Time and Nash Implementation

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In this paper, we study the full implementation problem using mechanisms that allow a delay. The delay on the equilibrium path may be zero, an infinitesimally small number or a fixed positive number. In all these three cases, implementable rules are fully characterized by a monotonicity condition. We provide examples to show that some delayed implementable social choice rules are not implementable in Nash-equilibrium refinements without a delay. As an application of our approach, we characterize delayed implementable rules in environments where only the discounting changes between states.

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1 Introduction

In this paper, we develop a general theory of mechanisms that use delay in the context of full implementation in Nash equilibrium. Many real-life mechanisms explicitly allow for a delay; some of the most prominent examples are the rule permitting filibusters in the United States Senate and suspensive veto power in the British parliament.¹

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¹Filibuster refers to a procedure that allows senators to make speeches and amendments indefinitely, thus delaying a vote. Suspensive veto power allows the House of Lords to delay the enactment of a law by one year. Mayhew (2003) argue that filibuster can be explained as a tool to test resoluteness of the parties. Wawro and Schickler (2006) elaborate on this argument.

1 We assume that delay is undesirable and is limited by a maximum allowable delay T .
2 We study three distinct restrictions for a delay Δ to occur on the equilibrium path: (i)
3 $\Delta = 0$; (ii) Δ is arbitrarily small, and (iii) Δ is a fixed number between zero and T .

4 Under restriction (i), $\Delta = 0$, delay can occur off the equilibrium path; this is the
5 most restrictive case for delayed implementation. For this case, we construct a novel
6 canonical mechanism that functions without relying on either no-veto-power conditions,
7 or restrictions on the environment beyond those implied by an added time dimension.

8 We call implementation under restriction (ii) *imminent implementation*. Restriction
9 (ii) allows us to deal with indifference in agents' preferences. In particular, we show
10 in Examples 1 and 2 that some SCRs are imminently implementable, but not imple-
11 mentable in a Nash equilibrium or its refinements.

12 Under restriction (iii), delayed implementation with $\Delta > 0$ allows us to exploit pref-
13 erence reversals that happen only “at a later time,” that is, around a social outcome
14 $(a, t + \Delta)$, but not around (a, t) . This is the most permissive case. As an illustration,
15 we consider an environment where only time preferences change and show, in Section 5,
16 that if there are “sufficiently many” outcomes, even a small delay would allow any SCR
17 to be Δ -delayed implementable.

18 Under the restriction that there is no delay on the equilibrium path, $\Delta = 0$, our
19 problem is similar to the problem studied by Bochet (2007), Benoit and Ok (2008), and
20 Sanver (2006).

21 Bochet (2007) and Benoit and Ok (Section 3, 2008) augment their environments by
22 lotteries instead of time. Both papers impose additional conditions on the environment
23 (strict preferences and top coincidence, respectively), that are not required in this paper.
24 The canonical mechanisms constructed in these two papers also differ in an essential way
25 from the mechanism constructed here, which exploits the undesirability of delay.

26 Benoit and Ok (Section 4, 2008) and Sanver (2006) augment an environment by
27 allowing the designer to give a payment (an award) to an agent off the equilibrium
28 path. There are two important differences between an award and a delay. First, an
29 award affects a single agent, while delay affects all agents. Separability has been shown
30 to be important in other implementation settings (Kunimoto and Serrano 2011). Second,
31 no-veto-power is vacuously satisfied in an environment with awards. As this is not the
32 case under our restriction (i), our mechanism that implements an SCR with zero delay
33 needs to account for this condition.

34 The necessary and sufficient condition derived by Sanver (2006) for implementation
35 by awards is weaker than our condition for zero-delayed implementation (see Example

1 in this paper). When an infinitesimally small delay is allowed on the equilibrium path (our restriction (ii)), our condition becomes very similar to Sanver's.²

In allowing to approximate an SCR, imminent implementation is the most similar to virtual implementation. Abreu and Sen (1991) and Matsushima (1988) show that any SCR can be virtually implemented; the results of this paper are not nearly as permissive. A mechanism that virtually implements an SCR delivers the social outcome with an arbitrarily high probability, but it relies critically on an occasional delivery of an incorrect outcome. This may be a serious practical drawback; arguments against schemes that deliver socially suboptimal outcomes ex-post are abundant in legal scholarly papers (Fried 2003). To give a particularly striking example, a virtual mechanism for King Solomon's problem of allocating a child to a true mother assigns positive probability to killing the baby *even if* King Solomon successfully determines who the true mother is (Serrano 2004). A mechanism that imminently implements the King Solomon problem relies on King Solomon having custody over the baby over a short period of time (Artemov 2006).³ We think that the latter inefficiency is more tolerable for society than the former.

In the next section, we define Δ -delayed implementation. In Section 3 we construct a canonical mechanism and provide the characterization of delayed implementable SCRs. We collect examples in Section 4. Section 5 contains an application of our characterization to the environment in which only time preferences change. We conclude in Section 6.

