A New Prospective for Learning Automata: A Machine Learning Approach

Wen Jiang^a, Bin Li^b, Shenghong Li^{a,*}, Yuanyan Tang^c, Chun Lung Philip Chen^c

^a*Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai, 200240, China*

^b*School of Information and Communication Engineering, Beijing University of Posts and Telecommunications, Beijing, 100876, China*

^c*Faculty of Science and Technology, University of Macau, Macau, China*

Abstract

In the field of Learning Automata (LA), how to design faster learning algorithms has always been a key issue. Among solutions reported in the literature, the stochastic estimator reward-inaction learning automaton (SE_{RI}), which belongs to the Maximum Likelihood estimator based LAs, has been recognized as the fastest ϵ -optimal LA. In this paper, we first point out the limitations of the traditional Maximum Likelihood estimator (MLE) based LAs and then introduce Bayesian estimator based approach, which is demonstrated to be equivalent to Laplace smoothing of the traditional method, to overcome these limitations. The key idea is that the Bayesian estimator, which estimates the probability of selecting each action in the LA, aims to reconstruct Bernoulli distribution from sequential data, and is formalized based on exponential conjugate family so that the LA has a relatively simple format for easy implementation. In addition, we also indicate that this Bayesian estimator could be applied to update almost all existing MLE estimator based LAs. Based on the proposed Bayesian estimator, a new LA, known as Generalized Bayesian Stochastic Estimator (GBSE) LA, is presented and proved to be ϵ -optimal. Finally, extensive experimental results

*Corresponding author

Email addresses: wenjiang@sjtu.edu.cn (Wen Jiang), stonebupt@gmail.com (Bin Li), shli@sjtu.edu.cn (Shenghong Li), yytang@umac.mo (Yuanyan Tang), philipchen@umac.mo (Chun Lung Philip Chen)

on benchmarks demonstrate our proposed learning scheme is more efficient than the current best LA SE_{RI} .

Keywords: Learning Automata, ϵ -optimal, Bayesian estimator, Maximum Likelihood estimator

1. Introduction

Learning Automata (LA), a reinforcement learning approach as one of the most powerful tools in adaptive learning systems, is an important research area of Artificial Intelligence (AI). It has a broad range of applications as been seen in [1] [2] [3] [4] [5] [6][7][8][9][10] and books [11] [12] [13]. The ultimate goal of LA is to learn the sole optimal action from the environment so as to achieve higher reward compared with other actions via interacting with the environment in terms of a sequence of repetitive feedback cycles. The functionality of LA can be generally summarized as follows. Firstly, it selects an action from a predefined action set according to a probability distribution vector defined as the probability of choosing an action at each instant. Then, this selected action will trigger the environment to generate a response to the LA according to the reward probability of each action. The LA takes this response and the corresponding selected action into account to update its probability vector. This process terminates when one element of the probability vector is equal to one. By learning to choose the optimal action, the automaton adapts itself to the environment. The LA has many merits such as being applicable to stochastic environment, having solid theoretical background, owning easily deployed stop criteria and providing online learning ability. At the meantime, it does not need a predefined metric space and achieves the global optimal solutions (for certain type of LA). These advantages have been playing a vital important role to extend its applications, but the slow convergence of the LA prohibits itself from applying to a wider range of applications. Hence, one of the major goals in the LA research area is to design fast LA algorithms.

The idea of LA was first introduced by Tsetlin to model biological learning

mechanism [14]. Since then LA has been extensively studied to develop various kinds of LA algorithms. These reported LA algorithms in the literature could be classified into two categories, i.e., Continuous Action-set Learning Automata (CALA) and Finite Action-set Learning Automata (FALA). In the category of FALA, one action is taken from a finite action set at each instant, while CALA takes the action from a real line (or an interval on the real line) so as to deal with continuous-valued parameters directly. So far, in the existing literatures, most of the Learning Automata (LA) we usually referred to is the FALA. Therefore, we will adopt this conventional indication in this paper.

FALA can also be further subdivided into two classes. The first one is known as Fixed Structure Stochastic Automata (FSSA) [14] [15], which has the property that its transition and output functions do not vary with time. The second class is named as Variable Structure Stochastic Automata (VSSA) whose transition and output functions change with time. So far, much effort has been made to design faster VSSA algorithms. As a pioneer researcher on LA, Oommen proposed discretized LA [16] with a convergence rate faster than their continuous counterpart. Meanwhile, Thathachar and Sastry introduced the class of Maximum Likelihood Estimator (MLE) learning automata [17] [18] [19]. Simulations have been demonstrated the superiority of the Maximum Likelihood estimator based algorithm over the one without estimator. Later, Georgios et.al presented the stochastic estimator reward-inaction learning automaton (SE_{RI}) based on stochastic MLE [20], which has been known to be the fastest LA so far. Another fast converging approach, known as structure LA has also been reported in [21] [22].

