$

Therefore, without loss of generality, we can assume that B is a full column rank matrix.

Remark 2: The one-sided Lipschitz condition possess inherent advantages with consideration of conservativeness. It extends the Lipschitz theory to a more general family of nonlinear systems. For example, the scalars ρ_1 , ρ_2 and ρ_3 can be positive, negative or zero. If ρ_2 and ρ_3 are restricted to positive, then, $f(\cdot)$ is Lipschitz. If $\rho_2 = 0$, then, the condition (6) is reduced to a sector constraint introduced in [25].

We want to solve the problem of robust output feedback finite-time H_∞ control less conservative based on the Finsler's lemma, which is given as follows.

Lemma 1: Let $x \in \mathcal{R}^n$, $\mathcal{D} \in \mathcal{S}^n$, $\mathcal{B} \in \mathcal{R}^{m \times n}$ such that $\text{rank}(\mathcal{B}) = r < n$. Let $\mathcal{B}^\perp$ a matrix satisfying $\mathcal{B}^\perp \in \mathcal{B}^{n \times (n-r)}$ and $\text{Im}(\mathcal{B}^\top) = \text{Ker}(\mathcal{B})$. The following conditions are equivalent:

1. $x^T \mathcal{D} x < 0$, $\forall x \in \mathcal{R}^n$ such that $x \neq 0$ and $\mathcal{B}x = 0$.
2. $(\mathcal{B}^\perp)^T \mathcal{D} \mathcal{B}^\perp < 0$.
3. $\exists \mu \in \mathcal{R}$ such that $\mathcal{D} - \mu \mathcal{B}^T \mathcal{B} < 0$.
4. $\exists \mathcal{F} \in \mathcal{R}^{n \times m}$ such that $\mathcal{D} + \mathcal{F} \mathcal{B} + \mathcal{B}^T \mathcal{F}^T < 0$.

III. ROBUST OUTPUT FEEDBACK FINITE-TIME H_∞ CONTROL WITH AUXILIARY MATRICES

In this section, we mainly address on solving finite-time H_∞ control by output feedback for the system (4) based on the Finsler's lemma.

From (4) and (7), we have the following closed-loop system

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) - BK e(t) + Gw(t) + f(x(t), u(t)), \\ \dot{e}(t) = \bar{A}_1 e(t) + Gw(t) + \tilde{f}(t), \end{cases} \quad (8)$$

where $e(t) = x(t) - \hat{x}(t)$, $\bar{A} = A + BK$, $\bar{A}_1 = A + LC$, $\tilde{f}(t) = f(x(t), u(t)) - f(\hat{x}(t), u(t))$. Firstly, the following sufficient conditions are given to guarantee that the closed-loop system (8) is FTB.

Lemma 2: The closed-loop system (8) is FTB with $(c_1, c_2, \mathcal{W}, T, R)$, if there exist two symmetric positive definite matrices $Q_1, Q_2 \in \mathcal{R}^{n \times n}$, and three matrices $U, X_1, X_2 \in \mathcal{R}^{n \times n}$, and eleven positive real numbers α_1, ϵ_i ($i = 1, \dots, 6$) and λ_i ($i = 1, \dots, 4$), such that the following inequalities

$$\Gamma_3 < 0, \quad (9)$$

and

$$\begin{bmatrix} \lambda_3 R & 0 \\ 0 & \lambda_4 R \end{bmatrix} < \begin{bmatrix} Q_1 & U \\ U^T & Q_2 \end{bmatrix} < \begin{bmatrix} \lambda_1 R & 0 \\ 0 & \lambda_2 R \end{bmatrix}, \quad (10)$$

and

$$(\lambda_1 + \lambda_2)c_1^2 + \alpha_1 d^2 < \lambda_4 c_2^2 e^{-\alpha_1 T}, \quad (11)$$

hold, where $\Gamma_3 =$

$$\begin{bmatrix} \Gamma_{31} & \frac{\epsilon_2 \rho_3}{2} I - \frac{\epsilon_1}{2} I + X_1 & X_1 G & \Gamma_{34} & -\alpha_1 U - K^T B^T X_2^T & 0 & U - \epsilon_6 K^T B^T X_2^T \\ \frac{\epsilon_2 \rho_3}{2} I - \frac{\epsilon_1}{2} I + X_1^T & -\epsilon_2 I & 0 & \epsilon_3 X_1^T & 0 & 0 & 0 \\ G^T X_1^T & 0 & -\alpha_1 I & \epsilon_3 G^T X_1^T & G^T X_2^T & 0 & \epsilon_6 G^T X_2^T \\ \Gamma_{34}^T & \epsilon_3 X_1 & \epsilon_3 X_1 G & -\epsilon_3 X_1 - \epsilon_3 X_1^T & U & 0 & 0 \\ -\alpha_1 U^T - X_2 B K & 0 & X_2 G & U^T & \Gamma_{32} & \Gamma_{33} & \Gamma_{35} \\ 0 & 0 & 0 & 0 & \Gamma_{33}^T & -\epsilon_5 I & \epsilon_6 X_2^T \\ U^T - \epsilon_6 X_2 B K & 0 & \epsilon_6 X_2 G & 0 & \Gamma_{35}^T & \epsilon_6 X_2 & -\epsilon_6 X_2 - \epsilon_6 X_2^T \end{bmatrix},$$

$$\Gamma_{31} = -\alpha_1 Q_1 + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I + X_1 \bar{A}_1 + \bar{A}_1^T X_1^T, \Gamma_{32} = -\alpha_1 Q_2 + \epsilon_4 \rho_1 I + \epsilon_5 \rho_2 I + X_2 \bar{A} + \bar{A}^T X_2^T, \Gamma_{33} = -\frac{\epsilon_4}{2} I + \frac{\epsilon_5 \rho_3}{2} I + X_2,$$

$$\Gamma_{34} = Q_1 - X_1 + \epsilon_3 \bar{A}_1^T X_1^T, \Gamma_{35} = Q_2 - X_2 + \epsilon_6 \bar{A}^T X_2^T.$$

