Pinning synchronization of nonlinearly coupled complex networks with time-varying delays using M-matrix strategies

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Abstract

In this paper, we investigate the global synchronization problem for a class of nonlinearly-coupled complex networks (NCCNs) with mixed time-varying delays by using an M-matrix approach. The M-matrix strategy allows many synchronization problems with much relaxed conditions to be considered and can be used to derive sufficient conditions for this new class of NCCNs by pinning control. We also propose an effective way of adapting the coupling strengths of complex networks and give numerical examples to illustrate the effectiveness of our theoretical results.

Keywords: nonlinearly-coupled complex networks, synchronization, M-matrix, time-varying delays, pinning control.

1. Introduction

A complex network is a structure that is made up of a large set of nodes that are connected by a set of edges and they are ubiquitously found in nature and the modern world (for example, the Internet, mobile sensor networks, food networks and human communication networks). Complex networks have been extensively studied in the physical sciences and mathematics in the past few decades and much is already known about their dynamical behavior (see [1, 2, 3, 4, 5, 6, 7] and the references therein). Synchronization (first discovered by Pecora and Carroll in 1990) is a process in which two or more dynamical systems seek to adjust a certain property of their motion to a common behavior in the limit as time tends to infinity [8]. Synchronized complex networks have wide-ranging applications to many fields (such as population dynamics, power systems and automatic control [8, 9, 10]) and many synchronization patterns have been discovered in the past (for example, complete synchronization [11, 12], cluster synchronization [13], phase synchronization [14] and so on). Techniques are also available that facilitate the attainment of synchronization such as by adaptive control [15, 16], intermittent control [17], impulsive control [18] and pinning control [19, 20, 21, 22].

As everyone knows, the real-world networks normally have a large number of nodes, and it is usually impractical to control a complex network by adding the controllers to all nodes [17]. Pinning control is a particularly useful technique that applies controllers to only a small fraction of the nodes in a network (thus greatly reducing the number of controlled nodes in real-world complex networks). In fact, pinning control is so effective in some networks that only one controller is required for synchronization [19]. And the pinning scheme of the most highly connected nodes induced that a significant reduction in required controllers as compared to the traditional control scheme [9]. In general, the exact number of nodes that are required to be pinned for any network and coupling strength is also completely known [23, 24].

Many networks in nature are nonlinearly-coupled (for example, neural and metabolic networks, in which the coupling configurations are continuously oscillating between two fixed states [25]). However, most synchronization studies that have been conducted so far have been confined to the smaller and less applicable class of complex networks that are linearly coupled [19, 20, 23] thus leaving the more important and naturally occurring nonlinearly-coupled ones virtually uninvestigated. This paper attempts to fill this gap in our knowledge by investigating the nonlinearly coupled problem via pinning control using the M-matrix approach.

Pinning control is a very effective scheme for synchronization in complex networks by controlling only a small number of the system nodes [26]. The M-matrix is one of the most common ways of expressing the structure of a problem in the biological, physical and social sciences and it is also commonly used in mathematics to

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establish eigenvalue bounds. Song *et al* recently proposed an M-matrix method to investigate the first-order leader-follower synchronization problem in pinning-controlled multiagent systems [27] and the leader-follower consensus problem for multi-agent systems with inherent nonlinear dynamics [28]. Consensus Tracking of multi-agent systems with Lipschitz-type node dynamics were studied by using tools M-matrix theory in [29]. The M-matrix provides a more general framework for the formulation and solution of nonlinearly-coupled systems and allows for the derivation of sufficient conditions for synchronization.

The main aim of this paper is to extend the results of [27] to the more general problem of synchronizing nonlinearly coupled complex networks. In particular, we study the synchronization of nonlinearly-coupled complex networks with time-varying delays by using the pinning control method using the M-matrix approach. We also rigorously construct an adaptive feedback controller that can be used to completely synchronize any real-world network. Our results will be applicable to many synchronization problems in different fields of science and technology.

This paper is organized as follows. In Sect. 2, we state some preliminary definitions and lemmas that will be needed for the rest of the paper. In Sect. 3, we introduce the general model for a nonlinearly coupled complex network with time-varying delayed dynamical nodes and establish some synchronization criteria for these complex dynamical networks. In Sect. 4, we discuss some numerical examples of the theoretical results. Sect. 5 is the conclusion of the paper.

2. Preliminaries

2.1. Notations

Throughout this paper, $\mathbb{R}^n$ shall denote the n -dimensional Euclidean space and $\mathbb{R}^{n \times n}$ the set of all $n \times n$ real matrices. For a real matrix A , let $A^\top$ be its transpose and $A^s = (A + A^\top)/2$ be its symmetric part. Let $\mathbf{1}_n = (1, 1, \dots, 1)^\top \in \mathbb{R}^n$ and I_n be the n -dimensional identity matrix. For symmetric matrices A , the notation $A > 0$ (respectively, < 0) shall mean that A is a positive-definite (respectively, negative-definite) matrix and $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ shall denote the greatest and least eigenvalues of a symmetric matrix respectively. For a square matrix $A \in \mathbb{C}^{n \times n}$, let $\lambda_i(A)$ be the i th eigenvalue, $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i(A)|$ be the spectral radius and $\mathcal{J}(A) = \min_{1 \leq i \leq n} \text{Re}(\lambda_i(A))$ be the minimum of the real parts of all the eigenvalues. The symbol $\otimes$ denotes the Kronecker product.

