The Laplacian polynomial and Kirchhoff index of graphs based on R -graphs

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Abstract

Let $R(G)$ be the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the corresponding edge. Let $I(G)$ be the set of newly added vertices. The R -vertex corona of G_1 and G_2 , denoted by $G_1 \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the i th vertex of $V(G_1)$ to every vertex in the i th copy of G_2 . The R -edge corona of G_1 and G_2 , denoted by $G_1 \ominus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the i th vertex of $I(G_1)$ to every vertex in the i th copy of G_2 . Liu et al. gave formulae for the Laplacian polynomial and Kirchhoff index of $RT(G)$ in [19]. In this paper, we give the Laplacian polynomials of $G_1 \odot G_2$ and $G_1 \ominus G_2$ for a regular graph G_1 and an arbitrary graph G_2 ; on the other hand, we derive formulae and lower bounds of Kirchhoff index of these graphs and generalize the existing results.

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1. Introduction

Throughout this article, all graphs considered are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. The adjacency matrix of G , denoted by $A(G)$, is an $n \times n$ symmetric matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent and 0 otherwise. Let $d_i = d_G(v_i)$ be the degree of vertex v_i in G and $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$. Denoted by $P_G(x)$ and $\mu_G(x)$ the adjacent characteristic polynomial $\det(xI - A(G))$ and the Laplacian characteristic polynomial $\det(xI - L(G))$ of G , respectively. Since $A(G)$ and $L(G)$ are all real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ (resp., $0 = \mu_1(G) \leq \mu_2(G) \leq \dots \leq \mu_n(G)$) are the adjacency (resp., Laplacian) eigenvalues of G . The collection of the adjacency (resp., Laplacian) eigenvalues of G together with their multiplicities forms the adjacency (resp., Laplacian) spectrum of G . For other undefined notations and terminology from graph theory, the readers may refer to [1] and the

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references therein. There are many applications for Laplacian eigenvalues of graphs. For example, there are many problems in physics and chemistry where the Laplacian eigenvalues play the central role. The Laplacian eigenvalues are in the segmentation of the combination optimization, method of design, parallel algorithm, solving linear systems, clustering and other aspects of a wide range of applications. See [33],[34].

In 1993, Klein and Randić [2] introduced a distance function named resistance distance on the basis of electrical network theory. They view a graph as an electrical network each edge of the graph is assumed to be a unit resistor, then take the resistance distance between vertices to be the effective resistance between them. Let G be a simple graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, and r_{ij} denote the effective resistance distance between vertices v_i and v_j as computed with Ohm's law when all the edges of G are considered to be unit resistors. The sum of resistance distance $Kf(G) = \sum_{i < j} r_{ij}(G)$ was proposed in [1], later called the Kirchhoff index of G in [3]. In electric theory, it is of interest to compute the effective resistance between any pair of vertices of a network, as well as the Kirchhoff index. The Kirchhoff index was introduced in chemistry as a better alternative to other parameters used for discriminating different molecules with similar shapes and structures. See [2]. The resistance distance and the Kirchhoff index attracted extensive attention due to its wide applications in physics, chemistry, etc. See [4]–[9]. For more information on resistance distance and Kirchhoff index of graphs, the readers are referred to the papers ([7]–[9]).

In [10], new graph operations based on $R(G)$ graphs: R -vertex corona and R -edge corona, are introduced, and their A -spectrum (resp., L -spectrum) are investigated. For a graph G , Let $R(G)$ be the graph obtained from G by adding a new vertex u_e and joining u_e to the end vertices of e for each $e \in E(G)$. The graph $R(G)$ appeared in [11] and we call it the R -graph of G . Let $I(G)$ be the set of newly added vertices, i.e. $I(G) = V(R(G)) \setminus V(G)$.

Let G_1 and G_2 be two vertex-disjoint graphs.

Definition 1.1 ([10]) The R -vertex corona of G_1 and G_2 , denoted by $G_1 \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the i th vertex of $V(G_1)$ to every vertex in the i th copy of G_2 .

Definition 1.2 ([10]) The R -edge corona of G_1 and G_2 , denoted by $G_1 \ominus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the i th vertex of $I(G_1)$ to every vertex in the i th copy of G_2 .

Note that if G_i has n_i vertices and m_i edges for $i = 1, 2$, then $G_1 \odot G_2$ has $n_1 + m_1 + n_1 n_2$ vertices and $3m_1 + n_1 m_2 + n_1 n_2$ edges, $G_1 \ominus G_2$ has $n_1 + m_1 + m_1 n_2$ vertices and $3m_1 + m_1 m_2 + m_1 n_2$ edges.

As the authors of [12] pointed out, it is an interesting problem to compute Kirchhoff index of large composition graphs in terms of parameters of small graph in the composition [13, 14]. The Kirchhoff index has been computed for some classes of graphs, such as cycles [15], complete graph [15], distance transitive graphs [16], and so on [5, 8, 15, 17, 20, 21, 22, 23, 24, 25, 32]. The Kirchhoff index of certain composite operations between two graphs was studied as well, such as product, lexicographic product [18] and join, corona, cluster [12]. Then recently Liu et al. [19] explore the Laplacian polynomial of $RT(G)$ of a regular graph G . Motivated by these results, in this paper we compute the Laplacian polynomial of $G_1 \odot G_2$ and $G_1 \ominus G_2$ for a regular graph G_1 and an arbitrary graph G_2 and derive formulae and low bounds of Kirchhoff index of these graphs and generalize their results in [19].

