A Fast SVD-Hidden-Nodes Based Extreme Learning Machine for Large-Scale Data Analytic

Wan-Yu Deng^{a,b}, Zuo Bai^c, Guang-Bin Huang^c, Qing-Hua Zheng^d

^a*School of Computer, Xian University of Posts & Telecommunications, Shaanxi, China*

^b*School of Computer Engineering, Nanyang Technological University, Singapore*

^c*School of Electrical & Electronic Engineering, Nanyang Technological University, Singapore*

^d*Department of Computer Science and Technology, Xi'an Jiaotong University, China*

Abstract

Big dimensional data is a growing trend that is emerging in many real world contexts, extending from web mining, gene expression analysis, protein-protein interaction to high-frequency financial data. Nowadays, there is a growing consensus that the increasing dimensionality poses impeding effects on the performances of classifiers, which is termed as the “peaking phenomenon” in the field of machine intelligence. To address the issue, dimensionality reduction is commonly employed as a preprocessing step on the Big dimensional data before building the classifiers. In this paper, we propose an Extreme Learning Machine (ELM) approach for large-scale data analytic. In contrast to existing approaches, we embed hidden nodes that are designed using singular value decomposition (SVD) into the classical ELM. These SVD nodes in the hidden layer are shown to capture the underlying characteristics of the Big dimensional data well, exhibiting excellent generalization performances. The drawback of using SVD on the entire data set, however, is the high computational complexity involved. To address this, a fast divide and conquer approximation scheme is introduced to maintain computational tractability on high volume data. The resultant algorithm proposed is labelled here as Fast Singular Value Decomposition-Hidden-nodes based Extreme Learning Machine or FSVD-H-ELM in short. In FSVD-H-ELM, instead of identifying the SVD hidden nodes directly from the entire data set, SVD hidden nodes are derived

*Corresponding author

Email address: wanyu.deng@gmail.com (Wan-Yu Deng)

from multiple random subsets of data sampled from the original dataset. Comprehensive experiments and comparisons are conducted to assess the FSVD-H-ELM against other *state-of-the-art* algorithms. The results obtained demonstrated the superior generalization performance and efficiency of the FSVD-H-ELM.

Keywords: extreme learning machine, singular value decomposition, Big data, Big dimensional data, fast approximation method

1. Introduction

The notion of “Big Dimensionality” was established recently in [1] to place emphasis on the growing phenomenon of problem dimensionality that is observed across diverse scenarios and applications in the real world. When addressing Big Data, volume often comes to mind naturally and take immediate precedence, since it presents the challenges pertaining to the scalability issue. Also, this is what directly comes to our mind when we refer to the term “Big”. However, it is worth highlighting that researchers have largely considered the “Big Instance Size” factor of “Volume” in the Big Data analytics community, which refers to the massive amounts of data that we continue to produce daily. In the context of data analytics, volume can be defined as a product of *instance size* and *dimensionality* of the data. The issue of “Big Dimensionality”, however, has received much lesser attention. For more coverage on this relatively under-explored topic of “Big Dimensionality” wherein the explosion of features (variables) brings about new challenges to computational intelligence and Big data, the reader is referred to [1]. *Big dimensional* data is a growing trend that is emerging in many real world contexts, extending from web mining, gene expression analysis, protein-protein interaction to high-frequency financial data. Today, there is a growing consensus that the increasing dimensionality poses impeding effects on the performances of classifiers, which is termed as the “peaking phenomenon” in the field of machine intelligence [2]. Previous methods [3] to address the high dimensionality characteristics of data have mainly concentrated on preprocessing the data with dimensionality reduction methods, keeping only the salient features and discarding the non-informative ones before building the classifier. In this paper, we present an attempt

to address both *Big dimensionality* and *Big instance size* simultaneously in the context
25 of Big Data analytic by embedding Singular Value Decomposition Hidden nodes into
the classical Extreme Learning Machine.

In spite of the extensive works on ELM research, existing efforts still cannot arrive
at representations and models that are elegant for Big data. Like all methods known to
date, ELM suffers from the effects of ‘curse of dimensionality and instance size’. The
30 performance of ELM has been shown to deteriorate substantially and becomes unstable
in the face of Big data. Taking this cue, this paper considers an implicit incorporation of
dimension reduction mechanism into the classical ELM to achieve high generalization
performances. In particular, we present a Fast Singular Value Decomposition-Hidden-
nodes based Extreme Learning Machine (or FSVD-H-ELM in short) to cope with *Big*
35 *dimensional & Big instance* data. We embed SVD hidden nodes into the classical Ex-
treme Learning Machine to arrive at SVD-H-ELM and provide theoretical proofs on
the optimality of using SVD hidden nodes over random hidden nodes. Further, instead
of identifying the SVD hidden nodes directly from the Big dataset, a fast divide and
conquer approximation scheme is also introduced to maintain computational tractabili-
40 ty and scalability by deriving SVD hidden nodes from multiple random subsets of data
sampled from the original Big dataset, in order to arrive at the Fast SVD-H-ELM or
FSVD-H-ELM. Comprehensive experiments conducted on commonly used Big dataset
indicate that the SVD nodes in the hidden layer are able to capture the underlying char-
acteristics of the Big dataset well, leading to excellent generalization performances and
45 efficiencies.

The rest of the paper is organized as follows: Section II gives a brief overview of
the classical ELM. Section III details the proposed Fast Singular Value Decomposition-
Hidden-nodes based Extreme Learning Machine (or FSVD-H-ELM). In Section IV,
A discussion on the feature learning ability of the fast divide-and-conquer scheme is
50 made. In Section V, we summarize the experimental studies on twelve commonly used
benchmark problems. Section VI summarizes the main conclusions.

2. Review of Extreme Learning Machine and Large-scale Data Analytics

2.1. Extreme Learning Machine

Feedforward neural networks play key roles in data analytic and have been widely
55 applied in many applications for its promising generalization ability[4, 5, 6, 7, 8, 9].
However, popular learning techniques face some challenging issues such as intensive
human intervene and slow learning speed[10]. Although many effective algorithms,
such as the back-propagation are available, training a neural network with all param-
eters adjustable is usually of high computational burden[11, 12]. To overcome such
60 issue, a useful learning scheme, the extreme learning machine (ELM), was suggest-
ed in [13] for single layer feedforward neural networks and subsequently extended
to different variations such as local connected structure[14] and multi hidden layer-
s structure[15]. ELM and its variations[16, 17, 18, 19, 20, 21, 22, 23] have been
successfully used in the fields as object recognition[14], terrain-based navigation [24]
65 activity recognition[25], time series[26], security assessment[27], written character
recognition[28], face recognition[29], gene selection and cancer classification [30]. In
essence ELM is a learning scheme whose hidden nodes need not be tuned and can be
randomly generated. Then the ELM transforms the training of the neural network into
a linear problem where the output weights of ELMs can be analytically determined
70 instead of being tuned[31]. For the sake of efficiency, in real applications they may be
determined in different ways such as with or without iterations, with or without incre-
mental implementations, etc. Xu etc.[32, 33] studied the theoretical feasibility of ELM
and proved that via suitable activation functions, such as polynomials, NadarayaWatson
and sigmoid functions, the ELMs can attain the theoretical generalization bound of the
75 neural networks with all connections adjusted. Before ELM, the similar idea has been
adopted earlier in[34][35] as the echo state network method (ESN), in [36] as the liq-
uid state machine (LSM), in[37] as RBF network with random centers, in[38] as the
feedforward neural networks with random weights and in [39] as the random vector
functional-link network (RVFL). The relationships between ELMs and these earlier
80 work please refer to the literatures[40, 41].

The output function of ELM with L hidden nodes for generalized SLFNs is:

$$f_L(\mathbf{x}) = \sum_{i=1}^L \beta_i h_i(\mathbf{x}) = \mathbf{h}(\mathbf{x})\boldsymbol{\beta} \quad (1)$$

where $\boldsymbol{\beta}=[\beta_1, \dots, \beta_L]$ is the output weights between the hidden nodes to output nodes. $\mathbf{h}(\mathbf{x})=[h_1(\mathbf{x}), \dots, h_L(\mathbf{x})]$ is the output vector of the hidden layer with respect to the input $\mathbf{x}$; $h_i(\mathbf{x}) = g(\mathbf{w}_i, b_i, \mathbf{x})$, $\mathbf{w}_i \in \mathbf{R}^d$, $b_i \in R$ is the output of the i -th *random hidden node* where the hidden node parameters (e.g. $\mathbf{w}$ and b) need not be tuned, in particular, can be *randomly* generated according to any continuous sampling distribution probability. $g(\mathbf{w}_i, b_i, \mathbf{x})$ is a nonlinear piecewise continuous function satisfying ELM universal approximation capability theorems[16]. For example, such nonlinear piecewise continuous function can be but are not limited to:

1) Sigmoid function

$$g(\mathbf{w}, b, \mathbf{x}) = \frac{1}{1 + \exp(-\lambda(\mathbf{w} \cdot \mathbf{x} + b))} \quad (2)$$

2) Fourier function

$$g(\mathbf{w}, b, \mathbf{x}) = \sin(\mathbf{w} \cdot \mathbf{x} + b) \quad (3)$$

3) Hardlimit function

$$g(\mathbf{w}, b, \mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{w} \cdot \mathbf{x} - b \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

4) Gaussian function

$$g(\mathbf{w}, b, \mathbf{x}) = \exp(-b\|\mathbf{x} - \mathbf{w}\|^2) \quad (5)$$

ELM theory[13, 17, 16, 42] shows that such $\mathbf{h}(\mathbf{x})$ has *universal approximation capability*, that is, $\lim_{L \rightarrow \infty} \|\mathbf{h}(\mathbf{x})\boldsymbol{\beta} - f(\mathbf{x})\|_F^2 = \lim_{L \rightarrow \infty} \|\sum_{i=1}^L \beta_i h_i(\mathbf{x}) - f(\mathbf{x})\|_F^2 = 0$ holds with a probability one for appropriate output weights $\boldsymbol{\beta}$. Given training samples $\boldsymbol{\Omega} = \{(\mathbf{x}_i, \mathbf{t}_i)\}_{i=1}^N$, ELM theory aims to reach the smallest norm $\min_{\boldsymbol{\beta}} \|\boldsymbol{\beta}\|_F^2$ of output weights and training error $\min_{\boldsymbol{\beta}} \|\mathbf{H}\boldsymbol{\beta} - \mathbf{T}\|_F^2$ where $\mathbf{H}$ is the hidden layer output

matrix (*randomized matrix*):

$$\mathbf{H} = \begin{bmatrix} g(\mathbf{w}_1, b_1, \mathbf{x}_1) & \cdots & g(\mathbf{w}_L, b_L, \mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ g(\mathbf{w}_1, b_1, \mathbf{x}_N) & \cdots & g(\mathbf{w}_L, b_L, \mathbf{x}_N) \end{bmatrix}$$

and

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_N \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots & t_{1m} \\ \vdots & \ddots & \vdots \\ t_{N1} & \cdots & t_{Nm} \end{bmatrix}$$

$\mathbf{h}(\mathbf{x})$ actually maps the data from the d -dimensional input space to the L -dimensional hidden layer random feature space. As a special case of $g(\mathbf{w}, b, \mathbf{x})$, $\mathbf{H}$ can be written as:

$$\mathbf{H} = g\left(\begin{bmatrix} \mathbf{X} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{b} \end{bmatrix}\right) = \begin{bmatrix} g(\mathbf{w}_1 \cdot \mathbf{x}_1 + b_1) & \cdots & g(\mathbf{w}_L \cdot \mathbf{x}_1 + b_L) \\ \vdots & \ddots & \vdots \\ g(\mathbf{w}_1 \cdot \mathbf{x}_N + b_1) & \cdots & g(\mathbf{w}_L \cdot \mathbf{x}_N + b_L) \end{bmatrix}$$

Numerous methods are now available for attaining the weight vector β , such as the *No-Prop* algorithm suggested by Widrow *et al.* [43] and the iterative algorithm developed in [44]. A popular and effective solution is given as follows:

$$\beta = \mathbf{H}^\dagger \mathbf{T} \quad (6)$$

where $\mathbf{H}^\dagger$ is the *Moore-Penrose generalized inverse* of $\mathbf{H}$.