Generally speaking, all estimator based algorithms mentioned above are actually derived by maximum likelihood estimation. The automaton records the number of times that each action has been selected and the number of times that each action has been rewarded. Then the reward estimates will be calculated as the ratio between the number of rewarded selections and the number of total selections. Recently, a new Bayesian LA (BLA) has been introduced in [23]. It is also demonstrated that the corresponding Bayesian pursuit algorithm

is superior to BLA. As mentioned in [23], both the BLA and corresponding pursuit algorithm have the beauty of relying on updating the hyper-parameters of sibling conjugate prior. Nevertheless, they still need sampling from the resulting
60 posterior.

In this paper, we point out the limitations of MLE based LAs, and introduce Bayesian estimator to calculate the deterministic estimates of each action based on Bayesian estimation and present a Bayesian stochastic estimator scheme, which is known as the Generalized Bayesian Stochastic Estimator Learning Au-
65 tomata (*GBSE*). The estimates of the reward probability of actions for the *GBSE* are computed by stochastically varying these Bayesian estimations to relax the relationship between the stochastic and the deterministic estimates which enables the actions selected for only a few times to be optimal. Consequently, it can be found that the *GBSE* converges rapidly and accurately.

70 The key contributions of this paper are summarized as follows.

- The new and novel Bayesian estimator proposed in this paper will play a fundamental role in the field of LA. To overcome the limitations from all existing MLE based LAs and construct faster LA, we propose the Bayesian estimator and demonstrate that it is equivalent to the MLE with Laplace
75 smoothing. We also show that this Bayesian estimator can be applied to all existing MLE based LAs in order to overcome the limitations from the MLE ones.
- We also propose a new algorithm, referred to as Generalized Bayesian Stochastic Estimator Learning Automata (*GBSE*) and prove that the
80 proposed scheme is ϵ -optimal in all stationary environments.
- The proposed *GBSE* is compared with by far the fastest LA, SE_{RI} , in various benchmark environments. Extensive experiments demonstrate that the proposed one is more efficient than SE_{RI} .

The rest of this paper is organized as follows. In Section 2, we shall introduce
85 the general idea of LA and the Maximum Likelihood Estimator. Section 3

introduces the concept of the proposed Bayesian estimator. The details of the proposed *GBSE* LA and the ϵ -optimality of this scheme are presented in Section 4. Extensive simulation results are presented to demonstrate the advantages of the proposed algorithm over the fastest known LA, *SE_{RI}*, in Section 5. The
90 last Section concludes this paper with some conclusive remarks.

2. Learning Automata and ML Estimator

Let us first introduce some notations and definitions, and then pointed out limitations of the MLE based LAs. Then the problems related to the MLE based LAs are identified.

95 2.1. LA and Stochastic Environment

The stochastic estimator LA framework is represented as a quintuple $\langle \mathbb{A}, \mathbb{B}, \mathbb{P}, \mathbb{E}, \mathbb{T} \rangle$, where

- $\mathbb{A} = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is a finite set of actions.
- $\mathbb{B} = \{0, 1\}$ is the response set of the P-model environment, while the Q-model and the S-model environments are defined as $\mathbb{B} = \{0, \dots, 1\}$ and
100 $\mathbb{B} = [0, 1]$, respectively.
- $\mathbb{P}$ is the probability distribution over actions. At instant t , the probability vector is defined as $\mathbf{P}(t) = [p_1(t), p_2(t), \dots, p_r(t)]^T$, where $p_i(t) = p\{\alpha(t) = \alpha_i\}$.
- $\mathbb{E}$ is the estimator. For each action α_i at instant t , $E_i(t) = (\hat{d}_i(t), R_i(t))$, in
105 which $\hat{d}_i(t)$ is deterministic estimates of the reward probability of action α_i and $R_i(t)$ is a random variable followed with some predefined distributions.
- $\mathbb{T}$ is the learning paradigm, $\mathbb{T}: [0, 1]^r \times \mathbb{A} \times \mathbb{B} \rightarrow [0, 1]^r$. The probability vector is updated by the following equation:

$$\mathbf{P}(t) = \mathbf{T}(\mathbf{P}(t), \alpha(t), \beta(t)). \quad (1)$$

110 The stochastic environment interacted with the LA is defined as $\langle \mathbb{A}, \mathbb{B}, \mathbb{D} \rangle$, where $\mathbb{A}$ and $\mathbb{B}$ are defined above; $\mathbb{D} = \{d_1, d_2, \dots, d_r\}$ is the probability distribution over actions and

$$p\{\beta(t) = 1 | \alpha(t) = \alpha_i\} = d_i \quad (2)$$

where $\beta(t)$ is the output of the stochastic environment at each instant t . Throughout this paper, $\beta(t) = 1$ means that the environment rewards the action selected
115 by the LA.

From the above definition, it is easy to understand that the output $\beta(t)$ follows Bernoulli distribution given a selected action α_i . It is well known that Bernoulli distribution is governed by two parameters: the number of trials and the parameter which characterizes each Bernoulli trial. On the LA counterpart,
120 the number of trials is actually the number of times that each action is selected. So our goal is to estimate another parameter, the reward probability d_i , of Bernoulli distribution.

