Nonlinear dynamic modeling of rotor system supported by angular contact ball bearings

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Abstract

In current bearing dynamic models, the displacement coordinate relations are usually utilized to approximately obtain the contact deformations between the rolling element and raceways, and then the nonlinear restoring forces of the rolling bearing could be calculated accordingly. Although the calculation efficiency is relatively higher, the accuracy is lower as the contact deformations should be solved through iterative analysis. Thus, an improved nonlinear dynamic model is presented in this paper. Considering the preload condition, surface waviness, Hertz contact and elastohydrodynamic lubrication, load distribution analysis is solved iteratively to more accurately obtain the contact deformations and angles between the rolling balls and raceways. The bearing restoring forces are then obtained through iteratively solving the load distribution equations at every time step. Dynamic tests upon a typical rotor system supported by two angular contact ball bearings are conducted to verify the model. Through comparisons, the differences between the nonlinear dynamic model and current models are also pointed out. The effects of axial preload, rotor eccentricity and inner/outer waviness amplitudes on the dynamic response are discussed in detail.

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Key words: angular contact ball bearing, nonlinear dynamic model, surface waviness, preload condition

Nomenclature

- $A_{in}, B_{in}, A_{out}, B_{out}, C_n$ Various waviness amplitudes
- $c_{\eta p}$ Viscosity-pressure coefficient
- $\mathbf{C}$ Damping matrix of the rotor-bearing system
- d_e Pitch diameter of the bearing
- d_b Ball diameter
- d_{ri}, d_{ro} Diameters of inner and outer races
- E Material elastic modulus
- E' Effective elastic modulus
- $f_{bL1}, f_{bL2}, f_{bL3}, f_{bL4}, f_{bL5}$ Restoring forces of bearing L
- $f_{bR1}, f_{bR2}, f_{bR3}, f_{bR4}, f_{bR5}$ Restoring forces of bearing R
- f_u, f_v, f_w, m^u, m^v Preloads applied on the inner race
- F_{cj}, M_{gj} Centrifugal and gyroscopic forces due to the rotation and revolution of the ball
- $\mathbf{f}_b$ Resultant force vector of the outer race
- $\mathbf{f}_p$ Preload vector of the inner race
- $\mathbf{F}_b$ Nonlinear restoring force vector induced by the two bearings
- $\mathbf{F}_e$ Unbalanced mass excitation vector of the rotor-bearing system
- $\mathbf{F}_g$ Self gravity vector of the rotor-bearing system
- G Dimensionless elastic modulus
- $\mathbf{G}$ Gyroscopic matrix of the rotor-bearing system
- h_{ci}, h_{co} Oil film thicknesses for the rolling ball contacting with the inner

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and outer raceways

IF_n Characteristic frequencies of the inner waviness

$k_{b1}, k_{b2}, k_{b3}, k_{b4}, k_{b5}$ Five linear stiffness coefficients of the rotor system

$k_{L1}, k_{L2}, k_{L12}, k_{R1}, k_{R2}, k_{R12}$ Stiffness coefficients of the bearing supports

K_i, K_o Contact stiffness coefficients between the ball and inner/outer races

$\mathbf{K}_n$ Stiffness matrix of the rotor-bearing system

l_1, l_2 Distances from rotor to bearing R and L

L_{ij}, l_{ij} Distance from inner groove curvature center to the ball mass center before and after deflection

L_{oj}, l_{oj} Distance from outer groove curvature center to the ball mass center before and after deflection

m_b, I_b Mass and moment of inertia of the ball

m_L, m_R Mass coefficients of the bearing supports

m_s, I_{sd}, I_{sp} Mass, diametral and polar moments of inertia of the rotor

m_{su}, e_s Unbalanced mass and eccentricity of the rotor

$\mathbf{M}$ Mass matrix of the rotor-bearing system

n_{in}, n_{out}, n_{ba} Harmonic orders of the waviness

N_b Number of rolling ball

OF_n Characteristic frequencies of the outer waviness

p_{ij}, p_{oj} Circumferential waviness of inner and outer raceways

q_{ij}, q_{oj} Axial waviness of inner and outer raceways

$\mathbf{q}$ System displacement vector

Q_{ij}, Q_{oj} Contact forces between the ball and inner/outer races

$Q_{yjL}, Q_{zjL}, Q_{\theta xjL}$ Reaction forces from the j th ball to the outer race of bearing L

$Q_{yjR}, Q_{zjR}, Q_{\theta xjR}$ Reaction forces from the j th ball to the outer race of

bearing R

r_{go} Groove curvature radius of the outer race

r_{gi} Groove curvature radius of the inner race

R_o Distance from the rotating axis center of outer race to the groove curvature center

R_x Equivalent radius along the ball rolling direction

t Time

$\mathbf{T}_j, \mathbf{T}_L, \mathbf{T}_R$ Transformation matrices

$u, v, w, \theta^u, \theta^v$ Five degrees of freedom of the inner race

u_{ent} Equivalent rotating speed

u_L, v_L, u_R, v_R Vibrational displacements of the bearing supports

$u_s, v_s, w_s, \theta_s^u, \theta_s^v$ Three translational and two angular displacements of the rotor

U Dimensionless speed

$\mathbf{u}$ Freedom vector of the inner race

w_{ij}, w_{oj} Rolling ball's waviness

W Dimensionless load

$x - y - z$ Rotating bearing coordinate

$X - Y - Z$ Fixed bearing coordinate

$X_s - Y_s - Z_s$ Rotor system coordinate

$X_{sL} - Y_{sL} - Z_{sL}$ Local coordinate for bearing L

$X_{sR} - Y_{sR} - Z_{sR}$ Local coordinate for bearing R

α_j Angle between self-rotation axis and the z axis

β_0 Nominal contact angle

δ_j Displacement vector of the outer groove curvature center

δ_{ij}, δ_{oj} Contact deflections between the ball and inner/outer races

$\Delta \mathbf{w}_j$ Additional displacements due to the waviness

$\zeta_j, \eta_j, \theta_j$	Displacements of the outer curvature center corresponding the j th ball
η_0	Lubricant viscosity at the atmospheric pressure and temperature
κ	Elliptical ratio
$\lambda_{ij}, \lambda_{oj}$	Friction coefficients between the ball and inner/outer races
ν	Poisson's ratio
ξ, ϵ	First and second kinds of elliptical integration
ς	Proportional damping coefficient
ϕ_0	Initial location angle of the ball
ϕ_{cj}	Location angle of the j th ball
$\psi_{in}, \varphi_{in}, \psi_{out}, \varphi_{out}, \vartheta_{ba}$	Phases of the waviness
ω_{cj}	Orbital speed of the j th ball
ω_o	Angular speed of outer race
ω_{sj}	Self-rotation angular speed of the j th ball
$\mathfrak{R}$	Equivalent curvature radius

1 Introduction

The rolling element bearing is the core supporting component of many rotating machinery. Its dynamic behavior plays a decisive role in the performance, operation reliability and service life of the entire equipment. It is well known that some key factors, including the Hertzian contacts, bearing radial clearance, surface waviness, preload condition and so on, would significantly affect the dynamic behaviors of rolling element bearings.