2.2. Graph Theory and Matrix Algebra

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph composed of a set of nodes $\mathcal{V} = \{v_1, \dots, v_n\}$, a set of edges $\mathcal{E} = \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]$ with non-negative adjacency elements a_{ij} . An edge of $\mathcal{G}$ is denoted by $e_{ij} = (v_i, v_j) \in \mathcal{E}$, which means that node v_j receives information from node v_i (v_i is called the parent of v_j). The adjacency elements associated with the edges of the graph are positive, i.e., $e_{ij} \in \mathcal{E} \Leftrightarrow a_{ij} > 0$. Moreover, we assume that $a_{ii} = 0$ for all $i \in 1, \dots, n$. The set of neighbors of node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. The in-degree and out-degree of node v_i are, respectively, defined as follows:

$$\deg_{in}(v_i) = \sum_{j=1}^n a_{ji}, \quad \deg_{out}(v_i) = \sum_{j=1}^n a_{ij}.$$

The degree matrix of digraph $\mathcal{G}$ is a diagonal matrix $\Delta = [\Delta_{ij}]$ where $\Delta_{ij} = 0$ for all $i \neq j$, and $\Delta_{ii} = \deg_{out}(v_i)$. The graph Laplacian associated with the digraph $\mathcal{G}$ is defined as

$$\mathcal{L}(\mathcal{G}) = L = \Delta - A.$$

A reversal of a graph is obtained by reversing the orientation of all the edges. If a graph has adjacency matrix A , then its reversal has adjacency matrix $A^\top$.

Firstly, some results on M-matrix are provided to study the pinning control synchronization of NCCNS.

Lemma 1. [27] The Laplacian matrix L has a simple zero eigenvalue and all the other eigenvalues have positive real parts if and only if the digraph associated with L has a directed spanning tree.

Lemma 2. [27] Let L be the Laplacian matrix of a digraph $\mathcal{G}$. The minimum number of directed trees in $\mathcal{G}$ is equal to the multiplicity of the zero eigenvalue of L .

Definition 1. [30] The set $Z_n \in \mathbb{R}^{n \times n}$ is defined by

$$Z_n \equiv \left\{ A = [a_{ij}] \in \mathbb{R}^{n \times n} : a_{ij} \leq 0, i \neq j, \quad i, j = 1, 2, \dots, n \right\}.$$

Definition 2. [30] A matrix $A \in \mathbb{R}^{n \times n}$ is said to be positive stable if $i_+(A) = n$, where $i_+(A)$ is the number of eigenvalues of A , counting multiplicities, with positive real part.

Definition 3. [30] A matrix A is called an M -matrix if $A \in Z_n$ and A is positive stable.

Lemma 3. [30] If $A \in Z_n$, the following statements are equivalent:

- 1) A is positive stable, that is, A is an M -matrix.
- 2) $A = \alpha I - P$, $P \geq 0$, $\alpha > \rho(P)$.
- 3) Every real eigenvalues of A is positive.
- 4) There is a positive diagonal matrix D such that $DA + A^\top D$ is positive definite; that is, there is a positive diagonal dominant.

Before proceeding further, the following essential lemmas are presented, which are used in proving the main result of this paper.

Lemma 4. [25] Let $\mathbf{1}_n = (1, 1, \dots, 1)^\top$, $I_n = \text{diag}\{1, 1, \dots, 1\} \in \mathbb{R}^n$ and $Q = (q_{ij}) = I_n - \frac{1}{N} \mathbf{1}_n \cdot \mathbf{1}_n^\top$. For all zero-row-sum matrices $M \in \mathbb{R}^{m \times n}$ and $\theta > 0$, we have

$$x^\top M y = x^\top M Q y \leq \frac{1}{2} \left(\frac{1}{\theta} x^\top M M^\top x + \theta y^\top Q y \right).$$

Lemma 5. [31] If $Q \in \mathbb{R}^{n \times n}$ is such that $q_{ij} = q_{ji}$ and $q_{ii} = -\sum_{j=1, i \neq j}^n q_{ij}$, $i, j = 1, 2, \dots, n$, then

$$u^\top Q \otimes P v = \sum_{i=1}^n \sum_{j=1}^n u_i q_{ij} P v_j = - \sum_{j>i} q_{ij} (u_i - u_j) P (v_i - v_j),$$

for all matrices $P \in \mathbb{R}^{m \times m}$ and vectors $u = (u_1^\top, u_2^\top, \dots, u_n^\top)^\top$ and $v = (v_1^\top, v_2^\top, \dots, v_n^\top)^\top$, where $u_i = (u_{i1}, u_{i2}, \dots, u_{im})^\top$ and $v_i = (v_{i1}, v_{i2}, \dots, v_{im})^\top$.

Lemma 6. [30] Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$. If $\lambda(A)$ is an eigenvalue of A and $\lambda(B)$ is an eigenvalue of B , then $\lambda(A)\lambda(B)$ is an eigenvalue of $A \otimes B$.

Lemma 7. For matrices A, B, C and D with appropriate dimensions, one has

- 1) $(A \otimes B)^\top = A^\top \otimes B^\top$;
- 2) $(A + B) \otimes C = A \otimes C + B \otimes C$;
- 3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

2.3. Model description

Consider a complex network consisting of N identical nodes with nondelayed and time-varying delayed nonlinear coupling