2.2. Maximum Likelihood Estimator

As mentioned above, stochastic estimator based LA makes use of the Maximum Likelihood Estimation to estimate d_i . It is known that the Maximum
125 Likelihood Estimation produces the estimate, $\hat{d}_i(t)$, of the probability parameter of Bernoulli distribution as

$$\hat{d}_i(t) = \frac{W_i(t)}{Z_i(t)} \quad (3)$$

which asymptotically converges to the true parameter of the Bernoulli distribution, i.e.,

$$\lim_{t \rightarrow \infty} \hat{d}_i(t) = d_i \quad (4)$$

130 where $W_i(t)$ and $Z_i(t)$ are the total numbers of action α_i having been selected and rewarded till time instant t , respectively.

A key point worthy of mentioning here is that the MLE based LAs suffer from the following two problems. The first one is the cold startup problem, i.e., there is no trial and observation as the learning system starts, $W_i(0) = 0$

135 and $Z_i(0) = 0$. However, Eq.(3) requires $Z_i(t) \neq 0$. The second potential problem underlying the traditional MLE based LA is more serious. Assuming that at time instant $t = 1$, the LA selects one action $\alpha_{selected}$ ($\alpha_{selected} \neq \alpha_{optimal}$). Then the estimate for the optimal action will not change. However, we have $\hat{d}_{selected}(1) = 1$ if the stochastic environment rewards the selected
140 action $\alpha_{selected}$. For a subclass of the MLE based LAs, i.e., pursuit LAs which update their probability vectors to follow the maximum ML estimators, since $\hat{d}_{selected}(1) = 1 > \hat{d}_{optimal}(1) = 0$, so the optimal action will never be selected.

The most widely used strategy to deal with these two drawbacks mentioned above is that those existing estimator based LAs MUST initialize their estimates
145 by selecting each action for several times to the stochastic environment and recording the corresponding environment responses. However, this initialization method still has several limitations as follows.

- Under the environment whose response is expensive or very difficult to get, this solution would not be feasible.
- 150 • Considering a game of ML estimator based LAs with N players or LAs, each of which has r actions, the number of total combinations of actions is N^r . In order to converge to the global optimal action, a possible solution is employing the estimator algorithms[24]. However, this initialization solution needs to select each combination of actions for several times in
155 advance, which means that there are *eachActionInitializingTimes* $\times N^r$ selections at the initialization stage. This is, of course, unbearable for practical applications.
- We further point out that this method can not fundamentally solve the above mentioned two problems since the MLE of optimal action remains
160 to be 0 with certain probability, even though this probability becomes smaller as the number of initial selections increases. Different with the deterministic estimator (MLE) based LAs, the stochastic MLE based LA can inherently overcome this problem since the probability vector is updated based on the MLE estimates plus a random variable which might

165 be negative. Thus the lowest estimate value used for updating probability
vector will never be 0. However, stochastic MLE based algorithms are
fundamentally distinguished from the MLE estimator based LAs and thus
could not be employed to update all MLE based LAs. This paper intro-
duces a new LA estimator based on Bayesian inference to solve the above
170 two problems, which is also applicable to all existing MLE based LAs.

3. Bayesian Estimator

The Bayesian and frequentist approaches are two broad types of statistical inference [25]. Bayesian approach usually treats the unknown parameter d_i as random variables. The frequentist approach interprets d_i as the fixed parameters and the frequency of an event approaches this parameter as the number of tries tends to infinity. We adopt exactly the Bayesian perspective in which the Bayesian theory plays a key role as:

$$\begin{aligned} p(d_i, \beta(t)|a(t) = a_i) \\ = p(d_i|\beta(t), a(t) = a_i) \cdot (\beta(t)|a(t) = a_i) \end{aligned} \quad (5)$$

where d_i is seen as a random variable and $\beta(t)$ is an observed value at each instant t . Meanwhile, the posterior distribution of d_i , given the observance of $\beta(t)$, is defined as:

$$\begin{aligned} p(d_i|\beta(t), a(t) = a_i) &= \frac{p(d_i, \beta(t)|a(t) = a_i)}{p(\beta(t)|a(t) = a_i)} \\ &= \frac{p(d_i|a(t) = a_i) \cdot (\beta(t)|d_i, a(t) = a_i)}{p(\beta(t), a(t) = a_i)}. \end{aligned} \quad (6)$$

175 The key quantities are $p(d_i|a(t) = a_i)$ and $p(\beta(t)|d_i, a(t) = a_i)$, which are referred to as the prior distribution and likelihood function, respectively. Hence, the above equation is often written as follows

$$posterior \propto prior \times likelihood. \quad (7)$$

With the above definition, we can introduce the conjugate prior distribution defined in [25]. Let $\mathcal{H}$ be a class of sampling distribution $p(\cdot|\theta)$ and $\mathcal{P}$ be a class

180 of prior distributions for θ , then the class $\mathcal{P}$ is said to be a conjugate for $\mathcal{H}$ if

$$p(\theta|\cdot) \in \mathcal{P} \text{ for all } p(\cdot|\theta) \in \mathcal{H} \text{ and } p(\cdot) \in \mathcal{P}. \quad (8)$$

