A NEW ANALYTICAL MODEL FOR VIBRATION OF A CYLINDRICAL SHELL AND CARDBOARD LINER WITH FOCUS ON INTERFACIAL DISTRIBUTED DAMPING

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ABSTRACT

This paper proposes a new analytical model for a thin cylindrical shell that utilizes a homogeneous cardboard liner to increase modal damping. Such cardboard liners are frequently used as noise and vibration control devices for cylindrical shell-like structures in automotive drive shafts. However, most prior studies on such lined structures have only investigated the associated damping mechanisms in an empirical manner. Only finite element models and experimental methods have been previously used for characterization, whereas no analytical studies have addressed sliding friction interaction at the shell-liner interface. The proposed theory, as an extension of a prior experimental study, uses the Rayleigh–Ritz method and incorporates material structural damping along with frequency-dependent viscous and Coulomb interfacial damping formulations for the shell-liner interaction. Experimental validation of the proposed model, using a thin cylindrical shell with three different cardboard liner thicknesses, is provided to validate the new model, and to characterize the damping parameters. Finally, the model is used to investigate the effect of the liner and the damping parameters on the modal attenuation of the shell vibration, in particular for the higher-order coupled shell modes.

Keywords: *Modal Experiments, Shell Vibration, Distributed Damping, Sliding Interfacial Damping, Rayleigh–Ritz Method, Automotive Applications*

1. Introduction

Koruk et al. [1] recently examined the modes of a thin cylindrical shell containing a cardboard liner. This study developed controlled laboratory experiments to estimate the modal loss factors based on

measured vibratory and acoustic transfer functions, and also investigated the effect of liner thickness on the natural frequencies and loss factors. However, their results are largely empirical with modal calculations provided by a finite element model. As such, this paper proposes a new analytical model for the vibratory motion of a thin cylindrical shell with a homogenous cardboard damping liner, as an extension of their prior work.

There has been considerable interest in the use of cardboard and other liners for damping of shell vibration modes, driven by the need to reduce noise and vibration from drive shafts in the automotive industry [2–5]. Koruk et al. [1] and Sun et al. [5] both cite wide usage of cardboard, rolled paper, and other hybrid treatments such as tuned absorbers to dampen beam, torsional, and shell modes. However, much of the prior literature addresses the “design-of-experiment” type of study and most investigations use finite element analyses with empirical damping values; a more extensive literature survey of such publications is available from Koruk et al. [1]. There is a large body of literature on constrained layer damping surface treatments [6–9], composite or sandwich structures [10], as well as foam or granular fillings [11], but no prior articles have examined relative sliding motion between material layers. Furthermore, these methods are often applied only to beams and plates [12–16] (rather than cylindrical shells), systems with only proportional damping [17], or cited in patents [18–19] with little scientific justification. In most publications, the primary damping mechanism is material (structural) damping, whereas for a cardboard liner, significant damping may arise due to friction at the sliding interface. This particular behavior has not been modeled by prior researchers, and as such, a new physics-based analytical model is needed to overcome a void in the literature.

2. Problem formulation

Consider a thin cylindrical shell of length l , thickness h , and radius a (from the central axis to the mid-plane of the shell) as depicted in Figure 1, with the assumed cylindrical coordinate system as shown; (see Appendix A for the identification of symbols). The shell is made of an elastic material with Young’s modulus E , mass density ρ , Poisson’s ratio ν , and structural (material) loss factor η . The system may also

have an internal cardboard liner of thickness h_L with length $l_L = l$ and radius $a_L = a - h_m$ (interference fit with sufficient “preload”) where $h_m = 0.5 \cdot (h + h_L)$ is the distance between mid-surfaces of the two layers. The liner, like the shell material, is assumed to be elastic with properties E_L , ρ_L , ν_L , and η_L (though E_L and η_L may be frequency dependent). The shell-liner interface is also assumed to have associated viscous and Coulomb damping coefficients, c_v and μ , respectively, possibly having frequency dependence as well. Finally, an external force, F_d is applied to the shell (normal to the surface) at (x_d, θ_d) , and in-plane (u , v) and transverse (w) motions are observed at location (x_0, θ_0) . Note that while other boundaries are possible, only free boundaries (as shown in Figure 1) are used for this study since they are most accurately replicated experimentally.

The objectives of this paper are threefold. First, propose a new analytical model for the vibratory response of a thin cylindrical shell with a cardboard liner; this model makes several simplifying assumptions and will incorporate the liner damping into a material loss factor, η_L , along with frequency-dependent interfacial viscous and Coulomb damping effects. Second, develop a laboratory experiment to estimate parameters (beyond the prior work by Koruk et al. [1]) and to validate the proposed model in terms of eigensolutions and harmonic responses for selected modes of interest to automotive applications [2, 5]. Finally, utilize this model to investigate the effects of liner properties on the damping of the system, particularly in the case of higher-order and coupled modes.

The analytical technique proposed here is the Rayleigh–Ritz method, and free boundaries are assumed at $x = 0$ and $x = L$ for $0 \leq \theta < 2\pi$. A thin cardboard liner will be used for distributed damping with no adhesion, but rather an assumed interference fit with sufficient “preload” such that the liner and shell remain in contact under out-of-plane motion with $w > 0$. The cardboard used for the liners is homogenous with assumed spatially uniform and isotropic properties (though potentially spectrally-varying), rather than corrugated or other nonhomogeneous material [20]. Furthermore, the cardboard liner is assumed to have the same length as the shell to allow for more tractable analysis. Energy methods and Lagrange’s equation are used to derive the governing equations in the time domain, but analyses are restricted to the frequency domain (at frequency ω [rad/s]). Frequency-dependent properties such as

$E_L(\omega)$ and $\eta_L(\omega)$ may be assumed for the cardboard liner, but their temperature-dependency is beyond the scope of this work. While these liner treatments are often used for noise reduction, this study is limited to only the vibratory, rather than the acoustic, response of the system.

Other assumptions include the following. Shape functions are to be separable [21], allowing for a simpler functional form. The cardboard liner is a separate layer with independent motions, but its out-of-plane displacement is assumed equal to that of the substructure. While the shell is thin relative to its length and radius (i.e. $h \ll a, L$), the coupling between out-of-plane motion, w , and in-plane motion, u and v , is not neglected as it is essential for the calculation of shell natural frequencies. Shear deformations and rotary inertia, however, are assumed negligible for both the shell and liner. Finally, the Flügge shell simplifications are employed for the strain energy formulation, and both the shell and liner (along with the combined system) are assumed to be governed by Flügge's linear elastic thin shell theory [21].

3. Development of analytical model

3.1. Motion and energy of a differential element

First, a differential element of nominal size $dx \times a \cdot d\theta$ is considered (as shown in Figure 2, along with corresponding motions) in order to characterize the motion of the shell-structure. Motions of the shell element may include transverse motion, w , in-plane motions u and v (in the x and θ directions, respectively), as well as shears γ_{xr} and $\gamma_{\theta r}$ (in the $x-r$ and $\theta-r$ planes, respectively). Motions of the liner include w_L (assumed equal to w), u_L , and v_L . For simplicity, all shears (in the shell or liner) are assumed to be negligible. Note that, as shown in Figure 2, there may be relative sliding motion, ξ , between the shell and liner with the sign convention such that motion of the liner in the $+x$ or $+\theta$ direction relative to the shell corresponds to $\xi > 0$. Constraint relationships are assumed such that:

$$\begin{aligned} w_L &= w + \xi_x \sin \psi_{xr} + \xi_\theta \sin \psi_{\theta r} \approx w \\ u_L &= u + \frac{1}{2}(h + h_L) \sin \psi_{xr} + \xi_x \cos \psi_{xr} \approx u + h_m \psi_{xr} + \xi_x \\ v_L &= v + \frac{1}{2}(h + h_L) \sin \psi_{\theta r} + \xi_\theta \cos \psi_{\theta r} \approx v + h_m \psi_{\theta r} + \xi_\theta, \end{aligned} \tag{1a-c}$$

where small angle approximations ($\psi_{xr} = \partial w / \partial x \ll 1$ and $\psi_{\theta r} = (1/a) \cdot \partial w / \partial \theta \ll 1$) are employed, and shell and liner transverse motions are assumed equal. Recall mid-surface to mid-surface distance is defined as h_m . In all, the independent unknown motions of interest are w , u , v , u_L , and v_L , all of which are continuous functions of x and θ .