2. Preliminaries

In this section, we determine the characteristic polynomials of graphs with the help of the *coronal* of a matrix. The M -coronal $T_M(\lambda)$ of an $n \times n$ matrix M is defined [26, 27] to be the sum of the entries of the matrix $(\lambda I_n - M)^{-1}$, that is

$$T_M(\lambda) = \mathbf{1}_n^T (\lambda \mathbf{I}_n - \mathbf{M})^{-1} \mathbf{1}_n,$$

where $\mathbf{1}_n$ denotes the column vector of dimension n with all the entries equal one.

If M has a constant row sum t , it is easy to verify that

$$T_M(\lambda) = \frac{n}{\lambda - t}. \quad (1)$$

The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This is an associate operation with the property that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exists.

Lemma 2.1 [29] Let M_1, M_2, M_3 and M_4 be respectively $p \times p, p \times q, q \times p$ and $q \times q$ matrices with M_1 and M_4 invertible, then

$$\begin{aligned} \det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} &= \det(M_4) \det(M_1 - M_2 M_4^{-1} M_3) \\ &= \det(M_1) \det(M_4 - M_3 M_1^{-1} M_2), \end{aligned}$$

where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the Schur complements of M_4 and M_1 , respectively.

3. The Laplacian polynomial of R -vertex and R -edge corona

For a regular graph G_1 , the next theorems give the representation of the Laplacian polynomial of $G_1 \odot G_2$ and $G_1 \ominus G_2$ by means of the characteristic polynomial and the Laplacian polynomial of G_1 and G_2 .

Theorem 3.1 Let G_1 be an r_1 -regular graph with n_1 vertices and m_1 edges and G_2 be an arbitrary graph with n_2 vertices. Then the Laplacian characteristic polynomial of $G_1 \odot G_2$ is given by

$$\begin{aligned} \text{(i)} \quad \mu_{G_1 \odot G_2}(x) &= \prod_{i=1}^{n_2} (x - 1 - \mu_i(G_2))^{n_1} (x - 2)^{m_1} (3 - x)^{n_1} P_{G_1} \left(\frac{(x-n_2)(x-2)}{3-x} + \frac{r_1(2x-3)}{x-3} + \frac{n_2(x-2)}{(x-3)(x-1)} \right). \\ \text{(ii)} \quad \mu_{G_1 \odot G_2}(x) &= \prod_{i=1}^{n_2} (x - 1 - \mu_i(G_2))^{n_1} (x - 2)^{m_1} (x - 3)^{n_1} \mu_{G_1} \left(\frac{x(x^2 - (3+n_2+r_1)x + (2n_2+r_1+2))}{(x-1)(x-3)} \right). \end{aligned}$$

Proof (i) Let B be the vertex-edge incidence matrix of G_1 . Since G_1 is an r_1 -regular graph, we have $D(G_1) = r_1 I_{n_1}$. By a pertinent labeling of the vertices of $G_1 \odot G_2$, then the Laplacian matrix of $G_1 \odot G_2$ can be written as

$$L(G_1 \odot G_2) = \begin{pmatrix} 2I_{m_1} & -B^T & 0_{m_1 \times n_1 n_2} \\ -B & (r_1 + n_2)I_{n_1} + L(G_1) & -I_{n_1} \otimes 1_{n_2}^T \\ 0_{n_1 n_2 \times m_1} & -I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes (I_{n_2} + L(G_2)) \end{pmatrix},$$

where 0_n denotes the length- n column vectors consisting entirely of 0's.

It follows that

$$\begin{aligned} \mu_{G_1 \odot G_2}(x) &= \det \begin{pmatrix} (x-2)I_{m_1} & B^T & 0_{m_1 \times n_1 n_2} \\ B & (x-r_1-n_2)I_{n_1} - L(G_1) & I_{n_1} \otimes 1_{n_2}^T \\ 0_{n_1 n_2 \times m_1} & I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes ((x-1)I_{n_2} - L(G_2)) \end{pmatrix} \\ &= \prod_{i=1}^{n_2} (x - 1 - \mu_i(G_2))^{n_1} \cdot \det(S), \end{aligned} \quad (2)$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x-2)I_{m_1} & B^T & 0_{m_1 \times n_1 n_2} \\ B & (x-r_1-n_2)I_{n_1} - L(G_1) & I_{n_1} \otimes 1_{n_2}^T \\ (I_{n_1} \otimes ((x-1)I_{n_2} - L(G_2)))^{-1} & 0_{n_1 n_2 \times m_1} & I_{n_1} \otimes 1_{n_2} \end{pmatrix} \\ &= \begin{pmatrix} (x-2)I_{m_1} & B^T & 0_{m_1 \times m_1} \\ B & (x-r_1-n_2)I_{n_1} - L(G_1) & I_{n_1} \otimes 1_{n_2}^T (xI_{n_2} - L(G_2))^{-1} 1_{n_2} \\ 0_{m_1 \times m_1} & 0_{m_1 \times n_1} & T_{L(G_2)}(x)I_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-2)I_{m_1} & B^T & 0_{m_1 \times m_1} \\ B & (x-r_1-n_2)I_{n_1} - L(G_1) & 0_{m_1 \times n_1} \\ 0_{n_1 \times m_1} & T_{L(G_2)}(x)I_{n_1} & 0_{n_1 \times n_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-2)I_{m_1} & B^T & 0_{m_1 \times m_1} \\ B & (x-r_1-n_2-T_{L(G_2)}(x))I_{n_1} - L(G_1) & 0_{m_1 \times n_1} \end{pmatrix}. \end{aligned}$$