Thus, appropriately modeling one or more key factors of the bearing has

been the focus of most current investigations. Jones [1] first analyzed the modeling of an elastically constrained ball and radial roller bearings, and proposed a general bearing model with five degrees of freedom considering gyroscopic moments and centrifugal forces of the balls. Tiwari et al. [2,3] proposed a widely used deep-groove ball bearing model taking into account both radial clearance and Hertzian contact characteristics. Considering the effect of radial internal clearance, Harsha et al. [4] presented a dynamic model for the deep-groove ball bearing based on the analytical mechanics method. Taking more factors into account, they subsequently set up an analytical model of the high-speed rotor supported on ball bearings [5]. Later, they also proposed a dynamic model of the rotor-bearing system with the effect of cage run-out and finite number of balls [6]. Based upon the model presented by Tiwari and Harsha, further research was carried out. Tomovic et al. [7] performed the response analysis of a rigid rotor supported by unloaded ball bearings. Lioulios and Antoniadis [8] investigated the effect of fluctuations of the rotational speed on dynamic behaviors of the rolling element bearings. It was concluded that even a small fluctuation of the rotating speed could cause significant changes of the dynamic behaviors of the rotating system. Ghafari et al. [9] proposed a nonlinear dynamic model of the balanced fault-free ball bearings based on the Hertzian contact theory. It was found that the equilibrium point of the bearing underwent a supercritical pitchfork bifurcation as the bearing internal clearance increased. Kappaganthu and Nataraj [10] developed a nonlinear model of the rotor-bearing system upheld on rolling element bearings with clearance. Considering the fault on the bearing, Tadina et al. [11] presented an improved dynamic model of the ball bearing. The centrifugal load effect of the ball and the clearance of the bearing were taken into account. Li and

Shin [12,13] set up an integrated dynamic model of high speed spindles. With that model, they found that more accurate thermal distribution could be predicted when the effects of bearing load were taken into account. Recently, Wang et al. [14] studied dynamic characteristics of angular ball bearing with non-linear vibration of rotor system, and found that the rotor vibration significantly affects the spin-to-roll ratio of rolling balls. Babu et al. [15] presented a dynamic model of a rigid shaft supported on angular contact ball bearings. They found that the effect of frictional moment on the vibrations of the system was significantly dependent on the bearing load.

Besides the radial clearance and Hertzian contact effect, the other key factors, such as the surface waviness, have also been paid close attention in the literature. Waviness is the global sinusoidally shaped imperfection on the surfaces of the bearing components. In general, these imperfections are due to the irregularities during the grinding and honing process of the bearing components [16]. Nowadays, the amplitude of the waviness in rolling element bearings is at the micrometer level. Despite that, waviness can still produce significant vibrations. In this field, one of the first systematic investigations was made by Tallian and Gustafson [17]. In the research, it was pointed out that race waviness with lower order affected the amplitude of the vibrations at the ball passage frequency. From the point of view of bearing manufacturing, Yhland [18] firstly focused on the experimental measurement of waviness. Subsequently, Yhland [19] proposed a linear analytical model to investigate the effect of geometrical imperfections on the vibration of the rotor-bearing assembly. Akturk [20] investigated the effect of bearing waviness on the vibration of the shaft. The research revealed that the relationship between waviness and vibration spectrum was very complicated.

Jang and Jeong [21–24] carried out a series of work to analyze the effects of bearing waviness on the vibration forces and frequencies and on the stability of the rotating system. Based on a five-degree-of-freedom model, they characterized the vibration frequencies due to the various types of waviness. Sopanen and Mikkola [25,26] proposed a detailed dynamic model of the deep-groove ball bearing taking various factors into consideration. Their model improved the dynamic analysis of the rotor-bearing system. Yet, the bearing model would be more comprehensive if the waviness on the ball was also considered. Bai and Xu [29] studied the effect of bearing waviness on the fluctuation of the cage speed. Cao and Xiao [30] developed a comprehensive model of the double-row spherical roller bearing. In their model, several excitations such as the waviness, clearance, surface defects were included.

With respect to the vibration analysis of the rotor-bearing system, only a few investigations considered the effect of bearing preload condition. In fact, the bearing load could remarkably affect the bearing supporting characteristics by changing the contact angle of the bearing. In terms of the bearing contact angle, there was also some noticeable work. Liao and Lin [31] proposed a new method for the analysis of deformation and load in a ball bearing with variable contact angle. Later, considering the centrifugal force and the angular misalignment of the bearing, they evaluated the variations of the contact angle [32]. Based on the contact angle analysis, they developed a new method for the analysis of ball bearing fatigue life [33]. Antoine et al. [34] analyzed the relationship between preload, speed and contact angle with a new analytical approach. It was pointed out that the contact forces and contact deformations of the bearing were directly dependent on the bearing contact angle. Recently, the authors [35,36]

presented a five-degree-of-freedom dynamic model of rotor-bearing system considering the bearing preload and the loads due to the rotor imbalance. In the model, the displacement coordinate relations are utilized to approximately obtain the contact deformations between the rolling element and raceways, and then the nonlinear restoring forces of the rolling bearing could be calculated accordingly. Although the calculation efficiency is relatively higher, the accuracy is lower as the contact deformations should be solved through iterative analysis.

Thus, an improved nonlinear dynamic model is presented in this paper. Considering the preload condition, surface waviness, Hertz contact and elastohydrodynamic lubrication, load distribution analysis is solved iteratively to more accurately obtain the contact deformations and angles between the rolling balls and raceways. The bearing restoring forces are then obtained through iteratively solving the load distribution equations at every time step. Dynamic tests upon a typical rotor system supported by two angular contact ball bearings are conducted to verify the model. Through comparisons, the differences between the nonlinear dynamic model and current models are also pointed out. The effects of axial preload, rotor eccentricity and inner/outer waviness amplitudes on the dynamic response are discussed in detail. Finally, some conclusions are given.

2 Load distribution analysis considering the preload condition

The contact deformations and angles between the rolling ball and inner/outer raceways are important parameters in modeling the nonlinear behaviors of rolling bearings. For the angular contact ball bearing,

appropriate value of preload should be applied to maintain the effective contact between the rolling element and the raceways, and then improve the bearing rotation accuracy and bearing stiffness. In this section, based upon the Hertz contact theory and elastohydrodynamic lubrication theory, load distribution analysis is conducted considering the preload condition, in order to obtain the contact deformations and angles.

2.1 Geometric relation

The inner race is fixed, and the outer race rotates at a constant angular speed ω_o . Two coordinates, of which one ($X - Y - Z$) is fixed at the rotating axis center of outer race and the other ($x - y - z$) is rotating with the Z axis in the orbital angular speed of ball ω_{cj} , are shown in Fig. 1. The center of the $x - y - z$ is at the mass center of the ball. The torsional degree of freedom for the inner race is ignored, and the remaining five degrees of freedom (including: two translations, two rotations and one axial motion) are considered. The freedom vector is expressed as: $\mathbf{u} = [u, v, w, \theta^u, \theta^v]^T$.

Accordingly, the preloads applied on the five directions are:

$\mathbf{f}_p = [f_u, f_v, f_w, m^u, m^v]^T$, in which f_u, f_v, f_w are three forces and m^u, m^v are two moments along the X, Y axes.

[Fig. 1 about here.]

At time t , location angle of the j th ball is $\phi_{cj} = \phi_0 + \omega_{cj}t + \frac{2\pi}{N_b}(j - 1)$, in which ϕ_0 and N_b are the initial angle and number of rolling ball. Without loss of generality, one has $\phi_0 = 0$. The external force causes the outer race deform. Thus, the displacements for the groove curvature center of the outer

race corresponding to the j th ball, including two translations along the y, z axes and one rotation with respect to the x axis, are expressed as $\boldsymbol{\delta}_j = [\zeta_j, \eta_j, \theta_j]^T$. Under the assumption of small deflection, the $\boldsymbol{\delta}_j$ could be obtained through coordinate transformations

$$\boldsymbol{\delta}_j = \mathbf{T}_j \mathbf{u} + \boldsymbol{\Delta} \mathbf{w}_j \quad (1)$$

where $\mathbf{T}_j$ is the transformation matrix, and is expressed as

$$\mathbf{T}_j = \begin{bmatrix} \cos \phi_{cj} & \sin \phi_{cj} & 0 & 0 & 0 \\ 0 & 0 & 1 & R_o \sin \phi_{cj} & -R_o \cos \phi_{cj} \\ 0 & 0 & 0 & \sin \phi_{cj} & -\cos \phi_{cj} \end{bmatrix} \quad (2)$$

in which $R_o = 0.5d_e - (r_{go} - 0.5d_b) \cos \beta_0$ denotes the distance from the rotating axis center of outer race to the groove curvature center, d_e is the pitch diameter of the bearing, r_{go} is the groove curvature radius, d_b is the ball diameter and β_0 is the nominal contact angle.

[Fig. 2 about here.]