The kinetic and potential energy expressions for the i th layer, T_i and U_i , respectively, of the shell-liner structure are then written in terms of the motion variables (as given by Soedel [22] and Sharma [23] from the Flügge shell equations):

$$\begin{aligned}
 T_i &= \frac{a_i}{2} \iint_{A_i} \rho_i h_i (\dot{w}^2 + \dot{u}_i^2 + \dot{v}_i^2) dx d\theta, \\
 U_i &= \frac{a_i}{2} \iint_{A_i} \frac{E_i h_i}{(1-\nu_i^2)} \left[u_{x,i}^2 + \frac{1}{a_i^2} (v_{\theta,i} + w)^2 + \frac{2\nu_i}{a_i} u_{x,i} (v_{\theta,i} + w) + \frac{1-\nu_i}{2} \left(v_{x,i} + \frac{1}{a_i} u_{\theta,i} \right)^2 \right] + \dots \\
 &\quad \frac{E_i h_i^3}{12(1-\nu_i^2)} \left[w_{xx}^2 + \frac{1}{a_i^4} (w + w_{\theta\theta})^2 - \frac{2\nu_i}{a_i} u_{x,i} w_{xx} + \frac{2\nu_i}{a_i^2} w_{xx} (w_{\theta\theta} - v_{\theta,i}) + \dots \right. \\
 &\quad \left. \frac{1-\nu_i}{2a_i^2} \left(\frac{1}{a_i^2} u_{\theta,i}^2 + 3v_{x,i}^2 + 4w_{x\theta}^2 + \frac{2}{a_i} u_{\theta,i} w_{x\theta} - 6v_{x,i} w_{x\theta} \right) \right] dx d\theta,
 \end{aligned} \tag{2a-b}$$

where integrals are over the entire surface area of the shell or liner, A_i , and subscripts x and θ refer to the partial spatial derivatives. Young's modulus, $\tilde{E}_i = E_i(1 + j\eta_i)$, may be complex-valued to account for structural damping. Energies are computed for each layer (shell and liner) using Eqs. (2).

The non-conservative interfacial force-per-unit-area is assumed to be of the form $F_L = c_v \dot{\xi} + \mu F_N \text{sgn}(\dot{\xi})$, where c_v is a viscous force coefficient, μ is a Coulomb friction coefficient, F_N is the normal force between shell and liner, and $\text{sgn}(\cdot)$ is the well-known signum function. This interfacial interaction is depicted for a differential interface element in the x - r plane in Figure 3. Under steady state harmonic motion (with frequency ω), and if the damping is relatively small, the Coulomb friction could be estimated by means of an “equivalent viscous damping coefficient,” c_μ , such that the cyclic energy dissipated by the assumed equivalent viscous force is equal to that dissipated by the Coulomb friction force. This procedure is detailed by Den Hartog [24] for a single-degree-of-freedom system, and the

resulting equivalent coefficient is both frequency and amplitude dependent. This formulation is adapted for a continuous system as:

$$c_{\mu,x}(\omega) = \frac{4\mu F_N}{\pi\omega |\xi_x(x, \theta, \omega)|}, \quad c_{\mu,\theta}(\omega) = \frac{4\mu F_N}{\pi\omega |\xi_\theta(x, \theta, \omega)|}, \quad (3a-b)$$

where the $|\cdot|$ operator is the spatial root-mean-square (RMS) value. This coefficient, in general, has different values associated with relative motion in the x and θ directions. The work done by the non-conservative force F_L can then be expressed as:

$$W_L = a_m \iint_A \vec{F}_L \cdot \vec{\xi}(x, \theta) dx d\theta = a_m \iint_A \left[(c_v + c_{\mu,x}) \dot{\xi}_x \xi_x + (c_v + c_{\mu,\theta}) \dot{\xi}_\theta \xi_\theta \right] dx d\theta. \quad (4)$$

where a_m is the radial distance to the shell/liner interface ($a_m = a_L + 0.5h_L = a - 0.5h$). Then, W_L can be expressed in terms of the 5 motion variables using the constraint relations for ξ from Eqs. (1).

3.2. Rayleigh–Ritz formulation

In order to determine the unknown motions, the Rayleigh–Ritz (Galerkin) method is used, where each of the unknown motions is assumed to be a linear combination of N_s weighted shape functions:

$$w(x, \theta; t) = \sum_{i=1}^{N_s} q_i^w(t) \phi_i^w(x, \theta) = e^{j\omega t} \sum_{i=1}^{N_s} q_i^w \phi_i^w(x, \theta) = e^{j\omega t} \{\mathbf{q}^w\}^T \mathbf{\Phi}^w(x, \theta). \quad (5)$$

Here, $\phi_i^w(x, \theta)$ is the i th assumed shape function for transverse motion (w), $q_i^w(t)$ is the corresponding unknown time-dependent weighting factor, and if harmonic motion is assumed (at angular frequency ω) the complex exponential time-varying function can be factored out resulting in a constant weighting factor q_i^w . Compact vector notation is ultimately used, where bold terms denote vectors (here of dimension $N_s \times 1$) and T is the transpose operator. Analogous formulations for u , v , u_L , and v_L can be developed, each with their own sets of shape functions ($\mathbf{\Phi}^u$, $\mathbf{\Phi}^v$, etc.) and weighting factors ($\mathbf{q}^u$, $\mathbf{q}^v$, etc.)

The only restrictions on the shape functions are that they are differentiable and conform to the desired boundary conditions [25]. Furthermore, separable shape functions are assumed for simplicity (i.e. $\phi_k(x, \theta) = X_m(x) \cdot \Theta_n(\theta)$) with N_x shape functions in the x -direction, N_θ in the θ direction, and mapping

$k = (m-1) \cdot N_\theta + n$, where $1 < m \leq N_x$, $1 < n \leq N_\theta$, and $1 < k \leq N_s = N_x \times N_\theta$. As such, the following five sets of shape functions are employed, adapted from [21]:

$$\begin{aligned}
\phi_k^w(x, y) &= X_m(x) \cos((n-1)\theta), \\
\phi_k^u(x, y) &= X_m(x) \cos((n-1)\theta), \\
\phi_k^v(x, y) &= \begin{cases} X_m(x), & n=1 \\ X_m(x) \sin((n-1)\theta), & n>1 \end{cases} \\
\phi_k^{u_L}(x, y) &= X_m(x) \cos(n(\theta - \vartheta_{x,n})), \\
\phi_k^{v_L}(x, y) &= \begin{cases} X_m(x), & n=1 \\ X_m(x) \sin((n-1)(\theta - \vartheta_{\theta,n})), & n>1. \end{cases}
\end{aligned} \tag{6a-e}$$

Here, ϑ is an arbitrary angular offset of the displacement shape of the liner relative to the shell; for the present ϑ_n will be assumed equal to zero for all n , as this results in minimal work W_L done by the interfacial forces. The ϕ^v and ϕ^{v_L} shape functions are modified for $n = 1$ to avoid the $\phi(x, \theta) = 0$ condition (which can result in singular matrices upon the integrations in Appendix B), and $X_m(x)$ is a beam-like shape function from [21], given by:

$$X_m(x) = \begin{cases} 1, & m=1 \\ \frac{1}{2} - \frac{x}{L_x}, & m=2 \\ (\sinh \beta_m x + \sin \beta_m x) - \frac{(\sinh \beta_m L_x - \sin \beta_m L_x)}{(\cosh \beta_m L_x - \cos \beta_m L_x)} (\cosh \beta_m x + \cos \beta_m x), & 2 < m \leq N_x. \end{cases} \tag{7a-c}$$

Here $m = 1$ and $m = 2$ are the well-known Rayleigh and Love inextensional modes and β_m is the wavenumber satisfying $\cos(\beta_m L_x) \cdot \cosh(\beta_m L_x) = 1$. In general for the functions of Eqs. 6, m is one more than the number of nodes (zero crossings) of the shape function and n is equal to the number of circumferential “lobes” (i.e. twice the number of zero crossings).