From (1), we have $T_{L(G_2)}(x-1) = \frac{n_2}{x-1}$ as each row sum of $L(G_2)$ is equal to 0. It is well known that $BB^T = A(G_1) + r_1 I_{n_1}$. Consequently,

$$\begin{aligned} \det(S) &= \det((x-2)I_{m_1}) \det \left((x-r_1-n_2-\frac{n_2}{x-1})I_{n_1} - L(G_1) - \frac{1}{x-2}BB^T \right) \\ &= (x-2)^{m_1} \det \left((x-2r_1-n_2-\frac{n_2}{x-1}-\frac{r_1}{x-2})I_{n_1} + (1-\frac{1}{x-2})A(G_1) \right) \\ &= (x-2)^{m_1} (3-x)^{n_1} \det \left(\left(\frac{(x-n_2)(x-2)}{3-x} + \frac{r_1(2x-3)}{x-3} + \frac{n_2(x-2)}{(x-3)(x-1)} \right) I_{n_1} - A(G_1) \right) \end{aligned}$$

$$= (x-2)^{m_1}(3-x)^{n_1} P_{G_1} \left(\frac{(x-n_2)(x-2)}{3-x} + \frac{r_1(2x-3)}{x-3} + \frac{n_2(x-2)}{(x-3)(x-1)} \right). \quad (3)$$

By virtue of (2) and (3) we have already established the statement (i) in Theorem 3.1.

(ii) Recall that $L(G) = rI_n - A(G)$. It follows from that

$$\begin{aligned} \det(S) &= (x-2)^{m_1-n_1} \det \left((x-2r_1-n_2-\frac{n_2}{x-1}-\frac{r_1}{x-2})I_{n_1} + \frac{x-3}{x-2}A(G_1) \right) \\ &= (x-2)^{m_1-n_1}(x-3)^{n_1} \det \left(\left(\frac{(x-n_2)(x-2)}{x-3} - \frac{r_1x}{x-3} - \frac{n_2(x-2)}{(x-3)(x-1)} \right) I_{n_1} - (r_1I_{n_1} - A(G_1)) \right) \\ &= (x-2)^{m_1-n_1}(x-3)^{n_1} \mu_{G_1} \left(\frac{x(x^2-(3+n_2+r_1)x+(2n_2+r_1+2))}{(x-1)(x-3)} \right). \end{aligned}$$

The proof is thus completed.

Remark 3.2: In [19], the authors gave the Laplacian polynomial of $G_1 \odot G_2$ when G_2 is K_2 , but we obtain the Laplacian polynomial of $G_1 \odot G_2$ when G_2 is an arbitrary graph, so we generalize Theorem 3.1 in [19].

Next, we consider the case for $G_1 \ominus G_2$. For a regular graph, the next theorem gives the representation of the Laplacian polynomial of $G_1 \ominus G_2$ by means of the characteristic polynomial and the Laplacian polynomial of G_1 and G_2 .

Theorem 3.3 Let G_1 be an r_1 -regular graph with n_1 vertices and m_1 edges and G_2 be an arbitrary graph with n_2 vertices. Then the Laplacian characteristic polynomial of $G_1 \ominus G_2$ is given by

$$\begin{aligned} (i) \mu_{G_1 \ominus G_2}(x) &= \prod_{i=2}^{n_2} (x-1-\mu_i(G_2))^{m_1} (-1)^{n_1} (x^2-(3+n_2)x+2)^{m_1-n_1} (x^2-(4+n_2)x+3)^{n_1} P_{G_1} \left(\frac{-x^3+(n_2+3+2r_1)x-(2+2r_1n_2+5r_1)x+3r_1}{x^2-(4+n_2)x+3} \right). \\ (ii) \mu_{G_1 \ominus G_2}(x) &= \prod_{i=2}^{n_2} (x-1-\mu_i(G_2))^{m_1} (x^2-(3+n_2)x+2)^{m_1-n_1} (x^2-(4+n_2)x+3)^{n_1} \mu_{G_1} \left(\frac{x[x^2-(3+n_2+r_1)x+(r_2n_2+r_1+2)]}{(x-1)(x-3)} \right). \end{aligned}$$