In Eq. (1), the $\boldsymbol{\Delta} \mathbf{w}_j$ represents the additional displacements due to the inner waviness, outer waviness and ball waviness. Fig. 2 shows the schematic diagram of the waviness appeared in each components, of which p_{ij}, p_{oj} and q_{ij}, q_{oj} denote circumferential and axial waviness of inner and outer raceways when contact with the j th ball. The w_{ij}, w_{oj} represent the rolling ball's waviness. These values all periodically vary with time, and could be expressed using the harmonic functions as follows

$$p_{ij} = \sum A_{in} \cos(n_{in} \omega_{cj} t + 2\pi(j-1)/N_b + \psi_{in}) \quad (3)$$

$$q_{ij} = \sum B_{in} \cos(n_{in}\omega_{cj}t + 2\pi(j-1)/N_b + \varphi_{in}) \quad (4)$$

$$p_{oj} = \sum A_{out} \cos(n_{out}(\omega_o - \omega_{cj})t + 2\pi(j-1)/N_b + \psi_{out}) \quad (5)$$

$$q_{oj} = \sum B_{out} \cos(n_{out}(\omega_o - \omega_{cj})t + 2\pi(j-1)/N_b + \varphi_{out}) \quad (6)$$

$$w_{ij} = \sum C_n \cos(n_{ba}\omega_{sj}t + \vartheta_{ba}) \quad (7)$$

$$w_{oj} = \sum C_n \cos(n_{ba}\omega_{sj}t + p_i + \vartheta_{ba}) \quad (8)$$

where $A_{in}, B_{in}, A_{out}, B_{out}, C_n, n_{in}, n_{out}, n_{ba}$ and $\psi_{in}, \varphi_{in}, \psi_{out}, \varphi_{out}, \vartheta_{ba}$ are, respectively, denote the amplitudes, harmonic orders and phases of the waviness, ω_{sj} is the self-rotation angular speed. For the pure rolling state and considering the fixed inner race and rotating outer race, one has the ball's self rotation speed $\omega_{sj} = \frac{0.5d_e}{d_b} \left(1 - \left(\frac{d_b \cos \beta_0}{d_e}\right)^2\right) \omega_o$, orbital speed of the ball $\omega_{cj} = 0.5 \left(1 + \frac{d_b \cos \beta_0}{d_e}\right) \omega_o$. According to the geometric relations in Figs. 1 and 2, it is easy to obtain

$$\Delta \mathbf{w}_j = \begin{bmatrix} p_{oj} - p_{ij} \\ q_{oj} - q_{ij} \\ 0 \end{bmatrix} \quad (9)$$

Without load, the ball just contacts with the inner and outer races. As the deflection is zero, so the ball center and groove curvature centers of inner and outer races lie in a straight line, which is shown as the dash line in Fig. 3. The contact angles for the ball with both inner and outer races are the same and equal to the nominal angle β_0 . As long as the load is applied, the outer race will first move and then the ball also moves due to the contact deflections. Equilibrium state, shown by the solid line in Fig. 3, could be achieved between the ball and inner/outer races. As the inner race is fixed, so its groove curvature center keeps unchanged. From Eq. (1), the

displacements of the groove curvature center of inner race are expressed by (ζ_j, η_j) . The displacements of ball center are unknown parameters and denoted by (v_{jy}, v_{jz}) . Due to the load on inner race, the inner and outer contact angles become different. The outer contact angle reduces to β_{oj} , while the inner contact angle increases to β_{ij} .

[Fig. 3 about here.]

Before deflection, the distance from the groove curvature center of inner race to the ball mass center is L_{ij} . After deflection, the distance becomes l_{ij} , as shown in Fig. 3. Similarly, the distance from the groove curvature center of outer race to the ball mass center changes from L_{oj} (before deflection) to l_{oj} (after deflection). For the given groove curvature radii of the inner/outer races (r_{gi}, r_{go}) , one has

$$L_{ij} = r_{gi} + p_{ij} - h_{ci} - (0.5d_b + w_{ij}) \quad (10)$$

$$L_{oj} = r_{go} + p_{oj} - h_{co} - (0.5d_b + w_{oj}) \quad (11)$$

in which h_{ci}, h_{co} are the oil film thicknesses for the rolling ball contacting with the inner and outer raceways, respectively. According to the result of Hamrock and Dowson [37], the oil film thickness is calculated by the following empirical equation

$$h_c = 2.69R_x U^{0.67} G^{0.53} W^{-0.067} (1 - 0.61e^{-0.73\kappa}) \quad (12)$$

where $U = \frac{\eta_0 u_{ent}}{E' R_x}$ denotes the dimensionless speed, in which η_0 is the lubricant viscosity at the atmospheric pressure and temperature of 20°C, $R_x = d_b/2(1 \pm d_b/d_e \cos \beta_0)$ represents the equivalent radius along the ball rolling direction. For the outer race, the "+" is selected. The "-" is chosen for the inner race. The equivalent rotating speed is $u_{ent} = \frac{d_{ri} d_{ro}}{2(d_{ri} + d_{ro})}$, where

the d_{ri}, d_{ro} denote the diameters of inner and outer races. The $G = E' c_{\eta p}$ is the dimensionless elastic modulus, in which $c_{\eta p}$ the viscosity-pressure coefficient. The dimensionless load is written by $W = \frac{Q_j}{E' R_x^2}$. Obviously, the oil film thickness h_c is related to the contact force Q_j . Thus, the value of h_c varies with time due to the variation of Q_j . In the following dynamic analysis, such variation will be considered in the iteration.

According to the geometric relations shown in Fig. 3, one can obtain

$$\tan \beta_{ij} = \frac{L_{ij} \sin \beta_0 + v_{jz}}{L_{ij} \cos \beta_0 - v_{jy}} \quad (13)$$

$$\tan \beta_{oj} = \frac{L_{oj} \sin \beta_0 + \eta_j - v_{jz}}{L_{oj} \cos \beta_0 - \zeta_j + v_{jy}} \quad (14)$$

$$l_{ij} = \frac{L_{ij} \sin \beta_0 + v_{jz}}{\sin \beta_{ij}} \quad (15)$$

$$l_{oj} = \frac{L_{oj} \sin \beta_0 + \eta_j - v_{jz}}{\sin \beta_{oj}} \quad (16)$$

Eqs. (13-16) are geometric equations, which should be satisfied in the load distribution analysis.

2.2 Force equilibrium equations of the ball

Besides the geometric equations, the force equilibrium equations for the rolling ball and outer race should also be established. Fig. 4 gives the forces applied on the j th rolling ball. In the figure, F_{cj} and M_{gj} denote the centrifugal and gyroscopic forces due to the rotation and revolution of the ball. Obviously, one has $F_{cj} = 0.5m_b d_e \omega_{cj}^2$ and $M_{gj} = I_b \omega_{sj} \omega_{cj} \sin \alpha_j$, where m_b, I_b the mass and moment of inertia of the ball, and α_j the angle between self-rotation axis and the z axis. For the pure rolling state, one has

$$\alpha_j = \tan^{-1} \left(\frac{\sin \beta_0}{\cos \beta_0 + d_b/d_e} \right).$$

[Fig. 4 about here.]

The contact forces between the ball and inner/outer races are denoted by Q_{ij} and Q_{oj} . According to the Hertz contact theory, one has $Q_{ij} = \chi_{ij} K_i \delta_{ij}^{3/2}$ and $Q_{oj} = \chi_{oj} K_o \delta_{oj}^{3/2}$, in which K_i, K_o and δ_{ij}, δ_{oj} the contact stiffness coefficients and contact deflections between the ball and inner/outer races, respectively. From the above geometric analysis, one has $\delta_{ij} = l_{ij} - L_{ij}$ and $\delta_{oj} = l_{oj} - L_{oj}$. When $\delta_{ij}, \delta_{oj} > 0$, $\chi_{ij} = 1, \chi_{oj} = 1$. Otherwise, $\chi_{ij} = 0, \chi_{oj} = 0$ for $\delta_{ij}, \delta_{oj} \leq 0$. The contact stiffness coefficients are determined by geometric dimensions of contact zone and material properties, i.e. $K = (\pi \kappa E' / 3 \xi \sqrt{3 \epsilon \Re / \xi})$, in which $E' = E / (1 - \nu^2)$ denote the effective elastic modulus, and E the material elastic modulus, ν the poisson's ratio, $\Re$ the equivalent curvature radius, κ the elliptical ratio, ξ, ϵ the first and second kinds of elliptical integration. From the monograph of Harris [38], the expressions for these parameters are given and the contact stiffness coefficients K_i, K_o could then be gained accordingly.

