A General Anisotropic Yield Criterion for Pressure-Dependent Materials

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Abstract

Many engineering materials exhibit strong anisotropy along with a pressure dependence on the plastic yield behavior. The ability to predict the effects of pressure dependence and plastic anisotropy of these materials on their respective yield surfaces is important for accurately analyzing their behavior during deformation. The current work develops a general anisotropic yield criterion that includes the capability of modeling pressure-dependent effects either additively or multiplicatively in the yield criterion at the onset and during the evolution of plastic deformation. The developed yield criterion can be used to represent both quadratic and non-quadratic yield surfaces and has no restrictions on the symmetricity of the plastic behavior, allowing the entire spectrum from isotropic to fully anisotropic. The pressure dependence of the yield criterion and the corresponding plastic rate of deformation equation are expressed as general functional forms to allow development and incorporation of new pressure dependence functions or to accept existing pressure dependence functions directly. Therefore, the developed yield criterion can be used as a framework for accurately modeling the plastic behavior of various material systems, largely reducing the complexity in yield surface description for different classes of materials and allowing a systematic approach for implementation in numerical analysis procedures such as the finite element method. Two existing pressure-dependent

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models are reformulated to show the applicability of the general yield criterion framework presented herein to developing new non-quadratic anisotropic models with a dependence on pressure. While the general yield criterion is developed as an extension of a specific non-quadratic anisotropic yield criterion, the same model development methodology can be applied to other novel or existing yield criteria that are defined in the deviatoric stress space.

Keywords: B. Anisotropic material, Pressure-dependent material, A. Yield condition, C. Finite elements

1. Introduction

Development of a yield criterion for plastically deformable materials in finite element analysis is of ample importance for accurately depicting the evolution of plastic state variables and the overall material behavior. The mathematical concept of the yield criterion greatly simplifies the computational aspects of analysis for crystalline materials, which would otherwise likely use the computationally demanding crystal plasticity modeling approach. However, there is certainly no restriction for applying the yield criterion concept only to crystalline materials as it has been used for describing composite (Camanho et al., 2013; Vogler et al., 2013), cementitious (Yu et al., 2010), geomechanical (Chakraborty et al., 2013; Lai et al., 2009; Yang et al., 2010; Yuanming et al., 2010), and polymeric (Caddell et al., 1973; Ghorbel, 2008) material classes as well. The evolution of plastic state variables for a given material class can be quite unique to its class, thus, deciding either which yield criterion to use or how to develop a new yield criterion may not be obvious. The present work is directed at establishing a general yield criterion that can be easily amended to include material sensitivity to multiple states of stress, e.g., deviatoric, hydrostatic, and triaxial, and that can be utilized for arbitrary symmetry of the plastic behavior.

Two of the most widely used yield criteria developed for crystalline materials are those of von Mises (1913) and Tresca (1864). In the von Mises yield criterion, it is assumed that the material will evolve to a new state of plasticity when the

second invariant (J_2) of the Cauchy stress deviator reaches a critical value. The Tresca yield criterion simply compares the maximum state of shear stress in the material with a critical value to determine whether plastic yield will occur. These two models are often taken as the foundation of models extended to include other mechanics such as a pressure or stress triaxiality dependence. For example, the Drucker-Prager yield criterion (Drucker and Prager, 1952) is an extension of the von Mises yield criterion to include pressure dependence via the first invariant of the Cauchy stress (I_1) in an additive fashion. Materials that are inherently pressure-dependent include those previously mentioned and also some that are perhaps less obvious, e.g., pharmaceutical materials (Han et al., 2008), showing the impact of these models on the general science community.

Although the von Mises and Tresca yield criteria can be used effectively for some materials, it has been observed (Stout et al., 1983) that there are limitations to these models because the yield surface for many materials should be represented with a higher degree of nonlinearity than quadratic, as proposed by von Mises, but not so far as to reach that proposed by Tresca, which is nonlinear to the highest degree. An attempt to model materials such as these was given initially by both Hershey (1954) and Hosford (1972). The Hershey/Hosford model is an extension of the von Mises yield criterion for arbitrary nonlinearity. This model added some generality to the yield criterion; however, the resulting yield criterion is pressure-independent and thus is limited to materials that are assumed to be plastically insensitive to this particular stress state.

Recently, rigorous upscaling methodologies have been used by Cazacu et al. (2014) and Revil-Baudard and Cazacu (2014) in order to describe yielding of porous solids where the matrix obeys the Tresca criterion. These works are developed using a strain-rate potential basis for evolution of the state of plasticity in the material. In contrast to existing heuristic yield criteria, the new porous Tresca criterion involves coupling between shear and mean stress effects, the yield locus being centrosymmetric. The 3D extension of the porous Tresca model indicated that the shape of the yield locus in the octahedral plane depends strongly on the mean stress, evolving from a hexagon with rounded corners for

low levels of mean stress to a triangle with rounded corners for higher levels of mean stress. These works also provide a very thorough study of the sensitivity of porous solids to the stress invariants for general multiaxial load states.