Proof: Consider the following Lyapunov function,

$$V(t) = \begin{bmatrix} e(t) \\ x(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & U \\ U^T & Q_2 \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \end{bmatrix}. \quad (12)$$

Then, the derivative of $V(t)$ defined by (12) along the system (8) is given as follows.

$$\begin{aligned} & \dot{V}(t) - \alpha_1 V(t) - \alpha_1 w^T(t) w(t) \\ &= \dot{e}^T(t) Q_1 e(t) + e^T(t) Q_1 \dot{e}(t) + \dot{x}^T(t) Q_2 x(t) + x^T(t) Q_2 \dot{x}(t) + 2\dot{e}^T(t) U x(t) + 2e^T(t) U^T \dot{x}(t) - \alpha_1 V(t) - \alpha_1 w^T(t) w(t) \\ &= \begin{bmatrix} e(t) \\ \tilde{f}(t) \\ w(t) \\ \dot{e}(t) \\ x(t) \\ f(x(t), u(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha_1 Q_1 & 0 & 0 & Q_1 & -\alpha_1 U & 0 & U^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 I & 0 & 0 & 0 & 0 \\ Q_1 & 0 & 0 & 0 & U & 0 & 0 \\ -\alpha_1 U^T & 0 & 0 & U^T & -\alpha_1 Q_2 & 0 & Q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U & 0 & 0 & 0 & Q_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{f}(t) \\ w(t) \\ \dot{e}(t) \\ x(t) \\ f(x(t), u(t)) \\ \dot{x}(t) \end{bmatrix}. \quad (13) \end{aligned}$$

Note that

$$\begin{bmatrix} e(t) \\ \tilde{f}(t) \end{bmatrix}^T \begin{bmatrix} \epsilon_1 \rho_1 I & -\frac{\epsilon_1}{2} I \\ -\frac{\epsilon_1}{2} I & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{f}(t) \end{bmatrix} \geq 0, \quad (14)$$

and

$$\begin{bmatrix} e(t) \\ \tilde{f}(t) \end{bmatrix}^T \begin{bmatrix} \epsilon_2 \rho_2 I & \frac{\epsilon_2 \rho_3}{2} I \\ \frac{\epsilon_2 \rho_3}{2} I & -\epsilon_2 I \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{f}(t) \end{bmatrix} \geq 0, \quad (15)$$

$$\begin{bmatrix} x(t) \\ f(x(t), u(t)) \end{bmatrix}^T \begin{bmatrix} \epsilon_4 \rho_1 I & -\frac{\epsilon_4}{2} I \\ -\frac{\epsilon_4}{2} I & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t), u(t)) \end{bmatrix} \geq 0, \quad (16)$$

and

$$\begin{bmatrix} x(t) \\ f(x(t), u(t)) \end{bmatrix}^T \begin{bmatrix} \epsilon_5 \rho_2 I & \frac{\epsilon_5 \rho_3}{2} I \\ \frac{\epsilon_5 \rho_3}{2} I & -\epsilon_5 I \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t), u(t)) \end{bmatrix} \geq 0. \quad (17)$$

From (13)-(17), we have

$$\dot{V}(t) - \alpha_1 V(t) - \alpha_1 w^T(t)w(t) \leq \mathcal{X}^T \Gamma_2 \mathcal{X},$$

$$\text{where } \mathcal{X} = \begin{bmatrix} e(t) \\ \tilde{f}(t) \\ w(t) \\ \dot{e}(t) \\ x(t) \\ f(x(t), u(t)) \\ \dot{x}(t) \end{bmatrix}, \text{ and}$$

$$\Gamma_2 = \begin{bmatrix} -\alpha_1 Q_1 + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I & -\frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & 0 & Q_1 & -\alpha_1 U & 0 & U^T \\ -\frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & -\epsilon_2 I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 I & 0 & 0 & 0 & 0 \\ Q_1 & 0 & 0 & 0 & U & 0 & 0 \\ -\alpha_1 U^T & 0 & 0 & U^T & -\alpha_1 Q_2 + \epsilon_4 \rho_1 I + \epsilon_5 \rho_2 I & -\frac{\epsilon_4}{2} I + \frac{\epsilon_5 \rho_3}{2} I & Q_2 \\ 0 & 0 & 0 & 0 & -\frac{\epsilon_4}{2} I + \frac{\epsilon_5 \rho_3}{2} I & -\epsilon_5 I & 0 \\ U & 0 & 0 & 0 & Q_2 & 0 & 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} \bar{A}_1 & I & G & -I & 0 & 0 & 0 \\ -BK & 0 & G & 0 & \bar{A} & I & -I \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{f}(t) \\ w(t) \\ \dot{e}(t) \\ x(t) \\ f(x(t), u(t)) \\ \dot{x}(t) \end{bmatrix} = 0,$$

and

$$\begin{aligned} \Gamma_3 &= \Gamma_2 + \begin{bmatrix} X_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \epsilon_3 X_1 & 0 \\ 0 & X_2 \\ 0 & 0 \\ 0 & \epsilon_6 X_2 \end{bmatrix} \begin{bmatrix} \bar{A}_1 & I & G & -I & 0 & 0 & 0 \\ -BK & 0 & G & 0 & \bar{A} & I & -I \end{bmatrix} \\ &+ \begin{bmatrix} \bar{A}_1 & I & G & -I & 0 & 0 & 0 \\ -BK & 0 & G & 0 & \bar{A} & I & -I \end{bmatrix}^T \begin{bmatrix} X_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \epsilon_3 X_1 & 0 \\ 0 & X_2 \\ 0 & 0 \\ 0 & \epsilon_6 X_2 \end{bmatrix}^T < 0, \end{aligned}$$

then, by Lemma 1, we have $\mathcal{X}^T \Gamma_2 \mathcal{X} < 0$, that is,

$$\dot{V}(t) < \alpha_1 V(t) + \alpha_1 w^T(t)w(t), \quad t \in [0, T].$$

Thus,

$$\frac{d}{dt}(e^{-\alpha_1 t} V(t)) < \alpha_1 e^{-\alpha_1 t} w^T(t)w(t), \quad t \in [0, T]. \quad (18)$$

Integrating (18) from 0 to t , we can obtain

$$e^{-\alpha_1 t} V(t) < V(0) + \alpha_1 \int_0^t e^{-\alpha_1 \tau} w^T(\tau)w(\tau) d\tau, \quad t \in [0, T]. \quad (19)$$

From the condition (10), it follows that

$$\lambda_4 e^{-\alpha_1 T} x^T(t) R x(t) < (\lambda_1 + \lambda_2) c_1^2 + \alpha_1 d^2, \quad t \in [0, T].$$

The condition (11) implies that $x^T(t) R x(t) \leq c_2^2$. Therefore, the proof is completed.