In the load distribution analysis, the friction force out of the plane is ignored. The gyroscopic moment just equals to the moment induced by the friction force in the $y - z$ plane, as shown in Fig. 4. $\lambda_{ij}, \lambda_{oj}$ denote the friction coefficients between the ball and inner/outer races. Here, the values of $\lambda_{ij}, \lambda_{oj}$ are set to be 1. From Fig. 4, the force equilibrium equations are obtained as follows

$$Q_{ij} \cos \beta_{ij} - Q_{oj} \cos \beta_{oj} + \frac{\lambda_{ij} M_{gj}}{d_b} \sin \beta_{ij} - \frac{\lambda_{oj} M_{gj}}{d_b} \sin \beta_{oj} + F_{cj} = 0 \quad (17)$$

$$-Q_{ij} \sin \beta_{ij} + Q_{oj} \sin \beta_{oj} + \frac{\lambda_{ij} M_{gj}}{d_b} \cos \beta_{ij} - \frac{\lambda_{oj} M_{gj}}{d_b} \cos \beta_{oj} = 0 \quad (18)$$

Eqs. (17) and (18) show the equilibrium state for the forces applied on the rolling ball.

2.3 Force equilibrium equations of the outer race

Not only the rolling ball is in equilibrium state, but also the outer race in analyzing the load distribution of rolling bearings. The j th ball provides contact force Q_{oj} and friction force $\frac{\lambda_{oj}M_{gj}}{d_b}$ to the outer race, and the directions are opposite with the directions shown in Fig. 4. By transforming these forces to the groove curvature center of the outer race, one has

$$\begin{bmatrix} Q_{yj} \\ Q_{zj} \\ Q_{\theta xj} \end{bmatrix} = \begin{bmatrix} Q_{oj} \cos \beta_{oj} + \frac{\lambda_{oj}M_{gj}}{d_b} \sin \beta_{oj} \\ -Q_{oj} \sin \beta_{oj} + \frac{\lambda_{oj}M_{gj}}{d_b} \cos \beta_{oj} \\ \frac{\lambda_{oj}M_{gj}}{d_b} r_{go} \end{bmatrix} \quad (19)$$

Thus, the force equilibrium equations of the inner race could be expressed as

$$\mathbf{f}_b = \sum_{j=1}^{N_b} \left([\mathbf{T}_j]^T \begin{bmatrix} Q_{yj} \\ Q_{zj} \\ Q_{\theta xj} \end{bmatrix} \right) \quad (20)$$

where $\mathbf{f}_b = [f_{bu}, f_{bv}, f_{bw}, m_{bu}, m_{bv}]^T$ denotes the resultant force vector of the outer race due to all of the contacting balls, which is just the restoring force vector of the rolling bearing. Obviously, the following equilibrium equation exists for the outer race

$$\mathbf{f}_p - \mathbf{f}_b = 0 \quad (21)$$

Eqs. (17), (18) and (21) are a set of nonlinear algebraic equations, which could be solved iteratively by the Newton-Raphson method. Obviously, the dimension of the nonlinear equations is $2N_b + 5$. After the unknown parameters are gained, the contact deformations $(\delta_{ij}, \delta_{oj})$ and inner/outer

contact angles (β_{ij}, β_{oj}) for every rolling ball could be computed by Eqs. (1) and (13-16). Moreover, the displacement vector $\mathbf{u}_o$ of outer race under given values of preload could be obtained accordingly.

3 Nonlinear dynamic model

Fig. 5 gives the schematic diagram for a rotor system supported by a pair of "face-to-face" angular contact ball bearings (B7004C). The inner race of the bearing is fixed on the stationary spindle, the outer race is connected with the rotor and rotates under constant angular speed ω_o . The axial preload is applied appropriately through the end cover, loading nut, sleeve and retainer ring, as shown in the figure. The nonlinear dynamic model of such a rotor-bearing system will be derived in this section.

[Fig. 5 about here.]

3.1 Nonlinear bearing force

The dynamic model of the rotor-bearing system is shown in Fig. 6. The system coordinate is denoted by $X_s - Y_s - Z_s$, and the local coordinates for bearing L and R are represented by $X_{sL} - Y_{sL} - Z_{sL}$ and $X_{sR} - Y_{sR} - Z_{sR}$, respectively. The mass, diametral moment of inertia and polar moment of inertia of the rotor are: m_s, I_{sd}, I_{sp} . Three translational u_s, v_s, w_s and two angular displacements θ_s^u, θ_s^v along the X_s, Y_s, Z_s are considered. The distances from rotor to bearing R and L are denoted by l_1, l_2 , respectively. Due to the Hertz contact, oil lubrication and unilateral contact between the ball and raceways (only compression nor tension), the relation between

restoring force and bearing displacement is nonlinear. The flexible stationary spindle is modeled by two mass-spring systems, of which m_L, m_R and $k_{L1}, k_{L2}, k_{L12}, k_{R1}, k_{R2}, k_{R12}$ are the mass and stiffness coefficients. The corresponding vibrational displacements are represented by u_L, v_L and u_R, v_R , as shown in Fig. 6.

[Fig. 6 about here.]

For given preloads, the pre-deflection of outer race is $\mathbf{u}_o$, which could be gained numerically by solving the nonlinear algebraic equations in the previous section. In operation, the rotor's vibrational displacement vector is $[u_s, v_s, w_s, \theta_s^u, \theta_s^v]^T$. Considering the spindle's deflection and pre-deflection of outer race under given preloads, the vibrational displacements of outer race of bearing L and R could be gained using coordinate transformation technique

$$\begin{bmatrix} u_{bL} \\ v_{bL} \\ w_{bL} \\ \theta_{bL}^u \\ \theta_{bL}^v \end{bmatrix} = \mathbf{T}_L \left(\begin{bmatrix} u_s \\ v_s \\ w_s \\ \theta_s^u \\ \theta_s^v \end{bmatrix} - \begin{bmatrix} u_L \\ v_L \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \mathbf{u}_o \quad (22)$$

$$\begin{bmatrix} u_{bR} \\ v_{bR} \\ w_{bR} \\ \theta_{bR}^u \\ \theta_{bR}^v \end{bmatrix} = \mathbf{T}_R \left(\begin{bmatrix} u_s \\ v_s \\ w_s \\ \theta_s^u \\ \theta_s^v \end{bmatrix} - \begin{bmatrix} u_R \\ v_R \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \mathbf{u}_o \quad (23)$$

where $[u_{bL}, v_{bL}, w_{bL}, \theta_{bL}^u, \theta_{bL}^v]^T$ and $[u_{bR}, v_{bR}, w_{bR}, \theta_{bR}^u, \theta_{bR}^v]^T$, respectively, denote the displacement vectors of outer race of bearing L and R. According to the definition in Fig. 6, the transformation matrices $\mathbf{T}_L, \mathbf{T}_R$ are derived as follows

$$\mathbf{T}_L = \begin{bmatrix} 1 & 0 & 0 & 0 & -l_1 \\ 0 & 1 & 0 & l_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1/l_1 & 0 & 1 & 0 \\ 1/l_1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (24)$$

$$\mathbf{T}_R = \begin{bmatrix} 1 & 0 & 0 & 0 & l_2 \\ 0 & -1 & 0 & l_2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1/l_2 & 0 & 1 & 0 \\ 1/l_2 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (25)$$

After the outer race displacements for the two bearings are obtained, the corresponding restoring forces could be gained through load distribution analysis presented in the previous section. In the following, the bearing L is taken as an example to show the solution process for nonlinear bearing restoring forces in detail.