The shape functions and weighting factors are then assembled into vectors of size $5N_s \times 1$ as:

$$\mathbf{\Phi} = \left\{ \mathbf{\Phi}^{wT} \quad \mathbf{\Phi}^{uT} \quad \mathbf{\Phi}^{vT} \quad \mathbf{\Phi}^{u_L T} \quad \mathbf{\Phi}^{v_L T} \right\}^T, \quad \mathbf{q} = \left\{ \mathbf{q}^{wT} \quad \mathbf{q}^{uT} \quad \mathbf{q}^{vT} \quad \mathbf{q}^{u_L T} \quad \mathbf{q}^{v_L T} \right\}^T. \tag{8a-b}$$

Substituting Eqs. (6–8) into Eq. (5) and the result into Eq. (2) yields the kinetic and potential energy, $T(\mathbf{q}, \Phi)$ and $U(\mathbf{q}, \Phi)$. Applying Lagrange's equation to the energies and non-conservative work (Eq. (4)) with respect to the unknown weighting factors, $\mathbf{q}$, results in the following system of equations:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \tilde{\mathbf{Q}}, \quad (9)$$

where $\tilde{\mathbf{Q}}$ is a complex, generalized forcing vector defined in Section 5, and $\mathbf{M}$, $\mathbf{C}$, and $\tilde{\mathbf{K}}$ are equivalent mass and stiffness matrices defined by:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & & & & \mathbf{0} \\ & \mathbf{M}_{22} & & & \\ & & \mathbf{M}_{33} & & \\ & & & \mathbf{M}_{44} & \\ \mathbf{0} & & & & \mathbf{M}_{55} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} & \mathbf{C}_{14} & \mathbf{C}_{15} \\ \mathbf{C}_{12}^T & \mathbf{C}_{22} & \mathbf{0} & \mathbf{C}_{24} & \mathbf{0} \\ \mathbf{C}_{13}^T & \mathbf{0} & \mathbf{C}_{33} & \mathbf{0} & \mathbf{C}_{35} \\ \mathbf{C}_{14}^T & \mathbf{C}_{24}^T & \mathbf{0} & \mathbf{C}_{44} & \mathbf{0} \\ \mathbf{C}_{15}^T & \mathbf{0} & \mathbf{C}_{35}^T & \mathbf{0} & \mathbf{C}_{55} \end{bmatrix} \quad (10a-c)$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} \tilde{\mathbf{K}}_{11} & \tilde{\mathbf{K}}_{12} & \tilde{\mathbf{K}}_{13} & \tilde{\mathbf{K}}_{14} & \tilde{\mathbf{K}}_{15} \\ \tilde{\mathbf{K}}_{12}^T & \tilde{\mathbf{K}}_{22} & \tilde{\mathbf{K}}_{23} & \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{K}}_{13}^T & \tilde{\mathbf{K}}_{23}^T & \tilde{\mathbf{K}}_{33} & \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{K}}_{14}^T & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{K}}_{44} & \tilde{\mathbf{K}}_{45} \\ \tilde{\mathbf{K}}_{15}^T & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{K}}_{45}^T & \tilde{\mathbf{K}}_{55} \end{bmatrix}.$$

The submatrices are given by integrals of various combinations of shape functions and their spatial derivatives, scaled by stiffness, inertial, and damping terms, as defined in Appendix B. The form of the submatrices is also similar to that derived by Chen and Huang [8]. Note here that transposes of the complex-valued $\tilde{\mathbf{K}}$ matrices in Eq. (10c) are standard, non-Hermitian type to ensure the positive value of material loss factor.

4. Nonlinear eigensolution and modal mapping

Setting $\tilde{\mathbf{Q}} = \mathbf{0}$ in Eq. (9) gives the free vibration response of the system (i.e. excluding external forcing) in terms of the weighting factors, $\mathbf{q}$. In order to compute the eigensolution of this system, the state-space formulation [25] is used with state vector $\mathbf{X} = \{\mathbf{q}^T \quad \dot{\mathbf{q}}^T\}^T$. This formulation results in the generalized eigenvalue problem $\tilde{\mathbf{A}}\tilde{\mathbf{X}} = \tilde{\lambda}\tilde{\mathbf{X}}$ where state matrix $\tilde{\mathbf{A}}$ and eigenvalues $\tilde{\lambda}_i$ are defined as:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\tilde{\mathbf{K}} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \quad (11a-b)$$

$$\tilde{\lambda}_i, \tilde{\lambda}_i^* = a \pm jb = -\zeta_i \omega_i \pm j \omega_i \sqrt{1 - \zeta_i^2}.$$

Here the complex eigensolution gives $2N_s$ eigenvectors ($\tilde{\Lambda}_i$) and eigenvalues and consisting of complex conjugate pairs (denoted by $*$), where ω_i and ζ_i are the i th natural frequency and modal damping ratio, given by:

$$\omega_i = \sqrt{(\text{Im } \tilde{\lambda}_i)^2 + (\text{Re } \tilde{\lambda}_i)^2}, \quad \zeta_i = \frac{-\text{Re } \tilde{\lambda}_i}{\sqrt{(\text{Im } \tilde{\lambda}_i)^2 + (\text{Re } \tilde{\lambda}_i)^2}}. \quad (12a-b)$$

The modal loss factor, η_i , can be related to the modal damping ratio by the at-resonance approximation, $\eta_i \approx 2\zeta_i$ [25].

If frequency-dependent properties are used for the cardboard liner and interface friction ($E_L(\omega)$, $\eta_L(\omega)$, and $c_v(\omega)$), the eigenvalue problem becomes nonlinear due to the frequency dependence of the state matrix, $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}(\omega)$. In this case, an iterative algorithm must be used as described by Lin and Lim [26] and by Plattenburg et al. [27]. Furthermore, since the equivalent Coulomb damping constant, c_μ , is amplitude dependent, this is not meaningful in the context of eigensolution (response amplitudes are not defined without excitation). As such, only the direct viscous damping (i.e. contribution from c_v) will be analyzed in terms of modal loss factors.

A modal mapping must be performed next to determine which eigenvalues are associated with which shape functions. Typically, transverse shell modes are of most interest as they fall in the mid-frequency acoustic radiation range, approximately 200–2000 Hz. An example of such a modal mapping is demonstrated here for the reduced case of an undamped shell with no liner. In this case, $\mathbf{M}$ and $\tilde{\mathbf{K}}$ are of dimension $3N_s \times 3N_s$. The dimensions and material properties of the example shell are included in Table 1. In this case, w , u , and v are the only motions of interest to the problem is reduced to size $3N_s$ (with $\mathbf{C} = \mathbf{0}$). With no interfacial friction in this case, the equations of motion are reduced to $\mathbf{M}\ddot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \mathbf{0}$ and the

standard complex eigenvalue problem can be solved for the matrix $[\mathbf{M}^{-1}\tilde{\mathbf{K}}]$ to yield eigenvalues $\tilde{\lambda}_i$, related to the modal properties as:

$$\omega_i = \sqrt{\text{Re} \tilde{\lambda}_i}, \quad \eta_i = \frac{\text{Im} \tilde{\lambda}_i}{\text{Re} \tilde{\lambda}_i}. \quad (13a-b)$$

Using the Rayleigh–Ritz method, there is no assurance that each given mode will be decoupled, i.e. fully represented by only one shape function. If this were the case, at some natural frequency ω_i there would be negligible u and v motions and w could be written as a single shape function. To determine the extent of modal coupling, a shape function participation factor is defined as from [27] by:

$$\Gamma_{i,k} = \frac{|\tilde{\Lambda}_i(k)|}{\sum_{r=1}^{\kappa N_s} |\tilde{\Lambda}_i(r)|} = \frac{|\tilde{\Lambda}_i((m-1)N_\theta + n)|}{\sum_{r=1}^{\kappa N_s} |\tilde{\Lambda}_i(r)|}, \quad (14)$$

where $0 \leq \Gamma_{i,k} \leq 1$ is a measure of how much the i th mode is represented by the k th shape function, and κN_s is the order of the problem (i.e. $3N_s$ for the reduced case or $5N_s$ for the full case). Furthermore, since the mid-frequency range and transverse modes are of interest, the modes are sorted by increasing natural frequencies (i.e. eigenvalues). If the modes were fully decoupled with all transverse modes at the lowest frequencies, a mapping would be expected as shown in Figure 4a. However, due to some coupling there are non-zero Γ values in the off-block-diagonal terms. This mapping of shape function index k vs. modal frequency ω_i (using the mapping from Section 3, $k = (m-1) \cdot N_\theta + n$, with k ranging from 1 to N_s for w shape functions, N_s+1 to $2N_s$ for u shape functions, etc.) is shown in Figure 4b for $N_s = 16$. Here, blank (white) entries correspond to $\Gamma_{i,k} = 0$, filled-in (black) entries correspond to $\Gamma_{i,k} = 1$, columns correspond to the i th eigenvector, and rows correspond to the k th shape function.

From Figure 4b, it can be seen that indeed most of the lower frequency modes contain strong participation from the w type shape functions. The lowest two modes (with largest participation from u and v type shape functions) correspond to rigid-body motion modes with $\omega_i = 0$ rad/s (permitted due to the free boundaries). Also of note, the $(0, 0)$ w type mode ($k = 1$) is seen to occur as a significantly higher mode (24th, for the case of $N_s = 16$). This is consistent as this is the “breathing” mode which typically has

a very high natural frequency [22]. Table 2 gives the natural frequencies along with mode number and modal index for the first 11 transverse modes, as calculated from the analytical model. Finite element model (FEM) results from Abaqus [28] are also given for the sake of model verification, and excellent agreement is seen in terms of natural frequency and modal indices. Natural frequencies (analytical, FEM, and experiment) are all rounded to the nearest 1 Hz. Note that the mode sequence in Table 2 is defined in terms of ascending ω_i from the analytical model, which is consistent with measurements. However, the finite element model predicts the (2, 2) mode to occur before the (0, 4) mode; this is of little concern due to the close spacing of these modes. Finally, Appendix C describes the results of two FEM models of a shell with a cardboard liner and interfacial friction in an attempt to simulate the engineering modeling practice.