2 Preliminaries

There is a finite set $N = \{1, \dots, n\}$ of agents and a known set A of *physical* outcomes. Let the set of outcomes $\mathcal{A} = A \times [0, T]$ be the set of physical outcomes augmented by time. The value $T > 0$ is an upper bound on a delay. If an arbitrary delay is allowed, then $\mathcal{A} = A \times \mathbb{R}_+$. We interpret an element $(a, t) \in \mathcal{A}$ as a physical outcome a delivered at time t .

Let Θ be the finite set of possible states. Associated with each $\theta \in \Theta$, there is a preference profile $\succeq^\theta = (\succeq_1^\theta, \dots, \succeq_n^\theta)$, where $\succeq_i^\theta$ represents agent i 's preference ordering over $\mathcal{A}$. The symbols $\succ_i^\theta$ and $\sim_i^\theta$ represent a strict preference relation and indifference,

²To be more precise, if this paper were to address Sanver's question and require implementation for any possible specification of preferences over time, the conditions would have been identical.

³Sanver (2004) uses the King Solomon problem to point out that adding outcomes to the environment may aid monotonicity.

respectively. We assume throughout the paper that preferences are complete and transitive. Let $A \subseteq \Theta$ and $B \subseteq N$; then $(a, t) \succ_B^A (b, t')$ means that for every state $\theta \in A$ and every agent $i \in B$, $(a, t) \succ_i^\theta (b, t')$. The notation $\succeq_B^A$ and $\sim_B^A$ is defined analogously.

We make two assumptions on time preferences: (1) the strict undesirability of delay and (2) continuity. Formally, we assume that for any $\theta \in \Theta$ and any $i \in N$, the following conditions hold: (1) for any $(a, t) \in \mathcal{A}$, and any $t' > t$, $(a, t) \succ_i^\theta (a, t')$; and (2) the upper contour set $\{(b, t') \in \mathcal{A} | \forall (a, t) \in \mathcal{A}, (b, t') \succeq_i^\theta (a, t)\}$ and the lower contour set $\{(b, t') \in \mathcal{A} | \forall (a, t) \in \mathcal{A}, (a, t) \succeq_i^\theta (b, t')\}$ are closed.

A *social choice rule* (SCR) $F : \Theta \rightarrow 2^{\mathcal{A}} \setminus \emptyset$ is a mapping from the set of states Θ to a subset of outcomes in $\mathcal{A}$.

Definition 1. An SCR F^Δ is Δ -*delayed* with respect to an SCR F if, for any θ , the following two conditions hold:

1. For all $(a, t) \in F(\theta)$, there exists $t' \in [t, t + \Delta] \cap [0, T]$ such that $(a, t') \in F^\Delta(\theta)$ and
2. For all $(a, t') \in F^\Delta(\theta)$, there exists $t \in [t' - \Delta, t'] \cap [0, T]$ such that $(a, t) \in F(\theta)$.

That is, a Δ -delayed SCR delivers the same physical outcome with a delay of no more than Δ relative to the original delay. We assume throughout the paper that $\Delta \geq 0$.

A *mechanism* (or a game form) is a pair (M, g) , consisting of an arbitrary message space $M = \prod_{j \in N} M_j$ and an outcome function $g : M \rightarrow \mathcal{A}$. Let $\mathcal{N}(M, g, \theta)$ be the set of pure-strategy Nash equilibria of a game induced by (M, g) in state θ . An SCR F is implemented by a mechanism (M, g) if $g(\mathcal{N}(M, g, \theta)) = F(\theta)$ for all $\theta \in \Theta$. If there exists a mechanism that implements a given SCR, this rule is called implementable.

Definition 2. An SCR F is Δ -delayed-implementable if there exists an implementable Δ -delayed SCR F^Δ .

Note that if an SCR F is Δ_1 -delayed-implementable and $\Delta_1 < \Delta_2$, then F is Δ_2 -delayed-implementable. Hence, the most stringent case is when $\Delta = 0$, where this definition coincides with the standard definition of a Nash-implementable SCR, applied to $\mathcal{A}$. The case in which an SCR can be implemented for any $\Delta > 0$ is of special interest here; we introduce the term *imminently implementable* for this case.

Definition 3. An SCR F is *imminently implementable* if, for any $\Delta > 0$, F is Δ -delayed-implementable.

3 Characterization of Δ -delayed Implementation

In this section we establish the condition, which we call Δ -delayed monotonicity, which is necessary and sufficient for Δ -delayed implementation if $N \geq 3$.

Definition 4. F is Δ -delayed-monotonic if, for every $\theta, \phi \in \Theta$ such that $(a, t) \in F(\theta)$ and $(a, t) \notin F(\phi)$, either of the following two conditions holds:

1. There exists $t' \in [t - \Delta, t + \Delta]$ such that $(a, t') \in F(\phi)$.
2. There exist $i \in N$, $(b, \bar{t}) \in \mathcal{A}$ and $t' \in [t, t + \Delta] \cap [0, T]$, such that

$$(a, t') \succeq_i^\theta (b, \bar{t}), (b, \bar{t}) \succ_i^\phi (a, t'). \quad (1)$$

The first part of Definition 4 takes care of the case in which the physical outcome is the same both in states θ and ϕ , but the delay in ϕ is longer than in θ by at most Δ . The second part is the key condition, which requires the existence of an agent with a preference reversal between states θ and ϕ .