The displacements of j th ball of bearing L δ_{jL} could be gained by substituting the outer race displacement vector of bearing L $\mathbf{u}_L = [u_{bL}, v_{bL}, w_{bL}, \theta_{bL}^u, \theta_{bL}^v]^T$ into Eq. (1). Similarly, the force equilibrium equations for every rolling balls of bearing L are obtained, as shown in Eqs. (17) and (18). Utilizing the Newton-Raphson method, such nonlinear algebraic equations are solved iteratively, and the displacements of every ball in the local coordinate are gained. According to Eqs. (13-16) and Eq. (19), the reaction force vector from the j th ball to the outer race of bearing L is $[Q_{yjL}, Q_{zjL}, Q_{\theta xjL}]^T$. Thus, in the system coordinate, the sum of reaction forces from all of the rolling ball, just the restoring forces of bearing L, are expressed as follows

$$\begin{bmatrix} f_{bL1} \\ f_{bL2} \\ f_{bL3} \\ f_{bL4} \\ f_{bL5} \end{bmatrix} = [\mathbf{T}_L]^T \left(\sum_{j=1}^{N_b} \left([\mathbf{T}_L]^T \begin{bmatrix} Q_{yjL} \\ Q_{zjL} \\ Q_{\theta xjL} \end{bmatrix} \right) \right) \quad (26)$$

Similarly, the restoring force vector of bearing R could also be gained as follows

$$\begin{bmatrix} f_{bR1} \\ f_{bR2} \\ f_{bR3} \\ f_{bR4} \\ f_{bR5} \end{bmatrix} = [\mathbf{T}_R]^T \left(\sum_{j=1}^{N_b} \left([\mathbf{T}_R]^T \begin{bmatrix} Q_{yjR} \\ Q_{zjR} \\ Q_{\theta xjR} \end{bmatrix} \right) \right) \quad (27)$$

3.2 Equations of motion for the rotor-bearing system

After nonlinear restoring forces of bearing L and R obtained, the equations of motion for the rotor-bearing system could be derived utilizing the energy method or Newton method

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \omega_o \mathbf{G})\dot{\mathbf{q}} + \mathbf{K}_n \mathbf{q} = \mathbf{F}_b + \mathbf{F}_e + \mathbf{F}_g \quad (28)$$

in which $\mathbf{q} = [u_s, v_s, w_s, \theta_s^u, \theta_s^w, u_L, v_L, u_R, v_R]^T$ denotes the system displacement vector, $\mathbf{M}, \mathbf{C}, \mathbf{G}, \mathbf{K}_n$ the mass, damping, gyroscopic and stiffness matrices, $\mathbf{F}_e, \mathbf{F}_g$ the unbalanced mass and self-gravity excitation vectors, and $\mathbf{F}_b$ the nonlinear restoring force vector induced by the two bearings. As the bearing is modeled as nonlinear force excitations, only linear stiffness of stationary spindle is included in the stiffness matrix $\mathbf{K}_n$. The above-mentioned matrices and vectors are expressed as follows

$$\mathbf{M} = \text{diag}([m_s, m_s, m_s, I_{sd}, I_{sd}, m_L, m_L, m_R, m_R]) \quad (29)$$

(30)

(31)

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{sp} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{sp} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (32)$$

$$\mathbf{F}_e = m_{su}e_s \begin{bmatrix} \cos(\omega_o t) & \sin(\omega_o t) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad (33)$$

$$\mathbf{F}_g = \begin{bmatrix} 0 & 0 & -m_s g & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad (34)$$

$$\mathbf{F}_b = \begin{bmatrix} (f_{bL1} + f_{bR1}) & (f_{bL2} + f_{bR2}) & (f_{bL3} + f_{bR3}) & (f_{bL4} + f_{bR4}) & & & & & \\ & (f_{bL5} + f_{bR5}) & -f_{bL1} & -f_{bL2} & -f_{bR1} & -f_{bR2} & & & \end{bmatrix}^T \quad (35)$$

where ς the proportional damping coefficient, m_{su}, e_s the unbalanced mass and eccentricity of the rotor. Due to the rotor is placed vertically, the direction of rotor's self weight is along the Z_s axis, as shown in Eq. (34).

Obviously, Eq. (28) is a series of nonlinear two order differential equations, and could be solved numerically. The solution process is given in Fig. 7. The

pre-deflections δ_{oj} and contact angles β_{oj} of outer race of the two bearings under given preloads are gained. Through solving nonlinear algebraic equations, the restoring forces of bearing L and R could be obtained. Then, the ordinary differential equation solver in MATLAB (i.e. ODE45) is utilized to compute vibrational responses at the current time. At every time interval, the nonlinear iteration analysis is conducted to obtain the bearing restoring forces more precisely. Thus, compared with the current models, the nonlinear dynamic model presented in this paper would need more computing time. The calculation will not stop until the time exceeded the end time.

[Fig. 7 about here.]

4 Current dynamic models

Two types of dynamic model for rolling bearings have been presented in current studies. One is called the linear model, which equivalently treats the bearing as a series of spring with linear stiffness. Such model is often used for analyzing the natural and frequency response characteristics of rotor-bearing system. The other model utilizes the displacement coordinate relations to approximately obtain the contact deformations between the rolling element and raceways, and then the nonlinear restoring forces of the rolling bearing could be calculated. Such model is called the displacement coordinate model here. Since no need for iterative calculation, the calculation efficiency is improved, but the accuracy is lower. In order for comparisons, the two current models will be introduced briefly in this section.

4.1 Linear model

As each bearing has five degrees of freedom, so the number of linear spring is five, including three translational stiffnesses k_{b1}, k_{b2}, k_{b3} and two angular stiffnesses k_{b4}, k_{b5} . The cross stiffnesses are ignored for simplicity. According to the definition, the linear bearing stiffness could be calculated approximately as follows

$$k_{bi} = \frac{\partial \mathbf{f}_p}{\partial \mathbf{u}} \approx \frac{\Delta \mathbf{f}_p}{\Delta \mathbf{u}} \quad i = 1, 2, \dots, 5 \quad (36)$$

Thus, each linear stiffness is expressed as

$$\begin{aligned} k_{b1} &= \frac{f_{u2} - f_{u1}}{u_2 - u_1}, \quad k_{b2} = \frac{f_{v2} - f_{v1}}{v_2 - v_1}, \quad k_{b3} = \frac{f_{w2} - f_{w1}}{w_2 - w_1} \\ k_{b4} &= \frac{m_{u2} - m_{u1}}{\theta_2^u - \theta_1^u}, \quad k_{b5} = \frac{m_{v2} - m_{v1}}{\theta_2^v - \theta_1^v} \end{aligned} \quad (37)$$

in which f_{u1}, f_{u2} are two values of preload for estimating the translational stiffness k_{b1} , and u_1, u_2 are the corresponding deformations of outer race. Similarly, the $f_{v1}, f_{v2}, f_{w1}, f_{w2}, m_{u1}, m_{u2}, m_{v1}, m_{v2}$ are values of preload for the other four stiffnesses, i.e. $k_{b2}, k_{b3}, k_{b4}, k_{b5}$. The $v_1, v_2, w_1, w_2, \theta_1^u, \theta_2^u, \theta_1^v, \theta_2^v$ are the deformations of outer race corresponding to the preloads.

The solution process is introduced here by taking the stiffness k_{b1} for example. For given preload f_{u1} , the deformation of outer race u_1 could be gained by iteratively solving the nonlinear algebraic equations (Eqs. (17),(18) and (force equilibrium equation3)). Changing the value of f_{u1} to f_{u2} (usually $f_{u2} = f_{u1} + \varrho$, ϱ is a small value), the corresponding deformation u_2 is then obtained similar with the solution process of u_1 . According to Eq. (37), the stiffness k_{b1} along the X axis is calculated numerically. The other values of $k_{b2}, k_{b3}, k_{b4}, k_{b5}$ could be, respectively, gained through the

above-mentioned solution process.

