Full Stress and Displacement Fields for Steel-Lined Deep Pressure Tunnels in Transversely Anisotropic Rock

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ABSTRACT

An analytical solution is derived that provides closed-form formulations for stresses and displacements for a deep pressure tunnel in a transversely anisotropic rock, with a steel liner, and subjected to a uniform internal pressure. For the derivation, it is assumed that the tunnel support includes a thin steel liner, concrete backfill and that there is an annulus of damaged rock around the concrete. It is also assumed that all materials remain elastic and that the concrete and the damaged rock cannot transmit shear or tangential stresses. The solution is verified by providing comparisons between its results and those from the Finite Element program ABAQUS. For thin steel liners, it can be assumed that the contact pressure between the different materials is uniform, and thus the bending moments in the liner are negligible. This is due to the low bending stiffness of the steel liner. The paper is inspired by and expands the work by Pachoud and Schleiss (2015) who conducted an extensive numerical parametric analysis to obtain correction factors that, when used with an analytical solution for isotropic materials, approximate the maximum principal stress in the liner and intact rock.

Key words: pressure tunnel, circular tunnel, steel liner, anisotropy, transversely anisotropic rock, water pressure.

1. Introduction

The purpose of pressure tunnels is to convey water under pressure with acceptable losses and without causing instabilities. Pressure tunnels are mostly built in hydroelectric power plant projects, but they are also used, albeit with smaller pressures, for water supply and wastewater conveyance.

Pressure tunnels usually have a circular cross section because of its structural advantages due to the large internal pressures, and also for hydraulic reasons. The cross section area

and the surface roughness (i.e. unlined tunnel or concrete or steel lining) depend on the head losses accepted over the length of the tunnel, provided that the opening is stable. The location of the ground water table, together with the hydraulic head in the tunnel, determines whether water will flow from the tunnel to the rock or vice versa (Merritt, 1999). If the hydraulic head inside the tunnel is larger than that from the water table, water will flow out from the tunnel, which requires that rock stresses around the opening are larger than the water pressures to prevent hydrofracturing or hydrojacking (Benson, 1989; Seeber, 1985a, b). If the tunnel is supported, the stiffness of the rock determines the contribution of the rock mass to support the internal pressure, as the load transfer from the liner to the rock depends largely on the stiffness of the liner relative to the surrounding rock (e.g. Einstein and Schwartz, 1979; Bobet, 2001, 2003; Bobet and Nam, 2007). Finally, the permeability of the rock mass is required to estimate the leakage flow and thus the pore pressures behind the support.

The layout of the pressure tunnel can be decided based on empirical methods and the information obtained during field exploration, if available. The empirical methods can provide initial estimates as to where a watertight liner is needed (Dann et al., 1964; Benson, 1989; Hartmaier et al., 1998; Bergh-Christensen, 1982; Broch, 1984a, b; Selmer-Olsen, 1985; Brekke and Ripley, 1989 and 1993; Alvarez et al. 1999). Following this, calculations regarding flow rate, stresses in the liner and in the ground should be made to assess whether the leakage is acceptable and that the stress state in the liner and ground is adequate, and to provide an estimate of the factor of safety (see e.g. Schleiss, 1986, 1997; Hendron et al., 1989; Fernandez, 1994; Fernandez and Alvarez, 1994 and Bobet and Nam, 2007).

A steel liner is the industry standard for the sections of the tunnel where the minimum principal stress of the rock around the excavation is smaller than the internal tunnel pressure with a suitable factor of safety (Fernandez, 1994). Steel liners are also needed when the internal pressure is large and the surrounding rock has low modulus, so leakage control with reinforced concrete is not possible. A concrete-encased steel cylinder is the most common type of liner (Eskilsson, 1999). There are two criteria that need to be satisfied for the design of the steel liner (Schleiss, 1988): (1) working stress and

deformation of the steel liner; and (2) load-bearing capacity of the rock mass. The first criterion requires a design of the steel liner to adequately support the internal pressures during operation, the external hydraulic pressures during grouting and dewatering, the handling loads during transportation and erection, and that limits local deformations, e.g. crack bridging in the backfill concrete (Schleiss, 1988; Brekke and Ripley, 1993). The second criterion is needed to guarantee sufficient safety against rock failure, and thus ensure the load sharing between liner, backfill concrete and rock assumed for the first criterion.

The design of a steel-lined pressure tunnel due to the internal pressure is generally based on an elastic analysis assuming that the support and rock mass are isotropic and elastic and that the concrete and an annulus of rock surrounding the concrete are damaged such that only radial pressures can be transmitted. Beyond the damaged rock, intact or undamaged rock is assumed (e.g. Schleiss, 1988; Moore, 1989; USACE, 1997; Hachem and Schleiss, 2009; ASCE 2012). Analytical solutions are available that account for these assumptions and are included in the next section for completeness. Pachoud and Schleiss (2015) provided a number of correction factors to the analytical solution to expand it to cases where the intact rock is transversely anisotropic. The factors were obtained after an extensive numerical analysis using a Finite Element Method and a genetic algorithm to minimize errors between the corrected analytical solution and the numerical results. The corrected factors were obtained to estimate the major principal stress in the liner and in the intact rock. This paper builds on the work from Pachoud and Schleiss (2015) and provides a full analytical solution for stresses and displacements for the steel liner, concrete, damaged rock and intact rock. It also expands the range of cases, as the solution accounts for the anisotropy of the damaged rock and does not have the limitation that the existing solution has of assuming that the shear modulus of the rock is equivalent to the empirical relation of Saint-Venant.

Clearly, analytical formulations are limited due to the assumptions that need to be made to reach the solution. In most cases, the design will require a numerical method that does not have the shortcomings of the analytical solutions, as it can consider the construction process, non-linear behavior, etc. Closed-form solutions however are invaluable to obtain

a better understanding of the interplay that exists between loads, rock and support, to identify what are the most critical parameters for the problem, and to provide first estimates or even a preliminary design. An added advantage is that they can be used with very little cost to conduct sensitivity analysis and, most importantly, to provide benchmark values to check the results of the more complex numerical models.

2. Steel-Lined Pressure Tunnel in Isotropic Rock

Figure 1 shows how the internal pressure is distributed between the steel, concrete and rock mass. As the internal pressure increases, the steel liner may initially take all the stress (this may occur if there is a gap between the steel and the concrete, e.g. due to differential thermal contraction between the steel and the concrete, creep of the rock, cycles of loading and unloading as the tunnel is pressurized and emptied, or the skin grouting of the steel-concrete interface is not done or is not effective). With further increase of internal pressure, as the steel liner expands, part of the loading is transferred to the concrete, to the rock damaged during excavation and to the undamaged/intact rock. It is generally assumed that the concrete and the damaged rock can only transmit radial deformations as the damage prevents transfer of tangential and shear stresses. This results in radial stresses that decrease with $1/r$ within the concrete and damaged rock zones, and $1/r^2$ in the undamaged rock zone. The magnitude of the radial stresses and corresponding radial displacements can be obtained by imposing equilibrium along the radial direction and compatibility of radial displacements at the liner-concrete, concrete-damaged rock, and damaged-undamaged rock boundaries (Moore, 1989; USACE, 1997; Hachem and Schleiss, 2009). Thus:

For the liner:

$$\begin{aligned} T_s &= (p_i + p_c) r_i \\ U_r^s &= (1 - n_s^2) \frac{r_i^2}{E_s t_s} (p_i + p_c) \end{aligned} \quad (1)$$