If $\Delta = 0$, this definition becomes the standard Maskin monotonicity condition, applied to $\mathcal{A}$ (note that part 1 is redundant).

For imminent implementability, an SCR needs to be Δ -delayed-monotonic for any $\Delta > 0$; we call such a condition *imminent monotonicity*. Then, condition 2 implies that either (i) $(a, t) \succeq_i^\theta (b, \bar{t}), (b, \bar{t}) \succ_i^\phi (a, t)$ (as in standard Maskin monotonicity applied to $\mathcal{A}$), or (ii) $(a, t) \succ_i^\theta (b, \bar{t}), (b, \bar{t}) \succeq_i^\phi (a, t)$.

Next, we establish the necessity of Δ -delayed monotonicity for Δ -delayed implementability.

Theorem 1. *If F is Δ -delayed-implementable, it is Δ -delayed-monotonic.*

It is important to note that Δ -delayed monotonicity is a sufficient condition for Δ -delayed implementability. When $\Delta > 0$, monotonicity is sufficient because the additional condition, “no veto power,” is trivially satisfied for a delayed SCR F^Δ . Hence, we provide a proof below for the case in which $\Delta = 0$. This proof, applied to F^Δ rather than F , implies sufficiency for $\Delta > 0$.

Theorem 2. *Suppose that $N \geq 3$ and an SCR F satisfies zero-delayed monotonicity. Then F is zero-delayed implementable.*

We first construct a canonical mechanism, which is used later in the proof of the sufficiency result.

For each player i , the strategy space is defined as $S_i = \Theta \times \mathcal{A} \times \{0, 1\}$, with a generic strategy for player i written as $s_i = (\theta_i, (a_i, t_i), \mu_i)$; $s \in \prod_{i \in N} S_i$ is a strategy profile.

Fix an arbitrary outcome $(\tilde{a}, T)$. The outcome function is defined as follows:

1. If, for all $i \in N$, $s_i = (\theta, (a, t), 0)$ with $(a, t) \in F(\theta)$, then

$$g(s) = (a, t).$$

2. If there exists agent k such that, for all $i, j \neq k$, $s_i = s_j = (\theta, (a, t), 0)$ with $(a, t) \in F(\theta)$, and $s_k \neq s_i$, then:

$$g(s) = \begin{cases} (a, T) & \text{if } (a_k, t_k) \succ_k^\theta (a, t) \text{ or if } t_k = 0; \\ (a_k, t_k) & \text{otherwise.} \end{cases} \quad (2a)$$

$$(2b)$$

3. If there is at least one agent k with $\mu_k = 1$ and case 2 does not apply, then use the following procedure to determine the outcome:

- a. In profile s , pick the agent k such that t_k is the largest among all agents such that $\{i \in N | \mu_i = 1 \text{ and } t_i < T\}$. If there are several such agents, or if all agents who announce $\mu_i = 1$ also announce $t_i = T$, pick the one with the lowest index. Denote this agent by k^* .

- b. Construct a profile of delays $\tilde{t}$ such that $\tilde{t}_i = T$ if $(a_i, t_i) = (a_i, 0) \notin F(\theta_i)$ and $\tilde{t}_i = t_i$ otherwise.

- c. Calculate the average delay t^* in profile $\tilde{t}$ for all agents who report $\mu_i = 1$ in profile s^* (formally, $t^* = \frac{1}{\#\{i \in N | \mu_i = 1\}} \sum_{\{i \in N | \mu_i = 1\}} \tilde{t}_i$) and define

$$g(s) = (a_{k^*}, t^*).$$

4. If all agents report $\mu_i = 0$ and cases 1 and 2 do not apply, $g(s) = (\tilde{a}, T)$.

Proof of Theorem 2. Let the true state be θ .

We first show that a message profile s^* such that for every $i \in N$, $s_i^* = (\theta, (a, t), 0)$, where $(a, t) \in F(\theta)$, is a Nash equilibrium. Suppose that there is an agent $j \in N$, who deviates from s_j^* . Then, rule 2 applies, which prescribes an outcome that is weakly worse than (a, t) for j in state θ . Hence, a deviation in the true state is not profitable.

Next, we show that if there are any other Nash equilibria, their outcomes are socially optimal.

Case 1: Suppose that a message profile s^* is such that $s_i^* = (\phi, (a, t), 0)$ for all $i \in N$ and $(a, t) \in F(\phi), (a, t) \notin F(\theta)$. By zero-delayed monotonicity, there exist $k \in N$ and $(a_k, t_k) \in \mathcal{A}$, such that $(a, t) \succeq_k^\phi (a_k, t_k), (a_k, t_k) \succ_k^\theta (a, t)$; moreover, by continuity, t_k can be chosen so that $t_k > 0$. Then $((\theta, (a_k, t_k), 1), s_{-k}^*)$ invokes rule 2b, with an outcome (a_k, t_k) : a profitable deviation for k .