It has also been observed that most materials exhibit, and in some cases are designed to exhibit (Blanc et al., 2006), anisotropic plastic behavior. Hill (1948), Hosford (1972), and Barlat et al. (1991) developed some of the earlier models specifically for analyzing this behavior. However, Hill’s 1948 criterion was developed with a limitation in that the yield surface was again quadratic in nature. This lack of generality was circumvented by both Barlat and Hosford, allowing the yield surface to be of arbitrary nonlinearity and anisotropic. These models were again sensitive only to the state of shear in the material. The Barlat 1991 model was an extension of Hosford’s model where the yield function was evaluated using a linear transformation of the Cauchy stress tensor such that a resultant yield surface could be evaluated and the stresses could be updated during plastic deformation using the transformed stress and then simply converted back to the actual Cauchy stress. The linear transformation of the Cauchy stress tensor has since been employed to incorporate anisotropic behavior in many yield criteria (Barlat et al., 1997, 2003, 2005, 2007; Bron and Besson, 2004; Dunand et al., 2012; Yoon et al., 2014; Yoshida et al., 2013).

An elegant approach was presented by Karafillis and Boyce (1993) that utilized the benefits of the aforementioned anisotropic models, e.g., the linear stress transformation and the arbitrary nonlinearity, while attempting to remove the shortcomings, i.e., the lack of both general dimensional stress states and general anisotropic behavior. The Karafillis-Boyce model (KB93) incorporates a linear stress transformation tensor of the Cauchy stress into an Isotropic Plasticity Equivalent (IPE) deviatoric stress, later incorporated in other anisotropic model developments (e.g., Barlat et al. (1997), Maniatty et al. (1999)), that is used to evaluate a yield criterion that is a weighted mixture of two yield surfaces. The two yield surfaces used for the resultant surface are such that the lower bound converges to the Tresca yield criterion and the upper bound converges to that described by Hershey and Hosford. The resultant surface can

also be set to the von Mises yield criterion when the nonlinearity is quadratic in nature. The yield surface mixing combined with the linear stress transformation leads to an extremely general modeling approach that can be tailored for a vast number of materials and material classes so long as they are characterized by shear-dominated plastic behavior. Successful utilization of this model, specifically for metal forming applications, can be found in works by Cao et al. (2000), Yao and Cao (2002), and Korkolis and Kyriakides (2008). However, this model lacks the ability to describe the pressure dependence of materials in that the IPE stress is a deviatoric tensor.

Cazacu and Barlat (2001) developed a generalization of Drucker’s yield criterion (Drucker, 1949) to orthotropy by first creating a method for extension of any isotropic yield function, which is expressed in terms of the second and third invariants of the Cauchy stress deviator (J_2 and J_3 , respectively) for orthotropic plasticity. While in the present work J_3 will not be formally considered, the generalization used in the work by Cazacu and Barlat is an excellent example of an attempt to create a procedure for extension of material models to include orthotropic behavior based on the linear transformation tensor presented by Barlat et al. (1991) and Karafillis and Boyce (1993). Likewise, the present work will utilize the linear transformation tensor in order to create a framework for developing new yield criteria based on the general dimensional KB93 yield criterion in which sensitivity of the plastic behavior can incorporate general stress states.

It should be noted that many models are not necessarily for arbitrary plastic anisotropy and in many cases were designed for orthotropic symmetry, as this is of particular importance for metal forming applications, and were not necessarily for general dimensional stress states. Many anisotropic models have been developed for analysis of sheet metal behavior in manufacturing processes by focusing on a plane-stress formulation, e.g., Barlat et al. (2003) and Vrh et al. (2014). While models developed for plane-stress conditions serve an important purpose in analysis of sheets, it is sometimes necessary to perform the analysis using a 3D stress tensor (Malhotra et al., 2012; Smith et al., 2013). The model

developed herein is targeted at removing this limitation while preserving the capability for plane-stress considerations. Similarly, models that have been developed specifically for orthotropic symmetry are quite common (e.g., Cazacu et al. (2010) and Nixon et al. (2010)). While the presently developed yield criterion will be implemented using orthotropic symmetry, it is still applicable to general anisotropy.