Lemma 3: The closed-loop system (8) is FTB with $(0, c_2, \mathcal{W}, T, R)$, and the performance index (3) is satisfied, if there exist two symmetric positive definite matrices $Q_1, Q_2 \in \mathcal{R}^{n \times n}$, and three matrices $U, X_1, X_2 \in \mathcal{R}^{n \times n}$, and ten positive real numbers $\alpha_1, \beta, \epsilon_i$ ($i = 1, \dots, 6$) and λ_2, λ_4 , such that the following inequalities

$$\Gamma_4 < 0, \quad (20)$$

$$\begin{bmatrix} \lambda_2 I & 0 \\ 0 & \lambda_4 I \end{bmatrix} < \begin{bmatrix} Q_1 & U \\ U^T & Q_2 \end{bmatrix}, \quad (21)$$

and

$$\alpha_1 d^2 < \lambda_4 c_2^2 e^{-\alpha_1 T}, \quad (22)$$

hold, where $\Gamma_4 =$

$$\begin{bmatrix} \Gamma_{31} & -\frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I + X_1 & X_1 G & Q_1 - X_1 + \epsilon_3 \bar{A}_1^T X_1^T & -\alpha_1 U - K^T B^T X_2^T & 0 & U - \epsilon_6 K^T B^T X_2^T \\ -\frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I + X_1^T & -\epsilon_2 I & 0 & \epsilon_3 X_1^T & 0 & 0 & 0 \\ G^T X_1^T & 0 & -\alpha_1 I & \epsilon_3 G^T X_1^T & G^T X_2^T & 0 & \epsilon_6 G^T X_2^T \\ Q_1 - X_1^T + \epsilon_3 X_1 \bar{A}_1 & \epsilon_3 X_1 & \epsilon_3 X_1 G & -\epsilon_3 X_1 - \epsilon_3 X_1^T & U & 0 & 0 \\ -\alpha_1 U^T - X_2 B K & 0 & X_2 G & U^T & \Gamma_{41} & \Gamma_{33} & \Gamma_{35} \\ 0 & 0 & 0 & 0 & \Gamma_{33}^T & -\epsilon_5 I & \epsilon_6 X_2^T \\ U^T - \epsilon_6 X_2 B K & 0 & \epsilon_6 X_2 G & 0 & \Gamma_{35}^T & \epsilon_6 X_2 & -\epsilon_6 X_2 - \epsilon_6 X_2^T \end{bmatrix},$$

$$\Gamma_{41} = -\alpha_1 Q_2 + \epsilon_4 \rho_1 I + \epsilon_5 \rho_2 I + X_2 \bar{A} + \bar{A}^T X_2^T + \beta C^T C.$$

Proof : The derivative of the Lyapunov function (12) along the states of the closed-loop system (8) is given as follows.

$$\dot{V}(t) - \alpha_1 V(t) < \alpha_1 w^T(t) w(t) - \beta y^T(t) y(t), \quad t \in [0, T].$$

Then,

$$\frac{d}{dt}(e^{-\alpha_1 t} V(t)) < e^{-\alpha_1 t} [\alpha_1 w^T(t) w(t) - \beta y^T(t) y(t)], \quad t \in [0, T].$$

Integrating the above differential inequality from 0 to T and noting that $x_0 = 0$ yields,

$$e^{-\alpha_1 T} V(T) < \int_0^T [e^{-\alpha_1 t} \alpha_1 w^T(t) w(t) - e^{-\alpha_1 t} \beta y^T(t) y(t)] dt, \quad t \in [0, T].$$

Therefore,

$$\int_0^T y^T(t) y(t) dt < \frac{\alpha_1 e^{\alpha_1 T}}{\beta} \int_0^T w^T(t) w(t) dt.$$

Thus, the performance index can be guaranteed by letting $\gamma = \sqrt{\frac{\alpha_1 e^{\alpha_1 T}}{\beta}}$, which completes the proof.

From Lemma 2 and Lemma 3, we have the following results.

Theorem 1: For the system (4), there exists an output feedback controller (7) such that the closed-loop system (8) is FTB with $(0, c_2, \mathcal{W}, T, R)$, and the performance index (3) is satisfied, if there exist two symmetric positive definite matrices $Q_1,$

$Q_2 \in \mathcal{R}^{n \times n}$, and five matrices $U, X_1, X_2, Y_2 \in \mathcal{R}^{n \times n}$, $Y_1 \in \mathcal{R}^{n \times p}$, and ten positive real numbers $\alpha_1, \beta, \epsilon_i$ ($i = 1, \dots, 6$) and λ_2, λ_4 , such that the conditions (21) and (22), and the following inequality