5. Harmonic responses

In order to account for an input to the shell system from an external force, the Rayleigh–Ritz method requires a generalized forcing term on the right-hand side of Eq. (9) [25] as:

$$\mathbf{Q}^\alpha(t) = a_i \int_0^{L_x} \int_0^{2\pi} \vec{F}^\alpha(x, \theta, t) \Phi^\alpha(x, \theta) d\theta dx, \quad (15)$$

where α again refers to one of the 5 sets of shape functions and F^α refers to a distributed force corresponding to one of those motions (i.e. a force normal to the surface corresponds to w , traction in the axial direction corresponds to u , etc.). A force to excite the structure is assumed as a point load normal to the surface (i.e. aligned with $\hat{e}_r$, the unit vector in the r direction) as this loading is easily experimentally replicated with an impulse hammer. As such, the force per unit area can be written as $\tilde{\tilde{F}}_d(x, \theta, t) = \tilde{F}_d \delta(x - x_d) \delta(\theta - \theta_d) \hat{e}_r e^{j\omega t}$, where F_d is the force amplitude. Applying this to Eq. (15)

results in the following for $\tilde{\mathbf{Q}}_d$:

$$\begin{aligned} \tilde{\mathbf{Q}}_d &= \left\{ \tilde{\mathbf{Q}}_d^{wT} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \right\}^T, \\ \tilde{\mathbf{Q}}_d^w &= \tilde{F}_d \Phi(x_d, \theta_d). \end{aligned} \quad (16a-b)$$

With the external loading from F_d , Eq. (9) becomes $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \tilde{\mathbf{K}}\mathbf{q} = \tilde{\mathbf{Q}}_d$, and assuming harmonic excitation and steady-state response, the weighting vector $\mathbf{q}$ is given as:

$$\tilde{\mathbf{q}}(\omega) = [\tilde{\mathbf{K}} + j\omega\mathbf{C} - \omega^2\mathbf{M}]^{-1} \{\tilde{\mathbf{Q}}_d\}. \quad (17)$$

If frequency dependent loss factor and elastic properties ($\eta_L(\omega)$, $E_L(\omega)$) are assumed for the cardboard, $\tilde{\mathbf{K}}$ becomes $\tilde{\mathbf{K}}(\omega)$ in Eq. 17. This equation can still be solved directly provided that the nature of $\eta_L(\omega)$, etc. is known. Recall, however, that the equivalent Coulomb friction coefficient is both frequency and amplitude dependent: $c_\mu = c_\mu(\omega, \zeta(\omega))$, where ultimately $\zeta = \zeta(\mathbf{q})$. In this case, Eq. 17 becomes implicitly defined as:

$$\tilde{\mathbf{q}}(\omega) = [\tilde{\mathbf{K}}(\omega) + j\omega\mathbf{C}(\omega, \tilde{\mathbf{q}}(\omega)) - \omega^2\mathbf{M}]^{-1} \{\tilde{\mathbf{Q}}_d\}. \quad (18)$$

In this case, the solution can be estimated by first computing $\mathbf{q}(\omega)$ for the case of no Coulomb damping, computing $\zeta_x(x, \theta, \omega)$ and $\zeta_\theta(x, \theta, \omega)$ from $\mathbf{q}(\omega)$, obtaining $|\zeta_x(\omega)|$ and $|\zeta_\theta(\omega)|$ (the surface-area averaged RMS values of ζ), and then using these RMS values as the input to $\mathbf{C}(\omega, \zeta(\omega))$ in Eq. 18. An iterative procedure could then be defined using the newly computed $\mathbf{q}(\omega)$ until convergence is achieved. A flowchart illustrating this algorithm is given in Figure 5.

From Eq. 17, the forced response spectrum in terms of out of plane motion $w(x, \theta; \omega)$ can be determined from Eq. 5 by $\{\tilde{\mathbf{q}}^w(\omega)\}^T \boldsymbol{\Phi}^w(x, \theta)$. This simulated result in terms of cross-point acceleration,

$\tilde{\mathcal{A}}_d(\omega) = \tilde{w}(\omega) / \tilde{F}_d(\omega) = -\omega^2 \tilde{w}(\omega) / \tilde{F}_d(\omega)$, is plotted in Figure 6 for a sample undamped shell (with dimensions from Table 1) at motion location $(\bar{x}_0, \bar{\theta}_0) = (0.02, 0)$ given forcing with unit force amplitude at $(\bar{x}_d, \bar{\theta}_d) = (0.44, 0)$. The case of a shell with a cardboard liner is also plotted in Figure 6 for comparison.

Here, overbars on the force and acceleration locations (e.g. $\bar{x}$ and $\bar{\theta}$) indicate normalization by l and 2π , respectively.

6. Experimental results of shells with and without liners

In order to validate the proposed theory, as well as to estimate the damping parameters (η_L , c_v , and μ), a laboratory experiment is designed and conducted, similar to those presented by Koruk et al [1]. A thin, aluminum shell is used with dimensions as detailed in Table 1. The free boundaries are approximated by suspending the shell from compliant bungee cords. Three different cardboard liners are used (again, with dimensions as listed in Table 1, and thicknesses $h_L = 0.31, 0.62$, and 0.98 mm), and the shell is excited with an impulse hammer at location $(\bar{x}_d, \bar{\theta}_d) = (0.44, 0)$. This location is chosen as near the center of the shell, but slightly offset such that $m = \text{odd}$ modes can be excited. Acceleration measurements are made at $(\bar{x}_0, \bar{\theta}_0) = (0.02, 0)$, chosen since free boundaries imply significant participation from all modes at the boundaries. All transducers used for the experiment (accelerometer and impulse hammer/load cell) are from PCB [29].

For the experimental study, all accelerance measurements are cross-point type at the previously specified excitation and measurement locations. A total of 4 cases are measured: the unlined shell, and the shell with the three liner thicknesses. For each case, 8 data-sets are collected and averaged to obtain frequency response functions (FRFs), and this procedure is repeated for 3 “runs” to quantify the variation from run to run. Sample plots of accelerometer measurements from three cases (unlined, as well as the thinnest and thickest liners), each for a single run, are shown in Figure 7. From the figure, it is seen that as predicted, the introduction of the liner significantly reduces the peak values the vibratory measurements, while also shifting the peaks to the left (due to the mass loading effect).

To determine the effect of the damping liner on the lightly damped shell, the aluminum structural (material) loss factor must first be characterized. In theory, all modal loss factors for the unlined shell should be equal to a single value, namely the material loss factor, η_{al} . These modal loss factors are extracted from the measurements of Figure 7 using a least-squares estimation algorithm [30] from the LMS Test.Lab software [31] with a 125-parameter curve fit. The mean and standard deviation of the 3 runs are then computed and plotted in Figure 8, along with the average loss factor of all modes. From

Figure 8, observe that indeed η_{al} is not truly independent of frequency. However the mean value of $\eta = 0.001$, or 0.1%, provides a reasonable approximation for the frequency range of interest, insomuch as the modal damping ratios induced by the presence of the liner are typically much higher than the variation in the unlined modal loss factors. Koruk et al. [1] also performed bench experiments to obtain a material loss factor value for the cardboard liner of $\eta_L = 0.034 = 3.4\%$. This value will be retained for the remainder of the analysis.

In order to quantify the damping effect of the liner, modes 3–6 and 9–11 (from Table 2) are analyzed. Note that modes 1, 2, 7, and 8 are intentionally omitted from the analysis to reduce the clutter of the figures and tables, and because the close spacing of some of these modes can result in modal overlap with increasing damping. The selected modes for the analysis are typically distinct, well isolated, and possess low modal damping in the unlined case. For each of these modes, the natural frequency ($f_i = \omega_i/2\pi$), modal loss factor (η_i), and peak $|\tilde{\mathcal{A}}(\omega_i)|$ values (averaged over 3 runs) are extracted from the FRFs. Then the change in natural frequency (Δf_i), relative loss factor (η_i [dB]), and accelerance insertion loss (IL_i) are evaluated for each mode and liner as:

$$\Delta f_i = f_{i,un} - f_{i,h_L}, \quad \eta_i [\text{dB}] = 20 \log_{10} \left(\frac{\eta_{i,h_L}}{\eta_{i,un}} \right), \quad IL_i = 20 \log_{10} \left(\frac{|\tilde{\mathcal{A}}_{h_L}(\omega_i)|}{|\tilde{\mathcal{A}}_{un}(\omega_{i,un})|} \right) \quad (19a-c)$$

where subscript *un* refers to the unlined shell and subscript h_L to a given liner thickness. The results for each case are then displayed in Figure 9. For all cases, the natural frequency shift, Δf , increases with liner thickness (due to the mass loading effect) in Figure 9a, though this effect is more pronounced at higher frequencies. From Figure 9b, the modal loss factor typically increases with liner thickness; the increase is about 10 dB with the inclusion of the liner, and roughly 10 dB more to the thickest liner case. The IL values typically increase with liner thickness (shown in Figure 9c), however the results suggest a “diminishing returns” trend with increasing h_L . Furthermore, it is clear that there is a “mode dependency” to the IL values (where modes such as 3, 5, and 9 undergo more attenuation than others), whereas the change in η_i with h_L appears to be more uniform across the selected modes.