Although the methodology behind the development of the succeeding framework will be general in nature, the primary purpose of this development is to incorporate a pressure dependence of materials in combination with general anisotropic behavior. Many crystalline (Bai and Wierzbicki, 2008; Stoughton and Yoon, 2004), geomechanical (Saksala and Ibrahimbegovic, 2014), cellular (Alkhader and Vural, 2010) and composite (Azizi et al., 2013) materials can be characterized by a pressure dependence and inherent anisotropy in their plastic yield behavior. Another important consideration for discussion is the inherent pressure dependence in material models that incorporate damage (Brunet and Morestin, 2001; Dunand and Mohr, 2011; Gurson, 1977; Khan and Liu, 2012a,b; Liao et al., 1997; Xue, 2007; Yoon et al., 2014). This is one of the primary motivations for the current research as the inclusion of pressure-sensitive damage has proven to be one of the most useful tools in analysis of many material classes for both final product performance (Hao et al., 2000, 2003, 2004; McVeigh et al., 2007; Srivastava et al., 2014; Tang et al., 2013a,b; Vernerey et al., 2006) and in material processing (Liewers et al., 2004; Malhotra et al., 2012). This motivated Benzerga and Besson (2001), further extended by Stewart and Cazacu (2011), to use limit-analysis methods in order to extend a pressure-dependent damage model for plastic anisotropy. The contribution of the developed framework to the aforementioned pressure-dependent damage models is that it is completely general in the non-quadratic nature of the yield surface, the anisotropic behavior, and also in the pressure dependence.

The development of the general pressure-dependent yield criterion is given in Section 2. Two examples, one based on the Drucker-Prager pressure-dependent yield criterion (Section 3.1) and the other based on the modified Cam-Clay

model Roscoe and Burland (1968) (Section 3.2), will be explored in Section 3. These two pressure-dependent models were selected due to vast differences of their yield loci, therefore illustrating the generality of the proposed framework. Moreover, the two example models are applicable to a variety of materials. Geomechanical materials (e.g., rocks and soils) can be modeled by Drucker-Prager-type models due to the fact that the yield limit in tension is orders of magnitude lower than that in compression, making the open yield surface inherent to this model physically sound. This model can also be used for metallic materials loaded in tension. The Cam-Clay-type models are applicable for both geomechanical materials as well as metallic materials without limitation on loading conditions since the yield surface is bounded for both compressive and tensile loading. An extension of a similar variant of the Cam-Clay model for the anisotropic behavior of rock material has been demonstrated by (Nova, 1986). Section 3 includes an investigation of the Drucker-Prager-based anisotropic yield loci in Section 3.3. Applications for the two developed models in finite element analysis are shown in Section 4. Application of the Drucker-Prager-based anisotropic model in a 3D finite element analysis framework using an Abaqus user-defined material subroutine is discussed in Section 4.1. A comparison of the model with experimental results is shown Section 4.2. Section 4.3 includes an application of the anisotropic modified Cam-Clay model to a 3D structure undergoing large deformation. Conclusions and future work will be discussed in Section 5. A brief summary of the KB93 yield criterion can be found in Appendix A. Readers who are less familiar with the KB93 yield criterion are encouraged to view Appendix A prior to reading Section 2 in order to get a better understanding of the developed general anisotropic pressure-dependent yield criterion.

2. Extension of the KB93 Yield Criterion for Pressure-Dependent Materials

In this section the new general pressure-dependent yield criterion will be developed based on the KB93 yield criterion. An overview of the KB93 yield criterion, including all relevant variables and parameters associated with it, is shown in Appendix A. The information presented therein is considered a prerequisite to the development of the new criterion. In order to extend the KB93 yield criterion for pressure-dependent materials, it is suggested to augment the linear transformation tensor to the following form,

$$\mathbf{L}^* = \mathbf{L} + \frac{1}{3}\mathbf{I} \otimes \mathbf{I} \quad (1)$$

where $\mathbf{L}^*$ is the new pressure-augmented linear isotropic equivalent projection tensor, $\mathbf{L}$ is the original KB93 linear isotropic deviatoric projection tensor defined in equation A.4, and $\mathbf{I}$ is the 2nd-order identity tensor. This augmentation ultimately produces a new form of the IPE tensor, which we will call the Pressure-augmented Isotropic Plasticity Equivalent (PIPE) tensor, that is non-deviatoric,

$$\mathbf{S}^* = \mathbf{L}^* : \boldsymbol{\sigma} \quad (2)$$

and

$$I_1^* = \mathbf{S}^* : \mathbf{I} = \boldsymbol{\sigma} : \mathbf{I} \quad (3)$$

where I_1^* is the first invariant of the PIPE stress, $\mathbf{S}^*$, and is equivalent to the first invariant of the actual Cauchy stress. We can now assert that the yield criterion can be a function of the first invariant of the actual Cauchy stress in the material such that the plastic behavior is pressure-dependent,

$$\phi^* = \bar{\phi}^\xi + \eta(I_1^*) - \beta(I_1^*)\bar{\sigma}^\psi = 0 \quad (4)$$

where $\bar{\phi}$ is a function of the original KB93 yield criterion, ϕ defined in equation A.5, and subsequently the IPE tensor, $\tilde{\mathbf{S}}$ defined in equation A.4, only,