Recently, the unified ELM was introduced in [17, 42], in which β is solved as a constrained optimization problem:

$$\text{Minimize}_{\beta} : \|\mathbf{H}\beta - \mathbf{T}\|_p^{\alpha_1} + \frac{1}{\gamma} \|\beta\|_q^{\alpha_2} \quad (7)$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $p, q = 0, \frac{1}{2}, 1, 2, \dots, F, +\infty$ and γ is control parameter for the tradeoff of structural risk and experimental risk. Numerous efficient methods can be used to calculate the output weights β including but not limited to orthogonal projection method, iterative methods[44], and eigenvalue decomposition method[45]. When

$p, q = F$ and $\alpha_1, \alpha = 2$, a popular and efficient closed-form solution[17] is:

$$\beta = \begin{cases} \mathbf{H}^\top \left(\frac{\mathbf{I}}{\gamma} + \mathbf{H}\mathbf{H}^\top \right)^{-1} \mathbf{T} & N \geq L \\ \left(\frac{\mathbf{I}}{\gamma} + \mathbf{H}^\top \mathbf{H} \right)^{-1} \mathbf{H}^\top \mathbf{T} & N \leq L \end{cases} \quad (8)$$

2.2. Large-scale Data Analytic

To process increasing amounts of data efficiently, many efficient and effective algorithms have been proposed. One kind of alternative methods directly avoids expensive
85 computation burden of nonlinear decision functions, and builds large-scale *linear* algorithms on original feature space such as [46, 47, 48, 49, 50, 51, 52, 53, 54]. This kind of algorithms has showcase competitive generalized performance at fast learning speed in some certain applications such as sparse high dimensional linear separated (or nearly) problems. On the other hand, arising number of researchers are devoting their
90 time and efforts in variants of nonlinear kernel approximation[55, 56, 57] to overcome the challenge of large scale learning. For instances, AMM[55] captures non-linearity by assigning a number of linear hyperplanes to each of classes, and then learned via Stochastic Gradient Descent (SGD). LLSVM[57] approximates kernel SVM optimization by a linear SVM using low-rank decomposition of the kernel matrix. BSGD-
95 SVM[56] maintains a fixed number of support vectors in the model, and incrementally updates them during the SGD training. Hsieh etc.[58] proposed a divide-and-conquer solver for kernel SVMs (DC-SVM) where the kernel SVM problem is partitioned into smaller subproblems by clustering the data, so that each subproblem can be solved independently. Random Kitchen Sinks algorithm[59] approximates kernel expansions
100 by means of multiplying the input with a Gaussian random matrix, followed by the application of a nonlinearity. From a survey of the literature[60], the core challenges of Big Data have been widely established and can be summarized under the popular 5Vs, namely, Volume, Velocity, Variety, Veracity and Value. The present work takes special interests on the volume aspect of Big data by focusing on the massive amounts
105 of data that have been generated across a wide range of sources. In particular, within the context of Big data analytics, volume can be defined by the product of instance size and dimensionality of the data. Taking this cue, in the rest of this section, we present the essential ingredients of our proposed Singular Value Decomposition-Hidden-nodes

based Extreme Learning Machine, which is designed to work with *Big dimensional & Big instance* data.

3. Proposed Method

Figure 1: Random hidden nodes in classical ELM and SVD hidden nodes in our proposed method. For random hidden nodes, the parameters of the hidden nodes are *randomly* generated according to some continuous sampling distribution probability, while for SVD hidden nodes, the parameters of the hidden nodes are generated under the conditions of optimal low rank approximation.

3.1. Problem of Interests

Given training set $\Omega = \{(\mathbf{x}_i, \mathbf{t}_i) | \mathbf{x}_i \in \mathbf{R}^d, \mathbf{t}_i \in \mathbb{R}^m\}_{i=1}^N$, where d is the data dimension, N is instance size and m is the number of outputs. In a single-output regression problem, $t_i \in R$ is a continuous real value ($\mathbb{R} = R$). For a m -class classification problem, $\mathbf{t}_i \in \{0, 1\}^m$ is a m -dimensional boolean vector ($\mathbb{R} = \{0, 1\}$). If the original class label is p , the expected output vector of the m outputs is $\mathbf{t}_i = [0, \dots, 0, \overset{p}{1}, 0, \dots, 0]_{1 \times m}$. In this case, only the p -th element of $\mathbf{t}_i = [t_1, t_2, \dots, t_m]$ is '1', while the rest of the elements are set to '0'. Binary classification is thus considered a special case of the multi-class problem where $m = 2$.

In the current work, our core interest is in the case where N and d are 'Big'. On *Big instance* data, where only N is big, many effective algorithms has been proposed and

studied [61, 55, 62, 63, 64, 65]. However, despite the many ongoing efforts, majority (78.8%) [1] of the studies on Big data analytic have been confined to *Big instance* data with features that are not so ‘Big’, i.e., typically lower than 100,000 in dimensions. On *Big instance* and *Big dimensional* data, both N and d are very large, and to date less than 4%[1] of the studies in the computational intelligence community have considered real world data with features that range in the millions[66, 67]. Thus, it is clear that the dimensionality of the algorithms under-studied is significantly lagging behind those being produced, mainly due to the many great challenges of *Big dimensional* and *Big instance* data.

3.2. Fast SVD-Hidden-Nodes Based Extreme Learning Machine for Big Dimensional & Big Instance Data

By treating the hidden layer of ELM as unsupervised generative model[68] with L hidden nodes, it plays the role of learning more useful (or discriminative) features from original data without losing much information. When the number of hidden nodes L is equal to or greater than the data dimensions, i.e., $L \geq d$, the input weights could be any unitary matrices to make reconstruction error approximating to zero, and thus it will become a trivial matrix; but by placing the constraint on the network by limiting the number of hidden nodes, we can learn essential embedding structure of the data. In special, the hidden layer as generative model is to re-represent the data with minimal reconstruction error $\left\| \begin{bmatrix} \mathbf{X} & \mathbf{1} \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{X}} & \mathbf{1} \end{bmatrix} \right\|_F^2$ and minimal hidden nodes L , that is:

$$\text{Minimize}_{\mathbf{W}, \mathbf{b}} : \left\| \begin{bmatrix} \mathbf{X} & \mathbf{1} \end{bmatrix} - \begin{bmatrix} \mathbf{X} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{b} \end{bmatrix}^\top \right\|_F^2 \quad (9)$$

where $\mathbf{W} \in \mathbf{R}^{d \times L}$ and $\mathbf{b} \in \mathbf{R}^{1 \times L}$ are input weights and bias, $\begin{bmatrix} \hat{\mathbf{X}} & \mathbf{1} \end{bmatrix}$ is the representation in L -dimension feature space of the original data.

In the classical ELM implementation, the above optimization condition is not considered and the hidden nodes have been randomly generated:

$$\begin{bmatrix} \mathbf{W}_{\text{rand}} \\ \mathbf{b}_{\text{rand}} \end{bmatrix} \leftarrow \text{Random values} \quad (10)$$

Figure 2: Fast *divide-and-conquer* approximation scheme for Big data (*Big instance size* and *Big dimensionality*). In the *divide-and-conquer* scheme, the complete data set is first sliced into multiple small subsets without overlapping according to a uniform distribution, and then each subset generates one part of the hidden nodes for FSVD-H-ELM such that reconstruction error is minimized independently.

The random hidden nodes architecture of the classical ELM is depicted in Fig.1-(a), which provides a simple approach for fast learning. However, when faced with *Big dimensional* data, the use of random hidden nodes tend to suffer since they fail to capture the true intrinsic characteristics of the *Big dimensionality*, resulting in a poor representation of the data in the hidden layer. Traditionally, researchers would use Principal Components Analysis (PCA) and other dimensionality reduction methods to work on the *Big dimensional* data as a form of pre-processing before building the classifiers [69, 3, 70, 71, 72]. However, such a scheme would require additional memory space for archiving the preprocessed data[73]. This can pose as an issue when dealing with

Big data. In contrast to previous studies in the literature, we embed hidden nodes that are designed using Singular Value Decomposition (SVD) directly into the classical ELM to arrive at the proposed SVD-H-ELM. Particularly, we generate the SVD hidden nodes based on the optimal rank- L SVD as shown in Fig.1-(b). As proved in the following, SVD hidden nodes can satisfy the optimization condition of Eq.(9).

We consider the SVD of $[\mathbf{X}, \mathbf{1}]$ as

$$[\mathbf{X}, \mathbf{1}] \stackrel{\text{SVD}}{=} \mathbf{U} \mathbf{S} \mathbf{V}^\top \quad (11)$$

where $\mathbf{U} \in \mathbf{R}^{N \times N}$ and $\mathbf{V} \in \mathbf{R}^{d \times d}$ are unitary matrices, and $\mathbf{S} \in \mathbf{R}^{N \times d}$ is diagonal matrix with non-negative real numbers with decent order $\sigma_1 > \sigma_2 > \dots > \sigma_N$ on the diagonal.

The optimal rank- L approximation of $[\mathbf{X}, \mathbf{1}]$ is given by zeroing out the $\min\{N, d\} - L$ trailing values of $\mathbf{S}$, that is:

$$[\mathbf{X}, \mathbf{1}] \triangleq \mathbf{U}_L \mathbf{S}_L \mathbf{V}_L^\top \quad (12)$$

where $\mathbf{U}_L$ and $\mathbf{V}_L$ denote the first $L < \min\{N, d\}$ columns of $\mathbf{U}$ and $\mathbf{V}$, and $\mathbf{S}_L = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_L)$.

Parameter of the hidden nodes can then be obtained as

$$\begin{bmatrix} \mathbf{W}_{\text{SVD}} \\ \mathbf{b}_{\text{SVD}} \end{bmatrix} \leftarrow \mathbf{V}_L \quad (13)$$

By replacing the random parameter of the hidden nodes with $(\mathbf{W}_{\text{SVD}}, \mathbf{b}_{\text{SVD}})$ and incorporating them as part of the ELM network thus arrive at the proposed SVD-H-ELM network. Consequently, the latent properties of the data can be appropriately represented via the SVD hidden nodes of the SVD-H-ELM.

Particularly, $(\mathbf{W}_{\text{SVD}}, \mathbf{b}_{\text{SVD}})$ gives the minimal *reconstruction error* with *Forbenius norm* according to the following theorem:

Theorem 1. The SVD nodes with parameter $(\mathbf{W}_{\text{SVD}}, \mathbf{b}_{\text{SVD}})$ can provide more accurate representation of input $\mathbf{X}$ in the measure of Forbenius norm than random nodes with $(\mathbf{W}_{\text{rand}}, \mathbf{b}_{\text{rand}})$.

Proof. Suppose that we have a classical ELM, in which parameter of the hidden nodes $(\mathbf{W}_{\text{Rand}}, \mathbf{b}_{\text{Rand}})$ is *randomly* generated according to any continuous sampling distribu-

tion probability, the *Forbenius norm* reconstruction error of the original data can be derived as:

$$\xi_{\text{Rand}} = \left\| [\mathbf{X}, \mathbf{1}] - [\mathbf{X}, \mathbf{1}] \begin{bmatrix} \mathbf{W}_{\text{Rand}} \\ \mathbf{b}_{\text{Rand}} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{\text{Rand}} \\ \mathbf{b}_{\text{Rand}} \end{bmatrix}^\dagger \right\|_F^2 \quad (14)$$

For the SVD hidden nodes in SVD-H-ELM where the parameter of hidden nodes are generated as (13), the reconstruction error of the original data is given:

$$\xi_{\text{SVD}} = \left\| [\mathbf{X}, \mathbf{1}] - [\mathbf{X}, \mathbf{1}] \begin{bmatrix} \mathbf{W}_{\text{SVD}} \\ \mathbf{b}_{\text{SVD}} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{\text{SVD}} \\ \mathbf{b}_{\text{SVD}} \end{bmatrix}^\top \right\|_F^2 = \left\| [\mathbf{X}, \mathbf{1}] - \mathbf{U}_L \mathbf{S}_L \mathbf{V}_L^\top \right\|_F^2 \quad (15)$$

where $\mathbf{U}_L \mathbf{S}_L \mathbf{V}_L^\top$ is optimal rank- L approximation of $[\mathbf{X}, \mathbf{1}]$ as shown in (12).

