Assessment of Stochastically Updated Finite Element Models Using Reliability Indicator

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Abstract: Finite element (FE) model updating techniques have been a viable approach to correcting an initial mathematical model based on test data. Validation of the updated FE models is usually conducted by comparing model predictions with independent test data that have not been used for model updating. This approach of model validation cannot be readily applied in the case of a stochastically updated FE model. In recognizing that structural reliability is a major decision factor throughout the lifecycle of a structure, this study investigates the use of structural reliability as a measure for assessing the quality of stochastically updated FE models. A recently developed perturbation method for stochastic FE model updating is first applied to attain the stochastically updated models by using the measured modal parameters with uncertainty. The reliability index and failure probability for predefined limit states are computed for the initial and the stochastically updated models, respectively, and are compared with those obtained from the ‘true’ model to assess the quality of the two models. Numerical simulation of a truss bridge is provided as an example. The simulated modal parameters involving different uncertainty magnitudes are used to update an initial model of the bridge. It is shown that the reliability index obtained from the updated model is much closer to true reliability index than that obtained from the initial model in the case of small uncertainty magnitude; in the case of large uncertainty magnitude, the reliability index computed from the initial model rather than from the updated model is closer to the true value. The present study confirms the usefulness of measurement-calibrated FE models and at the same time also highlights the importance of the uncertainty reduction in test data for reliable model updating and reliability evaluation.

Keywords: Stochastic FE model updating; Model validation; Structural reliability; Modal variability; Structural health monitoring.

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1. Introduction

Developing appropriate and credible finite element (FE) models is indispensable for design and evaluation of complex engineering structures. These models are approximate imitations of real-world systems and they never exactly imitate the real-world systems. Despite the high degree of sophistication in modeling techniques, practical applications often indicate great discrepancies between experimental modal data and the corresponding model predictions due to simplification of complex physical process, inappropriate assignment of parameter values, measurement errors, and incompleteness of available test data, among others. Model updating from test data has become a widely accepted procedure for enhancement of the test/analysis correlation and improvement in the quality of analytical model for a structure [1-4].

In reality, the modal parameters measured from an operational civil structure show significant variations which are caused by several factors such as natural variations of physical properties in a structure, measurement noise in data, and errors involved in modal identification process. Thus, they are best represented by random variables with uncertainty. Accordingly, stochastic model updating procedures taking into account modal parameter uncertainty have been developed by treating the model parameters as random variables and constructing their probability density functions (PDFs) from the measured uncertainty of modal parameters.

Depending on how the definition of probability is interpreted, two main alternatives of probability description, namely the Bayesian and frequentist description are available for selection. Beck and his colleagues developed a Bayesian-based probabilistic framework for stochastic model updating, where the prior distribution of the model parameters in an initial model and the measured modal data with uncertainty are used to quantify their posterior uncertainties [5-7]. And the results are often expressed in terms of the most probable value (MPV) of model parameters and their posterior variance/covariance. In the context of frequentist description of probability, the stochastic model updating aims at determining the ensemble statistics of the model parameters based on the modal data with uncertainty; only the mean and covariance of model parameters are of concern when the Gaussian distributions of random variables are followed [8-18]. Recently, Au [19] have explored the connections

between the Bayesian and frequentist quantification of uncertainty in model updating, and the author demonstrated that the updating results obtained with the two methods are quite close each other when there is no or little model error.

Due to the insensitivity of model parameters to modal data, sparseness of sensor locations and measurement noise, model updating is an ill-posed inverse problem such that small variation in input (modal data) can produce unnecessarily large variation in output (model parameters), which may deteriorate model's predictive performance. The updated finite element model usually produces a more accurate prediction of the low-order modal parameters used for model updating than the initial model, however it is not guaranteed that the updated model give an improved prediction of other independent data set, such high-order modal parameters, specific static deformation and local stress. Thus assessing the quality of the stochastically updated FE models is of significant value for assuring a reliable and useful model updating.

As pointed out in a recent verification and validation standard [20], calibrating parameters by minimizing the residuals between predictions and experiments only indicates model's fitting ability; there is no guarantee of future prediction capability. The assessment of quality of updated models is usually carried out by comparing model predictions with additional independent data set that have not been involved in model updating process [2, 21]. This approach serves well for aerospace and automotive engineering as global dynamic responses are of major concern. However, it may become inadequate for civil engineering applications where structural safety issues such as load-carrying capacity and stresses are of equal importance. For example, as indicated in a recent paper by Aktan and Brownjohn [22], the use of calibrated models with modal data for predictive simulations of local structure response may be problematic in civil engineering disciplines and this is analogous to danger of extrapolating from data that are only robust to interpolation. The problem is further complicated when quality of the stochastic FE models are concerned. Firstly, both the mean and variance/covariance of updating structural parameters contribute to model's predictive performance and model quality. Secondly, direct comparison between prediction and test data cannot be readily applied as the model predictions from the updated model are uncertain or

random. Wong and Yao [23] pointed out that structural reliability is a major decision factor for decision making throughout the entire life of a structure, and they emphasized that health monitoring, model updating and damage detection, and reliability evaluation are sequential components for implementation of monitoring-based risk management. Beck and Au [24] performed Bayesian updating of structural models and reliability using Markov Chain Monte Carlo simulation based on simulation data. Following the Wong and Yao's conceptual framework, Hua *et al.* [25] have explored the use of the structural reliability as a measure for assessing the quality of a stochastically updated model.