$$\begin{aligned} \dot{x}_i(t) = & f(t, x_i(t), x_i(t - \tau_1(t))) + \sum_{j=1, i \neq j}^N c_1 a_{ij} \Gamma[g(x_j(t)) - g(x_i(t))] \\ & + \sum_{j=1, i \neq j}^N c_2 b_{ij} \Gamma[h(x_j(t - \tau_2(t))) - h(x_i(t - \tau_2(t)))], \quad i = 1, 2, \dots, N, \end{aligned} \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^\top \in \mathbb{R}^n$ is the state vector of the i th node of the network, $f(t, x_i(t), x_i(t - \tau_1(t))) = [f_1(t, x_i(t), x_i(t - \tau_1(t))), f_2(t, x_i(t), x_i(t - \tau_1(t))), \dots, f_n(t, x_i(t), x_i(t - \tau_1(t)))]^\top$ is a continuous vector-valued function, $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$ is an inner coupling of the network that satisfies $\gamma_j > 0$, $j = 1, 2, \dots, n$, $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ and $B = [b_{ij}] \in \mathbb{R}^{n \times n}$ are the outer coupling matrices of the network at time t and $t - \tau_2(t)$ respectively such that $a_{ij} \geq 0$ for $i \neq j$, $a_{ii} = -\sum_{j=1, j \neq i}^N a_{ij}$, $b_{ij} \geq 0$ for $i \neq j$ and $b_{ii} = -\sum_{j=1, j \neq i}^N b_{ij}$, $\tau_1(t)$ is the inner time-varying delay, $\tau_2(t)$ is the coupling time-varying delay and the nonlinear coupling functions $h(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous and of the form $g(x_i(t)) = (g_1(x_{i1}(t)), g_2(x_{i2}(t)), \dots, g_n(x_{in}(t)))^\top$ and $h(x_i(t)) = (h_1(x_{i1}(t - \tau_2(t))), h_2(x_{i2}(t - \tau_2(t))), \dots, h_n(x_{in}(t - \tau_2(t))))^\top$.

The initial conditions associated with (1) are

$$x_i(s) = \varphi_i(s), \quad -\tilde{\tau} \leq s \leq 0, \quad i = 1, 2, \dots, N,$$

where $\check{\tau} = \max\{\tau_1(t), \tau_2(t)\}$ and $\varphi_i \in \mathcal{C}([-\check{\tau}, 0], \mathbb{R}^n)$ with the norm $\|\varphi_i\|^2 = \sup_{-\check{\tau} \leq s \leq 0} \varphi_i(s)^\top \varphi_i(s)$ and our objective is to control system (1) so that it stays in the trajectory $s(t) \in \mathbb{R}^n$ of the system

$$\dot{s}(t) = f(t, s(t), s(t - \tau_1(t))) \quad (2)$$

by adding pinning controllers to some of the nodes. Without loss of generality, let the first l nodes be controlled. Then the network is described by

$$\begin{aligned} \dot{x}_i(t) = & f(t, x_i(t), x_i(t - \tau_1(t))) + \sum_{j=1, i \neq j}^N c_1 a_{ij} \Gamma(g(x_j(t)) - g(x_i(t))) \\ & + \sum_{j=1, i \neq j}^N c_2 b_{ij} \Gamma(h(x_j(t - \tau_2(t))) - h(x_i(t - \tau_2(t)))) + u_i(t), \quad i = 1, 2, \dots, N, \end{aligned} \quad (3)$$

where $u_i(t)$ ($i = 1, 2, \dots, N$) are the linear state feedback controllers that are defined by

$$u_i(t) = \begin{cases} -c_1 \varepsilon_i \Gamma(x_i(t) - s(t)), & i = 1, 2, \dots, l, \\ 0, & i = l + 1, l + 2, \dots, N, \end{cases} \quad (4)$$

for some control gains $\varepsilon_i > 0$ ($i = 1, 2, \dots, l$), denoted by $\Xi = \text{diag}\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l, 0, \dots, 0\} \in \mathbb{R}^{n \times n}$. Define $e_i(t) = x_i(t) - s(t)$ ($i = 1, 2, \dots, N$) as the synchronization error. Then, according to the controller (4), the error system is

$$\begin{aligned} \dot{e}_i(t) = & f(t, x_i(t), x_i(t - \tau_1(t))) - f(t, s(t), s(t - \tau_1(t))) + \sum_{j=1}^N a_{ij} \Gamma(h(x_j(t)) - h(s(t))) \\ & + \sum_{j=1}^N b_{ij} \Gamma(g(x_j(t - \tau_2(t))) - g(s(t - \tau_2(t)))) + u_i(t), \quad i = 1, 2, \dots, N. \end{aligned} \quad (5)$$

Definition 4. The complex network (3) is said to be *synchronized* if the trivial solution of system (5) is such that

$$\sum_{i=1}^N \lim_{t \rightarrow \infty} \|e_i(t, t_0, \varphi_i)\| = 0,$$

for any initial data $\varphi_i \in \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$.

Now, we introduce the following assumptions to study the synchronization of network (1).

Assumption 1. [17] For each function $f(\cdot)$ in network (1), there exists a positive definite diagonal matrix $P = \text{diag}\{p_1, p_2, \dots, p_n\}$, a diagonal matrix $\Delta = \text{diag}\{\delta_1, \delta_2, \dots, \delta_n\}$ and constants $\eta > 0, \theta > 0$ such that

$$(x - y)^\top P((f(t, x, z) - f(t, y, w)) - \Delta \Gamma(x - y)) \leq -\eta(x - y)^\top (x - y) + \zeta(z - w)^\top (z - w) \quad (6)$$

for some given matrix Γ and all $x, y, z, w \in \mathbb{R}^n$.

Remark 1. A continuous function f is said to belong to the function class QUAD, denoted by $f \in \text{QUAD}(P, \Delta, \eta, \zeta)$, if $f(\cdot)$ satisfies Assumption 1. One can easily verify that the function class QUAD includes almost all the chaotic systems such as the Lorenz system, the Rössler system, the Chen system, the delayed Chua's circuit, the logistic delayed differential system, the delayed Hopfield neural network and the delayed CNNs.

Assumption 2. [25] For each nonlinear coupling function $h_i(\cdot)$ and $g_i(\cdot)$ in network (1), there exist two nonnegative scalars $\beta_i^h, \varepsilon_i^h, \beta_i^g$ and ε_i^g such that $h_i(x) - \beta_i^h x$ and $g_i(x) - \beta_i^g x$ satisfies the Lipschitz condition

$$|h_i(x_1) - h_i(x_2) - \beta_i^h(x_1 - x_2)| \leq \varepsilon_i^h |x_1 - x_2|$$

and

$$|g_i(x_1) - g_i(x_2) - \beta_i^g(x_1 - x_2)| \leq \varepsilon_i^g |x_1 - x_2|,$$

respectively, for all $x_1, x_2 \in R, i = 1, 2, \dots, n$.