$$\Gamma_5 < 0, \quad (23)$$

hold, where $\Gamma_5 =$

$$\begin{bmatrix} \Gamma_{51} & -\frac{\epsilon_1}{2}I + \frac{\epsilon_2\rho_3}{2}I + X_1 & X_1G & \Gamma_{54} & -\alpha_1U - Y_2^T & 0 & U - \epsilon_6Y_2^T \\ -\frac{\epsilon_1}{2}I + \frac{\epsilon_2\rho_3}{2}I + X_1^T & -\epsilon_2I & 0 & \epsilon_3X_1^T & 0 & 0 & 0 \\ G^TX_1^T & 0 & -\alpha_1I & \epsilon_3G^TX_1^T & G^TX_2^T & 0 & \epsilon_6G^TX_2^T \\ \Gamma_{54}^T & \epsilon_3X_1 & \epsilon_3X_1G & -\epsilon_3X_1 - \epsilon_3X_1^T & U & 0 & 0 \\ -\alpha_1U^T - Y_2 & 0 & X_2G & U^T & \Gamma_{52} & \Gamma_{53} & \Gamma_{55} \\ 0 & 0 & 0 & 0 & \Gamma_{53}^T & -\epsilon_5I & \epsilon_6X_2^T \\ U^T - \epsilon_6Y_2 & 0 & \epsilon_6X_2G & 0 & \Gamma_{55}^T & \epsilon_6X_2 & -\epsilon_6X_2 - \epsilon_6X_2^T \end{bmatrix},$$

$\Gamma_{51} = -\alpha_1Q_1 + \epsilon_1\rho_1I + \epsilon_2\rho_2I + X_1A + A^TX_1^T + Y_1C + C^TY_1^T$, $\Gamma_{52} = -\alpha_1Q_2 + \epsilon_4\rho_1I + \epsilon_5\rho_2I + X_2A + A^TX_2^T + Y_2^T + Y_2 + \beta C^TC$, $\Gamma_{53} = -\frac{\epsilon_4}{2}I + \frac{\epsilon_5\rho_3}{2}I + X_2$, $\Gamma_{54} = Q_1 - X_1 + \epsilon_3A^TX_1^T + \epsilon_3C^TY_1^T$ and $\Gamma_{55} = Q_2 - X_2 + \epsilon_6A^TX_2^T + \epsilon_6Y_2^T$. Moreover, if all of these conditions are satisfied, then, the observer gain $L = X_1^{-1}Y_1$ and the control gain $K = (B^TB)^{-1}B^TX_2^{-1}Y_2$ and $\gamma = \sqrt{\frac{\alpha_1 e^{\alpha_1 T}}{\beta}}$.

Proof: Let $Y_1 = X_1L$ and $Y_2 = X_2BK$. Then from Lemma 2 and Lemma 3, the results can be followed.

Once the parameters $c_2, T, d, \alpha_1, \epsilon_3$ and ϵ_6 are fixed, the inequalities (21), (22) and (23) are LMIs with respect to $Q_1, Q_2, U, X_1, X_2, Y_1, Y_2, \beta, \epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5, \lambda_2$ and λ_4 . Since γ reduces with the increase of β , therefore, the optimal value of γ can be obtained by solving the following optimization problem

$$\begin{aligned} \min \quad & -\beta \\ \text{s.t.} \quad & (21), (22) \text{ and } (23). \end{aligned} \quad (24)$$

The optimal value of γ^* is given by $\gamma^* = \sqrt{\frac{\alpha_1 e^{\alpha_1 T}}{-\beta^*}}$, where β^* denotes the optimal value of the optimization problem (24).

Remark 3: The main difficulties of this research paper are summarized as follows: auxiliary matrices are introduced into these sufficient conditions (9)-(11); and these conditions are computationally tractable and can be transformed into the feasibility problem of LMIs. By the Finsler's lemma, $\mathcal{X}^T\Gamma_2\mathcal{X} < 0$ in Lemma 2 is equivalent to the following matrix inequality

$$\Gamma_2 + \begin{bmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \\ X_{13} & X_{23} \\ X_{14} & X_{24} \\ X_{15} & X_{25} \\ X_{16} & X_{26} \\ X_{17} & X_{27} \end{bmatrix} \begin{bmatrix} \bar{A}_1 & I & G & -I & 0 & 0 & 0 \\ -BK & 0 & G & 0 & \bar{A} & I & -I \end{bmatrix} + \begin{bmatrix} \bar{A}_1 & I & G & -I & 0 & 0 & 0 \\ -BK & 0 & G & 0 & \bar{A} & I & -I \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \\ X_{13} & X_{23} \\ X_{14} & X_{24} \\ X_{15} & X_{25} \\ X_{16} & X_{26} \\ X_{17} & X_{27} \end{bmatrix}^T < 0,$$

where X_{ij} are auxiliary matrices. However, the above matrix inequality can not be transformed into LMI since the matrices K, L and X_{ij} are coupled together. For the purpose of simplicity and comparison, we set $X_{12} = X_{13} = X_{15} = X_{16} = X_{17} = 0$, $X_{21} = X_{22} = X_{23} = X_{24} = X_{26} = 0$, $X_{14} = \epsilon_3X_{11}$ and $X_{27} = \epsilon_6X_{25}$. Future works should investigate finite-time H_∞ control with different auxiliary matrices to reduce conservativeness further.

Remark 4: In [23], a static output feedback controller has been designed to ensure that the closed-loop system (8) is FTB and the H_∞ performance index is satisfied at the same time. Whereas, in this paper, a dynamic output feedback controller is proposed to solve finite-time H_∞ control for the one-sided Lipschitz nonlinear systems based on a classic Luenberger observer. The output feedback control gain and observer gain can be designed in an one-step optimization. Moreover, In order to derive less conservative sufficient conditions to ensure the closed-loop system (8) is FTB, note only an auxiliary matrix U is introduced in the Lyapunov function (12), but also additional matrices X_1 and X_2 are introduced to make the Lyapunov matrices Q_1 and Q_2 separate from the system matrix A . It is shown that these conditions are less conservative than other conditions in the literature on the basis of LMIs.

IV. FURTHER DISCUSSIONS AND COMPARISONS

In this section, further discussions and comparisons of the proposed methods are systematically given in two aspects: robust finite-time H_∞ control by state feedback for the system (4), and finite-time stabilization by output feedback for the system considered in [14].