This study examines the use of reliability index calculated from various finite element models for assessing the quality of the updated models in the presence of uncertain modal data. As the exact reliability index of an existing structure is never known, the motivation of this work is mainly to confirm the usefulness of model updating for improved reliability assessment and at the meanwhile to highlight the limitations in model updating in the case of large uncertainty of modal data. A recently developed method that combines the improved perturbation technique and the sensitivity-based model updating algorithm is first adopted for stochastic model updating by using the measured uncertain modal data. The updated results may be further revised according to Bayesian updating when the *prior* distribution about the model parameters is available. Then structural reliability analysis is conducted on the initial and the updated models to compute the reliability index and failure probability for predefined limit state functions. The reliability indices computed are next compared with those obtained from the 'true' model to assess the quality of various FE models. The proposed method is demonstrated by numerical simulations of a truss bridge. Simulated modal data with different uncertainty magnitudes are used to conduct stochastic model updating. The effect of uncertainty magnitudes in the simulated modal data on the quality of updated models is also addressed.

2. Stochastic Finite Element Updating

In the conventional sensitivity-based FE model updating, inferring model parameter from test data is achieved by solving an optimization problem where the model parameters in an initial

model are sought so that the updated model reproduces as closely as possible the test data. When modal data are used, the objective function in the optimization problem is expressed as the residuals between the test modal data and the corresponding model output, as follows

$$\text{Min } J_{\theta} = \boldsymbol{\varepsilon}^T \mathbf{W} \boldsymbol{\varepsilon} = \left\| \mathbf{W}^{1/2} (\tilde{\mathbf{z}} - \mathbf{z}(\boldsymbol{\theta})) \right\|_2^2 \quad (1)$$

where $\tilde{\boldsymbol{\varepsilon}}$ is the residuals of modal data between prediction and test, $\tilde{\boldsymbol{\varepsilon}} = \tilde{\mathbf{z}} - \mathbf{z}(\boldsymbol{\theta})$; $\tilde{\mathbf{z}}$ and $\mathbf{z}(\boldsymbol{\theta}) \in R^n$ are vectors of the experimental and predicted modal data with $n = n_f \times (n_m + 1)$; n_f and n_m are the numbers of measured natural frequencies and measured coordinates of each mode shape, respectively; $\boldsymbol{\theta} \in R^m$ is a vector consisting of m model parameters for updating; $\mathbf{W}_{\boldsymbol{\varepsilon}} = \mathbf{W}^{1/2}$ is the diagonal weighting matrix; and the subscript T represents the transpose of a matrix/vector. In order to obtain a unique solution, the number of known modal data n should be not less than the number of the unknown updating parameters m .

The nonlinear least-squares problem in Equation (1) is solved with the gradient-based optimization approach. Beginning with an initial estimate of updating parameters, the solution for updating parameters at the k th iteration step is obtained as

$$\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} + \Delta \boldsymbol{\theta}^{(k)} \quad (2a)$$

$$\mathbf{S}^{(k)} \Delta \boldsymbol{\theta}^{(k)} = \mathbf{e}^{(k)} \quad (2b)$$

where $\mathbf{S}^{(k)} = \mathbf{W}^{1/2} \partial \mathbf{z} / \partial \boldsymbol{\theta}$ is the weighted modal sensitivity matrix; $\mathbf{e}^{(k)} = \mathbf{W}^{1/2} (\tilde{\mathbf{z}} - \mathbf{z}(\boldsymbol{\theta}^{(k)}))$ is the weighted residual vector at the k th iteration step.

As already pointed out, the measured modal data shows statistical variations for a structure in changing operational/environmental conditions. In this study, they are treated as the Gaussian random variables with known statistics. And each measured modal component is expressed as the sum of a deterministic part and a random part, as

$$\tilde{z}_i = \bar{\tilde{z}}_i + Y_i \quad (i = 1, 2, \dots, n) \quad (3)$$

where $\bar{\tilde{z}}_i$ is the mean of each modal component $\tilde{z}_i$; Y_i represents the random part caused by the natural randomness and measurement errors as will be detailed in numerical example, and it is termed as basic random variable (BRV).