Case 2a: Suppose that a message profile s^* results in (a, T) under rule 2a, and $s_i^* = (\phi, (a, t), 0)$ for all $i \neq k$.

Let $t_j \in (t_k, T)$ if $t_k < T$ and $t_j = 0$ if $t_k = T$. Consider an announcement $s_j = (\phi, (a, t_j), 1)$ of player $j \neq k$. Note that rule 3 applies because there are three different announcements and $\mu_j = 1$. Note also that $\mu_i = 0$ for all $i \neq j, k$ (μ_k can be either 0 or 1). Hence, by the choice of t_j , j determines a physical outcome, which is a , and the delay is either $(t_j + t_k)/2$ or t_j . Either is less than T , so this is a profitable deviation for j .

Case 2b: Suppose that a message profile s^* results in (a_k, t_k) under rule 2b, and $s_i^* = (\phi, (a, t), 0)$ for all $i \neq k$. Then, if agent $j \neq k$ changes her strategy to $s_j = (\phi, (a_k, 0), 1)$, an outcome is either $(a_k, t_k/2)$ if $\mu_k = 1$ or $(a_k, 0)$ if $\mu_k = 0$. Either outcome is preferred by agent j , because $t_k > 0$ under rule 2b; hence j has a profitable deviation.

Case 3: Suppose that s^* is an equilibrium under rule 3 with an outcome (a^*, t^*) .

First, we make two observations.

Observation 1: Suppose that there is at least one agent, k , such that $\mu_k = 1$ and $\tilde{t}_k > 0$ (hence $t^* > 0$). Then, if s^* is an equilibrium, $\mu_i = 1$ for all $i \in N$. If not, agent j such that $\mu_j = 0$ has a profitable deviation to $s_j = (\theta, (a^*, 0), 1)$. The profile (s_j, s_{-j}^*) results in $(a^*, \frac{\#\{i \in N | \mu_i = 1\}}{\#\{i \in N | \mu_i = 1\} + 1} t^*) \succ_j^\theta (a^*, t^*)$; the physical outcome does not change because agent j , with $\mu_j = 0$, does not determine an outcome in s^* .

Observation 2: Suppose that there are two agents, k and j , with $\tilde{t}_k > \tilde{t}_j > 0$ (or $\tilde{t}_k = \tilde{t}_j > 0$ and $k < j$, so that agent j does not determine physical outcome). As rule 3 applies, there is at least one agent with $\mu_i = 1$; by observation 1, it implies that $\mu_i = 1$ for all $i \in N$. Then agent j has a profitable deviation to $(\theta, (a^*, 0), 1)$ because such a deviation results in outcome $(a^*, t^* - \tilde{t}_j/N) \succ_j^\theta (a^*, t^*)$.

By observation 2, if s^* is an equilibrium, then either (i) there is a single $\mu_k = 1, \tilde{t}_k > 0$ and for all $i \neq k, \mu_i = 1, \tilde{t}_i = 0$; or (ii) for each agent k such that $\mu_k = 1, \tilde{t}_k = 0$.

Note that in case (i), agent k has a profitable deviation to $(\theta_k, (a_k, \tilde{t}_k/2), 1)$, which results in $(a_k, \tilde{t}_k/(2N)) \succ_k^\theta (a_k, \tilde{t}_k/N) = (a^*, t^*)$.

In case (ii), any agent can enforce her most preferred outcome with arbitrarily small delay. That is, $s_k = (\theta_k, (a_k, t_k), 1)$ results in $(a_k, t_k / \#\{i \in N | \mu_i = 1\})$ for any $t_k > 0$. Hence, s^* is an equilibrium only if $(a^*, t^*) = (a^*, 0) \succeq_i^\theta (b, t)$ for any $i \in N$ and any $(b, t) \in \mathcal{A}$. Note that, for any agent i , $\tilde{t}_i = 0$ only if $(a_i, 0) \in F(\theta_i)$. Hence, there exists some $\theta_i \in \Theta$ such that $(a^*, 0) \in F(\theta_i)$. Then, since $(a^*, 0)$ is the best outcome for all $i \in N$ in state θ , zero-delayed monotonicity implies that $(a^*, 0) \in F(\theta)$. Hence, s^* is an equilibrium under rule 3 only if it results in a socially optimal outcome.

Case 4: If s^* does not result in cases 1 or 2, then there must exist agents i, j such that either (i) $s_i \neq s_j$ or (ii) $s_i = s_j$ and $(a_i, t_i) \notin F(\theta_i)$. Let a strategy of an agent $k \neq i, j$ be $s_k = (\theta, (\tilde{a}, T/2), 1)$. Note that (s_k, s_{-k}^*) can only result in rule 3. As for all agents $i \neq k$, $\mu_i = 0$, agent k determines both an outcome and a delay; hence, the outcome after k 's deviation is $(\tilde{a}, T/2) \succ_k^\theta (\tilde{a}, T)$. $\square$

4 Examples

Example 1 (Strong Pareto correspondence). The strong Pareto correspondence is not implementable either in an undominated Nash equilibrium (henceforth, UNE) or a subgame perfect equilibrium (henceforth, SPE), and it is not zero-delayed implementable. Yet it is imminently implementable and implementable by awards.