7. Experimental validation of proposed model and parametric studies

7.1. Natural frequency shift

In order to validate the new proposed model, first the effect of liner presence and liner thickness on natural frequency shift (Δf_i) will be compared. Koruk et al. [1] had made measurements for Δf_i due to liner thickness, however only a finite element model was used to predict this shift. Recall that the frequency shift is largely a mass loading phenomenon, and therefore should be controlled by ρ_L (which is calculated directly from the measured liner mass and volume, and tabulated in Table 1) and theoretically independent of C . The eigensolutions are computed for all three cases of liner thickness, and natural frequencies are extracted and compared to the undamped case. The same is done for the experimental results, using the same least-squares estimation algorithm [30] to extract the natural frequencies, and results are shown in Figure 10. Excellent agreement is seen for most modes for all three liner thicknesses. Some deviations include in modes 4 and 7 where the modes begin to become merged or transpose their order. However, these outliers notwithstanding, the analytical model is capable of predicting the shift in natural frequency with liner thickness reasonably well over the frequency range of interest.

7.2. Insertion loss and modal loss factor trends

Koruk et al. [1] measured only the modal loss factors from experiments and concluded that the damping mechanism was the combined effect of material dissipation, as well as viscous and/or Coulomb interfacial damping. The new proposed model is capable of predicting both modal loss factor as well as vibratory insertion loss from the harmonic forced response. In order to quantify the effects of the damping mechanisms, three simulations are performed: (i) several liner thicknesses with $\eta_L = 3.4\%$ and no interfacial damping ($c_v = \mu F_N = 0$); (ii) one liner thickness ($h_L = 0.62$ mm) and several values of c_v ($\mu F_N = 0$), and (iii) one liner thickness ($h_L = 0.62$ mm) and several values of μF_N ($c_v = 5 \times 10^2$ kg-m⁻²s⁻¹). The same seven modes are compared as from the experimental results in Section 6. In the first case, the only contribution is from the structural damping of the liner, and in the second and third, the effects of

increasing interfacial damping are seen. The results of these simulations in terms of modal loss factor and insertion loss are given in Figures 11–13.

From Figure 11, a clear trend of increasing η_i and IL is seen for all the analyzed modes. This confirms the dependence on h_L as seen in the experiment. However, there is not pronounced modal variation on h_L outside of modes 5 and 11, and neither the relative η_i nor the IL values are as large as those observed experimentally. Furthermore, the “diminishing returns” trend in insertion loss with increasing h_L is not observed. This suggests need of alternate damping mechanisms, as indicated by Koruk et al. [1]. From Figure 12, a stronger modal dependence is seen for the IL when the viscous interfacial damping is included. For larger c_v , significant IL is seen at modes 3, 5, and 9, with less IL at modes 4, 6, and 11, consistent with the experimental observations. Furthermore, it is seen that increasing c_v induces orders of magnitude increase (10–30 dB) in η_i rather than the modest increase induced by the liner structural damping from Figure 11a. This η_i increase relative to the unlined case corresponds more closely with the experimental results. Finally, a Coulomb interfacial damping effect is simulated with results shown in Figure 13. Here, a strong mode-dependence can again be seen, this time with more significant IL at modes 2, 4, and 10 (in disagreement with experiment). This would imply that the viscous interfacial effect may be more dominant than the Coulomb effect over the frequency range analyzed here, consistent with the conclusions of Foulkes et al. [2] based on time-domain and frequency response function methods. However, it could still be worthwhile to include some Coulomb contribution for the sake of the modal variation. Thus, the proposed model, with three different damping mechanisms (η_L , c_v , and μ), replicates many of the experimental observations, including liner thickness dependence and mode dependence of selected modes.

8. Effect of damping parameters on higher-order and coupled modes

As a final study, the effect of different damping parameters on the modal loss factors beyond the present frequency range of interest (up to 2000 Hz) is investigated. In particular, the higher-order extensional/torsional modes and those with significant coupling between shell and liner motions are

considered. Here, the unlined shell case (say, Case 1) will be compared with a lined shell and no interfacial effects, ($h_L = 0.62$ mm, $\eta_L = 3.4\%$, $c_v = 0$, Case 2), and a lined shell with interfacial effects ($h_L = 0.62$ mm, $\eta_L = 3.4\%$, $c_v = 5 \times 10^4$ kg-m⁻²-s⁻¹, Case 3). The Coulomb effects are omitted here, as no eigensolution is possible for this case. The simulations are performed for all cases, and modal loss factors are compared for just the w subset of modes, as well as for all modes, in Figure 14.

From Figure 14, observe that in the unlined case, all η_i are equal to the aluminum loss factor of 0.1%, as expected. For Case 2, many of the η_i are equal to either η_{al} (typically higher order extensional or torsional shell modes) or η_L . However, several of the modes, even some of those associated with the w subset, are somewhere in between these two values, suggesting some coupling exists between the shell and liner motion, even when they are not explicitly coupled by the viscous force. Finally, for Case 3, many modes fall into a category of either: “moderately damped” ($\eta_{al} < \eta_i < \eta_L$); or “highly damped” (close or equal to $\eta_i = 2.0 = 200\%$, the theoretical maximum value). The first category seems to contain more of the higher order shell modes, whereas the second category seems to be associated with the liner modes. This makes sense intuitively as in Case 2, high-order shell modes were not coupled with the liner damping. However when the viscous damping is introduced (which is proportional to in-plane motions, u and v), these modes (u and v shell motions) become highly coupled with the liner.

9. Conclusion

This article is an extension of the experimental work by Koruk et al. [1], and as such presents a new analytical model for vibratory motion of a thin cylindrical shell having a homogeneous cardboard liner as a resonance control device. Such a model is needed to guide dynamic design and analysis, as cardboard liners are used frequently in the automotive industry, but with little physics-based justification. The new theory explicitly models the relative motion between shell and liner and utilizes several damping mechanisms, including structural damping as well as frequency-dependent viscous and Coulomb interfacial damping. Results from a laboratory experiment validate the proposed model and highlight the significant vibratory attenuation that may be achieved by distributed damping liners. Many of the trends

observed experimentally are conceptually captured by the model, such as the shift in natural frequency and the mode-dependent damping the liner induces. The model is also shown to provide reasonable results and efficient computational time as compared to commercial finite element codes (as described briefly in Appendix C). The effects of shell/liner interaction on higher-order modes are also studied, showing that the interfacial viscous damping induces significant coupling between shell and liner motions at higher frequency.

While the proposed formulation is an improvement over previous work, the new model has the limitation that it still relies to some degree on empirical loss factor and friction coefficient values. Further refinement of the model could be provided by explicitly modeling the frequency-dependent effects and developing experiments to estimate damping parameters directly. Also, while cardboard liners are frequently used as noise and vibration control devices, this study is limited to only vibratory shell response; future work should include modeling and measurement of the acoustic insertion loss due to cardboard liners. The model also has some limitations in terms of its assumptions; for instance the $w = w_L$ condition may not always be satisfied, due to imperfections in the liner construction and/or placement. Nevertheless, the proposed theory provides some insight into the effect of such conditions, as well as to other complex cases (i.e. the behavior of higher-frequency modes). As a first attempt to analytically explain the physics of a lined-shell system, this model gives reasonable predictions for the natural frequencies, modal loss factors, and harmonic response, and should prove useful for developing dynamic design guidelines for damping of shell structures like drive shafts.