Due to the modal data uncertainty, it is obvious that the updating parameters and the resulting modal sensitivity matrix within the model updating iteration process will also be subject to uncertainty. Following the first-order Taylor series expansion, one can therefore express each term in Equation (2b) by an approximate linear function in terms of the BRVs around mean value point. Specifically, the change in structural parameters $\Delta\boldsymbol{\theta}^{(k)}$, weighted modal sensitivity $\mathbf{S}^{(k)}$, and weighted error $\mathbf{e}^{(k)}$ are given as follows,

$$\mathbf{S}^{(k)} = \bar{\mathbf{S}}^{(k)} + \sum_{i=1}^n \frac{\partial \mathbf{S}^{(k)}}{\partial Y_i} Y_i \quad (4a)$$

$$\Delta\boldsymbol{\theta}^{(k)} = \Delta\bar{\boldsymbol{\theta}}^{(k)} + \sum_{i=1}^n \frac{\partial \Delta\boldsymbol{\theta}^{(k)}}{\partial Y_i} Y_i \quad (4b)$$

$$\mathbf{e}^{(k)} = \bar{\mathbf{e}}^{(k)} + \sum_{i=1}^n \frac{\partial \mathbf{e}^{(k)}}{\partial Y_i} Y_i \quad (4c)$$

Substituting Equation (4) into Equation (2b) and comparing the zeroth-order and first-order terms of Y_i lead to two following recursive equations, i.e.,

$$\bar{\mathbf{S}}^{(k)} \Delta\bar{\boldsymbol{\theta}}^{(k)} = \bar{\mathbf{e}}^{(k)} \quad (5a)$$

$$\bar{\mathbf{S}}^{(k)} \frac{\partial \Delta\boldsymbol{\theta}^{(k)}}{\partial Y_i} = \left(\mathbf{W}^{1/2} \left(\frac{\partial \tilde{\mathbf{z}}}{\partial Y_i} - \frac{\partial \mathbf{z}^{(k)}}{\partial Y_i} \right) - \frac{\partial \mathbf{S}^{(k)}}{\partial Y_i} \Delta\bar{\boldsymbol{\theta}}^{(k)} \right) \quad (5b)$$

where the partial derivatives of $\mathbf{z}^{(k)}$ and $\mathbf{S}^{(k)}$ with respect to the BRV Y_i at the k th iteration step are evaluated using the chain rule of differentiation [13], as follows

$$\frac{\partial \mathbf{z}^{(k)}}{\partial Y_i} = \frac{\partial \mathbf{z}^{(k)}}{\partial \boldsymbol{\theta}^{(k)}} \frac{\partial \boldsymbol{\theta}^{(k)}}{\partial Y_i} \quad (6a)$$

$$\frac{\partial \mathbf{S}^{(k)}}{\partial Y_i} = \sum_{j=1}^m \frac{\partial \mathbf{S}^{(k)}}{\partial \theta_j^{(k)}} \frac{\partial \theta_j^{(k)}}{\partial Y_i} \quad (6b)$$

$$\frac{\partial \boldsymbol{\theta}^{(k)}}{\partial Y_i} = \frac{\partial (\boldsymbol{\theta}^{(k-1)} + \Delta\boldsymbol{\theta}^{(k-1)})}{\partial Y_i} = \frac{\partial \boldsymbol{\theta}^{(0)}}{\partial Y_i} + \sum_{j=1}^{k-1} \frac{\partial (\Delta\boldsymbol{\theta}^{(j)})}{\partial Y_i} \quad (6c)$$

where $\partial \mathbf{z}^{(k)} / \partial \boldsymbol{\theta}^{(k)}$ and $\partial \mathbf{S}^{(k)} / \partial \theta_j^{(k)}$ are the first-order and the weighted second-order modal sensitivities, respectively. Because the initial estimate $\boldsymbol{\theta}^{(0)}$ of updating parameters and the random part Y_i of the measured modal parameters is statistically independent, the first term in

right hand side of Equation (6c), $\partial \boldsymbol{\theta}^{(0)} / \partial Y_i$, vanishes.

The mean and variance-covariance matrix of updating parameters at the k th iteration step are obtained as follows

$$\bar{\boldsymbol{\theta}}^{(k+1)} = \bar{\boldsymbol{\theta}}^{(k)} + \Delta \bar{\boldsymbol{\theta}}^{(k)} \quad (7a)$$

$$\begin{aligned} \text{Var}(\boldsymbol{\theta}^{(k+1)}, \boldsymbol{\theta}^{(k+1)}) = & \left[\frac{\partial \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right] \boldsymbol{\Sigma}_Y \left[\frac{\partial \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right]^T + \left[\frac{\partial \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right] \boldsymbol{\Sigma}_Y \left[\frac{\partial \Delta \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right]^T \\ & + \left[\frac{\partial \Delta \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right] \boldsymbol{\Sigma}_Y \left[\frac{\partial \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right]^T + \left[\frac{\partial \Delta \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right] \boldsymbol{\Sigma}_Y \left[\frac{\partial \Delta \boldsymbol{\theta}^{(k)}}{\partial \mathbf{Y}} \right]^T \end{aligned} \quad (7b)$$

where $\boldsymbol{\Sigma}_Y$ is the variance-covariance matrix of the basic random vector $\mathbf{Y}$, which can be determined from the measured modal data; and $\partial \boldsymbol{\theta}^{(k)} / \partial \mathbf{Y}^{(k)}$ and $\partial \Delta \boldsymbol{\theta}^{(k)} / \partial \mathbf{Y}^{(k)}$ for all the successive iteration steps are evaluated from Equations (6c) and (5b), respectively. It is seen that the stochastic model updating reduces itself to determination of the first two statistical moments, namely the mean and variance of updating parameters provided that the updating parameters conform to Gaussian distribution. Gaussian distributions of measured modal data and updating model parameters are assumed throughout the present study, which warrants that the formulation based on the first-order approximation is most suited for the linear or weakly nonlinear structures subject to small uncertainty.