Proof (i) Let B be the vertex-edge incidence matrix of G_1 . Since G_1 is an r_1 -regular graph, we have $D(G_1) = r_1I_{n_1}$. By a pertinent labeling of the vertices of $G_1 \ominus G_2$, then the Laplacian matrix of $G_1 \ominus G_2$ can be written as

$$L(G_1 \ominus G_2) = \begin{pmatrix} (n_2+2)I_{m_1} & -B^T & -I_{m_1} \otimes 1_{n_2}^T \\ -B & r_1I_{n_1} + L(G_1) & 0_{n_1 \times m_1 n_2} \\ -I_{m_1} \otimes 1_{n_2} & 0_{m_1 n_2 \times n_1} & I_{m_1} \otimes (I_{n_2} + L(G_2)) \end{pmatrix}.$$

It follows that

$$\begin{aligned} \mu_{G_1 \ominus G_2}(x) &= \det \begin{pmatrix} (x-n_2-2)I_{m_1} & B^T & I_{m_1} \otimes 1_{n_2}^T \\ B & (x-r_1)I_{n_1} - L(G_1) & 0_{n_1 \times m_1 n_2} \\ I_{m_1} \otimes 1_{n_2} & 0_{m_1 n_2 \times n_1} & I_{m_1} \otimes ((x-1)I_{n_2} - L(G_2)) \end{pmatrix} \\ &= \prod_{i=1}^{n_2} (x-1-\mu_i(G_2))^{m_1} \cdot \det(S), \end{aligned} \quad (4)$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x-n_2-2)I_{m_1} & B^T & I_{m_1} \otimes 1_{n_2}^T \\ B & (x-r_1)I_{n_1} - L(G_1) & 0_{n_1 \times m_1 n_2} \\ (I_{m_1} \otimes ((x-1)I_{n_2} - L(G_2)))^{-1} & I_{m_1} \otimes 1_{n_2} & 0_{m_1 n_2 \times n_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-n_2-2)I_{m_1} & B^T & I_{m_1} \otimes 1_{n_2}^T (xI_{n_2} - L(G_2))^{-1} 1_{n_2} & 0_{m_1 \times n_1} \\ B & (x-r_1)I_{n_1} - L(G_1) & 0_{n_1 \times m_1} & 0_{n_1 \times n_1} \\ (x-n_2-2-T_{L(G_2)}(x-1))I_{m_1} & B^T & & \end{pmatrix} \\ &= \begin{pmatrix} (x-n_2-2-T_{L(G_2)}(x-1))I_{m_1} & B^T & & \\ B & (x-r_1)I_{n_1} - L(G_1) & & \end{pmatrix}. \end{aligned}$$

From (1), we have $T_{L(G_2)}(x-1) = \frac{n_2}{x-1}$ as each row sum of $L(G_2)$ is equal to 0. Note that $BB^T = A(G_1) + r_1I_{n_1}$. Consequently,

$$\begin{aligned} \det(S) &= \det((x-n_2-2-T_{L(G_2)}(x-1))I_{m_1}) \det \left((x-r_1)I_{n_1} - L(G_1) - \frac{BB^T}{x-n_2-2-T_{L(G_2)}(x-1)} \right) \\ &= (x-n_2-2-T_{L(G_2)}(x-1))^{m_1} \det \left((x-2r_1 - \frac{r_1}{x-n_2-2-T_{L(G_2)}(x-1)})I_{n_1} \right. \\ &\quad \left. + (1 - \frac{1}{x-n_2-2-T_{L(G_2)}(x-1)})A(G_1) \right) \\ &= (-1)^{n_1} (x^2-(3+n_2)x+2)^{m_1-n_1} (x^2-(4+n_2)x+3)^{n_1} P_{G_1} \left(\frac{r_1-(x-2r_1)(x-n_2-2-T_{L(G_2)}(x-1))}{x-n_2-3-T_{L(G_2)}(x-1)} \right) \end{aligned}$$

$$= (-1)^{n_1} (x^2 - (3 + n_2)x + 2)^{m_1 - n_1} (x^2 - (4 + n_2)x + 3)^{n_1} P_{G_1} \left(\frac{-x^3 + (n_2 + 3 + 2r_1)x - (2 + 2r_1n_2 + 5r_1)x + 3r_1}{x^2 - (4 + n_2)x + 3} \right). \quad (5)$$

By virtue of (4) and (5) we have already established the statement (i) in Theorem 3.2.

(ii) Recall that $L(G) = rI_n - A(G)$. It follows from that

$$\begin{aligned} \det(S) &= (x - n_2 - 2 - T_{L(G_2)}(x - 1))^{m_1} \det \left((x - 2r_1 - \frac{r_1}{x - n_2 - 2 - T_{L(G_2)}(x - 1)}) I_{n_1} + (1 - \frac{1}{x - n_2 - 2 - T_{L(G_2)}(x - 1)}) A(G_1) \right) \\ &= (x^2 - (3 + n_2)x + 2)^{m_1 - n_1} (x^2 - (4 + n_2)x + 3)^{n_1} \\ &\quad \det \left(\left(\frac{(x - 2r_1)(x - n_2 - 2 - T_{L(G_2)}(x - 1)) - r_1}{x - n_2 - 3 - T_{L(G_2)}(x - 1)} + r_1 \right) I_{n_1} - (r_1 I_{n_1} - A(G_1)) \right) \\ &= (x^2 - (3 + n_2)x + 2)^{m_1 - n_1} (x^2 - (4 + n_2)x + 3)^{n_1} \mu_{G_1} \left(\frac{x[x^2 - (3 + n_2 + r_1)x + (r_2n_2 + r_1 + 2)]}{x^2 - (4 + n_2)x + 3} \right). \end{aligned}$$

The proof is thus completed.