$$\bar{\phi} = \left(\frac{1}{2} \phi \right)^{\frac{1}{2k}} \quad (5)$$

The functions $\eta(I_1^*)$ and $\beta(I_1^*)$ in equation 4 are the additive and multiplicative pressure-dependent functions for the new anisotropic yield criterion, respectively. The exponents ξ and ψ allow a direct correspondence to be made with new or existing models that are nonlinear in the effective stress value and/or the matrix flow stress. In order for the units of the yield criterion to be consistent, the exponent ψ differs from ξ only when $\beta(I_1^*)$ is nonzero and has units of stress. The form of the isotropic equivalent projection tensor implies that the PIPE stress can be additively decomposed into the following two tensors,

$$\mathbf{S}^* = \tilde{\mathbf{S}} + \mathbf{P} \quad (6)$$

where $\tilde{\mathbf{S}}$ is the IPE tensor defined in equation A.4 and $\mathbf{P}$ is the pressure tensor defined by,

$$\mathbf{P} = \frac{1}{3} I_1^* \mathbf{I} = -p^* \mathbf{I} \quad (7)$$

and p^* is the pressure associated with the PIPE stress.

The evolution equation for the plastic dissipation is similar to what was found for the standard KB93 model (see equation A.6) in that the evolution of plastic dissipation is assumed to be equal between the general anisotropic material and the equivalent isotropic material,

$$\dot{w}^p = \boldsymbol{\sigma} : \mathbf{D}^p = \mathbf{S}^* : \mathbf{d}^{*p} = \bar{\sigma} \dot{\bar{\epsilon}}^p \quad (8)$$

where $\mathbf{d}^{*p}$ is the isotropic rate of plastic deformation tensor that may now be dependent on the pressure. The transformation from the isotropic rate of plastic

deformation tensor to its anisotropic counterpart follows from equation 8,

$$\mathbf{D}^p = \mathbf{L}^* : \mathbf{d}^{*p} \quad (9)$$

which constitutes the anisotropic flow rule. Assuming that the flow rule for the isotropic equivalent material is associative,

$$\mathbf{d}^{*p} = \dot{\lambda} \frac{\partial \phi^*}{\partial \mathbf{S}^*} \quad (10)$$

Based on the form of the pressure-dependent yield criterion, the flow rule for the isotropic equivalent material can be written as,

$$\mathbf{d}^{*p} = \dot{\lambda} \left(\frac{\partial \phi^*}{\partial \tilde{\mathbf{S}}} : \frac{\partial \tilde{\mathbf{S}}}{\partial \mathbf{S}^*} + \frac{\partial \phi^*}{\partial I_1^*} \frac{\partial I_1^*}{\partial \mathbf{S}^*} \right) = \dot{\lambda} \left(\frac{\partial \phi^*}{\partial \tilde{\mathbf{S}}} + \frac{\partial \phi^*}{\partial I_1^*} \mathbf{I} \right) \quad (11)$$

where the second equality comes from equations 3 and 6. The flow rule can be written in terms of the pressure based on equation 7 as,

$$\mathbf{d}^{*p} = \dot{\lambda} \left(\frac{\partial \phi^*}{\partial \tilde{\mathbf{S}}} - \frac{1}{3} \frac{\partial \phi^*}{\partial p^*} \mathbf{I} \right) \quad (12)$$

It is convenient to rewrite the flow rule of the isotropic equivalent material in the context of the standard KB93 model. This can be done quite easily by first recognizing that, based on equations 4 and 5,

$$\frac{\partial \phi^*}{\partial \tilde{\mathbf{S}}} = (\xi \bar{\phi}^{\xi-1}) \frac{\partial \bar{\phi}}{\partial \tilde{\mathbf{S}}} \quad (13)$$

where, according to equations 4, 5, and A.7,

$$\frac{\partial \bar{\phi}}{\partial \tilde{\mathbf{S}}} = \mathbf{Q}^T \cdot \frac{\partial \bar{\phi}}{\partial \boldsymbol{\Lambda}} \cdot \mathbf{Q} \quad (14)$$

It should be noted that the second comprising yield function, ϕ_2 defined in equation A.2, will cause the return direction in equation 14 to be non-deviatoric when $k \neq 1$. While this will cause the effective plastic strain value to inadvertently be dependent on a pressure-like term in the standard KB93 model during the stress-updating process, the overall effect on the anisotropic rate of plastic deformation tensor and the anisotropic Cauchy stress will be removed via

contraction with the deviatoric linear transformation tensor, $\mathbf{L}$. However, for the present development to be valid for implementation purposes the pressure-like term from this portion of the return direction must be removed so that the desired pressure term is accounted for properly. Therefore, the trace of the KB93 portion of the return direction is removed by incorporating a 4th-order deviatoric projection tensor defined by,

$$\mathbf{P}_4 = \mathbf{I}_4 - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \quad (15)$$

where $\mathbf{I}_4$ is the 4th-order identity tensor. Equation 14 can now be adjusted using the 4th-order deviatoric projection tensor of equation 15,

$$\frac{\partial \bar{\phi}}{\partial \bar{\mathbf{S}}} = \mathbf{Q}^T \cdot \left[\mathbf{P}_4 : \frac{\partial \bar{\phi}}{\partial \bar{\mathbf{\Lambda}}} \right] \cdot \mathbf{Q} \quad (16)$$

This ensures that the IPE stress from the KB93 model, as well as its contribution to the return direction, is always deviatoric during a stress updating procedure and that the only pressure dependence of the solution is introduced by the PIPE stress.