Having obtained the mathematical relationship between the mean/variance of the updating parameters and those of the modal data, we next combine the updating results above with a certain *prior* knowledge on the updating parameters to attain their posterior statistics by Bayesian analysis. Assume that the structural parameter θ was known a *prior* to obey normal distribution with a mean μ_1 and a variance σ_1^2 before performing stochastic model updating; and with the experimental modal data $\tilde{\mathbf{z}}$ the structural parameter is also identified following normal distribution with a mean μ_2 and a variance σ_2^2 . By Bayesian analysis of the Gaussian distributions, the posterior mean μ and posterior variance σ^2 of the updating structural parameters are given as (Appendix I)

$$\mu = \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2} \quad (8a)$$

$$\sigma^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad (8b)$$

It is seen that both the posterior mean and posterior variance of updating parameters are the weighted average of the prior distribution and the distribution determined from experimental modal data with uncertainty. In the case of large uncertainty in modal data which implies that the experiment data is not informative, the posterior statistics of updating parameters will be close to *prior* knowledge; on the other hand in case of an uninformative *prior* with large variance σ_1^2 , the effect of *prior* knowledge is considered negligible for the posterior statistics of updating parameters.

3. Structural Reliability Analysis

The stochastic model updating have been performed to reconcile the measured modal data and the corresponding model output, therefore the quality of the updated models have a large reliance on that of the measured modal data. Furthermore, large variability in the updating parameters can be induced due to modal insensitivity to updating parameters, sparseness of measurement locations and high level noise in measurement data. In this study, structural reliability computed from the stochastically updated models is used as a measure to evaluate the model quality under uncertainty. It is noted that the methods of structural reliability have been well documented [e.g. 26]. The basic formulation of reliability analysis is outlined here; details can be found in the reference.

Define the updating parameters $\boldsymbol{\theta}$ and external loading parameters as a combined random vector $\mathbf{X} = (x_1 \cdots x_l)^T$ with l being the number of random variables. The failure probability for a given limit-state function $g(\mathbf{x})$ is computed as the probability content that the random vector $\mathbf{x}$ falls in the failure domain $g(\mathbf{x}) < 0$, and is given by the following l -fold integrals

$$p_f = \int \cdots \int_{g(\mathbf{x}) < 0} f_{x_1, x_2, \dots, x_l}(x_1, x_2, \dots, x_l) dx_1 dx_2 \cdots dx_l \quad (9)$$

where $f_{x_1, x_2, \dots, x_l}(x_1, x_2, \dots, x_l)$ denotes the joint probability density function (PDF) of $\mathbf{x}$. An alternative to failure probability p_f is the reliability index β obtained as

$$\beta = \Phi^{-1}(1 - p_f) = -\Phi^{-1}(p_f) \quad (10)$$

in which $\Phi^{-1}(\cdot)$ is the inverse of cumulative distribution function (CDF) of the standard normal variate.

The first-order reliability method (FORM) is commonly used to approximate the probability integrals. In FORM, a transformation is first made to convert the variates $\mathbf{x}$ with arbitrary distribution into the equivalent uncorrelated standard normal variates $\mathbf{u}$, an iterative gradient-based search algorithm is then used in the transformed $\mathbf{u}$ -space to find the design point $\mathbf{u}^*$ which is the shortest-distance point from the origin to the limit-state surface in $\mathbf{u}$ -space. Next, a linearized hyperplane is used to approach the transformed limit-state surface $G(\mathbf{u}) = 0$ at the design point. The failure probability p_f is equal to the probability content for the linearized failure domain in $\mathbf{u}$ -space, as

$$p_f = \int_{\nabla G(\mathbf{u}^*) \cdot (\mathbf{u} - \mathbf{u}^*) \leq 0} \phi(\mathbf{u}) d\mathbf{u} = \Phi(-\beta) \quad (11a)$$

$$\beta = \boldsymbol{\alpha} \cdot \mathbf{u}^* = -\frac{\nabla G(\mathbf{u}^*)}{\|\nabla G(\mathbf{u}^*)\|_2} \cdot \mathbf{u}^* \quad (11b)$$

where $\boldsymbol{\alpha}$ is the normal vector (directional cosine) at the design point $\mathbf{u}^*$. For reliability analysis of a realistic structure, reliability analysis must be performed in connection with FE method to compute structural response and response gradient needed for finding the design point.