4. The Kirchhoff index of R -vertex and R -edge corona

In this section, we will explore the Kirchhoff index of the $G_1 \odot G_2$ and $G_1 \ominus G_2$ for a regular graph G_1 and an arbitrary graph G_2 .

Gutman and Mohar [30] and Zhu et al. [31] established the relationship between the Laplacian spectrum and Kirchhoff index as follows:

Lemma 4.1 [30] Let G be a connected graph with $n \geq 2$ vertices. Then

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

Denote by δ_i the degree of vertex $v_i \in V(G)$. Zhou and Trinajstić [22] proved that

Lemma 4.2 [17] Let G be a connected graph with $n \geq 2$ vertices. Then

$$Kf(G) \geq -1 + (n - 1) \sum_{v_i \in V(G)} \frac{1}{\delta_i}$$

with equality attained if and only if $G = K_n$ or $G = K_{t, n-t}$ for $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$.

The following lemma established a nice relationship between the Laplacian polynomial and Kirchhoff index as follows:

Lemma 4.3 [15] Let G be a connected graph with $n \geq 2$ vertices and $\mu_G(x) = x^n + a_1x^{n-1} + \dots + a_{n-1}x$. Then

$$\frac{Kf(G)}{n} = -\frac{a_{n-2}}{a_{n-1}} (a_{n-2} = 1 \text{ whenever } n = 2).$$

Let K_n denote the complete graph with n vertices. If $G = K_1$, we have nothing to discuss. So we assume that $G \neq K_1$ throughout this paper. The following Theorem 4.4 generalizes the Theorem 4.4 in [19] and we get the generalized results.

Theorem 4.4 Let G_1 be an r_1 -regular graph with n_1 vertices and m_1 edges and G_2 be an arbitrary graph with n_2 vertices. Then

$$\begin{aligned} Kf(G_1 \odot G_2) &= \frac{(2n_2 + r_1 + 2)^2}{6} Kf(G_1) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{2n_1(n_1 - 1)(2 + r_1 + 2n_2)}{3} \\ &\quad + \frac{n_1^2(r_1 - 2)(r_1 + 2 + 2n_2)}{8} + \frac{n_1^2(2 + r_1 + 2n_2)}{2} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}. \end{aligned}$$

Proof Let $\mu_G(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x$. It follows from Theorem 3.1 (ii) that

$$\begin{aligned}
 \mu_{G_1 \odot G_2}(x) &= \prod_{i=1}^{n_2} (x-1-\mu_i(G_2))^{n_1} (x-2)^{m_1-n_1} (x-3)^{n_1} \\
 &\quad [x^{n_1} \left(\frac{x^2 - (3+n_2+r_1)x + (2n_2+r_1+2)}{(x-1)(x-3)} \right)^{n_1} + \dots \\
 &\quad + a_{n_1-2}x^2 \left(\frac{x^2 - (3+n_2+r_1)x + (2n_2+r_1+2)}{(x-1)(x-3)} \right)^2 \\
 &\quad + a_{n_1-1}x \left(\frac{x^2 - (3+n_2+r_1)x + (2n_2+r_1+2)}{(x-1)(x-3)} \right)] \\
 &= \prod_{i=2}^{n_2} (x-1-\mu_i(G_2))^{n_1} (x-2)^{m_1-n_1} [x^{n_1} (x^2 - (3+n_2+r_1)x + (2n_2+r_1+2))^{n_1} \\
 &\quad + \dots + a_{n_1-2}x^2 (x-1)^{n_1-2} (x-3)^{n_1-2} (x^2 - (3+n_2+r_1)x + (2n_2+r_1+2))^2 \\
 &\quad + a_{n_1-1}x (x-1)^{n_1-1} (x-3)^{n_1-1} (x^2 - (3+n_2+r_1)x + (2n_2+r_1+2))],
 \end{aligned}$$