For the cracked backfill concrete:

$$S_r^c = \frac{r_i}{r} p_c$$

$$U_r^c = -w + (1 - \eta_s^2) \frac{r_i^2}{E_s t_s} (p_i + p_c) + (1 - \eta_c^2) \frac{r_i}{E_c} p_c \ln \frac{r}{r_i} \quad (2)$$

For the damaged rock:

$$S_r^{crm} = \frac{r_i}{r} p_c$$

$$U_r^{crm} = -w + (1 - \eta_s^2) \frac{r_i^2}{E_s t_s} (p_i + p_c) + (1 - \eta_c^2) \frac{r_i}{E_c} p_c \ln \frac{r_{crm}}{r_i} + (1 - \eta_{crm}^2) \frac{r_i}{E_{crm}} p_c \ln \frac{r}{r_{crm}} \quad (3)$$

For the intact rock:

$$S_r^{rm} = \frac{r_i r_{rm}}{r^2} p_c$$

$$S_q^{rm} = -\frac{r_i r_{rm}}{r^2} p_c$$

$$U_r^{rm} = -\frac{(1 + \eta_{rm})}{E_{rm}} \frac{r_i r_{rm}}{r} p_c \quad (4)$$

with

$$p_c = - \frac{\frac{1 - \eta_s^2}{E_s t_s} p_i r_i - \frac{w}{r_i}}{\frac{1 - \eta_s^2}{E_s t_s} r_i + \frac{1 - \eta_c^2}{E_c} \ln \frac{r_{crm}}{r_i} + \frac{1 - \eta_{crm}^2}{E_{crm}} \ln \frac{r_{rm}}{r_{crm}} + \frac{1 + \eta_{rm}}{E_{rm}}} \quad (5)$$

where U_r are the radial displacements (subscripts s, c, crm and rm denote steel, backfill concrete, damaged rock and intact rock, respectively); σ_r and σ_θ are the radial and tangential stresses (subscripts s, c, crm and rm denote steel, backfill concrete, damaged rock and intact rock, respectively); w is the gap between the steel and the concrete; r is the radial distance from the center of the tunnel; r_i , r_{crm} and r_m are the internal radius of the tunnel, the radial extent of the concrete and of the damaged rock, respectively; E_s , E_c , E_{crm} and E_{rm} are the Young's modulus of the steel, concrete, damaged rock and intact rock; ν_s , ν_c , ν_{crm} , ν_{rm} are the Poisson's ratios of steel, concrete, damaged rock and intact

rock; t_s is the thickness of the steel liner; p_i is the water pressure and p_c the radial stress at the liner-concrete interface. Note that because the problem has axial symmetry, shear stresses and tangential displacements are zero and, because of the assumption that a cracked material can only transmit radial stresses, the tangential stresses in the concrete and damaged rock are zero. Note also that the liner has been considered as a shell, i.e. the thickness is much smaller than the radius; in other words, $r_i \approx r_i + t_s$ or $t_s/r_i \ll 1$, which is typically the case for steel-lined pressure tunnels. For the above equations, it is assumed that all materials are elastic, and that tension is positive and compression is negative (the internal pressure p_i is taken as positive); thus, displacements are positive in the positive direction of the axes of the coordinate system.

Inspection of equations (1) to (4) indicates that stresses and displacements strongly depend on the radial stress at the liner-concrete contact, p_c (because of elasticity, all results are proportional, as they should, to the applied pressure p_i). Equation (5) provides the magnitude of p_c , which, neglecting the gap parameter w , can be written as:

$$p_c = - \frac{1-\nu_s^2}{E_{eq}} p_i \quad (6)$$

$$E_{eq} = \left(1-\nu_s^2\right) + \left(1-\nu_c^2\right) \frac{E_s}{E_c} \frac{t_s}{r_i} \ln \frac{r_{crm}}{r_i} + \left(1-\nu_{crm}^2\right) \frac{E_s}{E_{crm}} \frac{t_s}{r_i} \ln \frac{r_{rm}}{r_{crm}} + \left(1+\nu_{rm}\right) \frac{E_s}{E_{rm}} \frac{t_s}{r_i}$$

The equation shows that the result is inversely proportional to a weighted average of the relative stiffness between the steel liner and the different materials. The solution strongly depends on the ratio of the thickness of the steel liner to the internal tunnel radius, t_s/r_i .

The theoretical framework has been incorporated into recommendations for design of steel-lined pressure tunnels (e.g. USACE, 1997; ASCE 2012) and has been successfully used to estimate the load transfer between the steel liner and the backfill concrete and rock on a large number of pressure tunnels (Brekke and Ripley, 1989) and to provide estimates of the stress on the liner (e.g. Marulanda and Gutierrez, 1999).

3. Steel-Lined Pressure Tunnel in Transversely Anisotropic Rock

The problem solved is defined in Figure 2 that, as mentioned earlier, is identical to that considered by Pachoud and Schleiss (2015), with few exceptions that expand the scope of the investigation. It consists of a deep tunnel with circular cross section with internal radius r_i , subjected to an internal water pressure p_i , excavated in a transversely anisotropic rock. It is assumed that the steel-lined portion of the tunnel is very long, so plane-strain conditions apply.

The tunnel support includes a circular steel liner with thickness t_s and annular concrete backfill with thickness $r_{crm} - (r_i + t_s)$; with the assumption of $t_s/r_i \ll 1$, the concrete thickness is $\approx r_{crm} - r_i$. It is assumed that there is an annular volume of rock that is damaged during excavation, with constant thickness $r_{rm} - r_{crm}$. As explained by Pachoud and Schleiss (2015), a circular shape of the damaged rock is a reasonable approximation of actual, more complex, conditions and may be sufficiently accurate in the case of grouted near-rock zone. All materials are assumed to remain elastic, with the following properties: Steel liner: Isotropic material with Young's modulus E_s and Poisson's ratio ν_s ; concrete backfill: Isotropic material with E_c and ν_c ; damaged rock: Transversely anisotropic with properties E_x^{crm} , E_y^{crm} , G_{xy}^{crm} , ν_{xz}^{crm} , ν_{yx}^{crm} (note that this is different from Pachoud and Schleiss, 2015 who considered the damaged rock as isotropic); and intact rock: Transversely anisotropic with properties E_x^{rm} , E_y^{rm} , G_{xy}^{rm} , ν_{xz}^{rm} , ν_{yx}^{rm} . The x- and y-axes in Figure 2 denote the axes of elastic symmetry; the x axis, for convenience, is taken as horizontal. The concrete backfill and the damaged rock are assumed to be sufficiently cracked or damaged such that no tangential or shear stresses can be transmitted through them, which is a common assumption (e.g. Schleiss, 1988; Pachoud and Schleiss, 2015).

The problem is solved by imposing the condition of equilibrium on each of the four different materials and by satisfying the boundary conditions. The latter is accomplished by just ensuring compatibility of radial stresses and radial displacements at the contact between materials. This is the only condition needed since the shear stress and the

tangential stress in the concrete and damaged rock are zero, and thus at the interfaces the shear stress is also zero.