A special case of the above definition is the so called natural conjugate prior families. Every exponential distribution in this natural conjugate prior families, $p(d_i|\beta(t), a(t) = a_i)$, has a natural conjugate prior distribution. Taking the Bernoulli distribution as an example, let us consider the random variable
185 $\beta(t) \in \{0, 1\}$ so that

$$\begin{aligned} p(\beta(t) = 1|d_i, a(t) = a_i) &= d_i \\ p(\beta(t) = 0|d_i, a(t) = a_i) &= 1 - d_i. \end{aligned} \quad (9)$$

Therefore, the Bernoulli distribution over $\beta(t)$ can be formalized as

$$\begin{aligned} p(\beta(t)|d_i, a(t) = a_i) &= d_i^{\beta(t)}(1 - d_i)^{(1-\beta(t))} \\ &= (1 - d_i)e^{\log d_i^{\beta(t)}(1-d_i)^{-\beta(t)}} \\ &= (1 - d_i)e^{\beta(t) \log d_i} e^{-\beta(t) \log(1-d_i)} \\ &= (1 - d_i)e^{\beta(t) \log \frac{d_i}{1-d_i}}. \end{aligned} \quad (10)$$

It is easy to see from Eq.(10) that the Bernoulli distribution is actually a member of exponential family. It is well known that the conjugate prior distribution of Bernoulli distribution is Beta distribution, i.e.,

$$\begin{aligned} \text{Beta}(d_i|\mu_i, \nu_i, a(t) = a_i) \\ = \frac{\Gamma(\mu_i + \nu_i)}{\Gamma(\mu_i)\Gamma(\nu_i)} d_i^{(\mu_i-1)} (1 - d_i)^{(\nu_i-1)}. \end{aligned} \quad (11)$$

where $\Gamma(x)$ is the gamma function. We can verify the conjugate prior of Bernoulli distribution in Eq.(10) as follows:

$$\begin{aligned} p(d_i|\beta(t), a(t) = a_i) \\ \propto \text{Beta}(d_i|\mu_i, \nu_i, a(t) = a_i) p(\beta(t)|a(t) = a_i) \\ \propto d_i^{\mu_i-1+\beta(t)} (1 - d_i)^{\nu_i-1+1-\beta(t)}. \end{aligned} \quad (12)$$

As seen from Eq.(12), the posterior distribution has the same parametric
190 form as the prior distribution, i.e., the Beta distribution in Eq.(11). By comparing Eq.(11) with Eq.(12), the rules of updating the parameters of the posterior

distribution can be derived as:

$$\begin{aligned}\mu_i(t) &= \mu_i(t-1) + \beta(t) \\ \nu_i(t) &= \nu_i(t-1) + 1 - \beta(t).\end{aligned}\tag{13}$$

Considering the fact that the expectation of Beta distribution is $\frac{\mu_i(t)}{\mu_i(t) + \nu_i(t)}$, we choose the estimate of d_i as

$$\begin{aligned}\hat{d}_i(t) &= E\{Beta(d_i|\mu_i(t), \nu_i(t), a(t) = a_i)\} \\ &= \frac{\mu_i(t)}{\mu_i(t) + \nu_i(t)}.\end{aligned}\tag{14}$$

195 Meanwhile, let us define the corresponding stochastic estimator as

$$u_i(t) = \hat{d}_i(t) + R_i(t)\tag{15}$$

where $R_i(t)$ is a random number uniformly distributed in the range of $(-\frac{\gamma}{\mu_i(t) + \nu_i(t)}, \frac{\gamma}{\mu_i(t) + \nu_i(t)})$.

One more issue to be addressed is which type of prior distribution to be used, i.e., informative prior distribution or non-informative prior distribution. Non-information prior distribution does not have any prior belief about the
200 parameter d_i itself, for example, the uniform distribution and information prior hold the prior belief or knowledge on the parameter. Since the Beta distribution is reduced to a uniform distribution with $\mu_i(0) = 1, \nu_i(0) = 1$, the initial phase has a non-information prior distribution. Note that this initialization condition will lead to $\hat{d}_i(0) = \frac{1}{2} = 0.5$. As no observation is available at time instant 0
205 for the case of MLE approach, $\hat{d}_i(0)$ does not have any meaning and cannot be calculated. This is the fundamental difference between MLE and Bayesian estimators. Therefore, the LA based on the proposed Bayesian estimator will directly initialize $\hat{d}_i(0) = 0.5$. In this way, the above mentioned limitations of the MLE based LAs can be avoided. Because $\hat{d}_i(0) = \frac{1}{2} = 0.5$ is the middle
210 point in the range of (0,1), it is statistically closer to the random variable d_i than any other initial values in the range of (0,1), which leads to faster convergence.