A. Robust finite-time H_∞ control by state feedback

Robust finite-time H_∞ control by state feedback with auxiliary matrices can be easily obtained by the results of robust finite-time H_∞ control by output feedback based on Finsler's Lemma. In fact, let $e(t) = 0$, then, $\hat{x}(t) = x(t)$. Therefore, the output feedback controller (7) is reduced to the following state feedback controller

$$u(t) = Kx(t). \quad (25)$$

The closed-loop system (8) is also reduced to the following system

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + Gw(t) + f(x(t), u(t)), \\ y(t) = Cx(t), \end{cases} \quad (26)$$

where $\bar{A} = A + BK$. Therefore, we have the following results.

Lemma 4: The closed-loop system (26) is FTB with $(c_1, c_2, \mathcal{W}, T, R)$, if there exist a symmetric positive definite matrix $Q_1 \in \mathcal{R}^{n \times n}$ and a matrix $X_1 \in \mathcal{R}^{n \times n}$, six positive real numbers α, ϵ_i ($i = 1, 2, 3$) and λ_1, λ_2 , such that the following inequalities

$$\begin{bmatrix} -\alpha Q_1 + X_1 \bar{A} + \bar{A}^T X_1^T + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I & X_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & X_1 G & Q_1 - X_1 + \epsilon_3 \bar{A}^T X_1^T \\ X_1^T - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & -\epsilon_2 I & 0 & \epsilon_3 X_1^T \\ G^T X_1^T & 0 & -\alpha I & \epsilon_3 G^T X_1^T \\ Q_1 - X_1^T + \epsilon_3 X_1 \bar{A} & \epsilon_3 X_1 & \epsilon_3 X_1 G & -\epsilon_3 X_1 - \epsilon_3 X_1^T \end{bmatrix} < 0, \quad (27)$$

and

$$\lambda_2 I \leq \tilde{Q}_1 \leq \lambda_1 I, \quad (28)$$

$$\lambda_1 c_1^2 + \alpha_1 d^2 < \lambda_2 c_2^2 e^{-\alpha T}, \quad (29)$$

hold, where $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$.

Remark 5: In [23], it is shown that the system (26) is FTB, if the conditions (28), (29) and the following inequality

$$\begin{bmatrix} -\alpha Q_1 + Q_1 \bar{A} + \bar{A}^T Q_1^T + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I & Q_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & Q_1 G \\ Q_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & -\epsilon_2 I & 0 \\ G^T Q_1^T & 0 & -\alpha I \end{bmatrix} < 0, \quad (30)$$

are satisfied. We will show that the condition (27) are in the worst case as stringent as the condition (30) obtained in [23].

In fact, let $X_1 = Q_1$. The condition (27) becomes

$$\begin{bmatrix} -\alpha Q_1 + Q_1 \bar{A} + \bar{A}^T Q_1 + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I & Q_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & Q_1 G & \epsilon_3 \bar{A}^T Q_1 \\ Q_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & -\epsilon_2 I & 0 & \epsilon_3 Q_1 \\ G^T Q_1 & 0 & -\alpha I & \epsilon_3 G^T Q_1 \\ \epsilon_3 Q_1 \bar{A} & \epsilon_3 Q_1 & +\epsilon_3 Q_1 G & -2\epsilon_3 Q_1 \end{bmatrix} < 0, \quad (31)$$

which is equivalent to

$$\begin{bmatrix} -\alpha Q_1 + Q_1 \bar{A} + \bar{A}^T Q_1 + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I & Q_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & Q_1 G \\ Q_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & -\epsilon_2 I & 0 \\ G^T Q_1 & 0 & -\alpha I \end{bmatrix} + \frac{\epsilon_3}{2} \begin{bmatrix} \bar{A}^T Q_1 \\ Q_1 \\ G^T Q_1 \end{bmatrix} Q_1^{-1} \begin{bmatrix} \bar{A}^T Q_1 \\ Q_1 \\ G^T Q_1 \end{bmatrix}^T < 0. \quad (32)$$

If the condition (32) holds, then the condition (30) holds. On the other hand, if the condition (30) is satisfied, then there exists a sufficient small ϵ_3 such that the condition (32) is satisfied. That is the condition (31) is equivalent to the condition (30).

Lemma 5: The system (26) is FTB with $(0, c_2, \mathcal{W}, T, R)$, and the performance index (3) is satisfied, if there exists a symmetric positive definite matrix $Q_1 \in \mathcal{R}^{n \times n}$ and a matrix $X_1 \in \mathcal{R}^{n \times n}$, six positive real numbers $\alpha, \beta, \epsilon_i$ ($i = 1, 2, 3$) and λ_2 , such that the following inequalities

$$\begin{bmatrix} -\alpha Q_1 + X_1 \bar{A} + \bar{A}^T X_1^T + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I + \beta C^T C & X_1 - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I \\ X_1^T - \frac{\epsilon_1}{2} I + \frac{\epsilon_2 \rho_3}{2} I & -\epsilon_2 I \\ G^T X_1^T & 0 \\ Q_1 - X_1^T + \epsilon_3 X_1 \bar{A} & \epsilon_3 X_1 \\ X_1 G & Q_1 - X_1 + \epsilon_3 \bar{A}^T X_1^T \\ 0 & \epsilon_3 X_1^T \\ -\alpha I & \epsilon_3 G^T X_1^T \\ \epsilon_3 X_1 G & -\epsilon_3 X_1 - \epsilon_3 X_1^T \end{bmatrix} < 0, \quad (33)$$

$$\lambda_2 I \leq \tilde{Q}_1, \quad (34)$$

and

$$\alpha_1 d^2 < \lambda_2 c_2^2 e^{-\alpha T}, \quad (35)$$

hold, where $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$.