For the liner, given that it is treated as a shell, equilibrium is given by the following equations (e.g. Flügge, 1966):

$$\begin{aligned} r_i \frac{d T_s}{d q} - \frac{d M_s}{d q} &= -r_i^2 \tau^s \\ r_i T_s + \frac{d^2 M_s}{d q^2} &= r_i^2 S_r^s \end{aligned} \quad (7)$$

where r and θ are the polar coordinates, with origin at the center of the tunnel and θ measured counterclockwise from the x -axis; T_s and M_s are the axial force and moment acting on the liner; and σ_r^s and τ^s are the radial and shear stress at the contact with the concrete backfill. Deformations are expressed as (e.g. Flügge, 1966):

$$\begin{aligned} \frac{d^2 U_q^s}{d q^2} + \frac{d U_r^s}{d q} &= -\frac{(1-\eta_s^2)}{E_s A_s} r_i^2 \tau^s \\ \frac{d U_q^s}{d q} + U_r^s + \frac{I_s}{r_i^2 A_s} \frac{d^4 U_r^s}{d q^4} + 2 \frac{d^2 U_r^s}{d q^2} + U_r^s \frac{\ddot{\theta}}{\theta} &= \frac{(1-\eta_s^2)}{E_s A_s} r_i^2 S_r^s \end{aligned} \quad (8)$$

where U_r^s and U_θ^s are the radial and tangential displacements, in polar coordinates, and A_s and I_s are the area and moment of inertia, respectively, of the liner cross section ($A_s = t_s$ and $I_s = 1/12 t_s^3$ since the liner is assumed to have a constant thickness).

As the pressure inside the tunnel p_i increases, the liner deforms outwards. If there is a gap, w , between the liner and the concrete backfill, a contact pressure p_c between the liner and the concrete will not develop until the gap is closed. Thus, a distinction needs to be made between the total liner deformations and deformations at the liner-concrete interface, which are those needed to impose the boundary conditions. The following applies when the internal pressure is large enough to close the gap. In other words, when

$$p_i > \frac{E_s t_s}{1 - n_s^2} \frac{w}{r_i^2} \quad (9)$$

which is the pressure required to close the gap w .

Equations (7) and (8) can be solved when the stresses at the steel-concrete contact are known. As discussed, the shear stress at the contact is zero because the concrete cannot transmit shear or tangential stresses. For the radial stresses, the following expression is attempted, given the symmetry of the problem (about the x and y axes):

$$p_c = p_o + \sum_{n=2,4,6}^{\infty} p_n \cos nq \quad (10)$$

p_c is the radial stress at the liner-concrete contact, and p_o and p_n are constants that are obtained by imposing compatibility of radial displacements at each interface.

Thus,

$$\begin{aligned} T_s &= (p_o + p_i) r_i - \sum_{n=2,4,6}^{\infty} \frac{p_n r_i}{n^2 - 1} \cos nq \\ M_s &= - \sum_{n=2,4,6}^{\infty} \frac{p_n r_i^2}{n^2 - 1} \cos nq \end{aligned} \quad (11)$$

$$\begin{aligned} U_r^s &= -w + \frac{1 - n_s^2}{E_s t_s} (p_i + p_o) r_i^2 + \frac{1 - n_s^2}{E_s I_s} r_i^4 \sum_{n=2,4,6}^{\infty} \frac{p_n}{n^2 - 1} \cos nq \\ U_q^s &= - \frac{1 - n_s^2}{E_s I_s} r_i^4 \sum_{n=2,4,6}^{\infty} \frac{p_n}{n^2 - 1} \sin nq \end{aligned} \quad (12)$$

Note that the radial displacements in (12) are produced after the gap is closed. Thus, the total radial displacements of the liner are those given in (12) plus the gap w .

For the concrete backfill, with the assumption that only radial stresses can occur, the equation of equilibrium in polar coordinates takes the form:

$$\frac{dS_r}{dr} + \frac{S_r}{r} = 0 \quad (13)$$

which, after integration and compatibility of radial stresses at the contact with the liner, results in the following expression for the radial stresses in the concrete, σ_r^c :

$$S_r^c = \frac{r_i}{r} \left(\frac{\sigma}{\epsilon} p_o + \sum_{n=2,4,6}^{\infty} \frac{\sigma}{\epsilon} p_n \cos n\theta \right) \quad (14)$$

Integration of the radial strains associated with (14) gives the radial displacements U_r^c :

$$U_r^c = -w + \frac{1-\nu_s^2}{E_s t_s} (p_i + p_o) r_i^2 + \frac{1-\nu_s^2}{E_s I_s} r_i^4 \sum_{n=2,4,6}^{\infty} \frac{p_n}{n^2-1} \cos n\theta + \frac{1-\nu_c^2}{E_c} r_i \left(\frac{\sigma}{\epsilon} p_o + \sum_{n=2,4,6}^{\infty} \frac{\sigma}{\epsilon} p_n \cos n\theta \right) \ln \frac{r}{r_i} \quad (15)$$

For the damaged rock, equilibrium is given by (13). The radial stresses σ_r^{crm} , after imposing compatibility of radial stresses at the contact with the concrete are:

$$S_r^{crm} = \frac{r_i}{r} \left(\frac{\sigma}{\epsilon} p_o + \sum_{n=2,4,6}^{\infty} \frac{\sigma}{\epsilon} p_n \cos n\theta \right) \quad (16)$$

In plane strain, for a transversely anisotropic material, strains are expressed as (e.g. Detournay and Cheng, 1993; Cheng, 1998; Wang 2000):

$$\begin{aligned}
e_x &= a_1 \sigma_x - a_2 \sigma_y \\
e_y &= -a_2 \sigma_x + a_3 \sigma_y \\
g &= \frac{\tau_{xy}}{G_{xy}} \\
\sigma_z &= \nu_{xz} \sigma_x + \frac{E_x}{E_y} \nu_{yx} \sigma_y \\
a_1 &= \frac{1-\nu_{xz}^2}{E_x} \\
a_2 &= \frac{(1+\nu_{xz}) \nu_{yx}}{E_y} \\
a_3 &= \frac{1-\nu_{yx}^2}{E_y}
\end{aligned} \tag{17}$$

where σ_x , σ_y are the stresses in the x and y directions, E_x and E_y are the Young's modulus in the x and y directions, ν_{xz} and ν_{yx} are the Poisson's ratios in the xz and yx directions, respectively, and G_{xy} is the shear modulus. Note that because of the symmetry of the strain tensor, $\nu_{xy} = \nu_{yx} E_x/E_y$. Note also that the properties in the z and x directions are the same.

It is convenient to express the strains in polar coordinates and as a function of radial, tangential and shear stresses. After manipulation of equations (17), one gets:

$$\begin{aligned}
e_r &= a_{11} S_r + a_{12} S_q + a_{13} t_{rq} \\
e_q &= a_{21} S_r + a_{22} S_q + a_{23} S_{rq} \\
g_{rq} &= a_{31} S_r + a_{32} S_q + a_{33} S_{rq} \\
a_{11} &= \frac{1}{8} \frac{\dot{e}}{\dot{e}} 3a_1 - 2a_2 + 3a_3 + \frac{1}{G_{xy}} + 4(a_1 - a_3) \cos 2q + \frac{\ddot{x}}{\dot{e}} a_1 + 2a_2 + a_3 - \frac{1}{G_{xy}} \frac{\ddot{\theta}}{\dot{\theta}} \cos 4q \frac{\dot{u}}{\dot{u}} \\
a_{12} &= a_{21} = \frac{1}{8} \frac{\dot{e}}{\dot{e}} a_1 - 6a_2 + a_3 - \frac{1}{G_{xy}} - \frac{\ddot{x}}{\dot{e}} a_1 + 2a_2 + a_3 - \frac{1}{G_{xy}} \frac{\ddot{\theta}}{\dot{\theta}} \cos 4q \frac{\dot{u}}{\dot{u}} \\
a_{13} &= a_{31} = -\frac{1}{4} \frac{\dot{e}}{\dot{e}} 2(a_1 - a_3) \sin 2q + \frac{\ddot{x}}{\dot{e}} a_1 + 2a_2 + a_3 - \frac{1}{G_{xy}} \frac{\ddot{\theta}}{\dot{\theta}} \sin 4q \frac{\dot{u}}{\dot{u}} \\
a_{22} &= \frac{1}{8} \frac{\dot{e}}{\dot{e}} 3a_1 - 2a_2 + 3a_3 + \frac{1}{G_{xy}} - 4(a_1 - a_3) \cos 2q + \frac{\ddot{x}}{\dot{e}} a_1 + 2a_2 + a_3 - \frac{1}{G_{xy}} \frac{\ddot{\theta}}{\dot{\theta}} \cos 4q \frac{\dot{u}}{\dot{u}} \\
a_{23} &= a_{32} = -\frac{1}{4} \frac{\dot{e}}{\dot{e}} 2(a_1 - a_3) \sin 2q - \frac{\ddot{x}}{\dot{e}} a_1 + 2a_2 + a_3 - \frac{1}{G_{xy}} \frac{\ddot{\theta}}{\dot{\theta}} \sin 4q \frac{\dot{u}}{\dot{u}} \\
a_{33} &= \frac{1}{2} \frac{\dot{e}}{\dot{e}} a_1 + 2a_2 + a_3 + \frac{1}{G_{xy}} - \frac{\ddot{x}}{\dot{e}} a_1 + 2a_2 + a_3 - \frac{1}{G_{xy}} \frac{\ddot{\theta}}{\dot{\theta}} \cos 4q \frac{\dot{u}}{\dot{u}}
\end{aligned} \tag{18}$$

Given the assumption of no shear and tangential stresses and after integration of the radial strains, one gets the radial displacements U_{r}^{crm} as:

$$\begin{aligned}
U_r^{crm} &= -w + \frac{1-n_s^2}{E_s t_s} (p_i + p_o) r_i^2 + \frac{1-n_s^2}{E_s I_s} r_i^4 \sum_{n=2,4,6} \frac{\ddot{a}}{\dot{e}} \frac{p_n}{n^2-1} \cos nq + \frac{1-n_c^2}{E_c} r_i \frac{\ddot{x}}{\dot{e}} p_o + \sum_{n=2,4,6} \frac{\ddot{a}}{\dot{e}} p_n \cos nq \frac{\ddot{\theta}}{\dot{\theta}} \ln \frac{r_{crm}}{r_i} + \\
&\quad (b_1 + b_2 \cos 2q + b_3 \cos 4q) \frac{\ddot{x}}{\dot{e}} p_o + \sum_{n=2,4,6} \frac{\ddot{a}}{\dot{e}} p_n \cos nq \frac{\ddot{\theta}}{\dot{\theta}} r_i \ln \frac{r}{r_{crm}} \\
b_1 &= \frac{1}{8} \frac{\ddot{x}}{\dot{e}} 3a_1^{crm} - 2a_2^{crm} + 3a_3^{crm} + \frac{1}{G_{xy}^{crm}} \frac{\ddot{\theta}}{\dot{\theta}} \\
b_2 &= \frac{1}{2} (a_1^{crm} - a_3^{crm}) \\
b_3 &= \frac{1}{8} \frac{\ddot{x}}{\dot{e}} a_1^{crm} + 2a_2^{crm} + a_3^{crm} - \frac{1}{G_{xy}^{crm}} \frac{\ddot{\theta}}{\dot{\theta}}
\end{aligned} \tag{19}$$

Note that equation (19) satisfies compatibility of radial displacements at the concrete-damaged rock interface. Note also that equation (19) can be used when the damaged rock

is isotropic (which was the assumption by Pachoud and Schleiss, 2015) by making $E_x = E_y = E_{\text{crm}}$, $\nu_{xz} = \nu_{yx} = \nu_{\text{crm}}$ and $G_{xy} = E_{\text{crm}}/2(1+\nu_{\text{crm}})$, which results in $\beta_1 = (1-\nu_{\text{crm}}^2)/E_{\text{crm}}$ and $\beta_2 = \beta_3 = 0$.

The final problem is that of a circular opening of radius r_{rm} in a transversely anisotropic rock, with principal axes of elastic symmetry x and y (Figure 2). The solution can be obtained following Lekhnitskii (1963). Equilibrium is satisfied by finding a stress function $F(x,y)$ such that stresses are:

$$\begin{aligned} S_x &= \frac{\partial^2 F}{\partial y^2} \\ S_y &= \frac{\partial^2 F}{\partial x^2} \\ t_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} \end{aligned} \quad (20)$$

The compatibility equation is, in terms of the function $F(x,y)$:

$$\alpha_1 \frac{\partial^4 F}{\partial y^4} + \alpha_3 \frac{\partial^4 F}{\partial x^4} + \frac{1}{G_{xy}} \left(-2\alpha_2 \frac{\partial^4 F}{\partial x^2 \partial y^2} \right) = 0 \quad (21)$$

where $\alpha_1, \alpha_2, \alpha_3$ are obtained from equation (17) using the properties of the intact rock.

The solution of equation (21) gives $F(x,y)$. Lekhnitskii (1963) found a solution by introducing the complex variable $z_k = x + \mu_k y$, where μ_k is a complex number. Expressing (21) as a function of the complex variable z , one gets:

$$\alpha_1 m_k^4 + \frac{1}{G_{xy}} \left(-2\alpha_2 m_k^2 \right) + \alpha_3 = 0 \quad (22)$$

Defining $\phi(z_k) = F'(z_k) = \partial F / \partial z_k$, stresses can be obtained from:

$$\begin{aligned}
S_x &= 2 \operatorname{Re} \left[m_1^2 f_1'(z_1) + m_2^2 f_2'(z_2) \right] \\
S_y &= 2 \operatorname{Re} \left[f_1'(z_1) + f_2'(z_2) \right] \\
t &= -2 \operatorname{Re} \left[m_1 f_1'(z_1) + m_2 f_2'(z_2) \right]
\end{aligned} \tag{23}$$

where μ_1 and μ_2 are the roots of the equation:

$$a_1 m_k^4 + \frac{1}{G_{xy}} - 2a_2 m_k^2 + a_3 = 0 \tag{24}$$

The stress functions are, after Lekhnitskii (1963):

$$\begin{aligned}
f_1(z_1) &= \frac{1}{m_1 - m_2} \sum_{n=1}^{\infty} \frac{1}{2} \left(\bar{b}_n - m_2 \bar{a}_n \right) V_1^{-n} \\
f_2(z_2) &= -\frac{1}{m_1 - m_2} \sum_{n=1}^{\infty} \frac{1}{2} \left(\bar{b}_n - m_1 \bar{a}_n \right) V_2^{-n}
\end{aligned} \tag{25}$$

where $\bar{a}_n$ and $\bar{b}_n$ are constants and ζ_1 and ζ_2 are complex variables, related to the variables $z_k = x + \mu_k y$ using the conformal mapping relation:

$$z_k = \frac{1}{2} (1 - m_k i) r_{rm} V_k + \frac{1}{2} (1 + m_k i) r_{rm} V_k^{-1} \tag{26}$$

where i is the imaginary number $i = \sqrt{-1}$. The constants are obtained from:

$$\begin{aligned}
\sum_{n=1}^{\infty} \bar{a}_n e^{-nqi} &= -\frac{1}{2} S_r \sin q r_{rm} \, dq \\
\sum_{n=1}^{\infty} \bar{b}_n e^{-nqi} &= \frac{1}{2} S_r \cos q r_{rm} \, dq
\end{aligned} \tag{27}$$

where σ_r is the radial stress at $r = r_{rm}$; that is, from (16):

$$S_r = \frac{r_i}{r_{rm}} \frac{1}{G} p_o + \sum_{n=2,4,6}^{\infty} \frac{1}{2} p_n \cos nq \frac{1}{r} \tag{28}$$