Define the strong Pareto correspondence as $\Pi(\theta) = \{(a, 0) \in \mathcal{A} : \exists (b, 0) \in \mathcal{A} \text{ such that } \forall i \in N, (b, 0) \succeq_i^\theta (a, 0) \text{ and } \exists j \in N, (b, 0) \succ_j^\theta (a, 0)\}$ (note that if we have allowed positive delays, the definition of Π would be the same).

To see that the strong Pareto correspondence is imminently implementable, consider two states, $\theta, \phi \in \Theta$, such that $(a, 0) \in \Pi(\theta)$ and $(a, 0) \notin \Pi(\phi)$. If $(a, 0) \notin \Pi(\phi)$, then there exists $(b, 0) \in \mathcal{A}$ such that (1a) for all $i \in N$, $(b, 0) \succeq_i^\phi (a, 0)$; and (1b) there exists $j \in N$ such that $(b, 0) \succ_j^\phi (a, 0)$. As $(a, 0) \in \Pi(\theta)$, for this outcome $(b, 0)$ either (2a) there is an agent $k \in N$, such that $(a, 0) \succ_k^\theta (b, 0)$ or (2b) $(a, 0) \succeq_k^\theta (b, 0)$ for all $k \in N$. If condition (2a) holds, then, together with (1a), it implies condition (ii) of imminent monotonicity for agent k defined in (2a); if condition (2b) holds, then, together with (1b), it implies condition (i) of imminent monotonicity for agent j defined in (1b). Therefore, the strong Pareto correspondence is imminently monotonic; whenever $N \geq 3$, it implies imminent implementability.

The strong Pareto correspondence is not implementable in UNE and SPE, as shown in this example: Suppose there are three agents, two states θ and ϕ , and two physical outcomes $a, b \in A$. The preferences of agent 1 and agent 2 are identical: $a \succ_{1,2}^{\theta, \phi} b$. Agent

3 prefers outcome b in state θ , but is indifferent between the two outcomes in state ϕ :
 $b \succ_3^\theta a, a \sim_3^\phi b$.

The strong Pareto correspondence in this example is $\Pi(\theta) = \{a, b\}, \Pi(\phi) = a$.

An SCR is not implementable in UNE (hence, not in SPE or in Nash equilibrium either) unless it satisfies the following condition (Palfrey and Srivastava 1991):

Definition 5. An SCR F satisfies *Property Q* if for any pair of states θ, ϕ such that $x \in F(\theta)$ and $x \notin F(\phi)$, then either of the following two conditions hold:

- Condition 1: There exist an agent $i \in N$ and outcomes $a, b \in A$ with $a \succ_i^\theta b$ and $b \succeq_i^\phi a$ and there exists $c, d \in A$ with $c \succ_i^\theta d$.
- Condition 2: There exist an agent $i \in N$ and outcomes $a, b \in A$ with $a \succeq_i^\theta b$ and $b \succ_i^\phi a$.

It is easy to verify that neither condition 1 nor condition 2 of Definition 5 is satisfied in the example above, hence Π is not implementable in UNE.

Note also that the strong Pareto correspondence Π does not satisfy zero-delayed monotonicity. In the example above, the only agent whose preferences change between θ and ϕ is agent 3. However, for any $t \geq 0$, $(b, 0) \succeq_3^\phi (a, t)$, while zero-delayed monotonicity requires that $(a, t) \succ_3^\phi (b, 0)$ (the second part of equation 1). Hence Π is not zero-delayed monotonic. Sanver (2006) shows that the strong Pareto correspondence is implementable by awards, thus highlighting the difference between zero-delayed implementation and implementation by awards.

Example 2. Palfrey and Srivastava (1991) study an SCR which picks the Condorcet winner in a particular environment. They show that is not implementable in a subgame perfect equilibrium (we refer a reader to Palfrey and Srivastava (1991) for details). We demonstrate that this SCR is imminently implementable in this environment.

The environment consists of three alternatives, a, b , and c ; two states, θ and ϕ ; and five agents. The agents' preferences are the following: $a \succ_{1,2}^\theta b \succ_{1,2}^\theta c$; $a \succ_{1,2}^\phi b \sim_{1,2}^\phi c$; $b \succ_3^\theta c \succ_3^\theta a$; $b \sim_3^\phi c \succ_3^\phi a$; $c \succ_{4,5}^{\theta,\phi} b \succ_{4,5}^{\theta,\phi} a$. That is, the only difference between θ and ϕ is that agents 1, 2, and 3 prefer b to c in state θ and become indifferent between these two alternatives in state ϕ .