4. Numerical Example

A truss bridge is taken as example to illustrate the proposed methodology for assessing the quality of updated models with uncertainty by using reliability index. Three sets of analytical models are available to represent the bridge, namely the initial model, the updated models, and the true model. The initial model is established with structural design drawings and may not be able to reflect satisfactorily the condition of the bridge due to modeling error, incorrect model parameter values, and possible structural damage developed in the structure. The actual or ‘true’ model is the exact representation of the bridge but it is never known in practice; and the updated models are expected to give improved representations of the bridge as they have

been calibrated to the test data obtained from the bridge. In the following, the updated models will be obtained by stochastic FE model updating and the Bayesian updating, and then the failure probability and reliability index results obtained from the three models will be compared.

4.1 Description of the bridge

The truss bridge, as illustrated in Figure 1, is a simply-supported pin-jointed steel bridge consisting of top chord members, bottom chord members, vertical web members and diagonal web members. An initial model is assumed for structural analysis and condition assessment. In the initial model as shown in Figure 1, a total of 21 planar truss elements are used. The material properties used in this model are as follows: mass density $\rho = 7800 \text{ kg/m}^3$, area of cross section $A = 10^{-2} \text{ m}^2$ for each member, and the Young's modulus $E = 200 \text{ GPa}$ for each member.

Figure 1. Initial model for the truss bridge (dimension: m)

Figure 2. Actual model for the truss bridge

In order to provide the ‘measured’ modal data $\tilde{\mathbf{z}}$ to update the above initial model of the bridge, another analytical FE model with the same topology as the initial model is constructed to represent the actual or true condition of the bridge, which is referred to as the ‘true’ model in the present study. The Young's modulus of structural members in the ‘true’ model is set by different percentage of reduction of those in the initial model, as illustrated in Figure 2. The simulated modal data from the ‘true’ model will be taken as test data to update the chosen structural parameters in the initial model. The Young's modulus of each member is selected as structural parameter for updating ($m=21$).

4.2 Modeling of uncertainty in simulated modal parameters

As mentioned previously, the uncertainty in the measured modal parameters arises mainly from two sources, namely the measurement error uncertainty and normal modal variability (or natural randomness). Measurement uncertainty actually contains error involved in

measurements and in the associated data processing and modal identification. It commonly has a zero mean, and its magnitude depends on the experimental instruments, the test environments and data processing methods. The normal modal variability is contributed by the natural randomness of structural parameters, for example the varying environmental temperatures which at least alter the Young's modulus and thus induce changes in modal parameters of a structure. However, the probabilistic characteristic of the modal parameters in a real structure can be very complex and not amenable to further theoretical analysis. The most straightforward way to define their probabilistic characteristics appears using the long-term structural monitoring data [27, 28]. As a first approximation, it is assumed both the measurement uncertainty and natural randomness of structural parameters obey the normal distribution, and the magnitude/level of the uncertainty is described in terms of coefficients of variation (COV), defined as the ratio of standard deviation to mean of a random variable. Two cases of 'simulated' uncertainty in modal parameters are considered for stochastic model updating. In the first case (case 1), we investigate the stochastic model updating by using the uncertain modal parameters due to measurement error. The obtained covariance matrix in this case indeed gives the evidence how reliable the model updating results are in face of the measurement uncertainty. It is assumed that Gaussian uncorrelated random error with COV of 1% exists in each modal component, and the covariance matrix of the basic random variables $\mathbf{Y}$ are as follows

$$\text{Var}(Y_i, Y_j) = \begin{cases} 0 & \text{for } i \neq j \\ (1\%)^2 \bar{z}_i^2 & \text{for } i = j \end{cases} \quad (12)$$

where $\bar{z}_i$ is the mean of the 'measured' i th modal component $\tilde{z}_i$.

The second case (case 2) addresses the stochastic model updating by using the uncertain modal parameters due to the combination of natural randomness and measurement uncertainty. For simulating the natural randomness uncertainty in modal parameters, it is assumed that the Young's modulus of each member follows normal distribution with a given mean and a COV of 10%, and they are then used to compute the modal parameter uncertainty; for the measurement error uncertainty in modal parameters, Gaussian uncorrelated random error with

COV of 1% is again assumed for each modal component. Making use of the statistical independence between measurement uncertainty and natural randomness, the covariance of the simulated modal parameters is obtained as

$$\text{Var}(\tilde{\mathbf{z}}, \tilde{\mathbf{z}}) = \text{Var1} + \text{Var2} \quad (13)$$

where $\text{Var1} = \mathbf{S}_d \mathbf{\Sigma}_E \mathbf{S}_d^T$ is the covariance matrix of modal parameters caused by the natural randomness in the Young's modulus; and $\text{Var2} = (1\%)^2 \text{diag}(\bar{z}_1^2 \quad \dots \quad \bar{z}_i^2 \quad \dots \quad \bar{z}_n^2)$ is the covariance matrix of modal parameters due to the measurement uncertainty.

It should be noted that the first six modal frequencies ($n_f=6$) and the modal displacements ($n_m=5$) in vertical directions at nodes 7~11 in Figure 1 are used to establish the 'measured' modal parameters in order to simulate the incompleteness of measurement.