3. Main Results

In the section, we derive a sufficient condition for the attainment of synchronization by pinning control with the M-matrix approach.

Theorem 1. Under assumption 1 and 2, if $\tau_1(t)$ and $\tau_2(t)$ are bounded and continuously differentiable functions such that $\dot{\tau}_1(t) < \bar{\tau}_1 < 1$ and $\dot{\tau}_2(t) < \bar{\tau}_2 < 1$ and

$$\mathcal{J}(-c_1(A \otimes \Upsilon_g^\beta P\Gamma - \Xi \otimes P\Gamma)) > \rho, \quad (7)$$

where $\rho = \lambda_{\max}(-\eta\Psi \otimes I_n + \Psi \otimes P\Delta\Gamma + c_1\Psi AA^\top \Psi \otimes P\Gamma\Theta_g^{-1} + 2(1 - \frac{1}{N})I_N \otimes (\Upsilon_g^\varepsilon)^2 P\Gamma\Theta_g + c_2\Psi BB^\top \Psi \otimes P\Gamma\Theta_h^{-1} + \Psi \otimes Q + \Psi \otimes R)$, $\Upsilon_g^\varepsilon = \text{diag}\{\varepsilon_1^g, \dots, \varepsilon_n^g\}$, $\Upsilon_g^\beta = \text{diag}\{\beta_1^g, \dots, \beta_n^g\}$, $\Upsilon_h^\varepsilon = \text{diag}\{\varepsilon_1^h, \dots, \varepsilon_n^h\}$, $\Upsilon_h^\beta = \text{diag}\{\beta_1^h, \dots, \beta_n^h\}$, $Q = \frac{\zeta}{1-\bar{\tau}_1}I_n + \nu_1 I_n$, $R = 2c_2\lambda_{\max}(\Psi) \frac{N-1}{N(1-\bar{\tau}_2)}(\Upsilon_h^\varepsilon + \Upsilon_h^\beta)^2 P\Gamma\Theta_h + \nu_2 I_n$, Θ_h, Θ_g are positive definite matrices, Ψ is a positive definite diagonal matrix and ν_1 and ν_2 are positive scalars, then the pinning-controlled complex network (3) globally asymptotically synchronizes to the isolated node (2).

Proof. Let λ_i be an eigenvalue of $-c_1 A \otimes \Upsilon_g^\beta P\Gamma - \Xi \otimes P\Gamma$. Then the existence of a directed spanning tree in $\mathcal{G}$ ensures that $R(\lambda_i) > 0$ and the pinning condition (7) implies that $-c_1 A \otimes \Upsilon_g^\beta P\Gamma - \Xi \otimes P\Gamma$ is an M-matrix. Hence, there exists a positive definite diagonal matrix $\Psi = \text{diag}\{\xi_1, \dots, \xi_N\}$ such that

$$[\Psi(c_1(A \otimes \Upsilon_g^\beta P\Gamma - \Xi \otimes P\Gamma) - \rho I_{nN})]^S < 0.$$

Choose the Lyapunov-Krasovskii function as

$$V(t, e(t)) = \frac{1}{2}e(t)^\top \Psi \otimes P e(t) + \int_{t-\tau_1(t)}^t e(s)^\top \Psi \otimes Q e(s) ds + \int_{t-\tau_2(t)}^t e(s)^\top \Psi \otimes R e(s) ds,$$

where $e(t) = (e_1(t)^\top, e_2(t)^\top, \dots, e_N(t)^\top)^\top$ and let $e^k(t) = (e_{1k}(t), e_{2k}(t), \dots, e_{Nk}(t))^\top$, $\tilde{g}^k(x^k(t)) = (g_k(x_{1k}(t)), g_k(x_{2k}(t)), \dots, g_k(x_{Nk}(t)))^\top$, $\tilde{\epsilon}_g^k(x^k(t)) = (\epsilon_{g,k}(x_{1k}(t)), \epsilon_{g,k}(x_{2k}(t)), \dots, \epsilon_{g,k}(x_{Nk}(t)))^\top$ ($k = 1, 2, \dots, n$). Then differentiate $V(t)$ along system (5) to get

$$\begin{aligned} \dot{V}(t, e(t)) &= \sum_{i=1}^N \xi_i e_i(t)^\top P \left\{ f(t, x_i(t), x_i(t - \tau_1(t))) - f(t, s(t), s(t - \tau_1(t))) \right\} + \sum_{j=1}^N c_1 a_{ij} \Gamma g(x_j(t)) \\ &\quad + \sum_{j=1}^N c_2 b_{ij} \Gamma h(x_j(t - \tau_2(t))) - c_1 \varepsilon_i \Gamma e_i(t) \Big\} + e(t)^\top \Psi \otimes Q e(t) - (1 - \dot{\tau}_1) e(t - \tau_1(t))^\top \Psi \otimes Q e(t - \tau_1(t)) \\ &\quad + e(t)^\top \Psi \otimes R e(t) - (1 - \dot{\tau}_2) e(t - \tau_2(t))^\top \Psi \otimes R e(t - \tau_2(t)) \\ &= \sum_{i=1}^N \xi_i e_i(t)^\top P \left\{ f(t, x_i(t), x_i(t - \tau_1(t))) - f(t, s(t), s(t - \tau_1(t))) - \Delta \Gamma e_i(t) \right\} \\ &\quad + e(t)^\top \Psi \otimes P \Delta \Gamma e(t) + \sum_{i=1}^N \sum_{j=1}^N c_1 \xi_i a_{ij} e_i(t)^\top P \Gamma g(x_j(t)) + \sum_{i=1}^N \sum_{j=1}^N c_2 \xi_i b_{ij} e_i(t)^\top P \Gamma h(x_j(t - \tau_2(t))) \\ &\quad - c_1 e(t)^\top \Psi \Xi \otimes P \Gamma e(t) + e(t)^\top \Psi \otimes Q e(t) - (1 - \dot{\tau}_1) e(t - \tau_1(t))^\top \Psi \otimes Q e(t - \tau_1(t)) \\ &\quad + e(t)^\top \Psi \otimes R e(t) - (1 - \dot{\tau}_2) e(t - \tau_2(t))^\top \Psi \otimes R e(t - \tau_2(t)). \end{aligned} \quad (8)$$