Generally, Eq.(14) can also be written as

$$\begin{aligned}
\hat{d}_i(t) &= E\{Beta(d_i|\mu_i(t), \nu_i(t), a(t) = a_i)\} \\
&= \frac{\mu_i(t)}{\mu_i(t) + \nu_i(t)}, \mu_i(0) = 1, \nu_i(0) = 1 \\
&= \frac{W_i(t) + 1}{Z_i(t) + 2}
\end{aligned} \tag{16}$$

where $W_i(t)$ and $Z_i(t)$ are defined as the Maximum Likelihood Estimates mentioned in Eq.(3). Comparing Eq.(3) with Eq.(16), it is very interesting to see
215 that the Bayesian estimator is actually a 'smoothed' value of the MLE. In fact, as known to all, this is called as Laplace smoothing. Furthermore, it is trivial to prove that the Bayesian estimates or MLE with Laplace smoothing asymptotically converges to the probability parameter of an event characterizing the Bernoulli distribution, i.e.,

$$\lim_{t \rightarrow \infty} \hat{d}_i^{MLE}(t) = \lim_{t \rightarrow \infty} \hat{d}_i^{Bayesian}(t) = d_i \tag{17}$$

220 where the superscript indicates the corresponding estimator method.

In summary, the Bayesian estimator proposed here converges asymptotically similar to that of MLE and avoids the initialization problems of MLE based LAs. Hence, we suggest replacing MLE by Bayesian estimator for the LAs. In the following, we shall propose a new Bayesian estimator based Learning Automata.

225 4. Generalized Bayesian Stochastic estimator Learning Automata (GBSE)

The discretized generalized pursuit algorithm (DGPA) is demonstrated to be faster than the pursuit family of Learning algorithm with remarkable performance [26]. Also in [26], GPA and DGPA have been introduced to generalize the concepts of the pursuit algorithm by pursuing all the actions that have
230 higher estimates than the current chosen one. Actually, one explanation for this phenomenon could be found in [27] where the connection between the selectionism and Bayesianism is attempted for the Linear Reward-Inaction (L_{RI}) algorithm. It is concluded that in the L_{RI} scheme, the action probabilities

are redistributed towards the better-rewarded actions as the average reward in-
 creases steadily, which is very similar to the concepts underlying the GPA and
 235 DGPA. In this paper, we adopt this idea to design a new LA, i.e., Generalized
 Bayesian Stochastic estimator Learning Automata (*GBSE*) and then discuss
 its ϵ -optimality.

4.1. The GBSE Scheme

240 Our proposed *GBSE* LA is formally described as follows.

GBSEPROCEDURE

Input : the number of actions r , resolution parameters n , γ and $\Delta = \frac{1}{rn}$

Output : the optimal action

245 **Initialize** : $\mu_i(0) = 1, Z_i(0) = 2, i = 1, 2, \dots, r$

Repeat :

Step 1: At time t , choose an action, $a(t) = a_i$, according to the probability vector

$\mathbf{P}(t) = [p_1(t), p_2(t), \dots, p_r(t)]^T$.

Step 2: Receive a feedback from stochastic environment $\beta(t) \in \{0, 1\}$.

250 Step 3: Set $\mu_i(t) = \mu_i(t-1) + \beta(t), Z_i(t) = Z_i(t-1) + 1$.

Step 4: Compute $\hat{d}_i(t) = \frac{\mu_i(t)}{Z_i(t)}$.

Step 5: If $\beta(t) = 0$, go to Step 1.

Step 6: Compute stochastic estimates $u_i(t) = \hat{d}_i(t) + R_i(t)$, where $R_i(t)$ is a
 random number uniformly distributed within $(-\frac{\gamma}{Z_i(t)}, \frac{\gamma}{Z_i(t)})$.

255 Step 7: Denote $K(t)$ as the number of actions that $u_j(t) > u_i(t)$.

Step 8: Update the probability vector as follows

- For any $j, j \neq i$ such that $u_j(t) > u_i(t)$, $p_j(t+1) = \min\{p_j(t) + \frac{\Delta}{K(t)}\}$;
- For any $j, j \neq i$ such that $u_j(t) < u_i(t)$, $p_j(t+1) = \max\{p_j(t) - \frac{\Delta}{(r-K(t))}, 0\}$;
- 260 • For current action i , $p_i(t+1) = 1 - \sum_{j \neq i} p_j(t)$.

Step 9: If $\max\{p_j(t+1), \{j = 1 \dots r\}\} = 1$ then the convergence is reached, else
 go to Step 1.

Table 1: accuracy (number of correct convergences / number of experiments) of SE_{RI} and $GBSE$ repeated 50000 times for environments E_1 to E_5 , when using the 'best' parameters

	E_1	E_2	E_3	E_4	E_5
$GBSE$	0.999	0.999	0.997	0.999	0.998
SE_{RI}	0.997	0.997	0.995	0.998	0.997

ENDPROCEDURE

265 Note that one difference between the proposed $GBSE$ and SE_{RI} [20] learning algorithms is in the updating probability procedure (Step 8) where $GBSE$ increases those probabilities which have higher stochastic estimates than the current action while SE_{RI} just enhances the actions' probability with the highest estimates. Another difference is in the initialization phase where $GBSE$ sets
270 the total number of the action a_i rewarded and selected, to be 1 and 2, respectively; While for the MLE method, no observation is available at the beginning and SE_{RI} sets both the total number of action a_i rewarded, $W_i(t)$ and the total number of action a_i selected, $Z_i(t)$, to be 0. This improvement on the initialization phase results from Bayesian inference theory and leads to very different
275 performance and nice format without any additional computations. Next, we shall show the converging properties of the proposed $GBSE$ LA.