4.3 Model updating and reliability evaluation with measurement uncertainty only

Based on the mean and the covariance of simulated 'measured' modal parameters, stochastic model updating is carried out to obtain the mean and the covariance of the Young's modulus in the initial model of the bridge. The results are shown in Figure 3 and the mean of reduction factors rather than the Young's modulus are plotted for a clear comparison with the simulated exact value. It is seen that the mean of the simulated reduction factors have been successfully identified. In addition, the obtained coefficients of variation in this case actually provide the confidence level of updating parameters in the presence of measurement error, and can be used as a quality measure of model updating results for structures without natural randomness. Different COV in structural parameters of the updated model implies they have different sensitivities to the modal data used for model updating.

Figure 3. Comparison of updating results with simulated exact values for 1% measurement error: (a) mean of reduction factor; (b) COV

From the stochastic model updating results, the reliability of the bridge to expected future loadings are analyzed. In this study, the loadings imposed on the bridge include five vertical forces and one horizontal force, as shown in Figure 2. All loadings are taken as random variables and follow the lognormal distribution. The five vertical forces have the same mean

120 kN and the same standard deviation 24 kN, and the horizontal force has a mean 60 kN and a standard deviation 12 kN. The failure probability corresponding to a displacement limit-state function is considered in this example. The displacement limit state assumes that the bridge will be unserviceable when the vertical displacement at mid-span (node 9) exceeds 0.02 m.

For each analytical model, namely the initial model, the updated model, and the actual model, the failure probability and reliability index for the displacement limit-state function are determined by FORM and the results are summarized in Table 1. As expected, the updated model calibrated with the ‘measured’ modal parameter provides refined representations of the actual bridge, and therefore gives a more accurate estimate to the actual failure probability than the initial model. The difference in reliability index obtained from the three models is not quite significant as only minor difference in the Young’s modulus is assumed between the initial and the actual ‘true’ model.

Table 1. Reliability index and failure probability for the first case

We next consider that the stochastic model updating from modal data for different magnitudes of measurement error uncertainty with COV varying from 0.2% to 3.0%. The objective is to check the quality of the updated models under different uncertainty magnitudes. According to Equation (7), the means of updating parameters remain constant for varying uncertainty magnitudes while their variances are proportional to the uncertainty magnitudes for the perturbation method. In this regard, one notes that the perturbation method generally produces solutions with acceptable accuracy only for small uncertainty levels, it is important to investigate its accuracy for varying uncertainty magnitudes by using the Monte Carlo simulation (MCS). Specifically, 5000 independently and identically distributed (i.i.d.) samples of modal data are generated according to the given Gaussian distributions and they are then used to obtain the statistics of updating parameters through repeated deterministic FE model updating.

Figure 4 compares the coefficients of variation of selected updating parameters obtained from the MCS method and the perturbation method for different measurement error magnitudes. Due to inherent nonlinear relationship between modal frequencies and updating parameters, the MCS method essentially produces nonlinear variation of output with input. As seen in Figure 4, at the very small magnitude of measurement error, the two methods almost produce the same results. And the discrepancy between the two methods becomes apparent for increasing measurement error magnitudes. It may be concluded that that perturbation method usually yields comparable results with the MCS methods when the measurement error level is less than 2.0% for the present example.

Figure 4 Comparison of COVs between perturbation method and MCS method under varying measurement error levels

Figure 5 shows the variation of the reliability index calculated from the updated model for different measurement error magnitudes under the same loadings and the same limit state function. With the increase of measurement error magnitudes, the reliability index decreases gradually as a results of the increasing uncertainty in model parameters. The deviation from the true reliability index becomes significant which implies the updated model becomes increasingly unreliable. For example, the reliability index from the updated model drops to 5.394 for a 2% measurement error magnitude, which provides worse estimate than the initial model. A serious deterioration of model's predictive performance can be induced by the large measurement error. The results highlight the importance of uncertainty reduction for achieving reliable model updating and refined reliability assessment. Nevertheless, it is pointed out whether the initial model or the updated models performs better in the presence of data uncertainty shall be dealt with case by case rather than general conclusions.

Figure 5 Reliability index from updated models for different measurement error magnitudes

4.4 Model updating and reliability evaluation with both measurement uncertainty and natural randomness

The second case is the stochastic model updating using the uncertain modal data arising from both natural randomness and measurement uncertainty, as given in Equation (13). Figure 6

shows the mean and coefficients of variation of updating parameters computed from both the perturbation method and the MCS method. The COVs of updating parameters are again quite different for different structural members, ranging from 3% to almost 25%, which is caused by different sensitivities of updating parameters to modal parameters. The parameters with high sensitivity to modal parameters tend to have small variability after updating, and those with low sensitivity to modal parameters are subjected to large variation. This is also the reason why sensitive parameters are selected for model updating in most cases otherwise meaningless or erroneous results may be obtained.