where $x \neq 1, 3$. So the coefficient of x^2 in $\mu_{G_1 \odot G_2}(x)$ is

$$\begin{aligned}
 &(-2)^{m_1-n_1} \prod_{i=2}^{n_2} (-1 + \mu_i(G_2))^{n_1} [a_{n_1-2}(2n_2+r_1+2)^2(-1)^{n_1-2}(-3)^{n_1-2} \\
 &\quad + a_{n_1-1}(-(3+n_2+r_1))(-1)^{n_1-1}(-3)^{n_1-1} \\
 &\quad + a_{n_1-1}(2n_2+r_1+2)(n_1-1)(-1)^{n_1-2}(-3)^{n_1-1} \\
 &\quad + a_{n_1-1}(2n_2+r_1+2)(-1)^{n_1-1}(n_1-1)(-3)^{n_1-2}] \\
 &+ (m_1-n_1)(-2)^{m_1-n_1-1} \prod_{i=2}^{n_2} (1 + \mu_i(G_2))^{n_1} a_{n_1-1}(2n_2+r_1+2)(-1)^{n_1-1}(-3)^{n_1-1} \\
 &+ (-2)^{m_1-n_1} \sum_{j=2}^{n_2} n_1(-1 + \mu_j(G_2))^{n_1-1} \prod_{i=2, i \neq j}^{n_2} (-1 + \mu_i(G_2))^{n_1} a_{n_1-1} \\
 &\quad (2n_2+r_1+2)(-1)^{n_1-1}(-3)^{n_1-1}, \tag{6}
 \end{aligned}$$

and the coefficient of x in $\mu_{G_1 \odot G_2}(x)$ is

$$(-2)^{m_1-n_1} \prod_{i=2}^{n_2} (-1 + \mu_i(G_2))^{n_1} (2n_2+r_1+2)(-1)^{n_1-1}(-3)^{n_1-1} a_{n_1-1}. \tag{7}$$

Note that $G_1 \odot G_2$ has $n_1 + m_1 + n_1n_2$ vertices. It follows from Lemma 4.3, (6) and (7) that

$$\begin{aligned}
 \frac{Kf(G_1 \odot G_2)}{n_1 + m_1 + n_1n_2} &= -\frac{a_{n_1-2}}{a_{n_1-1}} \frac{2n_2+r_1+2}{3} + \frac{3+n_2+r_1}{2n_2+r_1+2} + \frac{4(n_1-1)}{3} \\
 &\quad + \frac{m_1+n_1}{2} + \sum_{i=2}^{n_2} \frac{n_1}{1+\mu_i(G_2)}.
 \end{aligned}$$

Substituting the result of Lemma 4.3 and $m_1 = \frac{n_1r_1}{2}$ into the above equation, we have

$$\begin{aligned}
 Kf(G_1 \odot G_2) &= \frac{2n_2+r_1+2}{3n_1} (n_1 + m_1 + n_1n_2) Kf(G_1) + \frac{3+n_2+r_1}{2n_2+r_1+2} (n_1 + m_1 + n_1n_2) \\
 &\quad + \frac{4(n_1-1)}{3} (n_1 + m_1 + n_1n_2) + \frac{m_1-n_1}{2} (n_1 + m_1 + n_1n_2)
 \end{aligned}$$

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$$\begin{aligned}
& +n_1(n_1 + m_1 + n_1 n_2) \sum_{i=2}^{n_2} \frac{n_1}{1 + \mu_i(G_2)} \\
= & \frac{(2n_2 + r_1 + 2)^2}{6} Kf(G_1) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{2n_1(n_1 - 1)(2 + r_1 + 2n_2)}{3} \\
& + \frac{n_1^2(r_1 - 2)(r_1 + 2 + 2n_2)}{8} + \frac{n_1^2(2 + r_1 + 2n_2)}{2} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}.
\end{aligned}$$

Summing up, we complete the proof.

Remark 4.5: Comparison to the Laplacian polynomial and its Kirchhoff index of $RT(G)$ in [19], our results of the graph $G_1 \odot G_2$ has generality. It is clear that handling the problems of Laplacian polynomial and Kirchhoff index are more difficult and complex, but we deduce those with a simple approach.

In what follows, we give a lower bound for the Kirchhoff index of $G_1 \odot G_2$.

Corollary 4.6 Let G_1 be an r_1 -regular graph with n_1 vertices and m_1 edges and G_2 be an arbitrary graph with n_2 vertices. Then

$$\begin{aligned}
Kf(G_1 \odot G_2) \geq & \frac{n_1(n_1 - 1)(2n_2 + r_1 + 2)^2}{6r_1} + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{n_1^2(r_1 - 2)(r_1 + 2 + 2n_2)}{8} \\
& + \frac{(4n_1^2 - 4n_1 - 2n_2 - r_1 - 2)(2 + r_1 + 2n_2)}{6} + \frac{n_1^2(2 + r_1 + 2n_2)}{2} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}.
\end{aligned}$$

Proof It follows from Lemma 4.2 and Theorem 4.4 that

$$\begin{aligned}
Kf(G_1 \odot G_2) \geq & \frac{(2n_2 + r_1 + 2)^2}{6} \left(\frac{n_1(n_1 - 1)}{r_1} - 1 \right) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{2n_1(n_1 - 1)(2 + r_1 + 2n_2)}{3} \\
& + \frac{n_1^2(r_1 - 2)(r_1 + 2 + 2n_2)}{8} + \frac{n_1^2(2 + r_1 + 2n_2)}{2} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)} \\
= & \frac{n_1(n_1 - 1)(2n_2 + r_1 + 2)^2}{6r_1} + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{n_1^2(r_1 - 2)(r_1 + 2 + 2n_2)}{8} \\
& + \frac{(4n_1^2 - 4n_1 - 2n_2 - r_1 - 2)(2 + r_1 + 2n_2)}{6} + \frac{n_1^2(2 + r_1 + 2n_2)}{2} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}.
\end{aligned}$$

The following result is proved in a way that is certainly similar in spirit to the proof of Theorem 4.4.