According to the Eckart-Young-Mirsky theorem [74], the rank- L approximation
 165 with the L largest singular values is the solution with the least reconstruction error,
 that is $\xi_{\text{SVD}} < \xi_{\text{Rand}}$ always holds. This means that SVD nodes yield more accurate
 representation of input $\mathbf{X}$ than random nodes in the hidden layer. $\square$

Remark 1: In practice, noise often exists in the data. Assume that the noise is
i.i.d. Gaussian noise. When dealing with data of low-to-medium dimensionality, the
 170 noise does not pose much of an issue[69]. However, when dealing with data of *Big*
dimensionality, the noise generally dominates in the last few principal components
 with smallest singular values, because the *i.i.d.* Gaussian noise is invariant across all
 principal components transformed by the input weight $[\mathbf{W}_{\text{SVD}}^\top, \mathbf{b}_{\text{SVD}}^\top]^\top = \mathbf{V}_L$, while
 the true signal variance is small in the last few components. In this case, the SVD
 175 hidden nodes in SVD-H-ELM provide more meaningful representations of the input
 $\mathbf{X}$ than random hidden nodes by disposing the last few components. The above thus
 summarizes the theoretical proof and analysis on the virtue of using SVD hidden nodes
 in SVD-H-ELM over the use of random hidden nodes in classic ELM.

3.3. A Fast Divide & Conquer Approximation Scheme

180 With the availability of SVD hidden nodes, improved representations of the *Big*
dimensional data can be achieved in the ELM over using random hidden nodes, thus

enabling the proposed SVD-H-ELM to attain improved generalization performance and stability. The drawback of SVD-H-ELM, however, is the high computational complexity of SVD, especially when working with *Big instance* problems. Therefore, in this subsection, we present a fast approximation scheme as shown in Fig.2, which takes inspiration from the concept of divide and conquer, to achieve computational tractability on large-scale problems. The proposed algorithm is then labelled here as the Fast Singular Value Decomposition-Hidden nodes-Extreme Learning Machine or FSVD-H-ELM. In FSVD-H-ELM, instead of identifying the SVD hidden nodes directly from the whole data, SVD hidden nodes are derived from multiple random subsets of data sampled from the original data, thus achieving improved scalability on *Big instance* and *Big dimensional* problems.

To date, many different methods to approximate the SVD at improved asymptotic running time have been described[75][76]. For instance, in [75], the decomposition was performed using QR to a random embedding matrix $\mathbf{C} = \mathbf{X}\mathbf{M}$, in which $\mathbf{M} \in \mathbf{R}^{d \times k}$ ($k < d$) is a random matrix and $\mathbf{X} \in \mathbf{R}^{N \times d}$ is the data matrix. The total computation burden is around $\mathcal{O}(Ndk + N^2k)$. In [76], sparsification was used on the data matrix by retaining an element x_{ij} with probability p_{ij} or replacing it with 0 otherwise. Although these approximation approaches can improve decomposition efficiency, since for *Big instances and Big dimension* data, the instances size N and the dimensions d are both very large, the approximate complexity is still considerable expensive. To tackle this problem, we adopt an efficient sampling scheme, where multiple subsets are constructed by picking random samples from the original data. The SVD hidden nodes derived from each of the multiple data subsets then serve as the approximation or generalization of the original data. Our scheme provides good approximation to the SVD of the original dataset with a bounded error. Intuitively, if the sampling is fair, these subsets can capture the general distribution of the data. In Section IV, theoretical proof and analysis were presented for our method. In summary, the SVD hidden nodes derived from these subsets can generally produce good representation of the *Big dimensional* data, while requiring tractable computational effort.

In our *divide-and-conquer* scheme, the complete data is first sliced (or divided) into $S = \lfloor \frac{N}{n} \rfloor$ subsets without overlapping such that each subset has around n samples.

Each subset produces $\ell = \rho \cdot n$ hidden nodes where $0 < \rho < 1$ is the ratio of the retained rank, then s subsets $\{\mathbf{X}_r\}_{r=1}^s$ are randomly selected from S subsets such that
215 $L = \sum_{r=1}^s \ell$ where L is the total number of hidden nodes in FSVD-H-ELM.

For each data subset in $\{\mathbf{X}_r\}_{r=1}^s$, the optimal rank- ℓ approximation is obtained similar to (12):

$$[\mathbf{X}_r, \mathbf{1}] \triangleq \mathbf{U}_\ell^r \mathbf{S}_\ell^r \mathbf{V}_\ell^{rT} \quad (16)$$

Simply assigning the parameter of ℓ hidden nodes, i.e. $\{(r-1) \cdot \ell + 1, (r-1) \cdot \ell + 2, \dots, r \cdot \ell\}$ -th superscript nodes with $\mathbf{V}_\ell^r$ and concatenating them together, then the parameter of the L hidden nodes can be obtained as following

$$\begin{bmatrix} \mathbf{W}_{\text{FSVD}} \\ \mathbf{b}_{\text{FSVD}} \end{bmatrix} \Leftarrow \begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 & \dots & \mathbf{W}_s \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_s \end{bmatrix} \Leftarrow [\mathbf{V}_\ell^1 \mid \mathbf{V}_\ell^2 \mid \dots \mid \mathbf{V}_\ell^s] \quad (17)$$

Therefore, the hidden layer output matrix of SVD hidden nodes can be derived as:

$$\mathbf{P}_L = [\mathbf{X}, \mathbf{1}] \begin{bmatrix} \mathbf{W}_{\text{FSVD}} \\ \mathbf{b}_{\text{FSVD}} \end{bmatrix} = [\mathbf{X}, \mathbf{1}] [\mathbf{V}_\ell^1 \dots \mathbf{V}_\ell^s] \quad (18)$$

$$\mathbf{H} = g(\mathbf{P}_L) = g(\mathbf{X} [\mathbf{V}_\ell^1 \dots \mathbf{V}_\ell^s]) \quad (19)$$

3.4. Computational Complexity

Here we analyze the computational complexities of the classical SVD (used in SVD-H-ELM) and our proposed divide-and-conquer SVD approximation scheme (used in FSVD-H-ELM). In particular, since the complexity of SVD on a $r \times c$ input data matrix is $\mathbf{X} \mathcal{O}(rc \cdot \min(r, c))$, it can be easily derived that:

$$\mathcal{O}_{\text{classical-SVD}} = \mathcal{O}(Nd \cdot \min(N, d)) \quad (20)$$

and

$$\mathcal{O}_{\text{Divide \& Conquer Approx-SVD}} = \mathcal{O}\left(\sum_{r=1}^s n^2 d\right) \quad (21)$$

Further, since the subset size n is typically set to be small ($n \ll N$) for the purpose of efficiency, it can be easily observed that the computational complexity of the fast *divide-and-conquer* approximation scheme $\mathcal{O}_{\text{Divide \& Conquer Approx-SVD}}$ is much lower than that
220 of the classical SVD $\mathcal{O}_{\text{classical-SVD}}$.

Remark 2: Since the SVD of each subset $\mathbf{X}_r, r = 1, 2, \dots, s$ is conducted independently, parallel computing of the scheme can be conveniently realized. Therefore, the memory requirement is significantly reduced and the training speed can also be greatly improved.

225 *Remark 3:* The proposed algorithm is also suitable for online model update than other two-stage algorithms, where the latter involves dimensionality reduction as a pre-processing stage before the learning stage. For any two-stage algorithm, with the arrival of a new data point, besides the mapping matrix of dimensionality reduction that needs to be performed again in order to update, it also requires the learning algorithms
230 to be recomputed or rebuilt. In the context of ELM, the number of hidden nodes L would need to be revised accordingly each time some new data points are received. On the contrary, with our proposed algorithm, L is the number of the retained components or the rank, i.e., $\sum_{r=1}^s \ell$. Thus, the value of L can be naturally adjusted by the rank- ℓ online updating in the online phase. Furthermore, with our scheme, there is no need
235 for additional memory to store the preprocessed data after performing dimensionality reduction.

4. A Discussion on the Feature Learning Ability of the Fast Divide & Conquer Approximation Scheme

In this section, we present an analysis on the *Frobenius reconstruction error* bound
240 of the fast divide-and-conquer approximation scheme and compares it to the classical SVD on the original complete dataset.

First, we define the following notations which are used in the analysis:

- $\mathbf{P}_L = [\mathbf{X}, \mathbf{1}] [\mathbf{V}_\ell^1 \cdots \mathbf{V}_\ell^s]$: The L -dimensional representation of the original dataset (see (18)) in the *divide-and-conquer* scheme with s subsets $\{\mathbf{X}_r\}_{r=1}^s$;
- 245 • $\mathbf{P}_\ell^r = [\mathbf{X}, \mathbf{1}] [\mathbf{V}_\ell^r]$: The ℓ -dimensional representation of the original dataset in the *divide-and-conquer* scheme with single subset $\mathbf{X}_r$;
- $\mathbf{P}_\ell = [\mathbf{X}, \mathbf{1}] [\mathbf{V}_\ell]$: The ℓ -dimensional representation of the original dataset in the optimal classical rank- ℓ SVD;

Algorithm 1: Fast SVD-Hidden-Nodes Based Extreme Learning Machine

Input: Given training dataset $\Omega = \{(\mathbf{x}_i, \mathbf{t}_i)\}_{i=1}^N$, activation function g , regularized parameter γ , number of hidden nodes L , subset instance size n and hidden nodes ℓ that each subset generates.

Output: The model of ELM with Fast SVD hidden nodes: input layer weight $\mathbf{W}_{\text{FSVD}}$, biases $\mathbf{b}_{\text{FSVD}}$, and output layer weight β .

- 1 Divide the training dataset into $\lfloor \frac{N}{n} \rfloor$ non-overlapping subsets, and then randomly select s subsets $\{\mathbf{X}_r\}_{r=1}^s$ such that $L = \sum_{r=1}^s \ell$;
 - 2 **for** $r = 1$ **to** s **do**
 - 3 Compute the optimal rank- ℓ approximation of $\mathbf{X}_r$: $\mathbf{X}_\ell^r \triangleq \mathbf{U}_\ell^r \mathbf{S}_\ell^r \mathbf{V}_\ell^{r\top}$;
 - 4 **end**
 - 5 Compute parameter of the hidden nodes:
$$\begin{bmatrix} \mathbf{W}_{\text{FSVD}} \\ \mathbf{b}_{\text{FSVD}} \end{bmatrix} \Leftarrow \begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_2 & \dots & \mathbf{W}_s \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_s \end{bmatrix} \Leftarrow \begin{bmatrix} \mathbf{V}_\ell^1 & \mathbf{V}_\ell^2 & \dots & \mathbf{V}_\ell^s \end{bmatrix};$$
 - 6 Compute the output matrix of hidden layer: $\mathbf{H} = g \left(\begin{bmatrix} \mathbf{X}, \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{\text{FSVD}} \\ \mathbf{b}_{\text{FSVD}} \end{bmatrix} \right)$;
 - 7 Compute the output weights: $\beta = \mathbf{H}^\top \left(\frac{\mathbf{I}}{\gamma} + \mathbf{H}\mathbf{H}^\top \right)^{-1} \mathbf{T}$ or $\left(\frac{\mathbf{I}}{\gamma} + \mathbf{H}^\top \mathbf{H} \right)^{-1} \mathbf{H}^\top \mathbf{T}$
 - 8 **return** $\mathbf{W}_{\text{FSVD}}$, $\mathbf{b}_{\text{FSVD}}$ and β
-

Theorem 2. Suppose N samples $\mathbf{X} = \{\mathbf{x}_i | \mathbf{x}_i \in \mathbf{R}^m\}_{i=1}^N$ are divided into $\lfloor \frac{N}{n} \rfloor$ subsets with subset size n , and each subset produces ℓ hidden nodes. The s subsets $\{\mathbf{X}_r\}_{r=1}^s$ are then randomly selected such that the $L = \sum_{r=1}^s \ell$ where L is the total number of hidden nodes in FSVD-H-ELM. Then with probability of at least $(1-\delta)$, the following inequality holds:

$$\|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_L \mathbf{P}_L^\top\|_F \leq \|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell \mathbf{P}_\ell^\top\|_F + N\theta_D \left(\frac{64\ell}{n}\right)^{\frac{1}{4}} \left\{ 1 + \left[\frac{2(N-sn)\theta_{dis}^2 \log \frac{1}{\delta}}{s(N-\frac{1}{2})(1-\frac{1}{2}\max\{sn, N-sn\})} \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

where

$$\theta_D = \max_i (\mathbf{x}_i \mathbf{x}_i^\top)^{\frac{1}{2}} \text{ and } \theta_{dis} = \max_{i,j} (\mathbf{x}_i \mathbf{x}_i^\top + \mathbf{x}_j \mathbf{x}_j^\top - 2\mathbf{x}_i \mathbf{x}_j^\top)^{\frac{1}{2}}.$$