The Condorcet winner is b in state θ and c in state ϕ . To see that this rule is imminently implementable, note that there exists $\epsilon > 0$ such that for any $\Delta \in (0, \epsilon)$ the preference reversal of agent 1 is strict: $(b, \Delta) \succ_1^\theta (c, 0)$ and $(c, 0) \succ_1^\phi (b, \Delta)$. Define $F^\Delta(\theta) = (b, \Delta), F^\Delta(\phi) = (c, 0)$. This SCR is implementable by the following simple

mechanism: agent 1 reports the state, θ or ϕ , and the designer implements (b, Δ) if the report is θ and $(c, 0)$ if the report is ϕ . As F^Δ is implementable for any $\Delta < \epsilon$, F is imminently implementable.

Example 3. The next example shows that the monotonicity of some rules defined for ordinal preferences may depend on the specification of preferences over time. This example is based on Borda count rule f^{BC} in Maskin (1999).

The environment consists of four physical alternatives, a, b, c , and d ; two states, θ and ϕ ; and two agents. The preferences over $\mathcal{A}$ are as follows: $(a, 3) \sim_1^\theta (d, 2) \sim_1^\theta (b, 1) \sim_1^\theta (c, 0)$; $(a, 3) \sim_1^\phi (b, 2) \sim_1^\phi (d, 1) \sim_1^\phi (c, 0)$; $(c, 3) \sim_2^\theta (b, 2) \sim_2^\theta (a, 1) \sim_2^\theta (d, 0)$; and $(b, 3) \sim_2^\phi (c, 2) \sim_2^\phi (a, 1) \sim_2^\phi (d, 0)$. We complete these preference relations by assuming that, for any $(a, t), (b, t') \in \mathcal{A}$, $i \in N$, and $\theta \in \Theta$, if $(a, t) \sim_i^\theta (b, t')$, then $(a, t + \tau) \sim_i^\theta (b, t' + \tau)$ for any $\tau > 0$. The implied ordinal preferences over undelayed outcomes are as in Maskin (1999).

In this example, $f^{BC}(\theta) = (a, 0)$, $f^{BC}(\phi) = (b, 0)$. This rule is not Maskin-monotonic (Maskin 1999). Note that each agent has strict preferences over undelayed outcomes. Sanver (2006) shows that almost monotonicity, the necessary condition for implementation by awards, coincide with Maskin monotonicity when preferences are strict.

This rule, however, is zero-delayed monotonic.⁴ There is a preference reversal around $(a, 0)$ when the state changes from θ to ϕ : $(a, 0) \sim_2^\theta (b, 1)$, $(b, 1) \succ_2^\phi (a, 0)$; and there is a preference reversal around $(b, 0)$ when the state changes from ϕ to θ : $(b, 0) \sim_1^\phi (a, 1)$, $(a, 1) \succ_1^\theta (b, 0)$.

Note that there exist many other specifications of preferences over $\mathcal{A}$ which result in the same ordinal preferences and a non-implementable SCR f^{BC} . Let the preferences over $\mathcal{A}$ be as follows: $(a, 3) \sim_1^\theta (d, 2) \sim_1^\theta (b, 1) \sim_1^\theta (c, 0)$; $(a, 4) \sim_1^\phi (b, 2) \sim_1^\phi (d, 1) \sim_1^\phi (c, 0)$; $(c, 4) \sim_2^\theta (b, 3) \sim_2^\theta (a, 1) \sim_2^\theta (d, 0)$; and $(b, 3) \sim_2^\phi (c, 2) \sim_2^\phi (a, 1) \sim_2^\phi (d, 0)$, with preferences over other delays derived as above. The key observation about these preference profiles is that if there are $x \in A$ and $t, t' \geq 0$ such that $(a, t) \sim_i^\theta (x, t')$, then $(a, t) \succeq_i^\phi (x, t')$.

Suppose that f^{BC} is Δ -delayed-monotone for some $\Delta \geq 0$. Since $(a, 0) \in f^{BC}(\theta)$, $(a, 0) \notin f^{BC}(\phi)$, either condition 1 or condition 2 of Definition 4 should hold for $(a, 0)$. However, condition 1 cannot hold because physical outcome is different in state ϕ and condition 2 cannot hold by the observation above. Hence, f^{BC} is not Δ -delayed-monotone for any $\Delta \geq 0$.

⁴Note that, as there are only two agents in this problem, we cannot use our characterization to conclude that f^{BC} is zero-delayed implementable.

5 An Application: Characterization when Only Time Preferences Change

In this section, we obtain a characterization of implementable rules when agents' preferences over outcomes delivered without a delay remain the same, but the patience of agents change. Such changes may occur, for instance, when a central bank changes its discount rate or when a government changes regulations on retirement age eligibilities.

For notational simplicity, we only consider SCRs that do not prescribe a delay; that is, for any $\theta \in \Theta$, if $(a, t) \in F(\theta)$, then $t = 0$.