The solution of (27) is:

$$\begin{aligned}
\bar{a}_1 &= \frac{\mathfrak{A}}{\mathfrak{C}} p_o - \frac{1}{2} p_2 \ddot{\theta} r_i \\
\bar{a}_n &= \frac{1}{2n} (p_{n-1} - p_{n+1}) r_i \quad n = 3, 5, 7 \dots \\
\bar{a}_n &= 0 \quad n = 2, 4, 6 \dots \\
\bar{b}_1 &= i \frac{\mathfrak{A}}{\mathfrak{C}} p_o + \frac{1}{2} p_2 \ddot{\theta} r_i \\
\bar{b}_n &= \frac{i}{2n} (p_{n-1} + p_{n+1}) r_i \quad n = 3, 5, 7 \dots \\
\bar{b}_n &= 0 \quad n = 2, 4, 6 \dots
\end{aligned} \tag{29}$$

Stresses are obtained from the derivatives of (25), which are:

$$\begin{aligned}
f'_1(z_1) &= \frac{1}{(m_1 - m_2) r_{rm}} \frac{1}{(1 + i m_1) - (1 - i m_1) V_1^2} \sum_{n=1}^{\infty} n (\bar{b}_n - m_2 \bar{a}_n) V_1^{-n+1} \\
f'_2(z_2) &= - \frac{1}{(m_1 - m_2) r_{rm}} \frac{1}{(1 + i m_2) - (1 - i m_2) V_2^2} \sum_{n=1}^{\infty} n (\bar{b}_n - m_1 \bar{a}_n) V_2^{-n+1}
\end{aligned} \tag{30}$$

Displacements are the result of the integration of the strains in (17) given the definition of stresses in (23). They are:

$$\begin{aligned}
U_x &= 2 \operatorname{Re} \left(\frac{\mathfrak{A}}{\mathfrak{C}} (a_1 m_1^2 - a_2) f_1(z_1) + (a_1 m_2^2 - a_2) f_2(z_2) \right) \dot{\theta} \\
U_y &= 2 \operatorname{Re} \left(\frac{\mathfrak{A}}{\mathfrak{C}} a_2 m_1 + \frac{a_3}{m_1} \ddot{\theta} f_1(z_1) + \frac{\mathfrak{A}}{\mathfrak{C}} a_2 m_2 + \frac{a_3}{m_2} \ddot{\theta} f_2(z_2) \right) \dot{\theta}
\end{aligned} \tag{31}$$

It is informative to expand the equations for displacements using the stress functions given in (25) with (29). The result is:

$$\begin{aligned}
U_x &= \operatorname{Re} \left(\sum_{n=1,3,5}^{\infty} \frac{\mathfrak{A}}{\mathfrak{C}} \dot{\theta} (m_1 + m_2) a_1^m \bar{b}_n - (a_2^m + m_1 m_2 a_1^m) \bar{a}_n \dot{\theta} [\cos nq - i \sin nq] \right) \dot{\theta} \\
U_y &= \operatorname{Re} \left(\sum_{n=1,3,5}^{\infty} \frac{1}{m_1 m_2} \frac{\mathfrak{A}}{\mathfrak{C}} \dot{\theta} (m_1 + m_2) a_3^m \bar{a}_n - (a_3^m + m_1 m_2 a_2^m) \bar{b}_n \dot{\theta} [\cos nq - i \sin nq] \right) \dot{\theta}
\end{aligned} \tag{32}$$

Given the form of the characteristic equation (24), the roots must satisfy the following:

$$\begin{aligned} m_1^2 + m_2^2 &= -\frac{1}{a_1} \frac{\partial}{\partial G_{xy}} \frac{1}{\partial} - 2 a_2 \frac{\partial}{\partial} \\ m_1^2 m_2^2 &= \frac{a_3}{a_1} \end{aligned} \quad (33)$$

which results in real numbers. The roots of the characteristic equation (24) are either complex numbers or pure imaginary numbers, but cannot be real numbers (Lekhnitskii, 1963). The roots $\mu_1 = a+ib$ and $\mu_2 = c+id$, given (33), can only have one of the two forms: $\mu_1 = ib$ and $\mu_2 = id$, or $\mu_1 = a+ib$ and $\mu_2 = -a+ib$. This is interesting because $\mu_1 + \mu_2 = i(b+d)$ or $\mu_1 + \mu_2 = 2ib$, and $\mu_1 \mu_2 = -bd$ or $\mu_1 \mu_2 = -(a^2 + b^2)$. In other words, the sum of the roots is a pure imaginary number and the product a negative real number. Using this property, the radial displacements, at the interface with the damage rock, can be expressed as:

$$\begin{aligned} U_r^{rm} &= \frac{1}{2} \frac{\partial}{\partial} \frac{\partial}{\partial} (m_1 + m_2) a_1^{rm} \bar{b}_n - (a_2^{rm} + m_1 m_2 a_1^{rm}) \bar{a}_n \frac{\partial}{\partial} [\cos (n+1)q + \cos (n-1)q] \\ &+ \frac{1}{2} \frac{i}{m_1 m_2} \frac{\partial}{\partial} \frac{\partial}{\partial} (m_1 + m_2) a_3^{rm} \bar{a}_n - (a_3^{rm} + m_1 m_2 a_2^{rm}) \bar{b}_n \frac{\partial}{\partial} [\cos (n+1)q - \cos (n-1)q] \end{aligned} \quad (34)$$

Finally, coefficients p_0, p_n are obtained by making the radial displacements in (19) equal to the radial displacements in (34), at $r = r_{rm}$. Doing this term by term, i.e. for the constant term, $\cos 2\theta$, $\cos 4\theta$, etc., results in the following system of linear equations:

$$\begin{aligned}
& p_0 \frac{\mathbb{A}}{\mathbb{C}} \frac{1-\eta_s^2}{E_s t_s} r_i + \frac{1-\eta_c^2}{E_c} \ln \frac{r_{cm}}{r_i} + b_1 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{2} A_3 \frac{\ddot{\theta}}{\theta} + \\
& p_2 \frac{1}{2} \frac{\mathbb{A}}{\mathbb{C}} b_2 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{2} A_4 \frac{\ddot{\theta}}{\theta} + p_4 \frac{1}{2} b_3 \ln \frac{r_{rm}}{r_{cm}} = \frac{w}{r_i} - \frac{1-\eta_s^2}{E_s t_s} p_i r_i \\
& p_0 \frac{\mathbb{A}}{\mathbb{C}} b_2 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{2} A_1 \frac{\ddot{\theta}}{\theta} + p_2 \frac{\mathbb{A}}{\mathbb{C}} \frac{1-\eta_s^2}{3 E_s I_s} r_i^3 + \frac{1-\eta_c^2}{E_c} \ln \frac{r_{cm}}{r_i} + b_1 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{4} A_2 + \frac{1}{12} A_3 \frac{\ddot{\theta}}{\theta} + \\
& p_4 \frac{1}{2} \frac{\mathbb{A}}{\mathbb{C}} b_2 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{6} A_4 \frac{\ddot{\theta}}{\theta} + p_6 \frac{1}{2} b_3 \ln \frac{r_{rm}}{r_{cm}} = 0 \\
& p_0 b_3 \ln \frac{r_{rm}}{r_{cm}} + p_2 \frac{1}{2} \frac{\mathbb{A}}{\mathbb{C}} b_2 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{6} A_1 \frac{\ddot{\theta}}{\theta} + p_4 \frac{\mathbb{A}}{\mathbb{C}} \frac{1-\eta_s^2}{15 E_s I_s} r_i^3 + \frac{1-\eta_c^2}{E_c} \ln \frac{r_{cm}}{r_i} + b_1 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{12} A_2 + \frac{1}{20} A_3 \frac{\ddot{\theta}}{\theta} + \\
& p_6 \frac{1}{2} \frac{\mathbb{A}}{\mathbb{C}} b_2 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{10} A_4 \frac{\ddot{\theta}}{\theta} + p_8 \frac{1}{2} b_3 \ln \frac{r_{rm}}{r_{cm}} = 0 \\
& p_{n-4} \frac{1}{2} b_3 \ln \frac{r_{rm}}{r_{cm}} + p_{n-2} \frac{1}{2} \frac{\mathbb{A}}{\mathbb{C}} b_2 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{2(n-1)} A_1 \frac{\ddot{\theta}}{\theta} + p_n \frac{\mathbb{A}}{\mathbb{C}} \frac{1-\eta_s^2}{n^2-1 E_s I_s} r_i^3 + \frac{1-\eta_c^2}{E_c} \ln \frac{r_{cm}}{r_i} + b_1 \ln \frac{r_{rm}}{r_{cm}} + \frac{A_2}{4(n-1)} + \frac{A_3}{4(n+1)} \frac{\ddot{\theta}}{\theta} + \\
& p_{n+2} \frac{1}{2} \frac{\mathbb{A}}{\mathbb{C}} b_2 \ln \frac{r_{rm}}{r_{cm}} + \frac{1}{2(n+1)} A_4 \frac{\ddot{\theta}}{\theta} + p_{n+4} \frac{1}{2} b_3 \ln \frac{r_{rm}}{r_{cm}} = 0 \quad \text{for } n > 4
\end{aligned} \tag{35}$$

with

$$\begin{aligned}
A_1 &= (m_1+m_2) a_1 + (a_2 - m_1 m_2 a_1) - \frac{m_1+m_2}{m_1 m_2} a_3 + \frac{a_3 - m_1 m_2 a_2}{m_1 m_2} \\
A_2 &= (m_1+m_2) a_1 - (a_2 - m_1 m_2 a_1) + \frac{m_1+m_2}{m_1 m_2} a_3 + \frac{a_3 - m_1 m_2 a_2}{m_1 m_2} \\
A_3 &= (m_1+m_2) a_1 + (a_2 - m_1 m_2 a_1) + \frac{m_1+m_2}{m_1 m_2} a_3 - \frac{a_3 - m_1 m_2 a_2}{m_1 m_2} \\
A_4 &= (m_1+m_2) a_1 - (a_2 - m_1 m_2 a_1) - \frac{m_1+m_2}{m_1 m_2} a_3 - \frac{a_3 - m_1 m_2 a_2}{m_1 m_2}
\end{aligned} \tag{36}$$

Note that in (36), given the properties of the roots of the characteristic equation previously discussed, i.e. $\mu_1+\mu_2 = b+d$ or $2b$, and $\mu_1\mu_2 = bd$ or (a^2+b^2) , the expressions $\mu_1+\mu_2$ and $\mu_1\mu_2$ are real numbers.

From a practical point of view, only a finite number of equations is needed, given that their contribution to the solution diminishes as n increases. It is important however to truncate the system of equations at a number that minimizes errors. Further comments are provided later in the next section.

The solution obtained is verified by comparing its results from those obtained using the numerical program ABAQUS. ABAQUS is a Finite Element code that has been extensively used and validated in geotechnical engineering. All the different cases are discretized with eight-node isoparametric elements, with only one quarter of the tunnel modeled due to the symmetry along the vertical and horizontal axes. A tied contact between the different materials is used, which effectively acts as frictionless since the shear stresses in the concrete and damaged rock are zero. No horizontal displacements are allowed along the vertical axis through the center of the tunnel and no vertical movements along the horizontal axis. A constant pressure equal to p_i is imposed to the internal perimeter of the tunnel. The external boundary is placed far from the center of the tunnel such that it does not have an effect on the solution. The two same cases used by Pachoud and Schleiss (2015) are chosen for the comparisons, which is done to facilitate a discussion on the advantages and limitations of the two approaches. The first case corresponds to a circular tunnel with internal radius $r_i = 1.5$ m, internal pressure $p_i = 10$ MPa, steel liner with a thickness of 0.03 m, concrete backfill 0.5 m thick, and damaged rock with a thickness of 0.5 m. The second case is analogous, with $r_i = 2.5$ m, $p_i = 15$ MPa, and thickness of steel, concrete and damage rock of 0.02 m, 0.5 m, and 0.7 m, respectively. Three scenarios are considered in each case: one, where all materials are isotropic, and the other two with transversely anisotropic intact rock. All scenarios assume that the damaged rock is isotropic and that there is no gap between the liner and the backfill concrete. The first scenario in each case considers, as mentioned, isotropic intact rock; case one, scenario two, includes properties of a transversely anisotropic rock taken from Tonon and Amadei (2003), and case two, scenario two, from Amadei (1996). Scenarios three from cases one and two are analogous to those with the anisotropic rock, except the shear modulus is estimated based on the following Saint-Venant empirical relation:

$$G'_{xy}{}^{rm} = \frac{E_y^{rm}}{1 + \frac{E_y^{rm}}{E_x^{rm}} + 2\nu_{yx}} \quad (37)$$

Table 1 lists the input properties used for both the analytical solution and for ABAQUS. In the numerical method, the condition of no shear and tangential stresses in the concrete and damaged rock is approximated by assuming that these materials have cylindrical anisotropy, with the radial Young's modulus equal to the value shown in Table 1, tangential Young's modulus equal to 10^{-12} MPa, the smallest number that the code accepts, and the Poisson's ratio in the radial and tangential directions equal to zero (this is necessary to have in the numerical model a positive definite stiffness matrix, given the small tangential Young's modulus used as input).

Figures 3 and 4 provide a comparison between the results obtained from ABAQUS and from the analytical solution for cases 1 and 2, respectively. More specifically, Figures 3(a) and 4(a) are plots of the normalized tangential stress at the interior fiber of the liner; Figures 3(b) and 4(b), of the normalized tangential stresses in the intact rock at its contact with the damaged rock; Figures 3(c) and 4(c), of the normalized radial displacements of the liner and Figures 3(d) and 4(d) of the normalized radial displacements of the intact rock at the contact with the damaged rock. In the figures, white symbols represent results from ABAQUS and black symbols from the analytical solutions. The comparison is excellent, with differences smaller than 1%-2% for all the values plotted. Pachoud and Schleiss (2015) only reported results for the maximum principal stress for the liner and the intact rock. For case 1, for the liner, they reported the values 24.87, 24.27, 22.27 for scenarios 1, 2, and 3, respectively; for the intact rock, 0.1, 0.156, and 0.133. For case 2, 5.44, 6.19, 4.99 for the liner and 0.237, 0.265 and 0.239 for the intact rock. The values given are normalized as in Figures 3 and 4. The differences between their results and the results obtained with the new simulations are all below 1%-4%, except for the maximum principal stress for the intact rock for the scenarios with anisotropic rock and G' from equation (37), where the errors are of the order of 20%-25%. These larger errors could be due to small differences in the models, and/or how the condition of no-shear and no-tangential stresses in the concrete and damaged rock are imposed.