The additional derivative for the return direction is taken as,

$$\frac{\partial \phi^*}{\partial I_1^*} = \frac{\partial \eta}{\partial I_1^*} - \bar{\sigma}^\psi \frac{\partial \beta}{\partial I_1^*} \quad (17)$$

which yields the following final form of the isotropic equivalent flow rule in terms of the first invariant of the PIPE stress,

$$\mathbf{d}^{*p} = \dot{\lambda} \left(\mathbf{Q}^T \cdot \left[(\xi \bar{\phi}^{\xi-1}) \mathbf{P}_4 : \frac{\partial \bar{\phi}}{\partial \bar{\mathbf{\Lambda}}} \right] \cdot \mathbf{Q} + \left[\frac{\partial \eta}{\partial I_1^*} - \bar{\sigma}^\psi \frac{\partial \beta}{\partial I_1^*} \right] \mathbf{I} \right) \quad (18)$$

Again, this can be written in terms of the pressure as,

$$\mathbf{d}^{*p} = \dot{\lambda} \left(\mathbf{Q}^T \cdot \left[(\xi \bar{\phi}^{\xi-1}) \mathbf{P}_4 : \frac{\partial \bar{\phi}}{\partial \bar{\mathbf{\Lambda}}} \right] \cdot \mathbf{Q} - \frac{1}{3} \left[\frac{\partial \eta}{\partial p^*} - \bar{\sigma}^\psi \frac{\partial \beta}{\partial p^*} \right] \mathbf{I} \right) \quad (19)$$

Depending on which is more convenient for a specific application, equation 18 or 19 will constitute the flow rule that would be used to update the PIPE stress within a stress update algorithm used for analysis.

3. Example Applications of the Yield Criterion Development Methodology

In the following subsections two examples of application of the yield criterion development methodology are shown. These examples were selected in order to give readers a more comprehensive understanding of what is required in order to create a new anisotropic pressure-dependent yield criterion using the developed methodology based on existing pressure-dependent yield criteria. While both of these yield criteria are novel developments afforded by the proposed methodology, the primary contribution of this manuscript to the research community is intended to be the yield criterion development framework and not the developed models themselves.

The reader who is interested in implementing the yield criterion development methodology into a numerical analysis code is encouraged to start with these examples as they are relatively simple and can be easily tested for the appropriate behavior of the model in question. From observation of equations 18 and 19 it should be clear that successful implementation of either of these models will result in an implementation template in which only the pressure-dependent functions and their derivatives need to be altered. This is yet another benefit of the form of the developed framework in that existing yield criteria can ideally be incorporated from an implementation point of view by simply leveraging selected portions of previously developed code.

3.1. Example 1: A Modified Drucker-Prager Model

The Drucker-Prager pressure-dependent plasticity yield criterion (Drucker and Prager, 1952) can be written as,

$$f = \sqrt{3J_2} - \alpha I_1 = \bar{\sigma} \quad (20)$$

where J_2 is the second invariant of the Cauchy stress deviator, I_1 is the first invariant of the Cauchy stress, and α is the pressure sensitivity coefficient. Recalling that the KB93 yield criterion converges to J_2 plasticity when $k = 1$, by

making a judicious choice of the functions η and β , along with the exponents ξ and ψ , the current pressure-dependent model will converge to the Drucker-Prager model when $\alpha \neq 0$ and to the J_2 plasticity model when $\alpha = 0$ assuming that $k = 1$. Note that when $k \neq 1$, the general pressure-dependent model will transform into a unique pressure-dependent yield function dictated also by the mixing coefficient c . In order to create a modified Drucker-Prager model, the functions η and β and the exponents ξ and ψ are selected to be,

$$\beta = 1 \quad (21a)$$

$$\eta = -\alpha I_1^* \quad (21b)$$

$$\xi = 1 \quad (21c)$$

$$\psi = 1 \quad (21d)$$

Therefore the pressure augmented yield criterion associated with equation 4 is written as,

$$\phi^* = \bar{\phi} - \alpha I_1^* - \bar{\sigma} = 0 \quad (22)$$

and the direction of plastic flow for an associative flow rule is written in terms of the standard KB93 model (see equation 18),

$$\frac{\partial \phi^*}{\partial \mathbf{S}^*} = \mathbf{Q}^T \cdot \left[\mathbf{P}_4 : \frac{\partial \bar{\phi}}{\partial \mathbf{\Lambda}} \right] \cdot \mathbf{Q} - \alpha \mathbf{I} \quad (23)$$