We define an environment described above as follows: $\mathcal{P} = \langle N, \mathcal{A}, \Theta \rangle$, with $\mathcal{A} = A \times \mathbb{R}_+$, is such that for any pair of states $\theta, \phi \in \Theta$:

- (i) For every agent $i \in N$ and any pair of outcomes $(a, 0), (b, 0) \in \mathcal{A}$, $(a, 0) \succeq_i^\theta (b, 0)$ if and only if $(a, 0) \succeq_i^\phi (b, 0)$;
- (ii) Let $(a, 0), (b, 0) \in \mathcal{A}$ be such that, for some agent $i \in N$, $(b, 0) \succ_i^\theta (a, 0) \sim_i^\theta (b, t)$ and $(a, 0) \succeq_i^\phi (b, t)$. Then for all $j \in N$ and for all $(c, 0), (d, 0) \in \mathcal{A}$ such that $(d, 0) \succ_j^\theta (c, 0) \sim_j^\theta (d, t')$, the preference relation in state ϕ is $(c, 0) \succeq_j^\phi (d, t')$.

In environment $\mathcal{P}$, define the following binary relation on the state space, which tracks changes in the agents' patience between two states:

Definition 6. For any two states $\theta, \phi \in \Theta$, the agents are said to be *weakly more patient* in state θ than in state ϕ , $\theta R \phi$, if and only if for every agent $i \in N$ and every pair of outcomes $(a, 0), (b, 0) \in \mathcal{A}$ such that $(b, 0) \succ_i^\theta (a, 0)$, if $(a, 0) \sim_i^\theta (b, t)$, then $(a, 0) \succeq_i^\phi (b, t)$.

Note that, because $\succeq$ is complete and time preferences are continuous, for any two states θ, ϕ and for any $(a, 0), (b, 0) \in \mathcal{A}$ such that $(b, 0) \succ_i^\theta (a, 0)$ for some $i \in N$ (hence, $(b, 0) \succ_i^\phi (a, 0)$ by property i), either $(a, 0) \sim_i^\theta (b, t)$, $(a, 0) \succeq_i^\phi (b, t)$ or $(a, 0) \sim_i^\phi (b, t)$, $(a, 0) \succeq_i^\theta (b, t)$. By property (ii) of $\mathcal{P}$, the same preference relation holds for any pair $(c, 0), (d, 0)$ and for any agent. Hence, binary relation R is complete. The binary relation P is defined as $\theta P \phi$ if and only if $\theta R \phi$ and $\neg \phi R \theta$.

The restrictions on environment $\mathcal{P}$ are consistent with intertemporal models where discount factor does not depend on an outcome and the patience of all agents changes consistently. Consider a model where agents are endowed with a utility function $u_i(a, t; \theta) = u_i(a, 0)\tau_i(t; \theta)$. The requirement (ii) on $\mathcal{P}$ is satisfied if for every two states $\theta, \phi \in \Theta$,

for all $i \in N$, and for any $t > 0$, either $\tau_i(t; \theta) > \tau_i(t; \phi)$; or $\tau_i(t; \theta) < \tau_i(t; \phi)$; or $\tau_i(t; \theta) = \tau_i(t; \phi)$.

Loewenstein and Prelec (1992) provide a particular form of a hyperbolic discount function. They assume that $\tau(t; (\beta, \alpha)) = (1 + \alpha t)^{-\beta/\alpha}$, where $\beta > 0$ indicates the degree of impatience; $\alpha > 0$ is a measure of how deeply the distant future is discounted relative to the immediate future (that is, “how hyperbolic” the agent is), with $\alpha \rightarrow 0$ gives exponential discounting in the limit. These preferences satisfy consistency if changes between states only involve β , while α is the same for all states; that is, if the patience changes, but the “behavioral component” is the same. Unless special assumptions are made on the relation of α and β , environment $\mathcal{P}$ does not allow for changes in α .

In this environment, the monotonicity of an SCR can be identified with the following simple condition.

Theorem 3. *In environment $\mathcal{P}$, a unanimous SCR F is zero-delayed monotonic if and only if for any $\theta P \phi$, $F(\theta) \subseteq F(\phi)$.*

That is, if an SCR contains some outcome $(a, 0)$ in state θ , that outcome can be dropped out of the SCR only when agents become more impatient; it must remain socially optimal when agents become more patient.

This condition is stringent; yet, when there are “many” outcomes (an environment is “rich”) and infinitesimal delays are allowed, any SCR becomes implementable.

Definition 7. An environment is *rich* if, for any $\epsilon > 0$, any $\theta \in \Theta$, and any $(a, 0) \in \mathcal{A}$, there is an agent i and an outcome $(b, 0)$ such that $(a, 0) \succ_i^\theta (b, 0) \succ_i^\theta (a, \epsilon)$.

Theorem 4. *If environment $\mathcal{P}$ is rich, any unanimous SCR is imminently monotonic.*

To prove Theorem 4 we first establish a similar result for an arbitrary $\Delta > 0$.

Definition 8. An environment is F^Δ -rich if, for any $\theta \in \Theta$ and any $(a, 0) \in F^\Delta(\theta)$, there is an agent i and an outcome $(b, 0)$ such that $(a, 0) \succ_i^\theta (b, 0) \succ_i^\theta (a, \Delta)$.