As assumption 2, we have

$$\begin{aligned} &\sum_{i=1}^N \sum_{j=1}^N \xi_i e_i(t)^\top P a_{ij} \Gamma (g(x_j(t)) - g(s(t))) \\ &= \sum_{k=1}^n p_k \gamma_k e^k(t)^\top \Psi A [\tilde{g}^k(x^k(t)) - \tilde{g}^k(s^k(t))] \\ &= \sum_{k=1}^n \beta_k^g p_k \gamma_k e^k(t)^\top \Psi A e^k(t) + \sum_{k=1}^n p_k \gamma_k e^k(t)^\top \Psi A [\tilde{\epsilon}_g^k(x^k(t)) - \tilde{\epsilon}_g^k(s^k(t))]. \end{aligned} \quad (9)$$

Furthermore, since A is a zero-row-sum matrix and assumption 2 (by Lemma 4 and Lemma 5), we have

$$\sum_{k=1}^n p_k \gamma_k e^k(t)^\top \Psi A [\tilde{\epsilon}_g^k(x^k(t)) - \tilde{\epsilon}_g^k(s^k(t))]$$

$$\begin{aligned}
&\leq \sum_{k=1}^n \frac{p_k \gamma_k}{\theta_k} e^k(t)^\top \Psi A A^\top \Psi e^k(t) + \sum_{k=1}^n p_k \gamma_k \theta_k [\tilde{\epsilon}_k(x^k(t)) - \tilde{\epsilon}_k(s^k(t))]^\top Q [\tilde{\epsilon}_k(x^k(t)) - \tilde{\epsilon}_k(s^k(t))] \\
&\leq \sum_{k=1}^n \frac{p_k \gamma_k}{\theta_k} e^k(t)^\top \Psi A A^\top \Psi e^k(t) \\
&\quad - \sum_{k=1}^n p_k \gamma_k \theta_k \sum_{j>i} q_{ij} ((\gamma(x_{kj}(t)) - \gamma(s_k(t))) - (\gamma(x_{ki}(t)) - \gamma(s_k(t))))^2 \\
&\leq \sum_{k=1}^n \frac{p_k \gamma_k}{\theta_k} e^k(t)^\top \Psi A A^\top \Psi e^k(t) \\
&\quad - 2 \sum_{k=1}^n p_k \gamma_k \theta_k \sum_{j>i} q_{ij} ((\gamma(x_{kj}(t)) - \gamma(s_k(t)))^2 + (\gamma(x_{ki}(t)) - \gamma(s_k(t)))^2) \\
&\leq \sum_{k=1}^n \frac{p_k \gamma_k}{\theta_k} e^k(t)^\top \Psi A A^\top \Psi e^k(t) - 2\varepsilon^2 \sum_{k=1}^n p_k \gamma_k \theta_k \sum_{j>i} q_{ij} (e_{kj}(t)^2 + e_{ki}(t)^2) \\
&= \sum_{k=1}^n \frac{p_k \gamma_k}{\theta_k} e^k(t)^\top \Psi A A^\top \Psi e^k(t) + 2 \sum_{k=1}^n \varepsilon_k^2 p_k \gamma_k \theta_k (1 - \frac{1}{N}) e^k(t)^\top e^k(t). \tag{10}
\end{aligned}$$

It therefore follows from inequalities (9) and (10) that

$$\begin{aligned}
&\sum_{i=1}^N \sum_{j=1}^N e_i(t)^\top P a_{ij} \Gamma (g(x_j(t)) - g(s(t))) \\
&\leq e(t)^\top (\Psi A \otimes \Upsilon_g^\beta P \Gamma + \Psi A A^\top \Psi \otimes P \Gamma \Theta_g^{-1} + 2(1 - \frac{1}{N}) I_N \otimes (\Upsilon_g^\varepsilon)^2 P \Gamma \Theta_g) e(t). \tag{11}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{i=1}^N \sum_{j=1}^N e_i(t)^\top P b_{ij} \Gamma (h(x_j(t - \tau_2(t))) - h(s(t - \tau_2(t)))) \\
&\leq e(t)^\top (\Psi B B^\top \Psi \otimes P \Gamma \Theta_h^{-1}) e(t) + e(t - \tau_2(t))^\top 2(1 - \frac{1}{N}) I_N \otimes (\Upsilon_h^\varepsilon + \Upsilon_h^\beta)^2 P \Gamma \Theta_h e(t - \tau_2(t)) \tag{12}
\end{aligned}$$

and after substituting inequalities (11-12) into equation (8), we have

$$\begin{aligned}
\dot{V}(t) &\leq e(t)^\top \left(-\eta \Psi \otimes I_n + c_1 \Psi A \otimes \Upsilon_g^\beta P \Gamma - c_1 \Psi \Xi \otimes P \Gamma + \Psi \otimes P \Delta \Gamma + c_1 \Psi A A^\top \Psi \otimes P \Gamma \Theta_g^{-1} \right. \\
&\quad \left. + 2(1 - \frac{1}{N}) I_N \otimes (\Upsilon_g^\varepsilon)^2 P \Gamma \Theta_g + c_2 \Psi B B^\top \Psi \otimes P \Gamma \Theta_h^{-1} + \Psi \otimes Q + \Psi \otimes R \right) e(t) \\
&\quad + e(t - \tau_1(t))^\top (\zeta \Psi \otimes I_n - (1 - \bar{\tau}_1) \Psi \otimes Q) e(t - \tau_1(t)) \\
&\quad + e(t - \tau_2(t))^\top (2c_2(1 - \frac{1}{N}) I_N \otimes (\Upsilon_h^\varepsilon + \Upsilon_h^\beta)^2 P \Gamma \Theta_h - (1 - \bar{\tau}_2) \Psi \otimes R) e(t - \tau_2(t)). \tag{13}
\end{aligned}$$