Proof. For $r \in [1, s]$, let $\mathbf{Z}_r = \sqrt{N/b} \mathbf{X} \mathcal{S}_r$, where $\mathcal{S}_r$ denotes the selection matrix corresponding to the subset $\mathbf{X}_r$. By definition of $\mathbf{P}_L$, the following condition holds:

$$\|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_L \mathbf{P}_L^\top\|_F^2 \leq \frac{1}{s} \left\| \sum_{i=r}^s \mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell^r \mathbf{P}_\ell^{rT} \right\|_F^2 \leq \frac{1}{s} \sum_{i=r}^s \|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell^r \mathbf{P}_\ell^{rT}\|_F^2 \quad (22)$$

Note that $\mathbf{P}_\ell^r \mathbf{P}_\ell^{rT} \approx (\mathbf{X}\mathbf{X}^\top) (\mathbf{X}_r \mathbf{X}_r)^\top (\mathbf{X}_r \mathbf{X}_r)^\top (\mathbf{X}_r \mathbf{X}_r)^\top$ is equivalent to the standard Nyström approximation of $\mathbf{X}\mathbf{X}^\top$. Hence, the following general inequality holds for the *Frobenius error* of the standard Nyström approximation method [5]:

$$\|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell^r \mathbf{P}_\ell^{rT}\|_F^2 \leq \|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell \mathbf{P}_\ell^\top\|_F^2 + N\theta_D \sqrt{64\ell} \|\mathbf{X}\mathbf{X}^\top - \mathbf{Z}_r \mathbf{Z}_r^\top\|_F \quad (23)$$

where $\|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell \mathbf{P}_\ell^\top\|_F^2$ is the *Frobenius error* bound of the classical SVD with reduced rank- ℓ . Substituting (23) into (22), we have:

$$\|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_L \mathbf{P}_L^\top\|_F^2 \leq \|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell \mathbf{P}_\ell^\top\|_F^2 + \frac{N\theta_D \sqrt{64\ell}}{s} \sum_{i=r}^s \|\mathbf{X}\mathbf{X}^\top - \mathbf{Z}_r \mathbf{Z}_r^\top\|_F \quad (24)$$

Denote

$$\phi(\mathcal{S}_r) = \frac{1}{s} \sum_{r=1}^s \|\mathbf{X}\mathbf{X}^\top - \mathbf{Z}_r \mathbf{Z}_r^\top\|_F \quad (25)$$

Let $\mathcal{S}'_r$ be a sampling matrix that comprises the same columns as $\mathcal{S}_r$ except for one, and let $\mathbf{Z}'_r$ denote $\sqrt{N/b} \mathbf{X} \mathcal{S}'_r$. Changing one column of the full sample $\mathcal{S}$ (the selection

matrix corresponding to $\{\mathbf{X}_r\}_{r=1}^s$ changes only one subsample $\mathcal{S}_r$ and thus only the single term $\|\mathbf{X}\mathbf{X}^\top - \mathbf{Z}_r\mathbf{Z}_r^\top\|_F$. Let $\mathbf{z}$ and $\mathbf{z}'$ denote the only differing column of $\mathbf{Z}_r$ and $\mathbf{Z}'_r$, then we have:

$$\begin{aligned} |\phi(\mathcal{S}'_r) - \phi(\mathcal{S}_r)| &\leq \frac{1}{s} \|\mathbf{z}'\mathbf{z}'^\top - \mathbf{z}\mathbf{z}^\top\|_F \\ &= \frac{1}{s} \|\mathbf{z}'\mathbf{z}'^\top - \mathbf{z}'\mathbf{z}'^\top + \mathbf{z}'\mathbf{z}'^\top - \mathbf{z}\mathbf{z}^\top\|_F = \frac{1}{s} \|(\mathbf{z}' - \mathbf{z})\mathbf{z}'^\top + \mathbf{z}(\mathbf{z}'^\top - \mathbf{z}^\top)\|_F \\ &\leq \frac{2}{s} \|\mathbf{z}' - \mathbf{z}\|_F \max\{\|\mathbf{z}\|_F, \|\mathbf{z}'\|_F\} \end{aligned} \quad (26)$$

Thus, the columns of $\mathbf{Z}_r$ are those of $\mathbf{X}$ as scaled by $\sqrt{N/n}$. The norm of the difference for two columns of $\mathbf{X}$ can be viewed as the norm of the difference for two feature vectors and thus can be derived to be bounded by θ_{dis} . Similarly, the norm of a single column of $\mathbf{X}$ is bounded by θ_{D} . This leads to the following inequality:

$$|\phi(\mathcal{S}'_r) - \phi(\mathcal{S}_r)| \leq \frac{2N\theta_{\text{dis}}\theta_{\text{D}}}{sn} \quad (27)$$

Further, following *Corollary 2* of [77], the expectation of $\phi(\mathcal{S}_r)$ can be bounded as follows:

$$E[\phi(\mathcal{S}_r)] = \frac{1}{s} \sum_{r=1}^s E[\|\mathbf{X}\mathbf{X}^\top - \mathbf{Z}_r\mathbf{Z}_r^\top\|_F] \leq \frac{N\theta_{\text{D}}}{\sqrt{n}} \quad (28)$$

And following *Theorem 1* of [78], we know:

$$\mathcal{P}[\phi(\mathcal{S}_r) - E[\phi(\mathcal{S}_r)] > \epsilon] \leq \exp\left(\frac{-2\epsilon^2}{\alpha(sn, N)|\phi(\mathcal{S}_r) - \phi(\mathcal{S}'_r)|^2}\right)$$

where

$$\alpha(sn, N) = \frac{sb(N - sn)}{(N - \frac{1}{2})(1 - 1/(2 \max\{sn, N - sn\}))} \quad (29)$$

Denote $\delta = \exp\left(\frac{-2\epsilon^2}{\alpha(sn, N)|\phi(\mathcal{S}_r) - \phi(\mathcal{S}'_r)|^2}\right)$ then we have:

$$\log \delta = \frac{-2\epsilon^2}{\alpha(sn, N)|\phi(\mathcal{S}_r) - \phi(\mathcal{S}'_r)|^2} \quad (30)$$

and then we have:

$$\epsilon = \left[\frac{\alpha(sn, N)}{2} |\phi(\mathcal{S}_r) - \phi(\mathcal{S}'_r)|^2 \log \frac{1}{\delta} \right]^{\frac{1}{2}} \quad (31)$$

From the above, it is noted that with probability $Pr[\phi(\mathcal{S}_r) - E(\phi(\mathcal{S}_r)) \geq \epsilon] \leq \delta$, the following condition holds:

$$\phi(\mathcal{S}_r) - E[\phi(\mathcal{S}_r)] \geq \left[\frac{\alpha(sn, N)}{2} |\phi(\mathcal{S}_r) - \phi(\mathcal{S}'_r)|^2 \log \frac{1}{\delta} \right]^{\frac{1}{2}} \quad (32)$$

then,

$$\begin{aligned} \phi(\mathcal{S}_r) &\geq \left[\frac{\alpha(sn, N)}{2} |\phi(\mathcal{S}_r) - \phi(\mathcal{S}'_r)|^2 \log \frac{1}{\delta} \right]^{\frac{1}{2}} + E[\phi(\mathcal{S}_r)] \\ &= \left[\frac{\alpha(sn, N)}{2} \left(\frac{2N\theta_{\text{dis}}\theta_{\text{D}}}{sn} \right)^2 \log \frac{1}{\delta} \right]^{\frac{1}{2}} + \frac{N\theta_{\text{D}}}{\sqrt{n}} \end{aligned} \quad (33)$$

In other words, with at least $(1-\delta)$ probability, the following condition holds:

$$\phi(\mathcal{S}_r) \leq \left[\frac{\alpha(sn, N)}{2} \left(\frac{2N\theta_{\text{dis}}\theta_{\text{D}}}{sn} \right)^2 \log \frac{1}{\delta} \right]^{\frac{1}{2}} + \frac{N\theta_{\text{D}}}{\sqrt{n}} \quad (34)$$

Combining (24), (25) and (34) we arrive at:

$$\begin{aligned} \|\mathbf{X}\mathbf{X}^\top - \mathbb{P}_L\mathbb{P}_L^\top\|_F^2 &\leq \|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell\mathbf{P}_\ell^\top\|_F^2 + \\ N\theta_{\text{D}}\sqrt{64\ell} &\left\{ \left[\frac{\alpha(sn, N)}{2} \left(\frac{2N\theta_{\text{dis}}\theta_{\text{D}}}{sn} \right)^2 \log \frac{1}{\delta} \right]^{\frac{1}{2}} + \frac{N\theta_{\text{D}}}{\sqrt{n}} \right\} \end{aligned} \quad (35)$$

Or

$$\begin{aligned} \|\mathbf{X}\mathbf{X}^\top - \mathbb{P}_L\mathbb{P}_L^\top\|_F &\leq \|\mathbf{X}\mathbf{X}^\top - \mathbf{P}_\ell\mathbf{P}_\ell^\top\|_F + \\ N\theta_{\text{D}} \left(\frac{64\ell}{n} \right)^{\frac{1}{4}} &\left\{ 1 + \left[\frac{2(N-sn)\theta_{\text{dis}}^2 \log \frac{1}{\delta}}{s(N-\frac{1}{2})(1-\frac{1}{2}\max\{sn, N-sn\})} \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} \end{aligned} \quad (36)$$

□

250 From the above, it is clear that the *divide-and-conquer* approximated scheme is capable of reconstructing the original input $\mathbf{X}$ with a bounded error. Further, it mitigates the ‘overfitting’ phenomena with features that tends to have improved generalized predictive performance since our current scheme resembles a form of ensemble of multiple SVD denoised data. Here we treat the random subsets as a result of rows-wise
255 denoising using random boolean values. It is worth noting that ensemble learning and denoising technology have been proven in the literature to learn more useful feature representation[79]. Our experimental results in the next section also verify the hypothesis.

Figure 3: Four groups of data: (a) *Small instance & Small dimension* (REAL-SIM, NEWS20.MUL, AUT_AVN, SECTOR); (b) *Small instance & Big dimension* (URL0, NEWS20.BIN); (c) *Big instance & Big dimension* (WEBSHAM, KDD2010, URL100); and (d) *Big instance & Small dimension* (RCV1.MUL, RCV1.BIN, EPSLION). $N \times d$: Instances $\times$ Dimensions. *Big dimension*: $d \geq 100,000$, *Big instance size*: $N \geq 100,000$.

5. Experimental Study and Performance Evaluation

260 In this section, we evaluate the performance of the proposed method, termed FSVD-H-ELM, and assess it against several *state-of-the-art* methods on 12 datasets which belong to 4 cases as show in Fig.3: Among them, all datasets except AUT_AVN¹ are obtained from the LIBSVM website². URL0 and URL100 are the URL datas from an anonymized 121-day subset of the ICML-09 URL data[80]. The original URL dataset
265 contains 121 independent subsets collected from 121 days. URL0 is the data from the first days using day-0 to predict day-1, while URL100 is the data from the first 100 days using them predict the rest days. These datasets have already been split into training and testing set as shown in Table 1.

¹<http://vikas.sindhwani.org/svmlin.html>

²<http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/>

All the experiments are conducted on a machine with the following configurations: MATLAB 2013a, Intel Xeon E5-2650, 2 GHz CPU (32 cores), 256G RAM. The comparison is conducted against other state-of-the-art large-scale machine learning algorithms including the Feature Generation Machine (FGM) [81], LibSVM[82], AMM[55], LLSVM[57], BSGD[56] and others. FGM proposed recently in [81] is used here for comparison, since it is one of the few *state-of-the-art* algorithms that has been designed to address feature selection and model learning of *Big dimensional* data and made available for download³. The main parameter of FGM is the number of selected features in each iteration. The larger selected features results in faster convergence while the smaller results in more accurate model. Here we set it to 100 for all problems as it can produce a good trading off between the efficiency and the effectiveness. For LibSVM, LLSVM and BSGD, the kernel is set to Gaussian kernel. The number of landmarks for LLSVM and the budgeted size of support vectors for BSGD-SVM is determined by cross-validation method. If the result is from other literature, it will be followed by the citation.