Theorem 2: For the system (4), there exists a state feedback controller (25) such that the closed-loop system (26) is FTB with $(0, c_2, \mathcal{W}, T, R)$, and the performance index (3) is satisfied, if there exists a symmetric positive definite matrix

$Q_1 \in \mathcal{R}^{n \times n}$, and two matrices $X_1, Y_2 \in \mathcal{R}^{n \times n}$, and seven positive real numbers $\alpha, \beta, \epsilon_i$ ($i = 1, 2, 3$) and λ_1, λ_2 , such that (34), (35), and the following inequality

$$\begin{bmatrix} \Gamma_1 & X_1 - \frac{\epsilon_1}{2}I + \frac{\epsilon_2 \rho_3}{2}I & X_1 G & Q_1 - X_1 + \epsilon_3 A^T X_1^T + \epsilon_3 Y_2^T \\ X_1^T - \frac{\epsilon_1}{2}I + \frac{\epsilon_2 \rho_3}{2}I & -\epsilon_2 I & 0 & \epsilon_3 X_1^T \\ G^T X_1^T & 0 & -\alpha I & \epsilon_3 G^T X_1^T \\ Q_1 - X_1^T + \epsilon_3 X_1 A + \epsilon_3 Y_2 & \epsilon_3 X_1 & \epsilon_3 X_1 G & -\epsilon_3 X_1 - \epsilon_3 X_1^T \end{bmatrix} < 0, \quad (36)$$

hold, where $\Gamma_1 = -\alpha Q_1 + X_1 A + A^T X_1^T + Y_2^T + Y_2 + \epsilon_1 \rho_1 I + \epsilon_2 \rho_2 I + \beta C^T C$, $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$. If all of these conditions are satisfied, then the state feedback control gain $K = (B^T B)^{-1} B^T X_1^{-1} Y_2$ and $\gamma = \sqrt{\frac{\alpha e^{\alpha T}}{\beta}}$.

Once the parameters c_2, T, d, α_1 and ϵ_3 are fixed, the inequalities (34), (35) and (36) are LMIs with respect to $Q_1, X_1, Y_2, \beta, \epsilon_1, \epsilon_2$ and λ_2 . Since γ reduces with the increase of β , therefore, the optimal value of γ can be obtained by solving the following optimization problem

$$\begin{aligned} \min \quad & -\beta \\ \text{s.t.} \quad & (34), (35) \text{ and } (36). \end{aligned} \quad (37)$$

The optimal value of γ^* is given by $\gamma^* = \sqrt{\frac{\alpha e^{\alpha T}}{-\beta^*}}$, where β^* denotes the optimal value of the optimization problem (37).

B. Finite-time stabilization by state feedback

Let $G = 0$ and $f(x(t)) \equiv 0$, then, the system (4) is reduced to the following system considered in [14]

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \quad x(t_0) = x_0, \end{cases} \quad (38)$$

where the state $x(t) \in \mathcal{R}^n$, the input $u(t) \in \mathcal{R}^m$, the output $y(t) \in \mathcal{R}^p$, $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, and $C \in \mathcal{R}^{p \times n}$. Without loss of generality, we assume that B has full column rank. An output feedback controller

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + LC(\hat{x}(t) - x(t)), \quad \hat{x}(0) = 0, \\ u(t) = K\hat{x}(t), \end{cases} \quad (39)$$

has been designed in [14] to make that the closed-loop system

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) - BK e(t), \quad x(0) = x_0, \\ \dot{e}(t) = \bar{A}_1 e(t), \quad e(0) = x_0, \end{cases} \quad (40)$$

FTS, where $e(t) = x(t) - \hat{x}(t)$, $\bar{A} = A + BK$, $\bar{A}_1 = A + LC$. The following sufficient conditions have also been derived.

Theorem 3 ([14]): There exists an output feedback control (39) such that the closed-loop system (40) is FTB, if there exist two symmetric positive definite matrices Q_1, Q_2 , a matrix M , a nonnegative scalar α_1 and positive scalars λ_k , ($k = 1, 2, 3$), such that

$$\begin{bmatrix} (A + BK)^T \tilde{Q}_1 + \tilde{Q}_1 (A + BK) - \alpha_1 \tilde{Q}_1 & -\tilde{Q}_1 B K \\ -K^T B^T \tilde{Q}_1 & A^T \tilde{Q}_2^T + \tilde{Q}_2 A + C^T M^T + M C - \alpha_1 \tilde{Q}_2 \end{bmatrix} < 0, \quad (41)$$

$$\lambda_3 I < Q_1 < \lambda_1 I, \quad 0 < Q_2 < \lambda_2 I, \quad (42)$$

$$c_1^2 (\lambda_1 + \lambda_2) \leq c_2^2 e^{-\alpha T} \lambda_3, \quad (43)$$

where $\tilde{Q}_1 = R^{1/2} Q_1 R^{1/2}$, $\tilde{Q}_2 = R^{1/2} Q_2 R^{1/2}$. In this case an output feedback controller (39) can make system (38) FTS with $L = \tilde{Q}_2^{-1} M$.

Based on our new Lyapunov function (12) and Finsler's lemma, we can obtain the following less conservative sufficient conditions such that the closed-loop system is FTS.

Theorem 4: There exists an output feedback controller (39) such that the closed-loop system (40) is FTS, if there exist two symmetric positive definite matrices $Q_1, Q_2 \in \mathcal{R}^{n \times n}$, four matrices $U, Y_2, X_1, X_2 \in \mathcal{R}^{n \times n}$, and matrix $Y_1 \in \mathcal{R}^{n \times p}$, and a nonnegative scalar α_1 and positive scalars $\lambda_k, (k = 1, 2, 3)$, such that

$$\begin{bmatrix} -\alpha_1 Q_1 + X_1 A_1 + A_1^T X_1^T + Y_1 C + C^T Y_1^T & Q_1 - X_1 + \epsilon_3 A_1^T X_1^T + \epsilon_3 C^T Y_1^T \\ Q_1 - X_1^T + \epsilon_3 X_1 A + \epsilon_3 Y_1 C & -\epsilon_3 X_1 - \epsilon_3 X_1^T \\ -\alpha_1 U^T - Y_2 & U^T \\ U^T - \epsilon_6 Y_2 & 0 \\ -\alpha_1 U - Y_2^T & U - \epsilon_6 Y_2^T \\ U & 0 \\ -\alpha_1 Q_2 + X_2 A + A^T X_2^T + Y_2 + Y_2^T & Q_2 - X_2 + \epsilon_6 X_2 A + \epsilon_6 Y_2 \\ Q_2 - X_2^T + \epsilon_6 A^T X_2^T + \epsilon_6 Y_2^T & -\epsilon_6 X_2 - \epsilon_6 X_2^T \end{bmatrix} < 0, \quad (44)$$