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Appendix A: List of Symbols

Nomenclature

a	radius
A	surface area (for integration)
$\mathbf{A}$	system matrix
$\mathcal{A}$	accelerance
c	viscous friction coefficient
D	bending stiffness
e	exponential constant
E	Young's modulus
f	frequency [Hz]
F	force amplitude
h	layer thickness
j	imaginary number, $\sqrt{-1}$
K	membrane stiffness
$\mathbf{K}$	stiffness matrix
l	length
$\mathbf{M}$	mass matrix
N	number (of shape functions, etc.)
$\mathbf{Q}$	generalized force vector
$\mathbf{q}$	shape function weighting vector
r	radial coordinate, endpoint, or location
t	time
T	kinetic energy
u, v	lateral displacements (in x, θ direction)
U	potential (strain) energy
w	transverse displacement
x	axial coordinate, endpoint, or location
X	axial shape function
$\mathbf{X}$	state vector
α	arbitrary motion type (transverse, in-plane, etc.)
β	wave number
γ	shear strain
Δ	change or shift
Γ	shape function participation factor
ζ	damping ratio
η	loss factor
θ	circumferential coordinate, endpoint, or location
ϑ	relative offset between shell and liner
Θ	circumferential shape function
κ	order of problem (full or reduced)
λ	eigenvalue

Λ	eigenvector
μ	Coulomb friction coefficient
ν	Poisson's ratio
ρ	mass density per unit volume
ϕ	Rayleigh–Ritz shape function
Φ	shape function vector
χ	error in iterative procedure
ψ	rotation angle of differential element
ω	angular frequency [rad/s]

Subscripts

0	motion location
d	harmonic force excitation
i	counting index for layers and modes
k	counting index for shape functions
L	cardboard liner
m	modal index (in x direction)
n	modal index (in θ direction)
N	normal (force)
s	shape function
un	unlined, or baseline case
v	viscous parameter
μ	equivalent Coulomb parameter

Superscripts

u, v, w	shape functions for different motions
$\sim$	complex valued
$-$	normalized or dimensionless (overbar)
$\rightarrow$	spatial vector
$*$	complex conjugate
T	matrix transpose

Operators

$ \cdot $	RMS value
$\delta(\cdot)$	Dirac delta function
$\text{sgn}(\cdot)$	signum function

Abbreviations

FEM	finite element method
FRF	frequency response function
IL	insertion loss
RMS	root mean square

Appendix B: Derivation of M, C, and K Matrices (in Section 3)

The total kinetic and potential energy for the full shell-liner system is written as:

$$T = \frac{a}{2} \iint_A \rho h (\dot{w}^2 + \dot{u}^2 + \dot{v}^2) dx d\theta + \frac{a_L}{2} \iint_{A_L} \rho_L h_L (\dot{w}^2 + \dot{u}_L^2 + \dot{v}_L^2) dx d\theta, \quad (\text{B1a})$$

$$\begin{aligned} U = & \frac{a}{2} \iint_A \frac{Eh}{(1-\nu^2)} \left[u_x^2 + \frac{1}{a^2} (v_\theta + w)^2 + \frac{2\nu}{a} u_x (v_\theta + w) + \frac{1-\nu}{2} \left(v_x + \frac{1}{a} u_\theta \right)^2 \right] + \dots \\ & \frac{Eh^3}{12(1-\nu^2)} \left[w_{xx}^2 + \frac{1}{a^4} (w + w_{\theta\theta})^2 - \frac{2\nu}{a} u_x w_{xx} + \frac{2\nu}{a^2} w_{xx} (w_{\theta\theta} - v_\theta) + \dots \right. \\ & \left. \frac{1-\nu}{2a^2} \left(\frac{1}{a^2} u_\theta^2 + 3v_x^2 + 4w_{x\theta}^2 + \frac{2}{a} u_\theta w_{x\theta} - 6v_x w_{x\theta} \right) \right] dx d\theta + \dots \\ & \frac{a_L}{2} \iint_{A_L} \frac{E_L h_L}{(1-\nu_L^2)} \left[u_{x,L}^2 + \frac{1}{a_L^2} (v_{\theta,L} + w)^2 + \frac{2\nu_L}{a_L} u_{x,L} (v_{\theta,L} + w) + \frac{1-\nu_L}{2} \left(v_{x,L} + \frac{1}{a_L} u_{\theta,L} \right)^2 \right] + \dots \\ & \frac{E_L h_L^3}{12(1-\nu_L^2)} \left[w_{xx}^2 + \frac{1}{a_L^4} (w + w_{\theta\theta})^2 - \frac{2\nu_L}{a_L} u_{x,L} w_{xx} + \frac{2\nu_L}{a_L^2} w_{xx} (w_{\theta\theta} - v_{\theta,L}) + \dots \right. \\ & \left. \frac{1-\nu_L}{2a_L^2} \left(\frac{1}{a_L^2} u_{\theta,L}^2 + 3v_{x,L}^2 + 4w_{x\theta}^2 + \frac{2}{a_L} u_{\theta,L} w_{x\theta} - 6v_{x,L} w_{x\theta} \right) \right] dx d\theta. \end{aligned} \quad (\text{B1b})$$

The work done by the non-conservative dissipative interfacial viscous and Coulomb forces is written as:

$$W_L = a_m \iint_A \left[(c_v + c_{\mu,x}) \dot{\xi}_x \xi_x + (c_v + c_{\mu,\theta}) \dot{\xi}_\theta \xi_\theta \right] dx d\theta. \quad (\text{B2})$$

Applying Lagrange's equation gives the following, where the Lagrangian is defined as: $L = T - U$:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q} \Rightarrow \mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \tilde{\mathbf{K}} \mathbf{q} = \mathbf{Q}. \quad (\text{B3a-b})$$

The mass matrix $\mathbf{M}$ can be written (utilizing the constraint relations from Eq. (1)) in terms of submatrices as:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & & & & \mathbf{0} \\ & \mathbf{M}_{22} & & & \\ & & \mathbf{M}_{33} & & \\ & & & \mathbf{M}_{44} & \\ \mathbf{0} & & & & \mathbf{M}_{55} \end{bmatrix}, \quad (\text{B4})$$

where the submatrices are defined as follows:

$$\begin{aligned}
\mathbf{M}_{11} &= a\rho h \int_0^l \int_0^{2\pi} \mathbf{\Phi}^w \mathbf{\Phi}^{wT} d\theta dx + a_L \rho_L h_L \int_0^l \int_0^{2\pi} \mathbf{\Phi}^w \mathbf{\Phi}^{wT} d\theta dx, \\
\mathbf{M}_{22} &= a\rho h \int_0^l \int_0^{2\pi} \mathbf{\Phi}^u \mathbf{\Phi}^{uT} d\theta dx, \quad \mathbf{M}_{33} = a\rho h \int_0^l \int_0^{2\pi} \mathbf{\Phi}^v \mathbf{\Phi}^{vT} d\theta dx, \\
\mathbf{M}_{44} &= a_L \rho_L h_L \int_0^l \int_0^{2\pi} \mathbf{\Phi}^{u_L} \mathbf{\Phi}^{u_L T} d\theta dx, \quad \mathbf{M}_{55} = a_L \rho_L h_L \int_0^l \int_0^{2\pi} \mathbf{\Phi}^{v_L} \mathbf{\Phi}^{v_L T} d\theta dx.
\end{aligned} \tag{B5a-e}$$

Similarly, $\mathbf{K}$ can be written as:

$$\tilde{\mathbf{K}} = \begin{bmatrix} \tilde{\mathbf{K}}_{11} & \tilde{\mathbf{K}}_{12} & \tilde{\mathbf{K}}_{13} & \tilde{\mathbf{K}}_{14} & \tilde{\mathbf{K}}_{15} \\ \tilde{\mathbf{K}}_{12}^T & \tilde{\mathbf{K}}_{22} & \tilde{\mathbf{K}}_{23} & \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{K}}_{13}^T & \tilde{\mathbf{K}}_{23}^T & \tilde{\mathbf{K}}_{33} & \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{K}}_{14}^T & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{K}}_{44} & \tilde{\mathbf{K}}_{45} \\ \tilde{\mathbf{K}}_{15}^T & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{K}}_{45}^T & \tilde{\mathbf{K}}_{55} \end{bmatrix}, \tag{B6}$$

where submatrices as defined as follows:

$$\begin{aligned}
\tilde{\mathbf{K}}_{11} &= \int_0^l \int_0^{2\pi} \left(\frac{\tilde{K}}{a} + \frac{\tilde{K}_L}{a_L} \right) \mathbf{\Phi}^w \mathbf{\Phi}^{wT} d\theta dx + \int_0^l \int_0^{2\pi} \left[(a\tilde{D} + a_L \tilde{D}_L) \mathbf{\Phi}_{xx}^w \mathbf{\Phi}_{xx}^{wT} + \left(\frac{\tilde{D}}{a^3} + \frac{\tilde{D}_L}{a_L^3} \right) (\mathbf{\Phi}_{\theta\theta}^w + \mathbf{\Phi}^w) (\mathbf{\Phi}_{\theta\theta}^w + \mathbf{\Phi}^w)^T + \dots \right. \\
&\quad \left. \left(\frac{2\tilde{D}\nu}{a} + \frac{2\tilde{D}_L\nu_L}{a_L} \right) \mathbf{\Phi}_{xx}^w \mathbf{\Phi}_{\theta\theta}^{wT} + \left(\frac{2\tilde{D}(1-\nu)}{a} + \frac{2\tilde{D}_L(1-\nu_L)}{a_L} \right) \mathbf{\Phi}_{x\theta}^w \mathbf{\Phi}_{x\theta}^{wT} \right] d\theta dx, \\
\tilde{\mathbf{K}}_{22} &= a\tilde{K} \int_0^l \int_0^{2\pi} \left[\mathbf{\Phi}_x^u \mathbf{\Phi}_x^{uT} + \frac{(1-\nu)}{2a^2} \mathbf{\Phi}_\theta^u \mathbf{\Phi}_\theta^{uT} \right] d\theta dx + a\tilde{D} \int_0^l \int_0^{2\pi} \frac{(1-\nu)}{2a^4} \mathbf{\Phi}_\theta^u \mathbf{\Phi}_\theta^{uT} d\theta dx, \\
\tilde{\mathbf{K}}_{33} &= a\tilde{K} \int_0^l \int_0^{2\pi} \left[\frac{1}{a^2} \mathbf{\Phi}_\theta^v \mathbf{\Phi}_\theta^{vT} + \frac{(1-\nu)}{2} \mathbf{\Phi}_x^v \mathbf{\Phi}_x^{vT} \right] d\theta dx + a\tilde{D} \int_0^l \int_0^{2\pi} \frac{3(1-\nu)}{2a^2} \mathbf{\Phi}_x^v \mathbf{\Phi}_x^{vT} d\theta dx, \\
\tilde{\mathbf{K}}_{44} &= a_L \tilde{K}_L \int_0^l \int_0^{2\pi} \left[\mathbf{\Phi}_x^u \mathbf{\Phi}_x^{uT} + \frac{(1-\nu_L)}{2a_L^2} \mathbf{\Phi}_\theta^u \mathbf{\Phi}_\theta^{uT} \right] d\theta dx + a_L \tilde{D}_L \int_0^l \int_0^{2\pi} \frac{(1-\nu)}{2a_L^4} \mathbf{\Phi}_\theta^u \mathbf{\Phi}_\theta^{uT} d\theta dx, \\
\tilde{\mathbf{K}}_{55} &= a_L \tilde{K}_L \int_0^l \int_0^{2\pi} \left[\frac{1}{a_L^2} \mathbf{\Phi}_\theta^v \mathbf{\Phi}_\theta^{vT} + \frac{(1-\nu_L)}{2} \mathbf{\Phi}_x^v \mathbf{\Phi}_x^{vT} \right] d\theta dx + a_L \tilde{D}_L \int_0^l \int_0^{2\pi} \frac{3(1-\nu_L)}{2a_L^2} \mathbf{\Phi}_x^v \mathbf{\Phi}_x^{vT} d\theta dx, \\
\tilde{\mathbf{K}}_{12} &= a\tilde{K} \int_0^l \int_0^{2\pi} \frac{\nu}{a} \mathbf{\Phi}_x^u \mathbf{\Phi}^{wT} d\theta dx + a\tilde{D} \int_0^l \int_0^{2\pi} -\frac{1}{a} \mathbf{\Phi}_x^u \mathbf{\Phi}_{xx}^{wT} + \frac{(1-\nu)}{2a^2} \mathbf{\Phi}_\theta^u \mathbf{\Phi}_{x\theta}^{wT} d\theta dx, \\
\tilde{\mathbf{K}}_{13} &= a\tilde{K} \int_0^l \int_0^{2\pi} \frac{1}{a^2} \mathbf{\Phi}_\theta^v \mathbf{\Phi}^{wT} d\theta dx + a\tilde{D} \int_0^l \int_0^{2\pi} -\frac{\nu}{a} \mathbf{\Phi}_\theta^v \mathbf{\Phi}_{xx}^{wT} - \frac{3(1-\nu)}{2a} \mathbf{\Phi}_x^v \mathbf{\Phi}_{x\theta}^{wT} d\theta dx, \\
\tilde{\mathbf{K}}_{14} &= a_L \tilde{K}_L \int_0^l \int_0^{2\pi} \frac{\nu_L}{a_L} \mathbf{\Phi}_x^{u_L} \mathbf{\Phi}^{wT} d\theta dx + a_L \tilde{D}_L \int_0^l \int_0^{2\pi} -\frac{1}{a_L} \mathbf{\Phi}_x^{u_L} \mathbf{\Phi}_{xx}^{wT} + \frac{(1-\nu_L)}{2a_L^2} \mathbf{\Phi}_\theta^{u_L} \mathbf{\Phi}_{x\theta}^{wT} d\theta dx, \\
\tilde{\mathbf{K}}_{15} &= a_L \tilde{K}_L \int_0^l \int_0^{2\pi} \frac{1}{a_L^2} \mathbf{\Phi}_\theta^{v_L} \mathbf{\Phi}^{wT} d\theta dx + a_L \tilde{D}_L \int_0^l \int_0^{2\pi} -\frac{\nu_L}{a_L} \mathbf{\Phi}_\theta^{v_L} \mathbf{\Phi}_{xx}^{wT} - \frac{3(1-\nu_L)}{2a_L} \mathbf{\Phi}_x^{v_L} \mathbf{\Phi}_{x\theta}^{wT} d\theta dx, \\
\tilde{\mathbf{K}}_{23} &= a\tilde{K} \int_0^l \int_0^{2\pi} \frac{\nu}{a} \mathbf{\Phi}_x^u \mathbf{\Phi}_\theta^{vT} + \frac{(1-\nu)}{2a} \mathbf{\Phi}_\theta^u \mathbf{\Phi}_x^{vT} d\theta dx, \\
\tilde{\mathbf{K}}_{45} &= a_L \tilde{K}_L \int_0^l \int_0^{2\pi} \frac{\nu_L}{a_L} \mathbf{\Phi}_x^{u_L} \mathbf{\Phi}_\theta^{v_L T} + \frac{(1-\nu_L)}{2a_L} \mathbf{\Phi}_\theta^{u_L} \mathbf{\Phi}_x^{v_L T} d\theta dx.
\end{aligned} \tag{B7a-k}$$

D and K are bending and membrane stiffness [22], respectively, and subscripts x and θ on Φ refer to partial spatial derivatives. Finally, the damping matrix $\mathbf{C}$ is a combination of a viscous and Coulomb damping matrix, $\mathbf{C} = \mathbf{C}_v + \mathbf{C}_\mu(\omega)$, where:

$$\mathbf{C}_v = \begin{bmatrix} \mathbf{C}_{v,11} & \mathbf{C}_{v,12} & \mathbf{C}_{v,13} & \mathbf{C}_{v,14} & \mathbf{C}_{v,15} \\ \mathbf{C}_{v,12}^T & \mathbf{C}_{v,22} & \mathbf{0} & \mathbf{C}_{v,24} & \mathbf{0} \\ \mathbf{C}_{v,13}^T & \mathbf{0} & \mathbf{C}_{v,33} & \mathbf{0} & \mathbf{C}_{v,35} \\ \mathbf{C}_{v,14}^T & \mathbf{C}_{v,24}^T & \mathbf{0} & \mathbf{C}_{v,44} & \mathbf{0} \\ \mathbf{C}_{v,15}^T & \mathbf{0} & \mathbf{C}_{v,35}^T & \mathbf{0} & \mathbf{C}_{v,55} \end{bmatrix}, \quad \mathbf{C}_\mu = \begin{bmatrix} \mathbf{C}_{\mu,11} & \mathbf{C}_{\mu,12} & \mathbf{C}_{\mu,13} & \mathbf{C}_{\mu,14} & \mathbf{C}_{\mu,15} \\ \mathbf{C}_{\mu,12}^T & \mathbf{C}_{\mu,22} & \mathbf{0} & \mathbf{C}_{\mu,24} & \mathbf{0} \\ \mathbf{C}_{\mu,13}^T & \mathbf{0} & \mathbf{C}_{\mu,33} & \mathbf{0} & \mathbf{C}_{\mu,35} \\ \mathbf{C}_{\mu,14}^T & \mathbf{C}_{\mu,24}^T & \mathbf{0} & \mathbf{C}_{\mu,44} & \mathbf{0} \\ \mathbf{C}_{\mu,15}^T & \mathbf{0} & \mathbf{C}_{\mu,35}^T & \mathbf{0} & \mathbf{C}_{\mu,55} \end{bmatrix} \quad (\text{B8a-b})$$