Let $y(t) = (e(t)^\top, e(t - \tau_1(t))^\top, e(t - \tau_2(t))^\top)^\top$ and recall that $Q = \frac{\zeta}{1 - \bar{\tau}_1} I_n + \nu_1 I_n$ and $R = 2c_2 \lambda_{\max}(\Psi) \frac{N-1}{N(1 - \bar{\tau}_2)} (\Upsilon_h^\varepsilon + \Upsilon_h^\beta)^2 P \Gamma \Theta_h + \nu_2 I_n$. Then inequality (13) implies that $\dot{V}(t) \leq 0$ and $\dot{V}(t) = 0$ if and only if $y(t) = 0_{3Nn}$. So the set $M = \{y(t) : y(t) = 0_{3Nn}\}$ is the largest invariant set contained in the set $D = \{y(t) | V(t) = 0\}$ for system (5) and thus every solution of system (5) (starting from any initial condition) approaches M as $t \rightarrow \infty$ (according to LaSalle's invariance principle). Therefore $\|e_i(t)\| \rightarrow 0, i = 1, 2, \dots, N$ and the pinning-controlled network (3) globally asymptotically synchronizes to the isolated node (1). $\square$

Remark 2. Even though the network (3) can be synchronized as long as the coupling strength satisfies $\mathcal{J}(-c_1(A \otimes \Upsilon_g^\beta P \Gamma - \Xi \otimes P \Gamma)) > \rho$ (see condition (7)), the theoretical value required for the coupling strength is usually much larger than that needed in practice. We shall next use an adaptive approach to find a sharp bound for this coupling strength.

To make Theorem 1 more applicative, we give the following corollaries.

When $h_i(x) = x$ and $g_i(x) = x$ are linear functions, system (3) is a linearly-coupled system and we have the following result.

Corollary 1. Under assumption 1, if $\tau_1(t)$ and $\tau_2(t)$ are bounded and continuously differentiable functions such that $\dot{\tau}_1(t) < \bar{\tau}_1 < 1$ and $\dot{\tau}_2(t) < \bar{\tau}_2 < 1$ and

$$\mathcal{J}(-c_1(A \otimes P\Gamma - \Xi \otimes P\Gamma)) > \rho,$$

where $\rho = \lambda_{\max}(-\eta\Psi \otimes I_n + \Psi \otimes P\Delta\Gamma + c_2\Psi BB^\top\Psi \otimes P\Gamma\Theta_h^{-1} + \Psi \otimes Q + \Psi \otimes R)$, $Q = \frac{\zeta}{1-\bar{\tau}_1}I_n + \nu_1 I_n$, $R = 2c_2\lambda_{\max}(\Psi)\frac{N-1}{N(1-\bar{\tau}_2)}P\Gamma\Theta_h + \nu_2 I_n$, Θ_h is a positive definite matrices, Ψ is a positive definite diagonal matrix and ν_1 and ν_2 are positive scales, then the pinning-controlled complex network (3) globally asymptotically synchronizes to the isolated node (2).

When $h_i(x) = x$ is a linear function and $t_1(t) = \tau$ and $c_2 = 0$ (without time-delay coupling), then system (3) is a linearly-coupled system without time-delay coupling and we have the following result.

Corollary 2. Under assumption 1, if

$$\mathcal{J}(-c_1(A \otimes P\Gamma - \Xi \otimes P\Gamma)) > \rho,$$

where $\rho = \lambda_{\max}(-\eta\Psi \otimes I_n + \Psi \otimes P\Delta\Gamma + \Psi \otimes Q)$, $Q = \zeta I_n + \nu_1 I_n$, Θ_h is a positive definite matrices, Ψ is a positive definite diagonal matrix and ν_1 and ν_2 are positive scalars, then the pinning-controlled complex network (3) globally asymptotically synchronizes to the isolated node (2)

Remark 3. Corollary 2 gives a sufficient condition for the synchronization of linearly coupled complex networks using pinning control that is the same as that obtained by Q. Song *et. al.* (Theorem 2 in Ref. [27]). We can therefore say that Theorem 1 is a nontrivial extension of the result obtained in [27].

Real-world complex networks have coupling strengths that are usually less than their theoretical values [19]. The following theorem, however, allows us to accommodate for relatively small coupling strengths by using an adaptive feedback control technique.