Figure 6 Comparison of perturbation method with MCS method for 10% randomness and 1% measurement error: (a) mean of reduction factor; (b) COV

For illustration of Bayesian updating, the prior distribution of updating parameters is assumed that all the updating parameters comply with a normal distribution with a mean μ_1 of 200GPa and a standard deviation σ_1 of 20GPa (or a COV of 0.1). It is quite intuitive to specify the parameter value in the initial model as the mean of the prior distribution, but others are also valid due to the subjective nature of prior distributions. According to Bayesian theorem, the posterior mean μ and covariance σ^2 of updating parameters are obtained from Equation (8). Table 2 summarizes the mean and coefficients of variations of updating parameters for the updated statistics, the prior statistics and the posterior statistics. The updated statistics, which is referred to as ‘updated model 1’, is the results obtained from the perturbation-based FE model updating (LHS of Equation (7)); and the posterior statistics, referred to as ‘updated model 2’, is obtained from the Bayesian analysis (LHS of Equation (8)). Figure 7 shows the updated, prior and posterior distribution functions of some selected updating parameters. It is evident that the posterior distribution functions of structural parameters are sharper than both the prior distribution function and the updated distribution function. This is expected because the variance of a variable reduces when more information is utilized. The Bayesian estimator pays more confidence to the information with small variance. When the variance in the assumed prior distribution is smaller than that in the updated distribution, the Bayesian estimator is closer to the prior distribution than to the updated distribution as shown in Figure

7(a); when the variance in the prior distribution is larger than that in the identified distribution, the Bayesian estimator is closer to the updated distribution than to the prior distribution as shown in Figure 7(b); and when the variance in the prior distribution is identical with that in the updated distribution, the Bayesian estimator is balanced between the prior distribution and the updated distribution as shown Figure 7(c-d).

Table 2. Updated, prior, and posterior statistics of reduction factor for updating parameters

Figure 7. Updated, prior and posterior PDFs of reduction factors for updating parameters:
(a) element 1; (b) element 5; (c) element 3; (d) element 21

The reliability index is again calculated from the various models for the second case. Two updated models are used for reliability analysis. The updated model 1 is the stochastically updated model from the perturbation algorithm without incorporating the prior knowledge. It is further revised to achieve the updated model 2 by incorporating the prior information through Bayesian updating. The reliability indices calculated from various models are listed in Table 3. By comparing column 3 and column 4 in Table 3, it is seen that the updated model 1 produces a closer result to the actual failure probability than the updated model 2, indicating that Bayesian updating is not of efficacy in this case. It is interesting to note: i) the initial model does not include the structural damage possibly developed in the structure, and it commonly gives a larger reliability index value than the actual model; ii) the updated model 1 always produce a smaller reliability index value than the actual model because the variances of updating parameters in this model are larger than the true values due to the measurement uncertainty, thus yielding a conservative estimate of reliability index. However, when the measurement uncertainty is significant, the variances of updating parameters will be very large and therefore the obtained reliability index will be too conservative and unreliable; and iii) the performance of updated model 2 is significantly affected by the prior information, and specification of a highly informative *prior* is crucial for the Bayesian updating.

Table 3. Reliability index and failure probability based on various models

The effect of modal uncertainty level on the quality of updated models is further investigated. According to Equation (7), it is obvious that the model parameter variance in the

stochastically updated model is linearly proportional to the variance in measured modal parameters. Stochastic finite element updating is performed for different uncertainty level in modal parameters ranging from 0.2% to 2.0% with an increment of 0.2% to obtain a series of updated model 1, as given the previous section. Figure 8 shows the variation of reliability index obtained with the initial model, the updated model 1, and the updated model 2, and the true model with different uncertainty levels in modal parameters. The initial and the true model are not updated with the measured modal parameters, and the corresponding reliability indices remain constant. The reliability index calculated with the updated model drastically reduces with increase of uncertainty level in modal parameters. For large measurement error magnitudes, the updated model may become unreliable as the reliability index computed from the initial model rather than from the updated model is closer to the true value. Therefore it is important to reduce the uncertainty level in measurement data for reliable model updating.

Figure 8. Variation of reliability index with different uncertainty magnitude in modal data

Concluding Remarks

This paper has investigated the use of structural reliability as a measure for assessing the quality of the stochastically updated models which are obtained from uncertain modal data. The numerical studies of stochastic model updating of a truss bridge involving different uncertainty magnitudes of modal data are used to update an initial model of the bridge. The results show that that the reliability index obtained from the updated model is much closer to true reliability index than that obtained from the initial model in the case of low uncertainty in modal data. For high level measurement uncertainty, the variability in updating parameters may be considerably large, and the updated model may also become unreliable. Since the exact reliability index of an existing structure is never known, the motivation of this work is mainly to confirm the usefulness of model updating calibrated with measurement data and at the meanwhile to highlight the importance of uncertainty reduction in measurement data for reliable model updating.