Lemma 1. If an environment $\mathcal{P}$ is F^Δ -rich, any unanimous SCR is Δ -delayed-monotonic.

Note that Theorem 4 is an immediate corollary of Lemma 1 because the rich environment is F^Δ -rich for an arbitrary F and Δ .

6 Conclusion

In this paper, we examine the effect of adding a delay to the implementation problem. We show that if we allow a delay off the equilibrium path, but allow no delay on the equilibrium path, we can dispense with the no-veto-power condition. We can also exploit new preference reversals provided by adding the time dimension, however their existence depends on the precise specification of the time preferences. If we allow an infinitesimally small delay on the equilibrium path, then a number of well-known non-implementable SCRs become implementable. Allowing a positive delay Δ expands the set of outcomes further. We show that in a certain environment where there is a continuum of outcomes and only time preferences change, any SCR is imminently implementable.

7 Appendix: Proofs

Proof of Theorem 1. Suppose there exist $\theta, \phi \in \Theta$ such that $(a, t) \in F(\theta)$, $(a, t) \notin F(\phi)$, and condition 2 of delayed monotonicity does not hold. If condition 2 does not hold, then for $\forall i \in N$, $(b, \bar{t}) \in \mathcal{A}$, $t' \in [t, t + \Delta] \cap [0, T]$,

$$(a, t') \succeq_i^\theta (b, \bar{t}) \text{ implies } (a, t') \succeq_i^\phi (b, \bar{t}). \quad (3)$$

Since F is Δ -delayed-implementable, there exists a Nash-implementable F^Δ , and a $t' \in [t, t + \Delta] \cap [0, T]$ such that $(a, t') \in F^\Delta(\theta)$. Hence, there exists a mechanism Γ and a profile m^* such that $g(m^*) = (a, t')$ and m^* is a Nash equilibrium in state θ .

Suppose that there exists a profitable deviation m_i of agent i in state ϕ ; that is, $g(m_i, m_{-i}^*) = (b, \bar{t}) \succ_i^\phi (a, t') = g(m^*)$. As m^* is a Nash equilibrium in state θ , then $(a, t') \succeq_i^\theta (b, \bar{t})$, which, according to equation (3) implies $(a, t') \succeq_i^\phi (b, \bar{t})$, a contradiction to m_i being a profitable deviation in state ϕ . We conclude that m^* must be a Nash equilibrium in state ϕ .

Since Γ implements F^Δ and m^* is an equilibrium in ϕ , it follows that $(a, t') \in F^\Delta(\phi)$. By definition 1, there exists $t'' \in [t' - \Delta, t'] \cap [0, T] \subset [t - \Delta, t + \Delta] \cap [0, T]$ such that $(a, t'') \in F(\theta)$, which is the first condition of Definition 4. $\square$

Proof of Theorem 3. 1. Suppose that for any $\theta R\phi$, $F(\theta) \subseteq F(\phi)$. First, note that if $\theta R\phi$ and $\phi R\theta$, then, by assumption, $F(\theta) = F(\phi)$: zero-delayed-monotonicity holds vacuously.

Hence, suppose $\theta P\phi$. Let $(a, 0) \in F(\phi)$ and $(a, 0) \notin F(\theta)$. Since $(a, 0) \notin F(\theta)$

and F is a unanimous SCR, there exists an agent j and an outcome $(b, 0)$, such that $(b, 0) \succ_j^\theta (a, 0)$. Let $t > 0$ be such that $(a, 0) \sim_j^\theta (b, t)$. Since $\theta P\phi$, $(a, 0) \succ_j^\phi (b, t)$. Hence, there exists an $\epsilon > 0$ such that for $\bar{t} = t - \epsilon$, $(a, 0) \succeq_j^\phi (b, \bar{t})$ and $(b, \bar{t}) \succ_j^\theta (a, 0)$. We conclude that F is zero-delayed monotonic.

2. Suppose that F is zero-delayed-monotonic, $(a, 0) \in F(\theta)$ and $\theta R\phi$. As $\theta R\phi$, for any $i \in N$ and any $(b, t) \in \mathcal{A}$, if $(a, 0) \succeq_i^\theta (b, t)$ then $(a, 0) \succeq_i^\phi (b, t)$. Then zero-delayed-monotonicity of F implies $(a, 0) \in F(\phi)$. $\square$

Proof of Lemma 1. Suppose that there are θ and ϕ such that $(a, 0) \in F(\theta)$, $(a, 0) \notin F(\phi)$. First, note that a preference reversal has already been identified in Theorem 3 for $\phi P\theta$. Thus, we consider the case $\theta P\phi$.

By definition 8, there exists an outcome $(b, 0)$ such that $(a, 0) \succ_i^\theta (b, 0) \succ_i^\theta (a, \Delta)$. Let $t < \Delta$ be such that $(a, t) \sim_i^\theta (b, 0)$. Since $\theta P\phi$, $(b, 0) \succ_i^\phi (a, t)$. This constitutes a preference reversal as required by Δ -delayed monotonicity. $\square$

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