After the linear stiffnesses of the bearing is solved, the system equations of motion for the rotor-bearing system could be easily derived based on the Lagrange's equation

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \omega_o \mathbf{G})\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}_e + \mathbf{F}_g \quad (38)$$

Obviously, the above equations are similar with that of the nonlinear dynamic model (see Eq. (28)). Due to the linear treatment, the nonlinear bearing force vector $\mathbf{F}_b$ disappears, and the linear stiffnesses for the bearing should also be reflected in stiffness matrix $\mathbf{K}$ of the system. The expression of $\mathbf{K}$ is given as follows

$$\mathbf{K} = \begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{15} & K_{16} & 0 & K_{18} & 0 \\ & K_{22} & 0 & K_{24} & 0 & 0 & K_{27} & 0 & K_{29} \\ & & K_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & K_{44} & 0 & 0 & K_{47} & 0 & K_{49} \\ & & & & K_{55} & K_{56} & 0 & K_{58} & 0 \\ & & & & & K_{66} & K_{67} & 0 & 0 \\ & & & & & & & K_{77} & 0 & 0 \\ & & & & & & & & K_{88} & K_{89} \\ & & & & & & & & & K_{99} \end{bmatrix} \quad (39)$$

where the elements are given as follows

$$\begin{aligned}
K_{11} &= 2k_{b1} + k_{b5}/l_1^2 + k_{b5}/l_2^2 \\
K_{15} &= k_{b1}(l_2 - l_1) + k_{b5}(1/l_1 - 1/l_2) \\
K_{16} &= -k_{b1} - k_{b5}/l_1^2 \\
K_{18} &= (-k_{b1} - k_{b5}/l_2^2 \\
K_{22} &= 2k_{b2} + k_{b4}/l_1^2 + k_{b4}/l_2^2 \\
K_{24} &= k_{b2}(l_1 - l_2) + k_{b4}(1/l_2 - 1/l_1) \\
K_{27} &= -k_{b2} - k_{b4}/l_1^2 \\
K_{29} &= (-k_{b2} - k_{b4}/l_2^2 \\
K_{33} &= 2k_{b3} \\
K_{44} &= 2k_{b4} + k_{b2}l_1^2 + k_{b2}l_2^2 \\
K_{47} &= -k_{b2}l_1 + k_{b4}/l_1 \\
K_{49} &= k_{b2}l_2 - k_{b4}/l_2 \\
K_{55} &= 2k_{b5} + k_{b1}l_1^2 + k_{b1}l_2^2 \\
K_{56} &= k_{b1}l_1 - k_{b5}/l_1 \\
K_{58} &= -k_{b1}l_2 + k_{b5}/l_2 \\
K_{66} &= k_{b4} + k_{L1} + k_{b5}/l_1^2 \\
K_{67} &= -k_{L12} \\
K_{77} &= k_{b2} + k_{L2} + k_{b4}/l_1^2 \\
K_{88} &= k_{b1} + k_{R1} + k_{b5}/l_2^2 \\
K_{89} &= -k_{R12} \\
K_{99} &= k_{b2} + k_{R2} + k_{b4}/l_2^2
\end{aligned} \tag{40}$$

4.2 Displacement coordinate model

Modeling the rolling bearing as linear springs is an approximate treatment. Due to the Hertz contact, lubrication oil film and one-direction contact between the rolling element and raceways, the relation between bearing displacement and restoring force is nonlinear, and utilizing the linear springs could not accurately describe such relation. Currently, most studies focused on the displacement coordinate model. The main idea is to obtain the contact deformations between rolling elements and raceways through geometric analysis and displacement coordinate relations. Then, the nonlinear restoring force of bearing could be calculated base upon Hertz contact theory.

The displacement vector of the rotor in system coordinate $X_s - Y_s - Z_s$ is denoted as $[u_s, v_s, w_s, \theta_s^u, \theta_s^v]^T$. Considering the deformation of stationary spindle, the vibrational displacement vectors for bearing L and R could be calculated through coordinate transformation as follows

$$\begin{bmatrix} u_{bL} \\ v_{bL} \\ w_{bL} \\ \theta_{bL}^u \\ \theta_{bL}^v \end{bmatrix} = \mathbf{T}_L \left(\begin{bmatrix} u_s \\ v_s \\ w_s \\ \theta_s^u \\ \theta_s^v \end{bmatrix} - \begin{bmatrix} u_L \\ v_L \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \quad (41)$$

$$\begin{bmatrix} u_{bR} \\ v_{bR} \\ w_{bR} \\ \theta_{bR}^u \\ \theta_{bR}^v \end{bmatrix} = \mathbf{T}_R \left(\begin{bmatrix} u_s \\ v_s \\ w_s \\ \theta_s^u \\ \theta_s^v \end{bmatrix} - \begin{bmatrix} u_R \\ v_R \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \quad (42)$$

where $[u_{bL}, v_{bL}, w_{bL}, \theta_{bL}^u, \theta_{bL}^v]^T$ and $[u_{bR}, v_{bR}, w_{bR}, \theta_{bR}^u, \theta_{bR}^v]^T$ are the displacement vectors of outer races of bearing L and R, respectively. The expressions of transformation matrices $\mathbf{T}_L, \mathbf{T}_R$ could be found in Eq. (24-25).

Taking the bearing L for example, the calculation process for the nonlinear restoring forces will be shown in the following section. According to Eq. (1), for given outer race's displacements, the displacements of outer race at the j th ball are given as

$$\begin{bmatrix} \xi_{bL} \\ \eta_{bL} \\ \theta_{bL} \end{bmatrix} = \mathbf{T}_j \begin{bmatrix} u_{bL} \\ v_{bL} \\ w_{bL} \\ \theta_{bL}^u \\ \theta_{bL}^v \end{bmatrix} \quad (43)$$

After load distribution analysis, the pre-deformation δ_{oj} and contact angle β_{oj} between ball and outer raceway are gained. From the geometry in Fig. 1, the contact deformation is calculated through displacement coordinate

relation

$$\delta_{ojL} = \eta_{bL} \sin \beta_{oj} - \xi_{bL} \cos \beta_{oj} + \delta_{oj} \quad (44)$$

Then, the restoring force provided by the j th ball is

$$\mathbf{f}_{bjL} = \lambda_{bj} [\mathbf{T}_j]^T \begin{bmatrix} -K_{oj}(\delta_{ojL})^{3/2} \cos \beta_{oj} \\ K_{oj}(\delta_{ojL})^{3/2} \sin \beta_{oj} \\ 0 \end{bmatrix} \quad (45)$$

in which λ_{bj} represents the contact state between the rolling ball and raceway. For $\delta_{ojL} > 0$, one has $\lambda_{bj} = 1$. Otherwise, one has $\lambda_{bj} = 0$. Thus, the nonlinear restoring force vector induced by bearing L is expressed as follows

$$\begin{bmatrix} f_{bL1} \\ f_{bL2} \\ f_{bL3} \\ f_{bL4} \\ f_{bL5} \end{bmatrix} = [\mathbf{T}_L]^T \left(\sum_{j=1}^{N_b} \mathbf{f}_{bjL} \right) \quad (46)$$

Similarly, the restoring force vector of bearing R is given as

$$\begin{bmatrix} f_{bR1} \\ f_{bR2} \\ f_{bR3} \\ f_{bR4} \\ f_{bR5} \end{bmatrix} = [\mathbf{T}_R]^T \left(\sum_{j=1}^{N_b} \mathbf{f}_{bjR} \right) \quad (47)$$

After the restoring forces of bearings are obtained, a dynamic model, similar with the nonlinear dynamic model shown in Eq. (28), could be derived.

Different with that of the nonlinear dynamic model, only the displacement coordinate relation is utilized to calculate the contact deformation at every time step. Obviously, such treatment would improve the computational efficiency, while the solution precision is reduced.

5 Model validation by dynamic experiment

Dynamic experiments are carried out to validate the nonlinear dynamic model. A rotor-bearing system is driven by a high-speed DC motor. The rotating speed varies in the range of 0-4600rpm. The test diagram is shown in Fig. 8. Two three-axis acceleration sensors are, respectively, located at the fixed and free ends of stationary spindle. Dynamic signal is amplified and acquired by data acquisition system, and then a high performance computer is utilized for post-analysis of the acceleration signal. Fig. 9 gives the physical picture of the whole experimental system.