Let the coupling strengths of the network functions be time-varying (we shall now design the corresponding coupling strength adaptive laws). System (3) can be written as

$$\begin{aligned} \dot{x}_i(t) = & f(t, x_i(t), x_i(t - \tau_1(t))) + \sum_{j=1, i \neq j}^N c_1(t) a_{ij} \Gamma(g(x_j(t)) - g(x_i(t))) \\ & + \sum_{j=1, i \neq j}^N c_2 b_{ij} \Gamma(h(x_j(t - \tau_2(t))) - h(x_i(t - \tau_2(t)))) + u_i(t) \quad i = 1, 2, \dots, N, \end{aligned} \quad (14)$$

where $c_1(t)$ is an adaptive coupling strength that can be suitably chosen to synchronize system (14). Design a local feedback controller

$$u_i(t) = \begin{cases} -c_1(t) \varepsilon_i \Gamma(x_i(t) - s(t)), & i = 1, 2, \dots, l, \\ 0, & i = l+1, l+2, \dots, N. \end{cases}$$

Theorem 2. Under assumption 1 and 2, if $\tau_1(t)$ and $\tau_2(t)$ are bounded and continuously differentiable functions such that $\dot{\tau}_1(t) < \bar{\tau}_1 < 1$ and $\dot{\tau}_2(t) < \bar{\tau}_2 < 1$. Let $c_1(t)$ be an adaptive coupling strength such that

$$\dot{c}_1(t) = \alpha \sum_{i=1}^N \xi_i e_i(t)^\top P e_i(t) = \alpha e(t)^\top \Psi \otimes P e(t)$$

for some $\alpha > 0$ and initial value $c_1(0) \geq 0$ and Ψ is a positive definite diagonal matrix. Then system (14) can be synchronized.

Proof. Choose a constant $\alpha > 0$ and let $e_i(t) = x_i(t) - s(t)$. Construct the following Lyapunov-Krasovskii function

$$V(t) = \frac{1}{2} e(t)^\top \Psi \otimes P e(t) + \int_{t-\tau_1(t)}^t e(s)^\top \Psi \otimes Q e(s) ds + \int_{t-\tau_2(t)}^t e(s)^\top \Psi \otimes R e(s) ds + \frac{1}{2\alpha\kappa} (c - \kappa c_1(t))^2$$

for some constants c and κ that will be determined later. Differentiating $V(t)$ gives

$$\frac{dV(t)}{dt} = \sum_{i=1}^N \xi_i e_i(t)^\top P \left\{ f(t, x_i(t), x_i(t - \tau_1(t))) - f(t, s(t), s(t - \tau_1(t))) \right\} + \sum_{j=1}^N c_1(t) a_{ij} \Gamma g(x_j(t))$$

$$\begin{aligned}
& + \sum_{j=1}^N c_2 b_{ij} \Gamma h(x_j(t - \tau_2(t))) - c_1 \varepsilon_i \Gamma e_i(t) \Big\} + e(t)^\top \Psi \otimes Q e(t) - (1 - \dot{\tau}_1) e(t - \tau_1(t))^\top \Psi \otimes Q e(t - \tau_1(t)) \\
& + e(t)^\top \Psi \otimes R e(t) - (1 - \dot{\tau}_2) e(t - \tau_2(t))^\top \Psi \otimes R e(t - \tau_2(t)) - c e(t)^\top \Psi \otimes P \Gamma e(t) + \kappa c_1(t) e(t)^\top \Psi \otimes P \Gamma e(t)
\end{aligned}$$

and so, by Theorem 1, we have

$$\begin{aligned}
\dot{V}(t) \leq & e(t)^\top \left(c_1(t) \Psi A \otimes \Upsilon_g^\beta P \Gamma - c_1(t) \Psi \Xi \otimes P \Gamma + \kappa c_1(t) \Psi \otimes P \Gamma + c_1(t) \Psi A A^\top \Psi \otimes P \Gamma \Theta_g^{-1} + 2(1 - \frac{1}{N}) I_N \otimes (\Upsilon_g^\varepsilon)^2 P \Gamma \Theta_g \right. \\
& - \eta \Psi \otimes I_n - c \Psi \otimes P \Gamma + \Psi \otimes P \Delta \Gamma + c_2 \Psi B B^\top \Psi \otimes P \Gamma \Theta_h^{-1} + \Psi \otimes Q + \Psi \otimes R \Big) e(t) \\
& + e(t - \tau_1(t))^\top (\zeta \Psi \otimes I_n - (1 - \bar{\tau}_1) \Psi \otimes Q) e(t - \tau_1(t)) \\
& + e(t - \tau_2(t))^\top (2c_2(1 - \frac{1}{N}) I_N \otimes (\Upsilon_h^\varepsilon + \Upsilon_h^\beta)^2 P \Gamma \Theta_h - (1 - \bar{\tau}_2) \Psi \otimes R) e(t - \tau_2(t)).
\end{aligned}$$

Hence, by choosing κ and c such that $\Psi A \otimes \Upsilon_g^\beta P \Gamma - \Psi \Xi \otimes P \Gamma + \kappa \Psi \otimes P \Gamma + \Psi A A^\top \Psi \otimes P \Gamma \Theta_g^{-1} \leq 0$ and $2(1 - \frac{1}{N}) I_N \otimes (\Upsilon_g^\varepsilon)^2 P \Gamma \Theta_g - \eta \Psi \otimes I_n - c \Psi \otimes P \Gamma + \Psi \otimes P \Delta \Gamma + c_2 \Psi B B^\top \Psi \otimes P \Gamma \Theta_h^{-1} + \Psi \otimes Q + \Psi \otimes R \leq 0$, where $Q = \frac{\zeta}{1 - \bar{\tau}_1} I_n + \nu_1 I_n$, $R = 2c_2 \lambda_{\max}(\Psi) \frac{N-1}{N(1-\bar{\tau}_2)} (\Upsilon_h^\varepsilon + \Upsilon_h^\beta)^2 P \Gamma \Theta_h + \nu_2 I_n$, and ν_1 and ν_2 are positive scalars, we have

$$\frac{dV(t)}{dt} \leq -\eta \sum_{i=1}^m e_i(t)^\top e_i(t) < 0$$

and so $x_i(t) \rightarrow s(t)$ and $\dot{c}(t) \rightarrow 0$ (by LaSalle's invariance principle). Therefore $c(t)$ converges to some final coupling strength c_0 . $\square$

4. Numerical Simulation

In this section, we present some numerical simulation results that validate the theorems of the previous section.