There is no limitation in using equations (35) for isotropic problems. When both the damaged rock and the intact rock are isotropic, $\beta_1 = (1-\nu_{\text{crm}}^2)/E_{\text{crm}}$ and $\beta_2 = \beta_3 = 0$ for the damaged rock; and $\mu_1=\mu_2=1$, $\alpha_3^{\text{rm}}=\alpha_1^{\text{rm}}=(1-\nu_{\text{rm}}^2)/E_{\text{rm}}$, $\alpha_2^{\text{rm}}=(1+\nu_{\text{rm}})/E_{\text{rm}}$, $A_1=A_4=0$, $A_2=$

$2(1+\nu_{rm})(3-5\nu_{rm})/E_{rm}$ and $A_3= 2(1+\nu_{rm})/E_{rm}$ for the intact rock. With these values, the system of equations (35) has a solution $p_n= 0$ for $n> 0$ and $p_c= p_o$, which has the same result as that given in (6).

4. Discussion

Figures 3 and 4 show that, when the rock is isotropic, stresses and displacements do not depend on the angular coordinate θ . This is expected because the problem is axisymmetric. What is interesting is the observation that the liner tangential stresses are insensitive to the tangential coordinate, even when the rock is anisotropic. The reason for this is the low stiffness of the steel liner, which then can deform with small or negligible moments. This notion is supported by the normalized plots of the radial stress between the liner and the backfill concrete in Figure 5. The figure shows a fairly constant pressure distribution on the liner. As a result, the liner experiences a uniform tangential stress (i.e. small bending moments). The observation is in agreement with the concept of flexible liner by Peck (1969), who suggested that a flexible liner would have a uniform pressure distribution and a deflected shape such that the bending moments at all points in the liner would be negligible; which is the case at hand due to the small thickness of the steel liner. Indeed, in the calculations performed, the values of the constants p_n in equation (10) are several orders of magnitude smaller than the value of p_o , which supports the notion that the radial stresses are dominated by the constant term p_o .

Given that the concrete and damaged rock, as hypothesized, cannot transmit shear or tangential stresses, the radial stresses at the contact between the damaged and the intact rock are also uniform. Because of the anisotropy of the intact rock, the radial deformations are not uniform even when the radial stress applied is uniform. Thus, larger radial displacements occur at the crown than at the springline given that in all cases considered the Young's modulus of the intact rock is larger in the horizontal direction than in the vertical direction.

The observation that the radial stress at the liner-concrete contact can be approximated as a uniform pressure can be used to simplify greatly the equations derived in the previous section, using for the radial stress at the liner-concrete interface:

$$p_c = p_o \quad (38)$$

Following the same procedure as that used before, the following results are obtained:

For the steel liner:

$$\begin{aligned} T_s &= (p_o + p_i) r_i \\ M_s &= 0 \\ U_r^s &= U_r^{crm} \Big|_{r=r_m} - \frac{1-\eta_c^2}{E_c} r_i p_o \ln \frac{r_{crm}}{r_i} - \frac{1-\eta_{crm}^2}{E_{crm}} r_i p_o \ln \frac{r_{rm}}{r_{crm}} + w \end{aligned} \quad (39)$$

where $U_r^{crm} \Big|_{r=r_m}$ is given by (42).

For the backfill concrete:

$$\begin{aligned} S_r^c &= \frac{r_i}{r} p_o \\ U_r^c &= U_r^s + \frac{1-\eta_c^2}{E_c} r_i p_o \ln \frac{r}{r_i} \end{aligned} \quad (40)$$

For the damaged rock, assuming isotropic properties:

$$\begin{aligned} S_r^{crm} &= \frac{r_i}{r} p_o \\ U_r^{crm} &= U_r^s + \frac{1-\eta_c^2}{E_c} r_i p_o \ln \frac{r_{crm}}{r_i} + \frac{1-\eta_{crm}^2}{E_{crm}} r_i p_o \ln \frac{r}{r_{crm}} \end{aligned} \quad (41)$$

and for the intact rock, at the interface with the damaged rock:

$$\begin{aligned}
S_r^{\text{rm}} \Big|_{r=r_m} &= \frac{r_i}{r_m} p_o \\
U_r^{\text{rm}} \Big|_{r=r_m} &= -\frac{1}{2} \frac{p_o r_i}{m_1 m_2} \left\{ (m_1 + m_2 - m_1 m_2) m_1 m_2 a_1^{\text{rm}} + 2 m_1 m_2 a_2^{\text{rm}} + (m_1 + m_2 - 1) a_3^{\text{rm}} + \right. \\
&\quad \left. \frac{1}{2} (m_1 + m_2 - m_1 m_2) m_1 m_2 a_1^{\text{rm}} - (m_1 + m_2 - 1) a_3^{\text{rm}} \right\} \cos 2q
\end{aligned} \quad (42)$$

And the tangential stresses, after Lekhnitskii (1963):

$$\begin{aligned}
S_q^{\text{rm}} \Big|_{r=r_m} &= - \frac{n - k + n(k-1)\cos^2 q + \frac{1}{2}(k+1)^2 - n^2 \frac{1}{2} \sin^2 q \cos^2 q}{\sin^4 q + \frac{1}{2}n^2 - 2k \frac{1}{2} \sin^2 q \cos^2 q + k^2 \cos^4 q} p_o \\
k &= \sqrt{\frac{E_x^{\text{rm}}}{E_y^{\text{rm}}}} \quad E_x^{\text{rm}} > E_y^{\text{rm}} \\
m &= \frac{E_x^{\text{rm}}}{G_{xy}^{\text{rm}}} - 2 n_{xy} \\
n &= \sqrt{2k + m}
\end{aligned} \quad (43)$$

The value of p_o is given by the following expression:

$$p_o \frac{1}{2} \frac{1-\eta_s^2}{E_s t_s} r_i + \frac{1-\eta_c^2}{E_c} \ln \frac{r_{\text{crm}}}{r_i} + \frac{1-\eta_{\text{crm}}^2}{E_{\text{crm}}} \ln \frac{r_m}{r_{\text{crm}}} + \frac{1}{2} A_3 = \frac{w}{r_i} - \frac{1-\eta_s^2}{E_s t_s} p_i r_i \quad (44)$$

with

$$A_3 = (m_1 + m_2) a_1^{\text{rm}} + (a_2^{\text{rm}} - m_1 m_2 a_1^{\text{rm}}) + \frac{m_1 + m_2}{m_1 m_2} a_3^{\text{rm}} - \frac{a_3^{\text{rm}} - m_1 m_2 a_2^{\text{rm}}}{m_1 m_2} \quad (45)$$

The differences between the results obtained from the simplified solution and the complete solution, for the cases and scenarios investigated, is smaller than 0.3%. Equation (44) provides a good estimate of the pressures behind the liner. In practice, the bending moments, even if they are small, should be evaluated, e.g. using the general formulation or numerical methods, due to the potential risk of failure in the case of high strength steel and flaws in weldings.