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Appendix I

According to the Bayes' theorem, the posterior probability density function (PDF) $p(\boldsymbol{\theta}|\mathbf{D}, M)$ of the model M specified by the parameter $\boldsymbol{\theta}$ is written as follows [5]:

$$p(\boldsymbol{\theta} | \mathbf{D}, M) = c p(\mathbf{D} | \boldsymbol{\theta}, M) p(\boldsymbol{\theta} | M) \quad (\text{A1})$$

where $p(\boldsymbol{\theta}|\mathbf{D}, M)$ is the posterior PDF of the model parameters given the data $\mathbf{D}$ and the model M ; c is a normalizing constant that ensures that the posterior PDF integrates to one; $p(\mathbf{D}|\boldsymbol{\theta}, M)$ is the likelihood function that represents the PDF of the data given the parameters $\boldsymbol{\theta}$; and $p(\boldsymbol{\theta}|M)$ is the prior PDF of the parameters $\boldsymbol{\theta}$.

Assume that the updating structural parameter $\boldsymbol{\theta}$ was known a prior to obey normal distribution with a mean μ_1 and a variance σ_1^2 before performing stochastic model updating; and with the data $\mathbf{D}$ (or $\tilde{\mathbf{z}}$), the structural parameter $\boldsymbol{\theta}$ is also identified following normal distribution with a mean μ_2 and a variance σ_2^2 . Therefore the prior PDF and the likelihood function can be written as follows:

$$p(\boldsymbol{\theta} | M) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(\boldsymbol{\theta} - \mu_1)^2}{2\sigma_1^2}\right) \quad (\text{A2})$$

$$p(\mathbf{D} | \boldsymbol{\theta}, M) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(\boldsymbol{\theta} - \mu_2)^2}{2\sigma_2^2}\right) \quad (\text{A3})$$

By substituting Eqs. (A2) and (A3) into (A1), the posterior PDF can be written as,

$$p(\boldsymbol{\theta} | \mathbf{D}, M) = c \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(\boldsymbol{\theta} - \mu_2)^2}{2\sigma_2^2}\right) \times \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(\boldsymbol{\theta} - \mu_1)^2}{2\sigma_1^2}\right) \quad (\text{A4})$$

By simplification, Eq. (A4) can be rewritten as follows

$$p(\boldsymbol{\theta} | \mathbf{D}, M) = c_1 \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\boldsymbol{\theta} - \mu)^2}{2\sigma^2}\right) \quad (\text{A5a})$$

$$\mu = \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2} \quad (\text{A5b})$$

$$\sigma = \sqrt{\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \quad (\text{A5c})$$

It is seen that the posterior probability density functions of the model M also conforms to the Gaussian distribution.

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Table 1. Reliability index and failure probability for the first case

	Initial model	Updated model	Actual model
Reliability index	6.648	6.264	6.316
Failure probability	1.428×10^{-11}	1.878×10^{-10}	1.339×10^{-10}

Table 2. Updated, prior, and posterior statistics of reduction factor for updating parameters

Element No.	Updated statistics (updated model 1)		Prior statistics		Posterior statistics (updated model 2)	
	Mean	COV	Mean	COV	Mean	COV
1	0.054	0.224	0	0.1	0.009	0.091
2	0.000	0.139	0	0.1	0.000	0.081
3	-0.007	0.102	0	0.1	-0.004	0.071
4	0.005	0.172	0	0.1	0.002	0.086
5	0.099	0.026	0	0.1	0.093	0.025
6	0.004	0.080	0	0.1	0.003	0.062
7	-0.001	0.060	0	0.1	0.000	0.052
8	0.149	0.208	0	0.1	0.028	0.090
9	0.050	0.078	0	0.1	0.031	0.062
10	0.001	0.127	0	0.1	0.000	0.079
11	0.099	0.053	0	0.1	0.077	0.047
12	0.003	0.116	0	0.1	0.001	0.076
13	0.000	0.064	0	0.1	0.001	0.054
14	-0.002	0.084	0	0.1	-0.001	0.064
15	0.102	0.058	0	0.1	0.076	0.050
16	0.114	0.072	0	0.1	0.075	0.058
17	0.085	0.077	0	0.1	0.054	0.061
18	0.046	0.091	0	0.1	0.025	0.067
19	0.003	0.115	0	0.1	0.001	0.075
20	0.004	0.117	0	0.1	0.002	0.076
21	-0.012	0.101	0	0.1	-0.006	0.071

Table 3. Reliability index and failure probability based on various models

	Nominal model	Updated model 1	Updated model 2	Actual model
Reliability index	6.457	6.121	6.340	6.124
Failure probability	5.345×10^{-11}	4.660×10^{-10}	1.145×10^{-10}	4.552×10^{-10}

Figure 1. Initial model for the truss bridge (dimension: m)

Figure 2. Actual model for the truss bridge

Figure 3. Comparison of updating results with simulated exact values for 1% measurement error: (a) mean of reduction factor; (b) COV

Figure 4 Comparison of COVs between perturbation method and MCS method under varying measurement error levels

Figure 5 Reliability index from updated models for different measurement error magnitudes

Figure 6 Comparison of perturbation method with MCS method for 10% randomness and 1% measurement error: (a) mean of reduction factor; (b) COV

Figure 7. Updated, prior and posterior PDFs of reduction factors for updating parameters:
(a) element 1; (b) element 5; (c) element 3; (d) element 21

Figure 8. Variation of reliability index with different uncertainty magnitude in modal data