Consider the isolated 2D chaotic delayed neural network

$$\begin{bmatrix} \dot{s}_1(t) \\ \dot{s}_2(t) \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} + \begin{bmatrix} 2 & -0.1 \\ -5 & 4.5 \end{bmatrix} \begin{bmatrix} \tanh(s_1(t)) \\ \tanh(s_2(t)) \end{bmatrix} + \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix} \begin{bmatrix} \tanh(s_1(t - \tau_1(t))) \\ \tanh(s_2(t - \tau_1(t))) \end{bmatrix},$$

where $\tau_1(t) = 1$. Taking $P = \text{diag}\{1, 1\}$ and $\Delta = \text{diag}\{5, 11.5\}$, we have $\eta = 0.15$ and $\zeta = 3.25$ so that condition (6) is satisfied.

Now consider a network with 50 nodes with a topological structure that is shown in Figure 1

$$\dot{x}_i(t) = f(t, x_i(t), x_i(t - \tau_1(t))) + c_1 \sum_{j=1}^{50} a_{ij} \Gamma g(x_j(t)) + c_2 \sum_{j=1}^{50} b_{ij} \Gamma h(x_j(t - \tau_2(t))) - c_1 \varepsilon_i \Gamma (x_i(t) - s(t)), \quad i = 1, 2, \dots, 50, \quad (15)$$

where $\tau_2(t) = 0.1 \frac{e^t}{1+e^t}$, $g_j(x) = 25x + \sin(x)$ and $h_j(x) = 5x + \cos(x)$ for $j = 1, 2$, $\Gamma = I_2$.

Computations then yield $\bar{\tau}_1 = 0$ and $\bar{\tau}_2 = 0.1$ for $i = 1, 2, \dots, 50$ and $j = 1, 2$, $\Upsilon_g^\beta = 25I_2$, $\Upsilon_h^\beta = 5I_2$, $\Upsilon_g^\varepsilon = I_2$, $\Upsilon_h^\varepsilon = I_2$. Let $l = 7$ and the control strength $\varepsilon_i = 100$ for $i = 1, 2, \dots, 7$. Then the solutions of inequalities (7) are (by using the Matlab LMI toolbox): $c_1 = 10$, $c_2 = 0.01$, $\rho = 180.3202$, $\Theta_h = 1.9082I_2$, $\Theta_g = 41.9134I_2$, $Q = 3.251I_2$, $R = 1.497$, $\nu_1 = 0.001$, $\nu_2 = 0.001$ and $\mathcal{J} = 183.1517$.

The initial conditions for this simulation are $x_{ij}(t_0) = \text{rand}_{ij} \times \cos(t_0)$ for some random integer rand_{ij} in $[0, 100]$ for $i = 1, 2, \dots, 50$, $j = 1, 2$ and $s(t_0) = [30 \cos(t_0), 2 \cos(t_0)]^\top$ for all $t_0 \in [-1, 0]$ and the trajectories of the states are shown in Figure 2. Figure 3 shows the time evolution of the synchronization errors with pinning control. Measuring the quality of the synchronization process by

$$E(t) = \frac{1}{50} \sum_{i=1}^{50} \|x_i(t) - s(t)\|^2$$

in Figure 4, it is clear that synchronization is achieved.

The above simulations show that large coupling and control strengths (c_1 and ε_i) are needed to meet condition (7). Now use the same parameters with $\alpha = 0.0001$ to simulate the controlled dynamic network (14) with adaptive coupling strength $c_1(t)$. The trajectories of the states are shown in Figure 5 and Figure 6 shows the time evolution of the synchronization errors with pinning control. Figure 7 and Figure 8 show the time evolution of $E(t)$ and $c_1(t)$ (note that $c_1(t)$ approaches 28.169) for the pinned complex network by adaptive control. It is clear that synchronization is achieved. (Note that the final coupling strength is smaller than its theoretical value.)

Figure 1: The topological structures of the undirected network with 50 nodes. We assume that $a_{ij} = a_{ji} = 1$ if there exists an edge between node i and node j and $a_{ij} = a_{ji} = 0$ otherwise in (a); $b_{ij} = b_{ji} = 1$ if there exists an edge between node i and node j and $b_{ij} = b_{ji} = 0$ otherwise in (b).

Figure 2: The trajectories of the state variables of x_{i1} (a) and x_{i2} (b) for $i = 1, 2, \dots, 50$ in system (15) using pinning control.

Figure 3: The time-evolution of e_{i1} (a) and e_{i2} (b) for $i = 1, 2, \dots, 50$ in system (15) using pinning control.

Figure 4: The synchronization process in system (15) using pinning control.

Figure 5: The trajectories of the state variables of x_{i1} (a) and x_{i2} (b) for $i = 1, 2, \dots, 50$ in system (15) using adaptive and pinning control.

Figure 6: The time-evolution of e_{i1} (a) and e_{i2} (b) for $i = 1, 2, \dots, 50$ in system (15) using adaptive and pinning control.

Figure 7: The synchronization process of system (15) using adaptive and pinning control.

Figure 8: The adaptive coupling strength $c_1(t)$ of the pinning-controlled complex network

5. Conclusion

In this paper, we investigated the synchronization problem for complex networks with non-delayed and time-varying delayed couplings. Specifically, we achieved global synchronization by applying a pinning control scheme to a small fraction of the nodes using the M-matrix strategies. Noting the big difference in the coupling strength magnitudes between theoretical and real-world systems, we also rigorously proved an adaptive feedback control technique that can be used to synchronize any real-world network. Finally, we considered some numerical examples that illustrate the theoretical analysis.

Acknowledgment

This work was supported by the National Science Foundation of China under Grant No.61273220, 61373087, 61472257 and 61401283. The authors are grateful to the reviewers and editors for their valuable comments and suggestions to improve the presentation of this paper.

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