One of the advantages of an analytical solution is that it provides insight into the variables that are most relevant. Inspection of equation (44) shows that the contact pressure between the liner and the concrete, neglecting the gap, is proportional to the applied pressure (expected for an elastic analysis), is inversely proportional to the weighed equivalent relative stiffness, E_{eq} , of the different materials, as defined in equation (46), and it strongly depends on the ratio t_s/r_i . Note that these observations are exactly the same as those obtained for the isotropic scenario (see equation (6)), and that the structure of equations (6) and (46) is the same.

$$p_c = - \frac{1-\nu_s^2}{E_{eq}} p_i \quad (46)$$

$$E_{eq} = (1-\nu_s^2) + (1-\nu_c^2) \frac{E_s}{E_c} \frac{t_s}{r_i} \ln \frac{r_{crm}}{r_i} + (1-\nu_{crm}^2) \frac{E_s}{E_{crm}} \frac{t_s}{r_i} \ln \frac{r_{rm}}{r_{crm}} + \frac{1}{2} E_s A_3 \frac{t_s}{r_i}$$

When the intact rock is isotropic, $A_3 = 2(1+\nu_{rm})/E_{rm}$, as already discussed, which makes equation (46) identical to equation (6), as it should.

5. Summary and Conclusions

An analytical solution is presented that provides close-form solutions for stresses and displacements for deep steel-lined pressure tunnels for a uniform internal pressure. The following assumptions are made in deriving the solution: plane strain conditions, circular cross section, tunnel support composed of a thin shell, concrete backfill, and an annulus of damaged rock around the concrete. It is assumed that all the materials remain elastic and that the damaged rock and intact rock are transversely anisotropic, with one of the axes of elastic symmetry parallel to the tunnel axis. It is also assumed that the concrete and damaged rock cannot develop shear or tangential stresses.

The paper is inspired by the work by Pachoud and Schleiss (2015), who conducted an extensive numerical parametric analysis for steel-lined pressure tunnels in transversely anisotropic rock to obtain correction factors that could be used with the analytical solution for isotropic rock. Only correction factors for the maximum stress in the liner and intact rock were provided. They showed that the maximum stresses obtained after the

corrections compared well with those obtained with the numerical method, except for the intact rock and for those cases when the shear modulus of the intact rock G was larger than the empirical relation of Saint-Venant G' . The solutions obtained in the present work do not have these limitations. They provide the complete stress and displacement fields for all the materials and they do not require that the damaged rock is isotropic.

The analytical solution presented has been verified by providing comparisons between its results and the results obtained from the Finite Element program ABAQUS, for a number of scenarios. The differences obtained are smaller than 1%-2%, which provides confidence in the formulation.

The equations provided may be of limited practical use because of the strong assumptions made for their derivation. There are clear advantages however of closed-form solutions over numerical methods. One of the advantages is that they can be used to readily identify the most important factors that determine the results. The paper shows that there are two parameters that strongly influence the results. The first one is the relative stiffness between the liner and the different materials, weighted with factors related to the geometry of the problem, and on the relative thickness of the steel liner with respect to the internal radius of the tunnel. Another important conclusion, given the results obtained from the cases analyzed, is that the tensile stress of flexible steel liners can be approximated assuming a uniform pressure. A simplified formulation is derived and presented that incorporates this observation. Other advantages over numerical methods are that they can be used for preliminary design, for sensitivity analysis, or to evaluate the results of complex numerical computations. This is supported in the technical literature by the widespread use of existing formulations for the analysis of steel-lined pressure tunnels.

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Notation

The following is a list of the variables used in the paper:

A_s, I_s	cross-section and moment of inertia of the liner
E_c, ν_c	Young's modulus and Poisson's ratio of the backfill concrete
E_{crm}, ν_{crm}	Young's modulus and Poisson's ratio of the damaged isotropic rock
E_{rm}, ν_{rm}	Young's modulus and Poisson's ratio of the intact isotropic rock
E_x^{rm}, E_y^{rm}	Young's modulus of intact rock in x- and y-axis ($E_x=E$ and $E_y=E'$ in Pachoud and Schleiss, 2015)
E_x^{crm}, E_y^{crm}	Young's modulus of damaged rock in x- and y-axis
E_s, ν_s	Young's modulus and Poisson's ratio of the liner
G_{xy}^{rm}	shear modulus of the intact rock
G_{xy}^{crm}	shear modulus of the damaged rock
p_c	radial stress at the steel-concrete contact
p_i	internal water pressure
r, θ	polar coordinates
r_i	radius of the tunnel
r_{crm}	limit of cracked concrete liner
r_{rm}	limit of damaged rock
t_s	liner thickness
T_s, M_s	axial force and moment of the liner
U_x, U_y	displacements in Cartesian coordinates; superscripts s, c, crm or rm denote steel, concrete, damaged rock or intact rock
U_r, U_θ	displacements in polar coordinates; superscripts s, c, crm or rm denote steel, concrete, damaged rock or intact rock
w	gap between steel liner and backfill concrete
x, y	Cartesian coordinates of axes of elastic symmetry
z_k	complex number, $z_k = x + \mu_k y$, $k=1, 2$
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	axial and shear strains in x- and y-axis
$\varepsilon_r, \varepsilon_\theta, \gamma_{r\theta}$	radial, tangential and shear stresses in polar coordinates

ζ_k	complex number that depends on z_k through conformal mapping
$\phi(z_k), \phi'(z_k)$	stress function and its derivative
μ_1, μ_2	roots of compatibility equation
$\nu_{xy}, \nu_{xz}, \nu_{yz}$	Poisson's ratios in x-y axes
$\sigma_x, \sigma_y, \tau_{xy}$	normal and shear stresses in Cartesian coordinates; superscripts s, c, crm or rm denote steel, concrete, damaged rock or intact rock
$\sigma_r, \sigma_\theta, \tau_{r\theta}$	radial, tangential, and shear stresses in polar coordinate; superscripts s, c, crm or rm denote steel, concrete, damaged rock or intact rock

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Table 1. Comparison between Analytical Solution and ABAQUS. Input Parameters

Case	P_i (MPa)	r_i (m)	E_s (GPa)	ν_s	t_s (m)	E_c (GPa)	ν_c	r_{crm} (m)	E_{crm} (GPa)	ν_{crm}	r_{rm} (m)	E_x^{rm} (GPa)	E_y^{rm} (GPa)	ν_{xz}	ν_{yx}	G_{xy}^{rm} (GPa)
1 Isotropic Rock	10	1.5	210	0.3	0.03	20	0	2.0	2	0	2.5	2.4	2.4	0.22	0.22	983
1 Anisotropic Rock	10	1.5	210	0.3	0.03	20	0	2.0	2	0	2.5	7.8	2.4	0.22	0.07	830
1 Anisotropic Rock & G'	10	1.5	210	0.3	0.03	20	0	2.0	2	0	2.5	7.8	2.4	0.22	0.07	1658
2 Isotropic Rock	15	2.5	210	0.3	0.02	20	0	3.0	20	0	3.7	23.9	23.9	0.18	0.18	10127
2 Anisotropic Rock	15	2.5	210	0.3	0.02	20	0	3.0	20	0	3.7	29.3	23.9	0.18	0.13	6200
2 Anisotropic Rock & G'	15	2.5	210	0.3	0.02	20	0	3.0	20	0	3.7	29.3	23.9	0.18	0.13	11514

Figure 1. Distribution of Radial Stresses in a Concrete-encased Steel Lining

Figure 2. Steel-Lined Pressure Tunnel in Transversely Anisotropic Rock

(a)

(b)

(c)

(d)

Figure 3. Comparison between ABAQUS and Analytical Solution. Case 1.

(a)

(b)

(c)

(d)

Figure 4. Comparison between ABAQUS and Analytical Solution. Case 2.

(a) Case 1

(b) Case 2

Figure 5. Radial Stress, p_c , at Liner-Concrete Contact