Table 1: Details of Datasets

Datasets	# Instances	# Dimensions	# Classes
REAL-SIM	72,309	20,958	2
RCV1.BIN	697,641	47,236	2
RCV1.MUL	534,135	47,236	53
NEWS20.MUL	19,928	62,061	20
URL0	20,000	3,231,955	2
NEWS20.BIN	19,996	1,355,191	2
WEBSpAM	350,000	16,609,143	2
KDD2010	20,012,498	29,890,095	2
EPSILON	500,000	2,000	2
AUT_AVN	71,175	20,707	2
URL100	2,396,130	3,231,961	2
SECTOR	9,619	55,197	105

³<http://www.tanmingkui.com/publications.html>

Table 2: Three different training/testing partitions

Datasets	Partition I		Partition II		Partition III	
	# Train set	# Test set	# Train set	# Test set	# Train set	# Test set
REAL-SIM	32,309	40,000	57,847	14,462	65,078	7,231
RCV1.BIN	677,399	20,242	558,110	139,531	627,880	69,761
RCV1.MUL	518,571	15,564	427,308	106,827	480,720	53,415
NEWS20.MUL	15,935	3,993	15,942	3,986	17,935	1,993
URL0	15,000	5,000	16,000	4,000	18,000	2,000
NEWS20.BIN	9,996	10,000	15,997	3,999	17,996	2,000
WEBSHAM	300,000	50,000	280,000	70,000	315,000	35,000
KDD2010	19,264,097	748,401	16,010,000	4,002,498	17,338,000	2,674,498
EPSILON	400,000	100,000	100,000	400,000	450,000	50,000
AUT_AVN	40,000	31,175	56,940	14,235	64,058	7,117
URL100	1,976,130	420,000	1,916,904	479,226	2,156,517	239,613
SECTOR	6,412	3,207	7,695	1,924	8657	962

5.1. Performance Comparison of FSVD-H-ELM, SVD-H-ELM and Classical ELM

285 For the classical ELM, SVD-H-ELM and the proposed FSVD-H-ELM algorithm, the Sigmoid activation function $g(x) = \frac{1}{1+e^{-\lambda x}}$ is considered, where λ is set to ‘1’ unless otherwise stated. The number of hidden nodes L is then chosen by cross-validation method with step size 100, and only the obtained optimized number of hidden nodes are reported. For FSVD-H-ELM, as the subset size $n = 200$ and $\rho = 0.5$ usually can
290 achieve the promising generalized performance at the lowest training time, we would set $n = 200$ and $\rho = 0.5$ unless otherwise sepecified. The details about the optimized parameter for settings for every datasets are reported in Table 3.

In order to perform a comprehensive comparison, the experimental studies are carried out on three different training/testing partitions as shown in Table 2. For each partition, 10 trials (in each trial the data will be shuffled before partition to attain different training and testing set) are conducted, and then, the experimental results including the average performance and standard derivation (mean $\pm$ std.) of the proposed FSVD-H-ELM and *state-of-the-art* methods considered in the study are summarized in Tables 4, 6 and 8. To evaluate the performance difference of the considered algorithms, paired t -test[83] is introduced to obtain t value and significant level p . The smaller the p value

Table 3: Parameters of FSVD-H-ELM, SVD-H-ELM and Classical ELM

Datasets	FSVD-H-ELM						SVD-H-ELM			Classical ELM		
	P. I		P. II		P. III		P. I	P. II	P. III	P. I	P. II	P. III
	L	(n, ρ)	L	(n, ρ)	L	(n, ρ)	L	L	L	L	L	L
REAL-SIM	4000	(200,0.5)	4000	(200,0.5)	4000	(200,0.5)	4000	4000	4000	5500	5500	5500
RCV1.BIN	3000	(200,0.5)	3000	(200,0.5)	3000	(200,0.5)	3000	3000	3000	4000	4000	4000
RCV1.MUL	3000	(200,0.5)	3000	(200,0.5)	3000	(200,0.5)	3000	3000	3000	4500	4500	4500
NEWS20.MUL	1000	(200,0.5)	1000	(200,0.5)	1000	(200,0.5)	1000	1000	1000	2000	2000	2000
URL0	600	(200,0.5)	600	(200,0.5)	600	(200,0.5)	600	600	600	1000	1000	1000
NEWS20.BIN	2000	(200,0.5)	2000	(200,0.5)	2000	(200,0.5)	2000	2000	2000	2500	2500	2500
WEBSHAM	1500	(200,0.5)	1500	(200,0.5)	1500	(200,0.5)	1500	1500	1500	2500	2500	2500
KDD2010	150	(200,0.5)	150	(200,0.5)	150	(200,0.5)	150	150	150	350	350	350
EPSILON	600	(200,0.5)	600	(200,0.5)	600	(200,0.5)	600	600	600	1500	1500	1500
AUT.AVN	2000	(200,0.5)	2000	(200,0.5)	2000	(200,0.5)	2000	2000	2000	3000	3000	3000
URL100	2000	(200,0.5)	2000	(200,0.5)	2000	(200,0.5)	2000	2000	2000	3000	3000	3000
SECTOR	2000	(200,0.5)	2000	(200,0.5)	2000	(200,0.5)	2000	2000	2000	3000	3000	3000

L : the number of hidden nodes, n : the subset size, ρ : the selection ratio.

Table 4: Performance of FSVD-H-ELM, SVD-H-ELM and Classical ELM on Partition I

Datasets	FSVD-H-ELM		SVD-H-ELM		Classical ELM	
	Training time (s)	Testing accuracy (%)	Training time (s)	Testing accuracy (%)	Training time (s)	Testing accuracy (%)
REAL-SIM	26	96.78 $\pm$ 0.046	153,413	96.8 $\pm$ 0.044	44	93.30 $\pm$ 0.054
RCV1.BIN	1440	97.10 $\pm$ 0.049	> 48h		1102	94.40 $\pm$ 0.053
RCV1.MUL	601	88.54 $\pm$ 0.052	> 48h		515	85.92 $\pm$ 0.044
NEWS20.MUL	45	84.94 $\pm$ 0.031	132,245	82.93 $\pm$ 0.024	79	81.07 $\pm$ 0.056
URL0	240	97.18 $\pm$ 0.042	114,261	96.70 $\pm$ 0.041	213	95.43 $\pm$ 0.048
NEWS20.BIN	414	95.80 $\pm$ 0.050	104,169	96.01 $\pm$ 0.045	400	87.85 $\pm$ 0.062
WEBSHAM	8,154	98.83 $\pm$ 0.034	OOM*		8,321	93.46 $\pm$ 0.041
KDD2010	1,769	88.74 $\pm$ 0.047	OOM		1,970	86.14 $\pm$ 0.055
EPSILON	452	89.99 $\pm$ 0.054	34,678	89.31 $\pm$ 0.047	365	87.13 $\pm$ 0.063
AUT.AVN	42	95.87 $\pm$ 0.048	14,822	95.48 $\pm$ 0.042	41	94.18 $\pm$ 0.052
URL100	610	99.01 $\pm$ 0.043	> 48h		589	97.73 $\pm$ 0.051
SECTOR	19	93.95 $\pm$ 0.054	108,334	93.88 $\pm$ 0.049	11	91.13 $\pm$ 0.063

* Out of memory.

Table 5: t value and significant level p of FSVD-H-ELM v.s. SVD-H-ELM and Classical ELM (Partition I)

	SVD-H-ELM (93.01%)	Classical ELM (90.64%)
FSVD-H-ELM (93.89%)	$t = 0.3714, p > 0.1$	$t = 1.6571, 0.01 > p > 0.001$

Table 6: Performance of FSVD-H-ELM, SVD-H-ELM and Classical ELM on Partition II

Datasets	FSVD-H-ELM		SVD-H-ELM		Classical ELM	
	Training time (s)	Testing accuracy (%)	Training time (s)	Testing accuracy (%)	Training time (s)	Testing accuracy (%)
REAL-SIM	46.4	97.59 $\pm$ 0.045	> 48h		79.2	93.86 $\pm$ 0.053
RCV1.BIN	1187.4	97.04 $\pm$ 0.046	> 48h		937.2	93.83 $\pm$ 0.054
RCV1.MUL	641	88.65 $\pm$ 0.053	> 48h		549.35	86.12 $\pm$ 0.043
NEWS20.MUL	72	86.33 $\pm$ 0.033	211,592	84.18 $\pm$ 0.022	126.40	82.37 $\pm$ 0.055
URL0	256.0	97.42 $\pm$ 0.040	121,880	96.82 $\pm$ 0.040	227	95.87 $\pm$ 0.046
NEWS20.BIN	662.4	96.16 $\pm$ 0.051	166670	96.21 $\pm$ 0.051	640	88.15 $\pm$ 0.060
WEBSHAM	7583.2	98.81 $\pm$ 0.032	<i>OOM</i> [*]		7738.5	93.16 $\pm$ 0.042
KDD2010	1470.2	88.72 $\pm$ 0.047	<i>OOM</i>		1637.3	86.11 $\pm$ 0.053
EPSILON	113	89.12 $\pm$ 0.051	21,655	89.12 $\pm$ 0.046	91.25	86.87 $\pm$ 0.063
AUT.AVN	59.17	95.98 $\pm$ 0.045	21099	95.63 $\pm$ 0.043	58.36	94.32 $\pm$ 0.054
URL100	581.1	89.97 $\pm$ 0.044	> 48h		571.3	97.75 $\pm$ 0.050
SECTOR	22.8	94.12 $\pm$ 0.051	118,653	93.96 $\pm$ 0.051	13.2	91.17 $\pm$ 0.062

^{*} *Out of memory.*

Table 7: t value and significant level p of FSVD-H-ELM v.s. SVD-H-ELM and Classical ELM (Partition II)

FSVD-H-ELM (92.65%)		SVD-H-ELM (90.80%)		Classical ELM (90.80%)	
FSVD-H-ELM (93.33%)	$t = 0.3622, p > 0.1$			$t = 1.3487, 0.01 > p > 0.001$	

Table 8: Performance of FSVD-H-ELM, SVD-H-ELM and Classical ELM on Partition III

Datasets	FSVD-H-ELM		SVD-H-ELM		Classical ELM	
	Training time (s)	Testing accuracy (%)	Training time (s)	Testing accuracy (%)	Training time (s)	Testing accuracy (%)
REAL-SIM	26	97.98 $\pm$ 0.045	> 48h		90.4	94.21 $\pm$ 0.052
RCV1.BIN	1076.1	97.08 $\pm$ 0.048	> 48h		1022.3	94.16 $\pm$ 0.055
RCV1.MUL	721.2	88.54 $\pm$ 0.053	> 48h		618	85.92 $\pm$ 0.043
NEWS20.MUL	81	84.94 $\pm$ 0.029	132,245	82.93 $\pm$ 0.023	142.2	81.07 $\pm$ 0.055
URL0	288	97.18 $\pm$ 0.043	114,261	96.70 $\pm$ 0.040	255.6	95.43 $\pm$ 0.047
NEWS20.BIN	745.2	95.80 $\pm$ 0.050	187500	96.01 $\pm$ 0.046	70	87.85 $\pm$ 0.063
WEBSHAM	8561.7	98.83 $\pm$ 0.032	<i>OOM</i> [*]		8737.1	93.46 $\pm$ 0.040
KDD2010	1532.6	88.74 $\pm$ 0.049	<i>OOM</i>		1706.8	86.14 $\pm$ 0.057
EPSILON	508.5	89.99 $\pm$ 0.052	39013	89.31 $\pm$ 0.049	410.625	87.13 $\pm$ 0.064
AUT.AVN	67.26	95.87 $\pm$ 0.049	23737	95.48 $\pm$ 0.043	65.6	94.18 $\pm$ 0.053
URL100	663	99.03 $\pm$ 0.042	> 48h		642.7	97.73 $\pm$ 0.052
SECTOR	25.6	94.25 $\pm$ 0.055	119,312	93.98 $\pm$ 0.047	14.85	91.19 $\pm$ 0.062

^{*} *Out of memory.*

Table 9: t value and significant level p of FSVD-H-ELM v.s. SVD-H-ELM and Classical ELM (Partition III)