[Fig. 8 about here.]

[Fig. 9 about here.]

5.1 System parameters

Tab. 1 gives the values of system parameters, including the rotor, supporting system and unbalanced mass excitation. The parameters of bearing B7004C and lubricant oil are listed in Tab. 2. Due to the limitation of experimental conditions, the surface waviness of raceways and rolling balls could not be accurately measured. In this study, the waviness amplitudes are set to be $0.1\mu\text{m}$, and the order are given according to the experimental results. The phase between various waviness is zero. According to Eqs. (3-8), the characteristic frequencies of waviness related to the inner and outer raceways are different. For the waviness of inner race, the characteristic frequencies IF_n are expressed as follows

$$IF_n = n\omega_c = \frac{n}{2}\left(1 + \frac{d_b \cos \beta_0}{d_e}\right)\omega_o \quad n = 1, 2, \dots \quad (48)$$

For the outer race, the characteristic frequencies of waviness are

$$OF_m = m(\omega_o - \omega_c) = \frac{m}{2}\left(1 - \frac{d_b \cos \beta_0}{d_e}\right)\omega_o \quad m = 1, 2, \dots \quad (49)$$

[Table 1 about here.]

[Table 2 about here.]

5.2 Model validation

Speed up and down tests in the range of 0-4600rpm are carried out, respectively. Waterfall plots of acceleration signal are plotted in Figs. 10 and 11, of which the former is about the transversal signal at the fix-end of stationary spindle and the latter is about the transversal signal at the free end of spindle. It is shown that besides the rotating frequency ω_o , the characteristic frequencies related to the inner and outer waviness also have distinct amplitudes. The $IF_3, IF_6, IF_9, IF_{12}$ for the inner race waviness are found in Fig. 10. Interestingly, these frequencies could be identified as the integer multiples of three times of cage rotating speed, i.e. $i * 3\omega_c, i = 1, 2, \dots$. Considering that the number of ball is 12, the appearance of such frequencies might be related to the processing of the inner race and rolling ball. For the outer race waviness, $OF_5, OF_{10}, OF_{21}, OF_{24}$ and even the relative higher order OF_{38}, OF_{76} could also be observed in Fig. 10. Unlike so many distinct spectra in Fig. 10, rotating frequency is the dominant spectra in the response at the free end of spindle, as shown in Fig. 11. Moreover, two spectral component independent of the rotating frequency are found around the 37Hz and 49.6Hz. From the following analysis, one will see that the two spectra are related the first two orders of natural frequency, as marked by ω_{n1}, ω_{n2} in the figure.

[Fig. 10 about here.]

[Fig. 11 about here.]

[Fig. 12 about here.]

[Fig. 13 about here.]

Both the displacement and velocity responses are obtained by solving the nonlinear dynamic model of rotor-bearing system. In the dynamic test, only the acceleration response is acquired. In order for comparisons, the numerical solved responses should be transformed to acceleration response by time difference. Then, the numerical acceleration responses at the fixed end and free end of stationary spindle are presented in Figs. 12 and 13, respectively. One can see that the rotating frequency, natural frequencies and characteristic frequencies of both inner and outer waviness exist in the response spectra, which is consistent with that of the test results in Figs. 10 and 11.

As the working speed of the rotor system is about 4600rpm, so the comparisons are also conducted and the results are shown in Figs. 14 and 15. One can see that the frequency components of the present and experimental results are basically consistent. Although the spectral amplitudes have some differences, they are in the same order of magnitude. Tab. 3 gives frequency and amplitude comparisons of the fixed end of stationary spindle for the system rotating at 4600rpm. It is shown that the maximal frequency difference is about 10%, while the amplitude difference is generally less than 30% (the maximum is about 60%). Considering that the spectral amplitudes is more sensitive than the frequency values, the nonlinear dynamic model presented in this study is believable and reasonable.

[Fig. 14 about here.]

[Fig. 15 about here.]

[Table 3 about here.]

5.3 *Model comparisons*

Steady-state responses obtained by three models (linear model, displacement coordinate model and nonlinear dynamic model) are shown in Fig. 16 for the axial preload $f_w = 100\text{N}$. The corresponding response spectra are given in Fig. 17. From the time-domain response, one can find that the response amplitude of displacement coordinate model is the maximum, the amplitude of nonlinear dynamic model is lower, and the amplitude of the linear model is the minimum. This means that the dynamic response might be overestimated using the displacement coordinate model, while utilizing the linear model might give underestimated results. The linear model does not consider the waviness effect, and thus it is easy to understand the difference between the linear model and the nonlinear dynamic model.

[Fig. 16 about here.]

[Fig. 17 about here.]

[Fig. 18 about here.]

[Fig. 19 about here.]

It is surprised to find so obvious difference between the displacement coordinate model and nonlinear dynamic model, as shown in Fig. 16. From the response spectra in Fig. 17, one can see that the spectral amplitudes obtained by nonlinear dynamic model are all lower than the ones obtained by displacement coordinate model, especially for the amplitudes of waviness

characteristic frequencies, i.e. $IF_3, IF_6, OF_5, OF_{10}$. Without considering the surface waviness, comparisons between displacement coordinate and nonlinear dynamic models are given in Figs. 18 and 19. The response amplitudes becomes slightly lower when the nonlinear dynamic model is used. This is reasonable as the angular contact ball bearing has a small size (inner race diameter is 20mm) and the rotation speed is moderate (4600rpm). In this case, the centrifugal force and gyroscopic effect are not excessive. Moreover, in the displacement coordinate model, the contact angles are assumed to be constant and the contact deformations between the rolling ball and outer race are obtained utilizing the displacement coordinate relation. In the nonlinear dynamic model, the contact angles and deformations are all time-variant and should be gained iteratively in every time step. The results of the contact angles and deformations are also presented in Fig. 20. From the figure, one can find that the simplified treatment in the displacement coordinate model would not bring great difference with the nonlinear dynamic model.

However, as long as the surface waviness is considered, Fig. 21 shows that the difference between the two models becomes distinct, especially for the contact deformations. This means that the actual contact angles and deformations should be gained by solving nonlinear load distribution equations iteratively, and could not be gained through the assumption of constant contact angle and displacement coordinate relation. In the displacement coordinate model, the contact deformations are greatly overestimated, which will lead to the obvious difference in the dynamic response, as shown in Fig. 16. Thus, one can conclude that the displacement coordinate model would not produce great errors only in the case of no

consideration of surface waviness. For the surface waviness considered, the nonlinear dynamic model should be used and utilizing the displacement coordinate model will lead to the overestimated dynamic response.

[Fig. 20 about here.]

[Fig. 21 about here.]

5.4 *Effect of axial load*

Figs. 22 and 23 give the time domain responses and corresponding spectra for the bearing under different axial preloads ($f_w = 100, 200\text{N}$). It is shown that increasing the value of preload increases the response amplitude, and the most obvious increase in amplitude is around the two natural frequencies. However, the amplitudes of inner/outer waviness characteristic frequencies are reduced with the increasing of preload, as shown in Fig. 23. Thus, increasing axial load would enhance the natural vibrations, and attenuate the dynamic response induced by surface waviness.

[Fig. 22 about here.]

[Fig. 23 about here.]

5.5 *Effect of rotor eccentricity*

The rotor eccentricity is also a key factor that will significantly influence the dynamic response of rotor-bearing systems. Choosing the rotor eccentricity equals to $1\mu\text{m}$ and $5\mu\text{m}$, the dynamic responses and spectra are obtained

and plotted in Figs. 24 and 25, respectively. One can see that the response amplitude is greatly increased with rotor eccentricity, and most obvious increase occurs at the rotating frequency and natural frequencies. Therefore, one can conclude that rotor eccentricity mainly affects the rotating frequency amplitude, and the attenuation of natural vibration becomes slowing down under the same level of damping.

[Fig. 24 about here.]

[Fig. 25 about here.]