$$\begin{bmatrix} \lambda_3 R & 0 \\ 0 & \lambda_4 R \end{bmatrix} < \begin{bmatrix} Q_1 & U \\ U^T & Q_2 \end{bmatrix} < \begin{bmatrix} \lambda_1 R & 0 \\ 0 & \lambda_2 R \end{bmatrix}, \quad (45)$$

$$(\lambda_1 + \lambda_2)c_1^2 < \lambda_4 c_2^2 e^{-\alpha_1 T}. \quad (46)$$

In this case an output feedback controller makes system (38) FTS in the form of (39) with $L = X_1^{-1} Y_1^{-1}$ and $K = (B^T B)^{-1} B^T X_2^{-1} Y_2$.

Remark 6: The system (38) considered in [14] is a special case of the system (4). In addition, the design in [14] is divided into two steps: a state feedback controller is firstly designed based on the procedure presented [26]; then a state observer is given in such a way that the FTS performance obtained at the first step is retained. By the analysis in the Remark 5, we can obtain that the condition (41) is equivalent to the condition (44) with $U = 0$. The auxiliary matrices X_1, X_2 and U are helpful to reduce conservatism in applying the criteria. Moreover, the observer gain and output feedback control gain can be designed in one step.

V. NUMERICAL SIMULATIONS

In this section, we use an example to illustrate the effectiveness of our new methods.

Example 1: For the purpose of comparison, the example illustrated in [23], [9] is borrowed. Consider the one-sided Lipschitz nonlinear system described by (4) with

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}, \quad G = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}, \quad f(x(t)) = \begin{bmatrix} \sin(x_1(t)) - 2x_1(t) \\ -2x_2(t) + \cos(x_2(t)) \end{bmatrix}.$$

From [23], [9], we can check the nonlinear function $f(x(t))$ satisfies the one-sided Lipschitz condition with $\rho_1 = -1$, $\rho_2 = 9$ and $\rho_3 = 0$. The initial state $x_0 = [0 \ 0]^T$ and the disturbance $w(t) = 0.5 \cos(20t)e^{-5t}$. The discussion can continue for two cases: (i) state feedback case (ii) output feedback case.

(i) State feedback case: The other parameters are given as: $d = 1$, $T = 5$, $c_2 = 1.4$, $R = I$, $\alpha = 0.6$ and $\epsilon_3 = 0.001$. With the aid of Matlabr LMI Toolbox [28], we solve the optimization problem (37) and obtain the following solutions

$$Q_1 = \begin{bmatrix} 6.4307 & -0.3092 \\ -0.3092 & 6.4954 \end{bmatrix}, \quad X_1 = \begin{bmatrix} -6.1320 & 0.1097 \\ 0.1173 & -6.1583 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 85.5860 & 15.6295 \\ 15.6168 & 83.0541 \end{bmatrix},$$

$$\epsilon_1 = 12.1175, \epsilon_1 = 0.0065, \lambda_2 = 6.1504, \beta = 88.1179.$$

The state feedback gain K is given as

$$K = \begin{bmatrix} -6.8230 & -9.4307 \end{bmatrix},$$

and the optimal $\gamma^* = 0.3698$. Fig 1. shows the trajectories of the closed-loop system. By using the methods in [23], we have the optimal $\gamma^* = 0.4383$. It is shown that our methods yield less conservative results.

Fig. 1. Trajectories of the states $x_i(t)$ ($i = 1, 2$) with the state feedback stabilizer.

(ii) Output feedback case: Set the other parameters as: $d = 1$, $T = 5$, $c_2 = 1$, $R = I$, $\epsilon_3 = 0.001$, $\epsilon_6 = 0.001$ and $\alpha = 0.5$. By solving the optimization problem (24), we have the following solutions

$$Q_1 = \begin{bmatrix} 0.8891 & 0.1581 \\ 0.1581 & 0.8546 \end{bmatrix}, Q_2 = \begin{bmatrix} 6.1572 & 0.0939 \\ 0.0939 & 6.2875 \end{bmatrix}, X_1 = \begin{bmatrix} 0.7090 & -0.0011 \\ -0.0033 & 0.7117 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 5.9965 & -0.0088 \\ -0.0342 & 6.0995 \end{bmatrix}, U = \begin{bmatrix} -0.1483 & -0.0913 \\ -0.1320 & -0.0817 \end{bmatrix}, Y_1 = \begin{bmatrix} -25.2525 \\ -19.0342 \end{bmatrix}, Y_2 = \begin{bmatrix} -45.8843 & -40.6549 \\ -29.4293 & -26.1182 \end{bmatrix},$$

$$\epsilon_1 = 1.4270, \epsilon_2 = 9.8432 \times 10^{-4}, \epsilon_4 = 12.0664, \epsilon_5 = 0.0116, \beta = 25.7434, \lambda_2 = 0.0073, \lambda_4 = 6.0914.$$

The observer gain is given by

$$L = \begin{bmatrix} -35.6567 \\ -26.9122 \end{bmatrix},$$

and the control gain is given as

$$K = \begin{bmatrix} -5.8045 & -5.1473 \end{bmatrix},$$

and the optimal $\gamma^* = 0.4864$. Using the methods proposed in [23], we have the optimal $\gamma^* = 0.6013$. It is shown the finite-time boundedness of the closed-loop system by the obtained output feedback controller in Fig. 3. Our methods also achieve less conservative results.

Fig. 3. Trajectories of the states $x_i(t)$ ($i = 1, 2$) with the output feedback stabilizer.