FSVD-H-ELM (92.4%)		SVD-H-ELM (90.7%)		Classical ELM (90.7%)	
FSVD-H-ELM (94.02%)	$t = 0.4112, p > 0.1$			$t = 1.7223, 0.01 > p > 0.001$	

the more significant the difference. t value can be computed as follows:

$$t = \frac{\bar{\mathbf{a}}_i - \bar{\mathbf{a}}_j}{\sqrt{\frac{v_i^2}{n_i} + \frac{v_j^2}{n_j}}} \quad (37)$$

where $\mathbf{a}_i = a_{i,1}, a_{i,2}, \dots, a_{i,12}$ denote the testing accuracies on the twelve datasets of i -th algorithm, $\bar{\mathbf{a}}_i$ and $\bar{\mathbf{a}}_j$ denotes the mean value of $\mathbf{a}_i$ and $\mathbf{a}_j$, v_i^2 and v_j^2 represent the variance of $\mathbf{a}_i$ and $\mathbf{a}_j$, and n_i and n_j denotes the number of datasets (here $n_i = n_j = 12$). Then, by checking t -table [83], the significant level p is obtained by t value. The obtained t -test results are summarized in Tables 5, 7 and 9. With respect to the classical ELM, the training time incurred by FSVD-H-ELM and the classical ELM are observed to be competitive to one another on the twelve data sets. This may be attributed to the fact that, although the SVD operation of FSVD-H-ELM is originally thought to be more computationally intensive than the random assignment of hidden nodes in the classical ELM, it turns out that as the classical ELM often needs many more hidden nodes than is required by the FSVD-H-ELM, the overall computation time incurred by the classical ELM is equivalently high due to the need to perform the *Moore-Penrose generalized inverse* on a larger matrix (in the latter) when deriving the output weights. From Tables 5, 7 and 9, we can find that, on three partitions, the average testing accuracy, t -value and significant level p of FSVD-H-ELM and classical ELM are 93.89% vs. 90.64% ($t = 1.6571$, $0.01 > p > 0.001$), 93.33% vs. 90.80% ($t = 1.3487$, $0.01 > p > 0.001$), 94.02% vs. 90.7% ($t = 1.7223$ and $0.01 > p > 0.001$) respectively. This indicates that FSVD-H-ELM showcased significant improvements in the test performance with respect to classical ELM. FSVD-H-ELM outperformed classical ELM with 2~8% improvements. For example, on the *Big dimensional* dataset NEWS20.BIN with Partition I, FSVD-H-ELM achieved a test performance of 95.80%, which is a significant improvement over the classical ELM accuracy of 87.85%. On the *Small dimensional* datasets REAL-SIM and RCV1.BIN, FSVD-H-ELM achieved test accuracy performances of 96.78% and 97.1%, respectively, which are also superior to that of the classical ELM at 93.3% and 94.4%, respectively. These indicate that the FSVD-H-ELM is able to appropriately capture the structural characteristic of the data much better than the classical ELM (which uses random hidden nodes).

320 With respect to the SVD-H-ELM, experimental results have only been successful-
 ly obtained on seven of the *small instance* datasets `REAL-SIM`, `NEWS20.MUL`, `URL0`,
`NEWS20.BIN` and `AUT_AVN`. On the *Big instance* datasets, `RCV1.BIN`, `URL100` and
`RCV1.MULT`, the simulations failed to complete despite 2 days of computations (48
 hours). On `WEBSPPAM` and `KDD2010`, for instance, SVD-H-ELM went into the state of
 325 "out of memory (OOM)". From these results, it is clear that SVD-H-ELM is incapable
 of handling the *Big instance* dataset well as expected, thus the need for our introduc-
 tion of the *divide-and-conquer approximation scheme* to arrive at the efficient FSVD-
 H-ELM. The results obtained indicate that FSVD-H-ELM is thousands of times more
 efficient than the SVD-H-ELM counterpart in most cases. For example, on `REAL-SIM`,
 330 SVD-H-ELM consumed a CPU wall clock time of 251,212 seconds to model the data,
 while FSVD-H-ELM only took 26 seconds to perform the same task. More important-
 ly, the higher efficiency of FSVD-H-ELM is attained at no form of degradation in the
 test accuracy performance over SVD-H-ELM, in spite of the approximation. Interest-
 ingly, FSVD-H-ELM exhibits excellent test accuracy performance in most cases. As
 335 shown in Tables 5, 7 and 9, the average testing accuracy of FSVD-H-ELM vs. SVD-
 H-ELM on three partitions are 93.89% vs. 93.01% ($t = 0.3714$, $p > 0.1$), 93.33%
 vs. 92.65% ($t = 0.3622$, $p > 0.1$), and 94.02% vs. 92.4% ($t = 0.4112$ and $p > 0.1$)
 respectively. These showcased the improved test accuracies over the SVD-H-ELM.
 For example, on the `URL0` dataset with Partition I, FSVD-H-ELM achieved a test ac-
 340 curacy of 97.18%, which is slightly superior to the SVD-H-ELM of 96.70%. These
 results thus highlights the high efficacy of the FSVD-H-ELM in handling large-scale
 datasets. It is worth noting that the reasons for the reported high performances of
 the FSVD-H-ELM may be attributed to the ensemble like structure of FSVD-H-ELM,
 which integrates multiple SVD de-noised data subsets. Both ensemble and de-noising
 345 method have been proven to be effective for improving the resultant test accuracy per-
 formance in the literature [78]. FSVD-H-ELM thus inherits their benefits to generate
 the reported stable and excellent performances. Moreover, FSVD-H-ELM also elimi-
 nates the 'out of memory' problem that exists when dealing with *Big instance & Big*
dimensional dataset such as `WEBSPPAM` and `KDD2010` and shown to work with them
 350 efficiently and elegantly. It also showcased a test accuracy of 88.74% on the `KDD2010`

dataset that is not only superior to all the other *state-of-the-art* algorithms considered, it also completed the task within a short time span of 1,769 seconds training time.

5.2. Performance Comparison of FSVD-H-ELM to the Other State-of-the-art Machine Learning Methods

355 In this subsection, a comprehensive performance comparison of the FSVD-H-ELM is conducted against other *state-of-the-art* large-scale machine learning algorithms including the FGM[81], LLSVM, BSGD[56] and others. The performance results in terms of training time and testing accuracy are then summarized in Table 10. Here we reference and report the results from the relevant literatures if available. From the
360 results, the following observations have been made:

1. With respect to the FGM, which is a recently proposed *state-of-the-art* algorithm for solving *Big dimensional* dataset, FSVD-H-ELM exhibited competitive generalization performances. Notably, although FGM is implemented in C++, FSVD-H-ELM is shown to incur shorter training time costs than FGM in many
365 cases, despite the latter operating under a MATLAB environment. For example on NEWS20.BIN, FSVD-H-ELM took 414 seconds to achieve a test accuracy of 95.80%, while FGM took 454 seconds to achieve a similar accuracy of 95.08%. Furthermore, FGM is specially designed for two-class problems and cannot support multi-class problems directly, unless some extension is made. A possible
370 solution is to first decompose the multi-class problem into multiple binary-class problems, train the classifiers to solve these problems, and then reconstruct the solution of the multi-class problem from the outputs attained by the classifiers. In contrast, FSVD-H-ELM can be convenient used for binary-class, multi-class and regression problems seamlessly.
- 375 2. With respect to the other *state-of-the-art* algorithms, FSVD-H-ELM has been observed to attain better generalization performances than TSVM[84], LLSVM[57], BSGD[56], DTSVM[89] and AMM[55] on REAL-SIM, RCV1.MUL, NEWS20.MUL, NEWS20.BIN and KDD2010 at much lower training time. Although on the RCV1.BIN, URL0 and WEBSHAM datasets, FSVD-H-ELM did not perform sig-
380 nificantly better than SVM (RBF), FGM and DTSVM, however, its test accu-

Table 10: Performance evaluation of FSVD-H-ELM and other state-of-the-art machine learning algorithms

Datasets	Methods	Trainingtime (s)	Testingaccuracy (%)
Big instance & Small dimension			
REAL-SIM	FSVD-H-ELM	26	96.78
	FGM	675	96.52
	TSVM[84]	373	93.10
	DA[84]	1,129	92.80
RCV1.BIN	FSVD-H-ELM	1440	97.10
	FGM	10,923	97.42
	LLSVM[85]	1,800	95.77
	BSGD-SVM[56]	5,400	97.08
	LibSVM (RBF)[86]	72,720	97.83
EPSILON	FSVD-H-ELM	452	89.89
	CDN[87]	3000	89.82
	newGLMNET[87]	500	89.82
	FGM	1,912	89.64
	LLSVM	3,870	88.71
	BSGD-SVM(5 pass)	6,896	89.58
URL100	FSVD-H-ELM	640	98.97
	AMM(Batch)	1214	98.13
	AMM(Online)	1465	98.10
	FGM	1,213	98.22
	LLSVM	2,512	98.17
	BSGD-SVM	1,701	98.32
Small instance & Small dimension			
RCV1.MUL	FSVD-H-ELM	610	88.54
	K-Pegasos[88]	N.A.	84.50
	K-binaryLR[88]	N.A.	83.00
	Multi-class LR[88]	N.A.	88.50
	FGM	Unsupported*	
NEWS20.MUL	FSVD-H-ELM	45	84.94
	LASVM[89]	23,339	83.10
	DTSVM[89]	3,053	83.22
	CBD[89]	39,590	75.23
	Bagging[89]	35,176	76.55
	FGM	Unsupported*	
AUT_AVN	FSVD-H-ELM	42	95.87
	FGM	218	95.72
	AMM(Batch)	321	94.23
	AMM(Online)	346	94.10
	FGM	206	95.22
	LLSVM	531	94.97
	BSGD-SVM	318	95.32
Small instance & Big dimension			
NEWS20.BIN	FSVD-H-ELM	414	95.80
	FGM	454	95.08
	Forgetron[90]	N.A.	90.00
	Projectron++[90]	N.A.	95.00
URL0	FSVD-H-ELM	240	97.18
	FGM	727	97.38
	AMM	224	97.13
Big instance & Big dimension			
WEBSHAM	FSVD-H-ELM	8154	98.83
	CART[89]	29,332	98.44
	DTSVM[89]	63,015	99.03
KDD2010	FSVD-H-ELM	1,769	88.74
	L1-SVM[81]	1,500	88.40
	FGM[81]	1,870	88.70
	PROX-SLR[81]	1,000	88.40
SECTOR	FSVD-H-ELM	19	93.95
	LibSVM(OVA)[91]	—	92.80
	NB(OVA)[91]	—	64.30
	LibSVM(BCH63)[91]	—	93.3
	NB(BCH63)[91]	—	87.2
	FGM	Unsupported*	

* FGM is especially designed for two-class problems and can not be applicable for multi-class problems directly. It needs some extending methods such as decomposing the multi-class problem to several two-class problems.

racies of 97.10% on RCV1.BIN, 97.18% on URL0 and 98.83% on WEBSHAM, are competitive to the optimal results of the *state-of-the-art* methods, which are 97.83%, 97.38% and 99.03%, respectively. More important, FSVD-H-ELM is able to attain the improved or competitive test accuracies at a much shorter training time than the *state-of-the-art* algorithms considered in most cases. In particular, on the *Big instance & Big dimensional* dataset KDD2010, which has more than 29 million features and 190 million training instance samples, FSVD-H-ELM is very competitive in terms of test accuracy. It showcased a testing accuracy of 88.74%, which is an improvement over the L1-SVM (88.4%), FGM (88.7%) and PROX-SLR (88.4%).

5.3. Stability of The Divide & Conquer Fast Approximation

In this subsection, the stability of the divide & conquer fast approximation scheme proposed is studied and visualized. For each dataset, 20 independent experimental trials have been conducted and the performance accuracies are summarized in Fig. 4. In all the trials, The parameter settings (including subset size and selection ratio) is set as in Table 3. The data subsets are constructed via random sampling from the original dataset and the SVD hidden nodes are derived subsequently. The results in Fig.4 demonstrate excellent robustness of the proposed FSVD-H-ELM, where the testing accuracies are observed to remains stable across all 20 independent experimental trails.