It should be noted that the Drucker-Prager plasticity model was chosen for this example because it is assumed to be highly recognizable by the research community. Despite the fact that this model has been successfully used for various applications, e.g., Pae (1977) in which the Drucker-Prager model was used to analyze polyoxymethylene and the results agreed well with experiments, it has inherent properties that make it a poor choice for most practical analyses

under general loading conditions. The most notable deterrent from use of this model is that the yield surface forms an infinitely tall cone (the base resides at infinity) in the Haigh-Westergaard stress space resulting in the yield stress under compressive loading to be non-existent. Therefore, it is only advised to use this particular model on a situational basis for specific material classes where this behavior has physical relevance.

3.2. Example 2: An Anisotropic Extension of the Modified Cam-Clay Model

The modified Cam-Clay pressure-dependent plasticity model (Roscoe and Burland, 1968) forms a closed ellipsoid in the Haigh-Westergaard stress space and therefore is not plagued by the same theoretical complications of the Drucker-Prager model. Moreover, this yield surface is continuously differentiable, which makes it convenient for analysis by numerical procedures, e.g., finite element analysis. This model could, for example, be implemented for analysis of soil materials with directional plastic properties. Furthermore, this model has been suggested on numerous occasions, e.g., de Borst and Groen (2000), to be the most widely used elasto-plastic model for the description of the mechanical behavior of clay.

The modified Cam-Clay pressure-dependent plasticity yield criterion can be written as,

$$f = \left(\sqrt{3J_2} \right)^2 + M^2 p(p - p_c) = 0 \quad (24)$$

where J_2 is the second invariant of the Cauchy stress deviator, M is the slope of the critical state line as described in the modified Cam-Clay model, p is the hydrostatic pressure associated with the Cauchy stress, and p_c is the pre-consolidated pressure. The modified Cam-Clay model does not explicitly include a matrix flow stress parameter (i.e., $\bar{\sigma}$ in the Drucker-Prager model) and thus we can choose our pressure dependent functions to coincide directly with the above model quite simply. The functions η and β and the exponents ξ and ψ

are selected to be,

$$\beta = 0 \quad (25a)$$

$$\eta = M^2 p^* (p^* - p_c) \quad (25b)$$

$$\xi = 2 \quad (25c)$$

$$\psi = \text{Not Applicable} \quad (25d)$$

Notice that ψ is not applicable because of the lack of a matrix flow stress in the modified Cam-Clay yield criterion. In practice, p_c can be used to control the hardening behavior (the model development could be constructed in such a way that p_c and $\bar{\sigma}$ are treated similarly) if it is allowed to evolve, but, for simplicity, the present study will not be expanded to that situation. Instead, the assumption that p_c is a constant value will be used in the present and subsequent sections. Therefore, the pressure augmented yield criterion associated with equation 4 is written as,

$$\phi^* = \bar{\phi}^2 + M^2 p^* (p^* - p_c) = 0 \quad (26)$$

and the direction of plastic flow for an associative flow rule is written in terms of the standard KB93 model (see equation 19),

$$\frac{\partial \phi^*}{\partial \mathbf{S}^*} = \mathbf{Q}^T \cdot \left[(2\bar{\phi}) \mathbf{P}_4 : \frac{\partial \bar{\phi}}{\partial \mathbf{\Lambda}} \right] \cdot \mathbf{Q} - \frac{M^2}{3} (2p^* - p_c) \mathbf{I} \quad (27)$$

The effect of the non-quadratic and anisotropic behavior on the modified Cam-Clay yield surface (as it has been defined in equation 26) will be to give a different perceived slope of the critical state line, M , as the loading direction and stress state changes. In this way, M can be thought of as a reference slope as the model approaches J_2 plasticity. As can be seen in equation 26, M is a ratio that is used to adjust the sensitivity of the yield surface to shear loading. This

sort of behavior was also shown in the work by Nova (1986) in the extension of a similar variant of the Cam-Clay model to anisotropy.

3.3. Investigation of the modified Drucker-Prager yield loci

The isotropic plasticity equivalent yield surface defined by equation 22 for the quadratic case ($k = 1$) with pressure dependence ($\alpha = 0.2$) and the mixing coefficient set to null ($c = 0.0$) in the plane stress space is shown in Figure 1. It can be seen that for this case each of the comprising yield functions, ϕ_1 and ϕ_2 , as well as the resulting yield function, ϕ^* , are equivalent and are also identical to the standard Drucker-Prager yield criterion. It should be noted that the actual material might exhibit plastically anisotropic behavior but we have assumed that the necessary transformation has been taken already and thus S_i^* are stress measures defined after transformation. The normalized pressure is varied from $p^*/\bar{\sigma} = -1.0$ to $p^*/\bar{\sigma} = 1.0$ in the figure and the pressure dependence on the yield surface can be easily observed with the surface growing while in a state of negative pressure (tension) and shrinking in a state of positive pressure (compression). The yield surface in the π -plane for the same pressure-dependent model parameters is shown in Figure 2. The radial nature of the yield loci should be clearly recognizable as that of the von Mises yield criterion. Again, the composing and resulting yield functions are equivalent when either $p^*/\bar{\sigma} = 0$ or $\alpha = 0$.