Where submatrices are defined as follows:

$$\begin{aligned} \mathbf{C}_{v,11} &= a_m c_v \int_0^l \int_0^{2\pi} h_m^2 \Phi_x^w \Phi_x^{wT} + \left(\frac{h_m}{a} \right)^2 \Phi_\theta^w \Phi_\theta^{wT} d\theta dx, \\ \mathbf{C}_{v,22} &= a_m c_v \int_0^l \int_0^{2\pi} \Phi^u \Phi^{uT} d\theta dx, \quad \mathbf{C}_{v,33} = a_m c_v \int_0^l \int_0^{2\pi} \Phi^v \Phi^{vT} d\theta dx, \\ \mathbf{C}_{v,44} &= a_m c_v \int_0^l \int_0^{2\pi} \Phi^{u_L} \Phi^{u_L T} d\theta dx, \quad \mathbf{C}_{v,55} = a_m c_v \int_0^l \int_0^{2\pi} \Phi^{v_L} \Phi^{v_L T} d\theta dx, \\ \mathbf{C}_{v,12} &= a_m c_v h_m \int_0^l \int_0^{2\pi} \Phi^u \Phi_x^{wT} d\theta dx, \quad \mathbf{C}_{v,13} = a_m c_v \frac{h_m}{a} \int_0^l \int_0^{2\pi} \Phi^v \Phi_\theta^{wT} d\theta dx, \\ \mathbf{C}_{v,14} &= -a_m c_v h_m \int_0^l \int_0^{2\pi} \Phi^{u_L} \Phi_x^{wT} d\theta dx, \quad \mathbf{C}_{v,15} = -a_m c_v \frac{h_m}{a} \int_0^l \int_0^{2\pi} \Phi^{v_L} \Phi_\theta^{wT} d\theta dx, \\ \mathbf{C}_{v,24} &= -a_m c_v \int_0^l \int_0^{2\pi} \Phi^u \Phi^{u_L T} d\theta dx, \quad \mathbf{C}_{v,35} = -a_m c_v \int_0^l \int_0^{2\pi} \Phi^v \Phi^{v_L T} d\theta dx, \end{aligned} \quad (\text{B9a-k})$$

and:

$$\begin{aligned} \mathbf{C}_{\mu,11} &= a_m \int_0^l \int_0^{2\pi} c_{\mu,x} h_m^2 \Phi_x^w \Phi_x^{wT} + c_{\mu,\theta} \left(\frac{h_m}{a} \right)^2 \Phi_\theta^w \Phi_\theta^{wT} d\theta dx, \\ \mathbf{C}_{\mu,22} &= a_m c_{\mu,x} \int_0^l \int_0^{2\pi} \Phi^u \Phi^{uT} d\theta dx, \quad \mathbf{C}_{\mu,33} = a_m c_{\mu,\theta} \int_0^l \int_0^{2\pi} \Phi^v \Phi^{vT} d\theta dx, \\ \mathbf{C}_{\mu,44} &= a_m c_{\mu,x} \int_0^l \int_0^{2\pi} \Phi^{u_L} \Phi^{u_L T} d\theta dx, \quad \mathbf{C}_{\mu,55} = a_m c_{\mu,\theta} \int_0^l \int_0^{2\pi} \Phi^{v_L} \Phi^{v_L T} d\theta dx, \\ \mathbf{C}_{\mu,12} &= a_m c_{\mu,x} h_m \int_0^l \int_0^{2\pi} \Phi^u \Phi_x^{wT} d\theta dx, \quad \mathbf{C}_{\mu,13} = a_m c_{\mu,\theta} \frac{h_m}{a} \int_0^l \int_0^{2\pi} \Phi^v \Phi_\theta^{wT} d\theta dx, \\ \mathbf{C}_{\mu,14} &= -a_m c_{\mu,x} h_m \int_0^l \int_0^{2\pi} \Phi^{u_L} \Phi_x^{wT} d\theta dx, \quad \mathbf{C}_{\mu,15} = -a_m c_{\mu,\theta} \frac{h_m}{a} \int_0^l \int_0^{2\pi} \Phi^{v_L} \Phi_\theta^{wT} d\theta dx, \\ \mathbf{C}_{\mu,24} &= -a_m c_{\mu,x} \int_0^l \int_0^{2\pi} \Phi^u \Phi^{u_L T} d\theta dx, \quad \mathbf{C}_{\mu,35} = -a_m c_{\mu,\theta} \int_0^l \int_0^{2\pi} \Phi^v \Phi^{v_L T} d\theta dx. \end{aligned} \quad (\text{B10a-k})$$

Appendix C: Finite Element Study

A finite element model with a cardboard liner and interfacial friction is developed in an attempt to simulate the engineering modeling practice. As such, two finite element models are developed in Abaqus [28]: the first using shell elements for both the liner and shell, and the second using solid elements with quadratic interpolation through the element thickness for both geometries. An interfacial Coulomb interaction is defined between the shell and the liner with $\mu = 0.5$. The results from the unlined shell case (shell elements) are also given in Table 2. For the first model, it is found that in the eigensolution and frequency response, there is no change in the natural frequencies or the acceleration of the structure relative to the unlined case. This suggests that first, the liner mass is not being coupled to the shell via the interaction; and that second, the frequency response solution is linearizing (i.e. omitting) the damping effects of the friction interaction to achieve a linear frequency response solution. In this case, the shell elements do not support the interaction behavior.

For the second model, the eigensolutions (in terms of natural frequencies, listed in Table C1) show a shift in natural frequency as well as “splitting” of repeated modes with the presence of the liner, with most of the natural frequencies shifting to a lower value due to mass loading. It should also be noted that many extraneous modes are obtained with $w \neq w_L$ (i.e. violating the assumptions of this paper). A second case is simulated with a “tied” interaction rather than Coulomb friction (i.e. inner shell and outer liner surfaces are explicitly constrained to remain “tied” together, with $w = w_L$, $u = u_L$, and $v = v_L$ at radial location $r = a_m$, the shell/liner interface) and in this case many of the selected modes are shifted to a higher natural frequency. This would suggest an artificial “stiffening” of the model, perhaps due to improper element choice (clearly shell elements are a more appropriate element for this application) or mismatching of nodes at the interface. Furthermore, due to the curvature of the shell and its thickness relative to other dimensions (i.e. $h \ll l, a$), a finer mesh is required. Thus, the computational expense of this model is much greater than for the case of shell elements and the required computational time for frequency response is larger by at least an order of magnitude.

From this brief study, it is seen that while commercial codes are often valuable tools for structural dynamics analysis, unexpected issues may arise with regard to the solution accuracy and computational time when complicating effects such as thin layers and friction interactions are included. In particular, “default” settings, elements, etc. (i.e. those often used in typical engineering practices) may not be appropriate for these analyses, and nonlinear simulations may be required [32]. Thus the proposed analytical model is beneficial for the sake of providing a reasonable, computationally efficient, and physics-based solution.

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(a)

(b)

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Table 1: Material properties and geometry of shell and liner, along with force and motion locations

		Shell	Liner
Material Properties	Material	Aluminum	Cardboard
	Young's Modulus, E [GPa]	68	4.9
	Density, ρ [kg-m ⁻³]	2606	649
	Poisson's Ratio, ν [-]	0.33	0.3
Material Loss Factor, η [%]		0.1	3.4
Geometry	Length, l [mm]	458	458
	Thickness, h [mm]	2.1	0.31–0.98
	Radius, a [mm]	75.4	73.6–73.9
Locations	Force Location, $(\bar{x}_d, \bar{\theta}_d)$	(0.44, 0.0)	NA
	Motion Location, $(\bar{x}_0, \bar{\theta}_0)$	(0.022, 0.0)	NA

Table 2: Validation of new analytical model in terms of natural frequencies of an unlined cylindrical shell

Mode #	Modal Index*	Natural Frequency [Hz]		
		Analytical	FEM	Experiment
1	(0, 2)	245	245	247
2	(1, 2)	256	255	259
3	(0, 3)	694	701	698
4	(1, 3)	709	714	712
5	(2, 3)	1006	1009	1013
6	(0, 4)	1330	1363	1319
7	(2, 2)	1334	1330	1350
8	(1, 4)	1350	1376	1367
9	(2, 4)	1460	1487	1464
10	(3, 4)	1822	1841	1810
11	(3, 3)	1865	1852	1864

*Rigid body and breathing modes not included

Table C1: Finite element study of shell-liner interaction, using solid elements in a commercial code [28]

Mode #	Modal Index	Natural Frequency [Hz]		
		Unlined	Lined with friction interaction	Lined with “tied” interaction
1*	(0, 2)	248	245, 249	252
2*	(1, 2)	258	256, 260	262
3	(0, 3)	700	697	711
4	(1, 3)	713	709	725
5	(2, 3)	1011	996	1010
6	(2, 2)	1340	1306	1310
7	(0, 4)	1341	1331	1362

*Denotes repeated shell modes (due to symmetry) that are split by presence of a liner and friction interaction