5.4. The Effects of Data Subset Size n on SVD-H-ELM Performance

Here we conduct further experimental study to evaluate the influence of data subset size n on FSVD-H-ELM performances, i.e., testing accuracy and computational time. Fig.5 summarizes the testing accuracy and training time performances of the FSVD-H-ELM for different data subset size $n = [5, 10, 50, 200, 700, 1100, 1500, 3000]$ on the six data sets. From the figures, it is worth noting that the testing accuracy can be observed to remain relatively stable for different data subset size n , which highlights the robustness of the proposed FSVD-H-ELM. The training time, however, varies significantly for different subset size n . In particular, when n gets too large or too small,

Figure 4: FSVD-H-ELM testing accuracies of 20 trials on 12 data sets. In every trials the subsets are different. The testing accuracies are observed to remain stable across all 20 independent experimental trails.

Figure 5: Training time(in seconds) and Testing accuracy (in %) w.r.t different data subset size n . In particular, the testing accuracy can be observed to remain relatively stable for different data subset sizes n . The training time, however, varies significantly with n .

the computational time is noted to greatly increase. This can be comprehended with a
410 complexity analysis of the FSVD-H-ELM.

Considering a data subset of size n , the original dataset of N data samples is first
divided into $\lfloor \frac{N}{n} \rfloor$ data subsets $\{\mathbf{X}_r\}_{r=1}^{\lfloor \frac{N}{n} \rfloor}$. Assuming l hidden nodes are generated by
each data subset of size n . A concatenation of the hidden nodes derived from each of
the $s = \lceil \frac{L}{l} \rceil$ randomly chosen data subsets in $\{\mathbf{X}_r\}_{r=1}^{\lfloor \frac{N}{n} \rfloor}$ then forms a trained FSVD-
415 H-ELM network of L hidden nodes. Note that the time complexity of SVD on a data
subset of size n and dimension d is $\mathcal{O}(n^2d)$, while other operations such as data subset
retrieval takes a constant cost of Θ . Thus, FSVD-H-ELM has overall time complexity
of $\mathcal{O}(\frac{L}{l} \cdot (n^2d + \Theta)) \approx \mathcal{O}(Ld \cdot n + L\Theta \cdot \frac{1}{n})$. From the analysis, it is clear that when
 n gets too large, the first term of the complexity, i.e., the cost of SVD on s data subsets
420 dominates. Conversely, when n gets too small, the second term, i.e., the data subset
retrieval time dominates in comparison. As a result, the total computational cost of
FSVD-H-ELM at $\mathcal{O}(Ld \cdot n + L\Theta \cdot \frac{1}{n})$ inflates when n is either too large or too small. In
this manner, an appropriate data subset size that accords to the computational require-
ments of the user should be chosen. Based on our experimental study, a data subset
425 size in the range of $n \in [50, 500]$ is noted to function well in general.

6. Conclusion

This paper has proposed the FSVD-H-ELM, which is designed to work with *Big*
dimensional data. In contrast to existing approaches, we have embedded singular val-
ue decomposition hidden nodes into the classical ELM, while preserving the benefits
430 of ELM where hidden nodes need not be tuned. In our study, SVD hidden nodes
have been shown to be capable of capturing the structural properties of *Big dimen-*
sional data, thus providing a more meaningful representation in the hidden layer of
ELM. To address the high computational complexity of SVD, especially when deal-
ing with large-scale problems, i.e., *Big instance* and *Big dimensional* data, a divide
435 and conquer fast approximation scheme is made available in FSVD-H-ELM. The SVD
hidden nodes which are derived from multiple random data subsets are used in place
of the SVD hidden nodes derived from the entire data set, thus making it highly s-

calable to Big data. Comprehensive experiments have been conducted to validate the proposed FSVD-H-ELM on benchmark data sets. Further comparisons against several other *state-of-the-art* approaches verified and confirmed the superior generalization performance, robustness and efficiency of the proposed FSVD-H-ELM for solving *Big dimensional* data. Here we use SVD to obtain the hidden nodes, however, besides the use of SVD, it is worth noting that one may also choose other feature learning schemes including Non-negative Matrix Factorization (NMF), Kernel Principle Analysis (KPCA) and so on according to the problem of interest and requirement. This thus forms the immediate interest of our future works on Big Data analytics.

Acknowledgment

This work is partially supported by the ASTAR Thematic Strategic Research Programme (TSRP) Grant No. 1121720013, the Computational Intelligence Research Laboratory at NTU and conducted in collaboration with the Innovation fund research group 61221063; National Science Foundation of China 61100166, 61202184, 91118005, 91218301; Shaanxi New Star of Science & Technology 2013KJXX-29; New Star Team of Xian University of Posts & Telecommunications; Provincial Key Disciplines Construction Fund of General Institutions of Higher Education in Shaanxi.

References

- [1] Y.-T. Zhai, Y.-S. Ong, I. W. Tsang, The emerging big dimensionality, *IEEE Computational Intelligence Magazine* 9 (3) (2014) 14–26.
- [2] R. Sarunas, P. Vitalijus, On dimensionality, sample size, classification error, and complexity of classification algorithm in pattern recognition, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 2 (3) (1980) 242–252.
- [3] P. Hanchuan, L. Fulmi, D. Chris, Feature selection based on mutual information criteria of max-dependency, max-relevance, and min-redundancy, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 27 (8) (2005) 1226–1238.

- [4] I. Chaturvedi, Y.-S. Ong, R. V. Arumugam, Deep transfer learning for classification of time-delayed gaussian networks, *Signal Processing* 110 (2015) 250 – 262, machine learning and signal processing for human pose recovery and behavior analysis. doi:http://dx.doi.org/10.1016/j.sigpro.2014.09.009.
URL http://www.sciencedirect.com/science/article/pii/S0165168414004198
- [5] C.-W. Seah, I. W. Tsang, Y.-S. Ong, Transfer ordinal label learning, *Neural Networks and Learning Systems*, IEEE Transactions on 24 (11) (2013) 1863–1876.
- [6] L. Feng, Y. Ong, M. Lim, I. Tsang, Memetic search with inter-domain learning: A realization between cvrp and carp, *Evolutionary Computation*, IEEE Transactions on PP (99) (2014) 1–1. doi:10.1109/TEVC.2014.2362558.
- [7] C.-W. Seah, Y.-S. Ong, I. Tsang, Combating negative transfer from predictive distribution differences, *Cybernetics*, IEEE Transactions on 43 (4) (2013) 1153–1165. doi:10.1109/TSMCB.2012.2225102.
- [8] C.-W. Seah, I. W. Tsang, Y.-S. Ong, Transductive ordinal regression, *Neural Networks and Learning Systems*, IEEE Transactions on 23 (7) (2012) 1074–1086.
- [9] G. Schaefer, B. Krawczyk, M. E. Celebi, H. Iyatomi, An ensemble classification approach for melanoma diagnosis, *Memetic Computing* 6 (4) (2014) 233–240.
- [10] J. A. Bullinaria, K. AlYahya, Artificial bee colony training of neural networks: comparison with back-propagation, *Memetic Computing* 6 (3) (2014) 171–182.
- [11] D. Rummelhart, Learning representations by back-propagation errors, *Nature* 323 (1986) 533–536.
- [12] M. A. Salama, A. E. Hassanien, K. Revett, Employment of neural network and rough set in meta-learning, *Memetic Computing* 5 (3) (2013) 165–177.
- [13] G.-B. Huang, Q.-Y. Zhu, C.-K. Siew, Extreme learning machine: theory and applications, *Neurocomputing* 70 (1) (2006) 489–501.

- [14] G.-B. Huang, Z. Bai, L. L. C. Kasun, C. M. Vong, Local receptive fields based extreme learning machine, *Computational Intelligence Magazine, IEEE* 10 (2) (2015) 18–29.
- [15] L. L. C. Kasun, H. Zhou, G.-B. Huang, C. M. Vong, Representational learning with extreme learning machine for big data, *IEEE Intelligent Systems* 28 (6) (2013) 1–4.
- [16] G.-B. Huang, L. Chen, C.-K. Siew, Universal approximation using incremental constructive feedforward networks with random hidden nodes, *IEEE Transactions on Neural Networks* 17 (4) (2006) 879–892.
- [17] G.-B. Huang, H. Zhou, X. Ding, R. Zhang, Extreme learning machine for regression and multiclass classification, *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics* 42 (2) (2012) 513–529.
- [18] H. Chen, J. Peng, Y. Zhou, L. Li, Z. Pan, Extreme learning machine for ranking: Generalization analysis and applications, *Neural Networks* 53 (0) (2014) 119 – 126.
- [19] A. Bueno-Crespo, P. J. Garca-Laencina, J.-L. Sancho-Gmez, Neural architecture design based on extreme learning machine, *Neural Networks* 48 (0) (2013) 19 – 24.
- [20] M. Fernandez-Delgado, E. Cernadas, S. Barro, J. Ribeiro, J. Neves, Direct kernel perceptron (dkp): Ultra-fast kernel elm-based classification with non-iterative closed-form weight calculation, *Neural Networks* 50 (0) (2014) 60 – 71.
- [21] G.-B. Huang, X. Ding, H. Zhou, Optimization method based extreme learning machine for classification, *Neurocomputing* 74 (1) (2010) 155–163.
- [22] H.-J. Rong, Y.-S. Ong, A.-H. Tan, Z. Zhu, A fast pruned-extreme learning machine for classification problem, *Neurocomputing* 72 (13) (2008) 359 – 366, machine Learning for Signal Processing (MLSP 2006) / Life System Modelling, Simulation, and Bio-inspired Computing (LSMS 2007).

doi:<http://dx.doi.org/10.1016/j.neucom.2008.01.005>.

URL <http://www.sciencedirect.com/science/article/pii/S0925231208000738>

520

[23] E. Cambria, G.-B. Huang, L. L. C. Kasun, H. Zhou, C. M. Vong, J. Lin, J. Yin, Z. Cai, Q. Liu, K. Li, V. C. Leung, L. Feng, Y.-S. Ong, M.-H. Lim, A. Akusok, A. Lendasse, F. Corona, R. Nian, Y. Miche, P. Gastaldo, R. Zunino, S. Decherchi, X. Yang, K. Mao, B.-S. Oh, J. Jeon, K.-A. Toh, A. B. J. Teoh, J. Kim, H. Yu, Y. Chen, J. Liu, Extreme learning machines [trends controversies], Intelligent Systems, IEEE 28 (6) (2013) 30–59. doi:10.1109/MIS.2013.140.

525

[24] E. Kan, M. Lim, Y. Ong, A. Tan, S. Yeo, Extreme learning machine terrain-based navigation for unmanned aerial vehicles, Neural Computing and Applications 22 (3-4) (2013) 469–477. doi:10.1007/s00521-012-0866-9.

530

URL <http://dx.doi.org/10.1007/s00521-012-0866-9>

[25] W.-Y. Deng, Q.-H. Zheng, Z.-M. Wang, Cross-person activity recognition using reduced kernel extreme learning machine, Neural Networks 53 (0) (2014) 1 – 7.

[26] J. Butcher, D. Verstraeten, B. Schrauwen, C. Day, P. Haycock, Reservoir computing and extreme learning machines for non-linear time-series data analysis, Neural Networks 38 (0) (2013) 76 – 89.

535

[27] Y. Xu, Z. Y. Dong, J. H. Zhao, P. Zhang, K. P. Wong, A reliable intelligent system for real-time dynamic security assessment of power systems, Power Systems, IEEE Transactions on 27 (3) (2012) 1253–1263.

[28] B. P. Chacko, V. V. Krishnan, G. Raju, P. B. Anto, Handwritten character recognition using wavelet energy and extreme learning machine, International Journal of Machine Learning and Cybernetics 3 (2) (2012) 149–161.

540

[29] A. Mohammed, R. Minhas, Q. J. Wu, M. Sid-Ahmed, Human face recognition based on multidimensional pca and extreme learning machine, Pattern Recognition 44 (10) (2011) 2588 – 2597.