Figure 1: Plane stress yield surfaces for ϕ^* , ϕ_1 , and ϕ_2 with $k = 1$, $c = 0.0$, and $\alpha = 0.2$.

Figure 2: Yield surfaces for ϕ^* , ϕ_1 , and ϕ_2 in the π -plane with $k = 1$, $c = 0.0$, and $\alpha = 0.2$.

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An example using the full potential of the pressure-dependent isotropic plas-

ticity equivalent yield surface can be seen in Figure 3. The model parameters,
 referring again to equation 22, used for this case are such that it is highly
 non-quadratic ($k = 15$) with pressure dependence ($\alpha = 0.2$) and the mixing
 coefficient set to allow both composing yield functions to have an impact on
 the resulting yield surface ($c = 0.85$). For this case, it can be seen that the
 two composing yield functions approach their respective maximum deviations
 from the von Mises yield surface, i.e., the Tresca yield criterion for ϕ_1 and the
 upper bound surface of ϕ_2 . However, for this model the resulting yield surface
 is of course a mixture of the two composing surfaces as dictated by the mixing
 coefficient, c . By simply adjusting c we can now adjust the resulting pressure-
 dependent isotropic plasticity equivalent yield surface as desired, using c as a
 shape parameter and α as the pressure sensitivity parameter. The same yield
 loci for varying pressure in the π -plane can be seen in Figure 4.

Figure 3: Plane stress yield surfaces for ϕ^* , ϕ_1 , and ϕ_2 with $k = 15$, $c = 0.85$, and $\alpha = 0.2$.

Figure 4: Yield surfaces for ϕ^* , ϕ_1 , and ϕ_2 in the π -plane with $k = 15$, $c = 0.85$, and $\alpha = 0.2$.

4. Applications in Finite Element Analysis

Finite element analysis is an excellent tool for analyzing complex nonlinear behavior of structures. Therefore, the ability to implement the developed plasticity laws into a finite element framework is highly desirable. This section will showcase the ability to implement the two developed yield criteria in finite element analysis through a verification study of the modified Drucker-Prager model, **a comparison of the modified Drucker-Prager model with experimental results**, and a large deformation study using the anisotropic modified Cam-Clay model. The details of implementation for the plasticity laws are beyond the scope of the present work, which is directed at providing a framework for development of new pressure-dependent anisotropic plasticity laws, and will not be discussed.

4.1. Application of the Modified Drucker-Prager Model in 3D Finite Element Analysis

A 3D finite element implementation of the modified Drucker-Prager model described in Section 3.1 was developed for use in the finite element analysis software package, Abaqus. A fully implicit stress update algorithm for the model was developed and used to create a VUMAT, i.e., a user-defined material for explicit time integration, within the aforementioned software. Although the algorithmic details of the stress update procedure are beyond the interest of this paper, it is important to mention that the numerical scheme employed for the present work utilizes an orthogonal return to the yield surface in the 3D stress space, i.e., including the contribution of the volumetric component to the return direction, and not a return to the π -plane projection of the yield surface.

In order to utilize the modified Drucker-Prager model for analysis, a selection must be made as to the elastic and plastically anisotropic behavior that will be used. Therefore, the form of the projection tensor for the IPE stress must be determined. If one assumes that the material of interest is characterized by orthotropic symmetry, as seen in thin metallic sheets, the isotropic plasticity

equivalent projection tensor corresponding to the IPE stress can then be written in matrix form as,

$$\mathbf{L} = C \begin{bmatrix} 1 & \beta_1 & \beta_2 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & \beta_3 & 0 & 0 & 0 \\ \beta_2 & \beta_3 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_3 \end{bmatrix} \quad (28)$$

400 For this specific case,

$$\beta_1 = \frac{\alpha_2 - \alpha_1 - 1}{2} \quad (29a)$$

$$\beta_2 = \frac{\alpha_1 - \alpha_2 - 1}{2} \quad (29b)$$

$$\beta_3 = \frac{1 - \alpha_1 - \alpha_2}{2} \quad (29c)$$

The anisotropy parameters contained in $\mathbf{L}$ can be selected such that isotropic behavior is recovered. To obtain isotropic plastic behavior from the actual Cauchy stress,

$$C = \frac{2}{3} \quad (30a)$$

$$\alpha_1 = \alpha_2 = 1 \quad (30b)$$

$$\gamma_1 = \gamma_2 = \gamma_3 = \frac{3}{2} \quad (30c)$$

405 The material properties and the KB93 model parameters related to the deviatoric stress-transformation tensor chosen for this case study are shown in Tables 1 and 2, respectively. Note that the material is assumed to exhibit perfectly plastic behavior and therefore the matrix flow stress is constant.