Theorem 4.7 Let G_1 be an r_1 -regular graph with n_1 vertices and m_1 edges and G_2 be an arbitrary graph with n_2 vertices. Then

$$\begin{aligned}
Kf(G_1 \odot G_2) = & \frac{(r_1 n_2 + r_1 + 2)^2}{6} Kf(G_1) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{(n_1^2 - n_1)(n_2 + 4)(2 + r_1 + r_1 n_2)}{6} \\
& + \frac{n_1^2(r_1 - 2)(3 + n_1)(r_1 + 2 + r_1 n_2)}{8} + \frac{n_1^2 r_1(2 + r_1 + r_1 n_2)}{4} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}.
\end{aligned}$$

Proof Let $\mu_G(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x$. It follows from Theorem 3.2 (ii) that

$$\begin{aligned}
\mu_{G_1 \odot G_2}(x) = & \prod_{i=2}^{n_2} (x - 1 - \mu_i(G_2))^{m_1} (x^2 - (3 + n_2)x + 2)^{m_1 - n_1} (x^2 - (4 + n_2)x + 3)^{n_1} \\
& [x^{n_1} \left(\frac{x^2 - (3 + n_2 + r_1)x + (r_1 n_2 + r_1 + 2)}{x^2 - (4 + n_2)x + 3} \right)^{n_1} + \dots \\
& + a_{n_1-2} x^2 \left(\frac{x^2 - (3 + n_2 + r_1)x + (r_1 n_2 + r_1 + 2)}{x^2 - (4 + n_2)x + 3} \right)^2]
\end{aligned}$$

$$\begin{aligned}
& + a_{n_1-1} x \left(\frac{x^2 - (3 + n_2 + r_1)x + (r_1 n_2 + r_1 + 2)}{x^2 - (4 + n_2)x + 3} \right)] \\
& = \prod_{i=2}^{n_2} (x - 1 - \mu_i(G_2))^{m_1} (x^2 - (3 + n_2)x + 2)^{m_1 - n_1} \\
& \quad [x^{n_1} (x^2 - (3 + n_2 + r_1)x + (r_1 n_2 + r_1 + 2))^{n_1} + \dots + a_{n_1-2} x^2 (x^2 - (3 + n_2 \\
& \quad + r_1)x + (r_1 n_2 + r_1 + 2))^2 (x^2 - (4 + n_2)x + 3)^{n_1-2} + a_{n_1-1} x (x^2 - (3 + n_2 \\
& \quad + r_1)x + (r_1 n_2 + r_1 + 2)) (x^2 - (4 + n_2)x + 3)^{n_1-1}],
\end{aligned}$$

so the coefficient of x^2 in $\mu_{G_1 \ominus G_2}(x)$ is

$$\begin{aligned}
& 2^{m_1 - n_1} \prod_{i=2}^{n_2} (-1 + \mu_i(G_2))^{m_1} [a_{n_1-2} (r_1 n_2 + r_1 + 2)^2 3^{n_1-2} \\
& \quad + a_{n_1-1} (-(3 + n_2 + r_1)) 3^{n_1-1} \\
& \quad + a_{n_1-1} (r_1 n_2 + r_1 + 2) (n_1 - 1) (-1)^{n_1-2} 3^{n_1-2} (-(4 + n_2)) (n_1 - 1)] \\
& - a_{n_1-1} (m_1 - n_1) (3 + n_2) 2^{m_1 - n_1 - 1} \prod_{i=2}^{n_2} (-1 + \mu_i(G_2))^{m_1} (r_1 n_2 + r_1 + 2) 3^{n_1-1} \\
& + 2^{m_1 - n_1} \sum_{j=2}^{n_2} m_1 (-1 + \mu_j(G_2))^{m_1 - 1} \prod_{i=2, i \neq j}^{n_2} (-1 + \mu_i(G_2))^{m_1} a_{n_1-1} (r_1 n_2 + r_1 + 2) 3^{n_1-1},
\end{aligned}$$

and the coefficient of x in $\mu_{G_1 \ominus G_2}(x)$ is

$$2^{m_1 - n_1} \prod_{i=2}^{n_2} (-1 + \mu_i(G_2))^{m_1} (r_1 n_2 + r_1 + 2) 3^{n_1-1} a_{n_1-1}.$$

Note that $G_1 \ominus G_2$ has $n_1 + m_1 + n_1 n_2$ vertices. It follows that

$$\begin{aligned}
\frac{Kf(G_1 \ominus G_2)}{n_1 + m_1 + m_1 n_2} &= -\frac{a_{n_1-2} r_1 n_2 + r_1 + 2}{a_{n_1-1} 3} + \frac{3 + n_2 + r_1}{r_1 n_2 + r_1 + 2} + \frac{(n_1 - 1)(4 + n_2)}{3} \\
& \quad + \frac{(m_1 - n_1)(3 + n_2)}{2} + \sum_{i=2}^{n_2} \frac{m_1}{1 + \mu_i(G_2)}.
\end{aligned}$$