4.2. Proof of ϵ -optimality

In the LA contexts, whether or not ϵ -optimality for a given algorithm is the golden standard for showing the theoretical convergence of the algorithm, which
280 is given as follows.

Definition 1. ϵ -optimality: Given any arbitrarily small $\epsilon > 0$ and $\delta > 0$, there exist large enough $n_0 < \infty$ (that depends on ϵ and δ) and $t_0 < \infty$ such that for all time $t \geq t_0$ and for any resolution parameter $n > n_0$: $\Pr\{|p_m(t)-1| < \epsilon\} > 1-\delta$.

The flaws in the proof of ϵ -optimality for pursuit learning algorithm were

revealed in [28]. Based on [28], we shall be easily able to prove the ϵ -optimality of the proposed *GBSE*. Let us first consider three important Lemmas related to the proof of the ϵ -optimality.

Lemma 1. *For each action α_i , if $p_i(0) \neq 0$, then for any given constants $\delta > 0$ and $M > 0$, there exist $n_0 < 0$ and $T_1 < \infty$ such that for all resolution parameters $n > n_0$ and all time $t > T_1$, $\Pr[\alpha_i \text{ is selected more than } M \text{ times at time } t] \geq 1 - \delta$.*

For the details of the proof, please refer to [18]. This Lemma essentially means $Z_i(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then, the MLE is to asymptotically converge, i.e., $\hat{d}_i(t) \rightarrow d(t)$ as $t \rightarrow \infty$, which is formally presented in the following Lemma.

Lemma 2. *For each action α_i , if $p_i(0) \neq 0$, then for any arbitrarily small $\delta > 0$ and arbitrarily small $\eta > 0$, there exists $T_2 < \infty$, such that, for all time $t > T_2$ and $i \in \{1, 2, \dots, r\}$, there have [28]:*

$$\Pr\{|\hat{d}_i(t) - d_i| > \eta\} < \delta \quad (18)$$

Lemma 3. *Suppose there exist an index m and a time instant $T_3 < \infty$ such that $u_m(t) > u_j(t)$ for all $j, j \neq m$ and all $t \geq T_3$, then $p_m \rightarrow 1$ with a unit probability as $t \rightarrow \infty$.*

Proof. Lemma 3 is referred to as monotonicity property of learning algorithm and its proof is the same as Theorem 7 in [26], which is included here for convenience.

Firstly, we shall illustrate that the sequence of random variable $\{p_m(t)\}_{t \geq 0}$ is a submartingale. The convergence will be reached from the submartingale convergence theorem [11].

Based on the aforementioned probability updating scheme in the *GBSE*

algorithm, we have:

$$\begin{aligned}
p_m(t+1) &= p_m(t) + \frac{\Delta}{K(t)}, & \text{if } \alpha(t) = \alpha_i, i \neq m \\
p_m(t+1) &= 1 - \sum_{i \neq m} (p_i(t) - \frac{\Delta}{r-1}) \\
&= p_m(t) + \Delta, & \text{if } \alpha(t) = \alpha_m.
\end{aligned} \tag{19}$$

Then, for $t \geq T_3$, we have

$$\begin{aligned}
&E[p_m(t+1) - p_m(t) | \mathbf{Q}(t)] \\
&= \sum_{i \neq m} \left(\frac{\Delta}{K(t)} \right) \cdot p_i(t) + \Delta \cdot p_m(t) \\
&= \frac{\Delta}{K(t)} + \Delta \cdot p_m(t) \cdot \left(1 - \frac{1}{K(t)} \right) \geq 0
\end{aligned} \tag{20}$$

where $\mathbf{Q}(t) = (\mathbf{P}(t), \mathbf{R}(t), \hat{\mathbf{d}}(t))$ is the state vector of our stochastic estimator algorithm.

Thus, $p_m(t)$ is a submartingale. From the submartingale convergence theorem[11],
310 it can be found that the process $\{p_m(t)\}_{t \geq 0}$ is converged. $E[p_m(t+1) - p_m(t) | \mathbf{Q}(t)] \rightarrow 0$ with the probability of 1, abbreviated as w.p.1 and $p_m(t) \rightarrow 1$ w.p.1. Hence, Lemma 3 is proved.

With the above three Lemmas, we shall easily prove that the *BGSE* algorithm is ϵ -optimal.