5.6 *Effect of waviness amplitudes*

In this section, the inner and outer waviness amplitudes are analyzed for their effects upon the system dynamic response in both time domain and frequency domain. Figs. 26, 27 and 28, 29, respectively, plot the response comparisons for waviness amplitudes increasing from $0.1\mu m$ to $0.2\mu m$. It is shown from Figs. 26 and 27 that increasing the outer waviness amplitude greatly increases the dynamic response, and the most obvious amplitude increase arises at the characteristic frequencies of outer race waviness (OF_5, OF_{10}). Similarly, increasing the inner waviness amplitude also increase the time-domain response. The difference is that the most obvious amplitude enhancement occurs at the inner waviness characteristic frequencies (IF_3, IF_6), as shown in Figs. 28 and 29.

[Fig. 26 about here.]

[Fig. 27 about here.]

[Fig. 28 about here.]

[Fig. 29 about here.]

6 Conclusions

A nonlinear dynamic model of rotor systems supported by angular contact ball bearings is presented in this paper. Both the Hertz contact, elastohydrodynamic lubrication and preload condition is considered, and the restoring forces provided by the bearings are solved through interactive analysis. The model is verified through dynamic experiments. Through model comparisons, the differences between nonlinear dynamic model and current models (including the linear model and displacement coordinate model) are pointed out. In addition, some important design parameters, such as the axial preload, rotor eccentricity and inner/outer waviness amplitudes, are discussed for their effects on the vibrational response of rotor-bearing systems. The results would be helpful for understanding dynamic behaviors of rotor-bearing systems, and provide the basis for the vibration control of such dynamic systems.

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List of Figures

1	Schematic diagram for the bearing coordinates.	47
2	Schematic diagram for the bearing waviness.	48
3	Geometric relations for the bearing before and after deflection.	49
4	Forces applied on the j th rolling ball [38].	50
5	Schematic diagram for a rotor system supported by two angular contact ball bearings.	51
6	Nonlinear dynamic model of the rotor-bearing system.	52
7	Solution process for the nonlinear dynamic model.	53
8	Schematic diagram for the dynamic experiment.	54
9	Physical picture of the whole experimental system.	55
10	Waterfall plot for the tested acceleration signal at the fixed end of stationary spindle.	56
11	Waterfall plot for the tested acceleration signal at the free end of stationary spindle.	57
12	Numerical acceleration signal at the fixed end of stationary spindle.	58
13	Numerical acceleration signal at the free end of stationary spindle.	59
14	Spectral comparisons between present and experimental results at the fixed end of stationary spindle for rotating speed equal to 4600rpm.	60
15	Spectral comparisons between present and experimental results at the free end of stationary spindle for rotating speed equal to 4600rpm.	61
16	Model comparison for steady-state responses obtained by three models.	62
17	Model comparison for response spectra	63

18	Model comparison between displacement coordinate and nonlinear dynamic models without surface waviness.	64
19	Model comparison of response spectra between displacement coordinate and nonlinear dynamic models without surface waviness.	65
20	Contact angles and deformations obtained by the displacement coordinate model and nonlinear dynamic model without surface waviness.	66
21	Contact angles and deformations obtained by the displacement coordinate model and nonlinear dynamic model with surface waviness.	67
22	The effect of axial load upon time response.	68
23	The effect of axial load upon response spectra.	69
24	The effect of unbalanced mass upon time response.	70
25	The effect of unbalanced mass upon response spectra.	71
26	The effect of outer waviness upon time response.	72
27	The effect of outer waviness upon response spectra.	73
28	The effect of inner waviness upon time response.	74
29	The effect of inner waviness upon response spectra.	75

Fig. 1. Schematic diagram for the bearing coordinates.

Fig. 2. Schematic diagram for the bearing waviness.

Fig. 3. Geometric relations for the bearing before and after deflection.

Fig. 4. Forces applied on the j th rolling ball [38].

Fig. 5. Schematic diagram for a rotor system supported by two angular contact ball bearings.

Fig. 6. Nonlinear dynamic model of the rotor-bearing system.

Fig. 7. Solution process for the nonlinear dynamic model.

Fig. 8. Schematic diagram for the dynamic experiment.

Fig. 9. Physical picture of the whole experimental system.

Fig. 10. Waterfall plot for the tested acceleration signal at the fixed end of stationary spindle.

Fig. 11. Waterfall plot for the tested acceleration signal at the free end of stationary spindle.

Fig. 12. Numerical acceleration signal at the fixed end of stationary spindle.

Fig. 13. Numerical acceleration signal at the free end of stationary spindle.

Fig. 14. Spectral comparisons between present and experimental results at the fixed end of stationary spindle for rotating speed equal to 4600rpm.

Fig. 15. Spectral comparisons between present and experimental results at the free end of stationary spindle for rotating speed equal to 4600rpm.

Fig. 16. Model comparison for steady-state responses obtained by three models.

Fig. 17. Model comparison for response spectra

Fig. 18. Model comparison between displacement coordinate and nonlinear dynamic models without surface waviness.

Fig. 19. Model comparison of response spectra between displacement coordinate and nonlinear dynamic models without surface waviness.

Fig. 20. Contact angles and deformations obtained by the displacement coordinate model and nonlinear dynamic model without surface waviness.

Fig. 21. Contact angles and deformations obtained by the displacement coordinate model and nonlinear dynamic model with surface waviness.

Fig. 22. The effect of axial load upon time response.

Fig. 23. The effect of axial load upon response spectra.

Fig. 24. The effect of unbalanced mass upon time response.

Fig. 25. The effect of unbalanced mass upon response spectra.

Fig. 26. The effect of outer waviness upon time response.

Fig. 27. The effect of outer waviness upon response spectra.

Fig. 28. The effect of inner waviness upon time response.

Fig. 29. The effect of inner waviness upon response spectra.

List of Tables

1	Physical parameters of the tested rotor-bearing system.	77
2	Parameters of the bearing B7004C and lubricant oil.	78
3	Frequency and amplitude comparisons between experimental and numerical results (fixed end response and rotating speed 4600rpm).	79

Table 1
Physical parameters of the tested rotor-bearing system.

Description	Value
Rotor mass (kg)	1.15
Diametral moment of inertia (kgm^2)	3.02e-3
Polar moment of inertia (kgm^2)	0.5e-3
Mass eccentricity (m)	1e-6
Rotating speed (rpm)	0-4600
Bearing distance (m)	10e-3/15e-3
Mass of supporting structure of bearing L (kg)	6.5
Supporting stiffness of bearing L (N/m)	8e5,1e6,2e5
Mass of supporting structure of bearing R (kg)	6.5
Supporting stiffness of bearing R (N/m)	2e5,2.5e6,1e5

Table 2
Parameters of the bearing B7004C and lubricant oil.

Description	Value
Number of balls	12
Nominal contact angle	15^0
Pitch diameter (mm)	31
Diameter of inner raceway (mm)	20
Diameter of outer raceway (mm)	42
Curvature radius of inner race (mm)	3.1
Curvature radius of outer race (mm)	3.1
Ball diameter (mm)	6
Elastic modulus (N/m^2)	2.075e11
Poisson's ratio	0.3
Ball density (kg/m^3)	7800
Lubricating oil viscosity $\text{Pa} \cdot \text{s}$	1.2e-3
Viscosity-pressure coefficient Pa^{-1}	2.2e-8

Table 3

Frequency and amplitude comparisons between experimental and numerical results
(fixed end response and rotating speed 4600rpm).

	Frequency comparisons (Hz)			Amplitude comparisons (m)		
	experiment	numerical	relative difference	experiment	numerical	relative difference
rotating frequency	76	77	1.3%	0.00077	0.001035	25%
natural frequency	37	37	0.0%	0.00012	5.657e-5	53%
	45	50	10%	8.9e-5	6.769e-5	24%
inner waviness frequency	134.5	137(IF_3)	1.8%	0.000191	0.00014	27%
	275	273(IF_6)	0.7%	0.000197	0.000278	29%
outer waviness frequency	152	156(OF_5)	2.5%	0.000145	0.000368	60%
	311.5	312(OF_{10})	0.2%	0.00035	0.000466	25%