VI. CONCLUSION

In this paper, we investigated robust finite-time H_∞ control for one-sided Lipschitz systems by using the Finsler's lemma. Finite-time boundedness conditions were firstly provided for the one-sided Lipschitz system. The proposed conditions were less conservative since a new Lyapunov function was constructed, and auxiliary matrices were introduced to make the Lyapunov matrix separate from the system matrix. Then, finite-time H_∞ controller was designed by dynamic output feedback. The observer gain and output feedback control gain could be designed in one step optimization. Further discussions and comparisons were also presented in two aspects. At last, numerical simulations were given to verify the validity of the proposed methods.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation of China (61374028, 61273183), the Grant National Science Foundation of Hubei Provincial (2013CFA050), the Scientific Innovation Team Project of Hubei Provincial Department of Education (T201504).

References

- [1] M.S. Chen and C. C. Chen, Robust nonlinear observer for Lipschitz nonlinear systems subject to disturbances, IEEE Trans. Autom. Control, 52 (2007) 2365-2369.
- [2] A. Zemouche and M.Boutayeb, On LMI conditions to design observers for Lipschitz nonlinear systems, Automatica 49 (2013) 585-591.
- [3] S. Tong, B. Huo, and Y. Li, Observer-based adaptive decentralized fuzzy fault-tolerant control of nonlinear large-scale systems with actuator failures, IEEE Trans. On Fuzzy Systems, 22(1) (2014) 1-15.
- [4] S. Tong, B. Huo, and Y. Li, Fuzzy adaptive actuator failure compensation control of uncertain stochastic nonlinear systems with unmodeled dynamics, IEEE Trans. On Fuzzy Systems, 22(3) (2014) 563-574.
- [5] S. Tong, B. Huo, and Y. Li, Fuzzy adaptive output feedback control of MIMO nonlinear systems with partial tracking errors constrained, IEEE Trans. On Fuzzy Systems, 23(4) (2015) 729-742.
- [6] M. Abbaszadeh and H. Marquez, Nonlinear observer design for one-sided Lipschitz systems. in Proceedings of American Control Conference, USA, (2010) 5284-5289.
- [7] G. Hu, A note on observer for one-sided Lipschitz nonlinear systems. IMA J. Math. Control Inf. 25(3) (2008) 297-303.
- [8] M. Boutayeb, M. Benallouch and M. Zasadzinski, Observer design for one-sided Lipschitz discrete-time systems. Syst. Control Lett. 61(9) (2012) 879-886.
- [9] W. Zhang, H. Su, Y. Liang and Z. Han, Non-linear observer design for one-sided Lipschitz systems: an linear matrix inequality approach. IET Control Theory Appl. 6(9) (2012) 1297-1303.

- [10] Y. Zhao, J. Tao and N. Shi, A note on observer design for one-sided Lipschitz nonlinear systems. *Syst. Control Lett.* 59 (2010) 66-71.
- [11] W. Zhang, H. Su, H. Wang and Z. Han, Full-order and reduced-order observers for one-sided Lipschitz nonlinear systems using Riccati equations. *Commun. Nonlinear Sci. Numer. Simul.* 17(12) (2012) 4968-4977.
- [12] P. Dorato, Short time stability in linear time-varying systems. In *Proceeding of the IRE International Convention Record Part 4*, (1961) 83-87.
- [13] L. Weiss and E. F. Infante, Finite time stability under perturbing forces and on product spaces. *IEEE Trans. Automat. Control*, 12 (1967) 54-59.
- [14] F. Amato, M. Ariola and P. Dorato, Finite-time stabilization via dynamic output feedback. *Automatica*, 42 (2006) 337-342.
- [15] F. Amato and M. Ariola, Finite-time control of discrete-time linear system. *IEEE Trans. Automat. Control*, 50(5) (2005) 724-729.
- [16] F. Amato, M. Ariola and P. Dorato, Finite-time control of linear systems subject to parameteric uncertainties and disturbances. *Automatica*, 37 (2001) 1459-1463.
- [17] L. Zhu, Y. Shen, and C. Li, Finite-time control of discrete-time systems with time-varying exogenous disturbance. *Commun. Nonlinear Sci. Numer. Simul.*, 14 (2) (2009) 361-370.
- [18] Y. Shen, Finite-time control of linear parameter-varying systems with norm-bounded exogenous disturbance. *Control Theory and Applications*, 6 (2) (2008) 1884-1888.
- [19] F. Amato, M. Ariola and C. Cosentino, Finite-time stability of linear time-varying systems: analysis and controller design. *IEEE Trans. Automat. Contr.* 55 (4) (2010) 1003-1008.
- [20] Q. Meng and Y. Shen, Finite-time H_∞ control for linear continuous system with norm-bounded disturbance. *Commun. Nonlinear Sci. Numer. Simul.*, 14(4) (2009) 1043-1049.
- [21] H. Kussaba, R. Borges and J. Ishihara, A new condition for finite time boundedness analysis. Submitted to *Journal of Franklin Institute*, 2015.
- [22] J. Song and S. He, Finite-time H_∞ control for quasi-one-sided Lipschitz nonlinear systems. *Neurocomputing*, 149 (2015) 1433-1439.
- [23] J. Song and S. He, Robust finite-time H_∞ control for one-sided Lipschitz nonlinear systems via state feedback and output feedback. *Journal of the Franklin Institute* (in press), 2015.
- [24] H. Liu and X. Lin, Finite-time H_∞ control for a class of nonlinear system with time-varying delay. *Neurocomputing*, 149 (3) (2015) 1481-1489.
- [25] A. Howell and J.K. Hedrick, Nonlinear observer design via convex optimization. In *Proceeding of American Control Conference, USA*, (2002) 2088-2093.
- [26] F. Amato, M. Ariola, and P. Dorato, Finite time control of linear systems subject to parametric uncertainties and disturbances. *Automatica*, 37 (2001) 1459-1463.
- [27] B. S. Gottfried and J. Weisman, *Introduction to optimization theory*. Englewoods Cliffs: Prentice-Hall, 1973.
- [28] S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishnan, *Linear matrix inequality in systems and control theory*. SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1994.