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□

Theorem 1. *In every stationary random environment, the GBSE is ϵ -optimal. Formally, given any arbitrarily small $\epsilon > 0$ and $\delta > 0$, there exist large enough $n_0 < \infty$, $t_0 < \infty$ and a small enough $\gamma_0 > 0$ such that for all time $t \geq t_0$ and for any resolution parameter $n > n_0$, $\gamma < \gamma_0$, there have $\Pr\{|p_m(t) - 1| < \epsilon\} > 1 - \delta$.*

320 *Proof.* Let us first clarify the relationship among $u_i(t)$, $\hat{d}_i(t)$ and $d_i(t)$. Set $\theta > 0$ to be the difference between the largest and the second largest environment reward probabilities, d_m and d_i . Since the reward probability of the best action is unique, it follows that $d_m - d_i \geq \theta$ for any $i \neq m, i \in \{1, 2, \dots, r\}$.

From Lemma 2, it is known that for any action α_i and a given $\delta > 0$, there
 325 exist $T_1^i < \infty$ and $G_1^i < \infty$, such that if $Z_i(t) > G_1^i$ or $t > T_1^i$, then:

$$\Pr\{|\hat{d}_i(t) - d_i| < \frac{\theta}{4}\} < \delta. \quad (21)$$

Denote $G_2^i \triangleq \frac{4\gamma}{\theta}$. Considering that $R_i(t)$ is a random variable following
 uniform distribution, $U(-\frac{\gamma}{Z_i(t)}, \frac{\gamma}{Z_i(t)})$, if $Z_i(t) > G_2^i$, we have:

$$|R_i(t)| < \frac{\gamma}{Z_i(t)} < \frac{\gamma}{G_2^i} < \frac{\gamma}{4\gamma/\theta} = \frac{\theta}{4}. \quad (22)$$

Therefore, both Eq.(21) and Eq.(22) will hold if $Z_i(t) > \max\{G_1^i, G_2^i\} \triangleq G_3^i$.
 In addition, it is very trivial to verify that if $Z_i(t) > G_3^i$, then there have [20]:

$$\Pr\{|u_i(t) - d_i| < \frac{\theta}{2}\} \geq \Pr\{|\hat{d}_i(t) - d_i| < \frac{\theta}{4}\} > 1 - \delta \quad (23)$$

330 which means that $u_i(t)$ and $\hat{d}_i(t)$ will be away from $d_i(t)$ by $\frac{\theta}{2}$ and $\frac{\theta}{4}$ with a
 high probability, respectively.

Next, we shall focus on proving the main Theorem. Let us define the events
 as follows:

$$\begin{aligned} A_\epsilon(t) &= \{p_m(t) > 1 - \epsilon\}; \\ B(t) &= |u_m(t) - d_m| < \eta, t > 0; \\ \overline{B}(T') &= \sup_{t \geq T'} |u_m(t) - d_m| < \eta = \bigcap_{t \geq T'} B(t). \end{aligned}$$

For any t and T' , the following holds,

$$\Pr\{A_\epsilon(t + T')\} \geq \Pr\{A_\epsilon(t + T') | \overline{B}(T')\} \Pr\{\overline{B}(T')\}. \quad (24)$$

It is known that, for $Z_i(t) > G_3^i$, $\Pr\{|u_i(t) - d_i| < \frac{\theta}{2}\} > 1 - \delta$. Then if
 $Z_i(t) > G_3^i$, there have:

$$\Pr\{u_m(t) > u_i(t)\} > 1 - \delta, i \neq m. \quad (25)$$

335 With Lemma 3, we can derive that for those t satisfying $Z_i(t) > G_3^i$, there
 have:

$$\Pr\{A_\epsilon(t + T') | \overline{B}(T')\} \rightarrow 1. \quad (26)$$

On the other hand, $\Pr\{\overline{B}(T') \rightarrow 1\}$ can be arrived by the same steps given in [28]. With Lemma 2, it follows that $d_m \in (0, 1)$ leads to $u_m(t) \in (0, 1)$ for some $T_4 < \infty$ and $t > T_4$. Then, from the inequality in Eq. (24), we have
340 $\Pr\{A_\epsilon(t + T')\} \rightarrow 1$. Hence, this Theorem 1 is proved. $\square$

Table 2: the comparison of the average number of iterations required for convergence of SE_{RI} and $GBSE$ in environments E_1 to E_5 , when using the 'best' parameters (5000 experiments were performed for each scheme in each environment)

Environment	$GBSE$		SE_{RI}	
	Parameter	Iterations	Parameter	Iterations
E_1	$n = 2, \gamma = 7$	461	$n = 16, \gamma = 8$	426
E_2	$n = 5, \gamma = 9$	840	$n = 32, \gamma = 12$	834
E_3	$n = 17, \gamma = 17$	2514	$n = 105, \gamma = 25$	2540
E_4	$n = 2, \gamma = 5$	355	$n = 13, \gamma = 6$	325
E_5	$n = 6, \gamma = 8$	692	$n = 33, \gamma = 12$	729

Table 3: The number of iterations required to achive the same accuracy as SE_{RI} shown in table 1 in environments E_1 to E_5 (5000 experiments were performed for each scheme in each environment)

Environment	$GBSE$	
	Parameter	Iterations
E_1	$n = 1, \gamma = 7$	401
E_2	$n = 3, \gamma = 9$	772
E_3	$n = 6, \gamma = 17$	2262
E_4	$n = 1, \gamma = 5$	306
E_5	$n = 3, \gamma = 8$	612