Substituting the result of Lemma 4.3 and $m_1 = \frac{n_1 r_1}{2}$ into the above equation, we have

$$\begin{aligned}
Kf(G_1 \ominus G_2) &= \frac{(r_1 n_2 + r_1 + 2)^2}{6} Kf(G_1) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{(n_1^2 - n_1)(n_2 + 4)(2 + r_1 + r_1 n_2)}{6} \\
& \quad + \frac{n_1^2(r_1 - 2)(3 + n_1)(r_1 + 2 + r_1 n_2)}{8} + \frac{n_1^2 r_1(2 + r_1 + r_1 n_2)}{4} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}.
\end{aligned}$$

Summing up, we complete the proof.

Corollary 4.8 Let G_1 be an r_1 -regular graph with n_1 vertices and m_1 edges and G_2 be an arbitrary graph with n_2 vertices. Then

$$\begin{aligned}
Kf(G_1 \ominus G_2) &\geq \frac{(n_1^2 - n_1)(r_1 n_2 + r_1 + 2)^2}{6 r_1} + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{(n_1^2 - n_1)(n_2 + 4)(2 + r_1 + r_1 n_2)}{6} \\
& \quad + \frac{(3 n_1^2 r_1 - 2 r_1 n_2 - 2 r_1 - 4)(2 + r_1 + r_1 n_2)}{12} + \frac{n_1^2(r_1 - 2)(3 + n_1)(r_1 + 2 + r_1 n_2)}{8} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}.
\end{aligned}$$

Proof It follows from Lemmma 4.2 and Theorem 4.6 that

$$\begin{aligned}
 Kf(G_1 \ominus G_2) &\geq \frac{(r_1 n_2 + r_1 + 2)^2}{6} \left(\frac{n_1(n_1 - 1)}{r_1} - 1 \right) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{n_1^2 r_1(2 + r_1 + r_1 n_2)}{4} \\
 &\quad + \frac{(n_1^2 - n_1)(n_2 + 4)(2 + r_1 + r_1 n_2)}{6} + \frac{n_1^2(r_1 - 2)(3 + n_1)(r_1 + 2 + r_1 n_2)}{8} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)} \\
 &= \frac{(n_1^2 - n_1)(r_1 n_2 + r_1 + 2)^2}{6r_1} + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{(n_1^2 - n_1)(n_2 + 4)(2 + r_1 + r_1 n_2)}{6} \\
 &\quad + \frac{(3n_1^2 r_1 - 2r_1 n_2 - 2r_1 - 4)(2 + r_1 + r_1 n_2)}{12} + \frac{n_1^2(r_1 - 2)(3 + n_1)(r_1 + 2 + r_1 n_2)}{8} \sum_{i=2}^{n_2} \frac{1}{1 + \mu_i(G_2)}.
 \end{aligned}$$

At last, we give an example.

Example 4.9 Let K_n a complete graph on n vertices. Note that the eigenvalues of L_{K_n} are 0 and n repeated $n - 1$ times. Then by Theorem 4.4 and Theorem 4.7, we have

$$\begin{aligned}
 Kf(K_n \odot K_n) &= \frac{(2n_2 + r_1 + 2)^2}{6} (n_1 - 1) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{2n_1(n_1 - 1)(2 + r_1 + 2n_2)}{3} \\
 &\quad + \frac{n_1^2(r_1 - 2)(r_1 + 2 + 2n_2)}{8} + \frac{n_1^2(2 + r_1 + 2n_2)}{2} \left(\frac{n_2 - 1}{1 + n_2} \right). \\
 Kf(K_n \ominus K_n) &= \frac{(r_1 n_2 + r_1 + 2)^2}{6} (n_1 - 1) + \frac{n_1(3 + n_2 + r_1)}{2} + \frac{(n_1^2 - n_1)(n_2 + 4)(2 + r_1 + r_1 n_2)}{6} \\
 &\quad + \frac{n_1^2(r_1 - 2)(3 + n_1)(r_1 + 2 + r_1 n_2)}{8} + \frac{n_1^2 r_1(2 + r_1 + r_1 n_2)}{4} \frac{n_2 - 1}{1 + n_2}.
 \end{aligned}$$

5. Conclusions

In this paper, we explore the Laplacian polynomial of $G_1 \odot G_2$ and $G_1 \ominus G_2$ for a regular graph G_1 and an arbitrary graph G_2 and derive formulae for Kirchhoff index of these graphs. By utilizing the spectral graph theory, we establish the explicit formulas for $Kf(G_1 \odot G_2)$ and $Kf(G_1 \ominus G_2)$ in terms of $Kf(G_1)$, $Kf(G_2)$, the number of vertices and the vertex degree of regular graph G_1 and G_2 , based on which we propose a lower bound for the Kirchhoff index for $G_1 \odot G_2$ and $G_1 \ominus G_2$ with respect to the number of vertices and the vertex degree.

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