- 545 [30] S. Saraswathi, S. Sundaram, N. Sundararajan, M. Zimmermann, M. Nilsen-Hamilton, Icga-pso-elm approach for accurate multiclass cancer classification resulting in reduced gene sets in which genes encoding secreted proteins are highly represented, *IEEE/ACM Transactions on Computational Biology and Bioinformatics* 8 (2) (2011) 452–463.
- 550 [31] J. Luo, C.-M. Vong, P.-K. Wong, Sparse bayesian extreme learning machine for multi-classification, *Neural Networks and Learning Systems, IEEE Transactions on* 25 (4) (2014) 836–843.
- [32] X. Liu, S. Lin, J. Fang, Z. Xu, Is extreme learning machine feasible? a theoretical assessment (part i), *Neural Networks and Learning Systems, IEEE Transactions on* 26 (1) (2015) 7–20.
- 555 [33] S. Lin, X. Liu, J. Fang, Z. Xu, Is extreme learning machine feasible? a theoretical assessment (part ii), *Neural Networks and Learning Systems, IEEE Transactions on* 26 (1) (2015) 21–34.
- [34] H. Jaeger, The echo state approach to analysing and training recurrent neural networks-with an erratum note, Bonn, Germany: German National Research Center for Information Technology GMD Technical Report 148 (2001) 1–47.
- 560 [35] H. Jaeger, H. Haas, Harnessing nonlinearity: Predicting chaotic systems and saving energy in wireless communication, *Science* 304 (5667) (2004) 78–80.
- [36] W. Maass, T. Natschläger, H. Markram, Real-time computing without stable states: A new framework for neural computation based on perturbations, *Neural computation* 14 (11) (2002) 2531–2560.
- 565 [37] D. Lowe, Adaptive radial basis function nonlinearities, and the problem of generalisation, in: *Artificial Neural Networks, 1989., First IEE International Conference on (Conf. Publ. No. 313)*, 1989, pp. 171–175.
- 570 [38] W. Schmidt, M. Kraaijveld, R. Duin, Feedforward neural networks with random weights, in: *Pattern Recognition, 1992. Vol.II. Conference B: Pattern Recognition*

Methodology and Systems, Proceedings., 11th IAPR International Conference on, 1992, pp. 1–4.

- 575 [39] Y. Pao, Adaptive pattern recognition and neural networks, Reading, MA (US); Addison-Wesley Publishing Co., Inc., 1989.
- [40] G.-B. Huang, What are extreme learning machines? filling the gap between frank rosenblatt’s dream and john von neumann’s puzzle, Cognitive Computation in press.
- [41] G. Huang, G.-B. Huang, S. Song, K. You, Trends in extreme learning machines: 580 A review, Neural Networks 61 (0) (2015) 32 – 48.
- [42] G.-B. Huang, An insight into extreme learning machines: Random neurons, random features and kernels, Cognitive Computation 6 (3) (2014) 376–390.
- [43] B. Widrow, A. Greenblatt, Y. Kim, D. Park, The no-prop algorithm: A new learning algorithm for multilayer neural networks, Neural Networks 37 (2013) 182– 585 188.
- [44] Z. Bai, G.-B. Huang, D. Wang, H. Wang, M. B. Westover, Sparse extreme learning machine for classification, IEEE Transactions on Cybernetics 44 (10) (2014) 1858–870.
- [45] G. G. H, V. L. C. F, Matrix computations, Vol. 3, JHU Press, 2012.
- 590 [46] T. Zhang, Solving large scale linear prediction problems using stochastic gradient descent algorithms, in: Proceedings of the twenty-first international conference on Machine learning, ACM, 2004, p. 116.
- [47] I. W. Tsang, J. T. Kwok, P.-M. Cheung, Core vector machines: Fast svm training on very large data sets, J. Mach. Learn. Res. 6 (2005) 363–392.
- 595 [48] T. Joachims, Training linear svms in linear time, in: Proceedings of the 12th ACM SIGKDD international conference on Knowledge discovery and data mining, ACM, 2006, pp. 217–226.

- [49] C.-J. Hsieh, K.-W. Chang, C.-J. Lin, S. S. Keerthi, S. Sundararajan, A dual coordinate descent method for large-scale linear svm, in: Proceedings of the 25th international conference on Machine learning, ACM, 2008, pp. 408–415.
- [50] A. Bordes, L. Bottou, P. Gallinari, Sgd-qn: Careful quasi-newton stochastic gradient descent, *J. Mach. Learn. Res.* 10 (2009) 1737–1754.
- [51] Z. Zhu, W. Chen, G. Wang, C. Zhu, Z. Chen, P-packsvm: Parallel primal gradient descent kernel svm, in: Data Mining, 2009. ICDM '09. Ninth IEEE International Conference on, 2009, pp. 677–686.
- [52] Y.-W. Chang, C.-J. Hsieh, K.-W. Chang, M. Ringgaard, C.-J. Lin, Training and testing low-degree polynomial data mappings via linear svm, *J. Mach. Learn. Res.* 11 (2010) 1471–1490.
- [53] H.-F. Yu, C.-J. Hsieh, K.-W. Chang, C.-J. Lin, Large linear classification when data cannot fit in memory, in: Proceedings of the 16th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, KDD '10, ACM, New York, NY, USA, 2010, pp. 833–842.
- [54] S. Shalev-Shwartz, Y. Singer, N. Srebro, A. Cotter, Pegasos: Primal estimated sub-gradient solver for svm, *Mathematical programming* 127 (1) (2011) 3–30.
- [55] Z. Wang, N. Djuric, K. Crammer, S. Vucetic, Trading representability for scalability: Adaptive multi-hyperplane machine for nonlinear classification, in: Proceedings of the 17th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining(KDD11), 2011, pp. 24–32.
- [56] Z. Wang, C. Koby, V. Slobodan, Breaking the curse of kernelization: Budgeted stochastic gradient descent for large-scale SVM training, *Journal of Machine Learning Research* 13 (1) (2012) 3103–3131.
- [57] K. Zhang, L. Lan, Z. Wang, F. Moerchen, Scaling up kernel SVM on limited resources: A low-rank linearization approach., in: Proceedings of the 15 th International Conference on Artificial Intelligence and Statistics (AISTATS), Vol. 22, 2012, pp. 1425–1434.

- [58] C.-J. Hsieh, S. Si, I. S. Dhillon, A divide-and-conquer solver for kernel support vector machines, arXiv preprint arXiv:1311.0914.
- [59] A. Rahimi, B. Recht, Weighted sums of random kitchen sinks: Replacing minimization with randomization in learning, in: D. Koller, D. Schuurmans, Y. Bengio, L. Bottou (Eds.), *Advances in Neural Information Processing Systems 21*, Curran Associates, Inc., 2009, pp. 1313–1320.
- [60] Y. Zhai, Y.-S. Ong, I. Tsang, The emerging "big dimensionality", *Computational Intelligence Magazine*, IEEE 9 (3) (2014) 14–26.
- [61] Y. Mu, G. Huay, W. Fan, S.-F. Chang, Hash-SVM: Scalable kernel machines for large-scale visual classification, in: *Conference on Visualization and Pattern Recognition(CVPR)*, 2014, pp. 1–8.
- [62] A. Verdaldi, A. Zisserman, Efficient additive kernels via explicit feature maps, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 34 (3) (2012) 480–492.
- [63] S. Paisitkriangkrai, C. Shen, A. van den Hengel, A scalable stagewise approach to large-margin multiclass loss-based boosting, *IEEE Transactions on Neural Networks and Learning Systems* 25 (5) (2014) 1002–1013.
- [64] S. Shalev-Shwartz, T. Zhang, Accelerated proximal stochastic dual coordinate ascent for regularized loss minimization, in: *Proceedings of the 31st International Conference on Machine Learning*, 2014, pp. 64–72.
- [65] A. Krizhevsky, I. Sutskever, G. E. Hinton, Imagenet classification with deep convolutional neural networks, in: *Advances in Neural Information Processing Systems(NIPS'2012)*, 2012, pp. 1–9.
- [66] M. Tan, I. W. Tsang, L. Wang, Minimax sparse logistic regression for very high-dimensional feature selection, *IEEE Transactions on Neural Networks and Learning Systems* 24 (10) (2013) 1609–1622.

- [67] Q. Mao, I. W. Tsang, Efficient multi-template learning for structured prediction, *IEEE Transactions on Neural Networks and Learning Systems*. 24 (2) (2013) 248–261.
- 655 [68] A. Jordan, On discriminative vs. generative classifiers: A comparison of logistic regression and naive bayes, *Advances in neural information processing systems* 14 (2002) 841.
- [69] S. Günter, N. N. Schraudolph, S. V. N. Vishwanathan, Fast iterative kernel principal component analysis, *Journal of machine learning Research* 8 (2007) 1893–
660 1918.
- [70] A. K. Jain, R. P. W. Duin, J. Mao, Statistical pattern recognition: A review, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 22 (1) (2000) 4–37.
- [71] J. Fan, S. Richard, Y. Wu, Ultrahigh dimensional feature selection: beyond the linear model, *Journal of Machine Learning Research* 10 (2009) 2013–2038.
- 665 [72] C. Pierre, Independent component analysis, a new concept?, *Signal processing* 36 (3) (1994) 287–314.
- [73] E. Bingham, H. Mannila, Random projection in dimensionality reduction: applications to image and text data, in: *Proceedings of the seventh ACM SIGKDD international conference on Knowledge discovery and data mining (KDD '01)*,
670 2001, pp. 245–250.
- [74] E. Carl, Y. Gale, The approximation of one matrix by another of lower rank, *Psychometrika* 1 (3) (1936) 211–218.
- [75] N. Halko, P.-G. Martinsson, J. A. Tropp, Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions, *SIAM review* 53 (2) (2011) 217–288.
675
- [76] D. Achlioptas, F. Mcsherry, Fast computation of low rank matrix approximations, *Journal of the ACM* 54 (2) (2007) 611–618.

- [77] S. Kumar, M. Mohri, A. Talwalkar, Sampling methods for the Nystrom method, *Journal of Machine Learning Research* 13 (4) (2012) 981–1006.
- 680 [78] S. Kumar, M. Mohri, A. Talwalkar, Ensemble Nystrom method, in: *Advances in Neural Information Processing Systems* 22 (NIPS 2009), 2009, pp. 1–9.
- [79] P. Vincent, H. Larochelle, I. Lajoie, Y. Bengio, P.-A. Manzagol, Stacked denoising autoencoders: Learning useful representations in a deep network with a local denoising criterion, *Journal of Machine Learning Research* 11 (2010) 3371–3408.
- 685 [80] J. Ma, L. K. Saul, S. Savage, G. M. Voelker, Identifying suspicious urls: an application of large-scale online learning, in: *Proceedings of the 26th Annual International Conference on Machine Learning*, ACM, Montreal, Canada, 2009, pp. 681–688.
- [81] M. Tan, I. W. Tsang, L. Wang, Towards ultrahigh dimensional feature selection
690 for big data, *Journal of Machine Learning Research* 15 (2014) 1371–1429.
- [82] C.-C. Chang, C.-J. Lin, LIBSVM: A library for support vector machines, *ACM Transactions on Intelligent Systems and Technology* 2 (2011) 1–27.
- [83] J. Rice, *Mathematical statistics and data analysis*, Cengage Learning, 2006.
- [84] V. Sindhwani, S. S. Keerthi, Large scale semi-supervised linear SVMs, in: *Proceedings of the 29th Annual International ACM SIGIR Conference on Research and Development in Information Retrieval*, ACM, New York, NY, USA, 2006,
695 pp. 477–484.
- [85] W.-Y. Deng, Q.-H. Zheng, L. Chen, Regularized extreme learning machine, in: *Proceedings of IEEE Symposium on Computational Intelligence and Data Mining*,
700 IEEE, 2009, pp. 389–395.
- [86] K. Choi, K.-A. Toh, H. Byun, Incremental face recognition for large-scale social network services, *Pattern Recognition* 45 (8) (2012) 2868–2883.
- [87] G.-X. Yuan, C.-H. Ho, C.-J. Lin, An improved glmnet for l1-regularized logistic regression, *J. Mach. Learn. Res.* 13 (1) (2012) 1999–2030.

- 705 [88] D. Chen, W. Chen, Q. Yang, Characterizing inverse time dependency in multi-class learning, in: Proceedings of the 11th IEEE International Conference on Data Mining (ICDM),, 2011, pp. 1020–1025.
- [89] F. Chang, C.-Y. Guo, X.-R. Lin, C.-J. Lu, Tree decomposition for large-scale SVM problems, *Journal of Machine Learning Research* 11 (2010) 2935–2972.
- 710 [90] F. Orabona, J. Keshet, B. Caputo, Bounded kernel-based online learning, *Journal of Machine Learning Research* 10 (2009) 2643–2666.
- [91] J. D. M. Rennie, R. Rifkin, Improving multiclass text classification with the support vector machine, Tech. rep. (2001).