5. Experiments

In this section, we shall compare the convergence speed of the proposed *GBSE* with that of *SE_{RI}* which has been demonstrated to be the fastest LA in the literature [20]. As mentioned above, one of the differences between *GBSE* and *SE_{RI}* lies on the stochastic estimator, i.e., the *GBSE* is based on Bayesian inference while the *SE_{RI}* is a maximum likelihood estimator. Thus, the performance comparison between them will also shed light on the role of Bayesian inference played in LA. Within the context of LA, the speed of convergence is compared in terms of iterations needed to converge under benchmark environments, and the widely used five environments are from E_1 to E_5 . The actions' reward probabilities for each environment were set to be the same as those in [20] which are listed as follows:

$$\begin{aligned}
E_1 : D &= \{0.65, 0.50, 0.45, 0.40, 0.35, 0.30, 0.25, 0.20, \\
&\quad 0.15, 0.10\} \\
E_2 : D &= \{0.60, 0.50, 0.45, 0.40, 0.35, 0.30, 0.25, 0.20, \\
&\quad 0.15, 0.10\} \\
E_3 : D &= \{0.55, 0.50, 0.45, 0.40, 0.35, 0.30, 0.25, 0.20, \\
&\quad 0.15, 0.10\} \\
E_4 : D &= \{0.70, 0.50, 0.30, 0.20, 0.40, 0.50, 0.40, 0.30, \\
&\quad 0.50, 0.20\} \\
E_5 : D &= \{0.10, 0.45, 0.84, 0.76, 0.20, 0.40, 0.60, 0.70, \\
&\quad 0.50, 0.30\}.
\end{aligned} \tag{27}$$

It has been demonstrated that *SE_{RI}* performs better than the discretized generalized pursuit algorithm (*DGPA*) and discretized pursuit estimator reward-inaction algorithm (*DP_{RI}*) under the above five environments [20]. In all the simulations performed, we have the same setting as [20]. For example, the computation of an algorithm is considered to converge if the probability of choosing an action is greater or equal to a threshold T ($0 < T \leq 1$). Before

the performance comparison of different methods, many evaluation tests were
 360 executed to determine the 'best' learning parameters for each scheme. The value
 is considered as 'best' if it yields the fastest convergence and the automaton
 always converges to the correct action in a sequence of NE experiments. The
 values of T and NE are taken to be equal to the ones used in [20], that is,
 $T = 0.999$ and $NE = 750$. The variance coefficient, defined as $\frac{\sigma\{p_1(t)\}}{E\{p_1(t)\}}$, where
 365 $\sigma\{p_1(t)\}$ is the standard deviation and $E\{p_1(t)\}$ is the mean of random variable
 $p_1(t)$, should be small enough since a high variance coefficient results in an
 unfair comparison.

On one hand, the average numbers of iterations for both schemes of the
 $GBSE$ and the SE_{RI} are summarized in Table ???. As mentioned above, the
 370 proposed scheme do not need sample each action while SE_{RI} needs sample
 process and these sample times are included in the table, which shows that our
 proposed $GBSE$ algorithm converges with the same speed as the SE_{RI} .

On the other hand, Table 1 demonstrates that the accuracy of the proposed
 $GBSE$ is higher than that of SE_{RI} in all benchmark environments. For example,
 375 in Environment E_3 , $GBSE$ takes 2514 iterations to converge while SE_{RI} takes
 2538 iterations; However, the accuracy of $GBSE$ is 0.997 while that of SE_{RI}
 is 0.995. Similarly, $GBSE$ converges in 840 iterations while SE_{RI} converges in
 831 iterations, and the accuracy of $GBSE$ is 0.999 compared to that of SE_{RI} is
 0.997 in environment E_4 .

380 In summary, $GBSE$ is more efficient than SE_{RI} . For all the environments,
 $GBSE$ converges with a comparable number of iterations with SE_{RI} while
 achieving a higher accuracy, which leads to unfair comparison regarding the
 number of iterations.

To verify the number of iterations required with the same accuracy, a set of
 385 experiments have been conducted. The results are demonstrated in Table ??,
 which shows that the average numbers of iterations of $GBSE$ is less than SE_{RI}
 with the same accuracy.

Hence, the advantage of the proposed algorithm is obvious.

6. Conclusions

390 In this paper, we first pointed out the drawbacks of the MLE based LAs. The Bayesian estimator was proposed to overcome these limitations for a faster Learning Automata algorithm. It is also demonstrated that our proposed method is the Laplace smoothing version of the Maximum Likelihood estimates. A new LA algorithm based on Bayesian estimator, referred as *GBSE*, was presented
395 and proved to be ϵ -optimal in all stationary environments. Experiments have confirmed that the Bayesian estimator plays an important role in LA and shown that our proposed *GBSE* is more efficient than the *SE_{RI}*, which is known to be the fastest LA reported in the past literatures. Since this Bayesian estimator is also the Laplace smoothing Maximum Likelihood estimates, we are currently
400 studying how to apply this method to all other existing MLE based LAs.

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