Table 1: Elastic and plastic material properties used in the modified Drucker-Prager case study.

E (MPa)	ν	$\bar{\sigma}$ (MPa)
73312	0.33	260

Table 2: KB93 model parameters for the deviatoric stress-transformation tensor.

	C	α_1	α_2	β_1	β_2	β_3	γ_1	γ_2	γ_3
isotropic	0.667	1.0	1.0	-0.5	-0.5	-0.5	1.5	1.5	1.5
anisotropic	0.639	1.103	1.120	-0.492	-0.509	-0.612	1.5	1.5	0.73

It should be noted that the elastic and plastic anisotropy parameters for this case study were arbitrarily chosen in order to showcase the potential applications of the present theoretical development. Therefore, these parameters are not calibrated for any specific material (see Section 4.2 for an experimental case study with calibrated parameters). The elastic properties and yield stress shown in Table 1 are in the typical range for 2000 series aluminum alloys while the anisotropic parameters shown in Table 2 are taken from Karafillis and Boyce (1993) for 2008-T4 aluminum with the exception of the parameter C as this must correspond to the selected matrix flow stress indicated in Table 1.

The finite element model used for this study is shown below in Figure 5. The model consists of a single C3D8R linear hexahedral element with reduced integration and hourglass control. It can be seen in the figure that symmetry boundary conditions were applied on the back face, bottom face, and one side face of the single element model while the other side face and the top face were traction free. A displacement boundary condition was applied to the front face of the element such that either tension or compression could be achieved. The angle θ shown in the figure corresponds to the loading direction in comparison to the rolling direction of a metallic sheet or perhaps the longitudinal direction of the fibers in a composite material.

Figure 5: FEA model and boundary conditions for the pressure-dependent case study.

In order to analyze and verify the ability of the model to predict pressure-dependent anisotropy, three case studies were conducted using the finite element model shown in Figure 5. The results from the first of these case studies are shown below in Figure 6. This case study uses a linear transformation tensor that is plastically anisotropic (see Table 2) but with no sensitivity to pressure and thus the standard KB93 yield criterion is recovered, as shown in Figure 6. The purpose of this case study is to verify that the KB93 model implementation is working correctly. It can be seen from the figure that this results in a plastic yield behavior that varies depending on the load direction, the single element model being loaded between 0° and 90° . The figure indicates that the plastic yield behavior reaches a maximum allowable equivalent stress at 0° with the minimum corresponding to 90° , verifying the plastically anisotropic behavior of the material and the finite element implementation. It should be noted that for the present case only one yield stress needs to be defined despite the fact that the material behavior has a varying equivalent stress at yield. This is because there is a correlation between the parameter C in equation 28 and the yield stress at 0° , 45° , and 90° such that it is the ratio of the average yield stress obtained from simulation for $C = 1$ to the average stress obtained for the same cases in experiments, as explained in the original manuscript for the KB93 model (Karafillis and Boyce, 1993).

Figure 6: Normalized von Mises equivalent stress as a function of loading direction ($^{\circ}$) with $\mathbf{L}$ set for anisotropic plasticity behavior and a non-quadratic yield function with $k = 15$ and $c = 0.85$.

The second case study uses a linear transformation tensor that is set for a plastically isotropic (see Table 2) material with pressure sensitivity varying from $\alpha = 0.0$ to $\alpha = 0.2$ in increments of 0.05 and the yield surface parameters set to $k = 15$ and $c = 0.85$. The results of this case study can be seen in Figure 7. This set of input parameters will yield a non-quadratic Drucker-Prager model with plastic isotropy (if the parameter k were set to unity then the standard Drucker-Prager model would be recovered). The non-quadratic yield surface for the results shown in Figure 7 corresponds to those observed in Figures 3 and 4. It can also be seen in Figure 7 that there is now a pressure sensitivity of the material, indicated by the difference in plastic yield in the case of tension and compression, as dictated by α . When $\alpha = 0.0$ the non-quadratic KB93 yield surface will be retrieved and combining this fact with the uniaxial

loading condition is the reason that the normalized von Mises equivalent stress is unity for that specific case, as shown in the figure. On the other hand, as the sensitivity parameter α is increased the solution in tension and compression deviates further from the standard von Mises solution. The behavior of the material for varying load directions yields no difference in the response for a given pressure sensitivity as the values of the transformation tensor are set to the isotropic case.

Figure 7: Normalized von Mises equivalent stress as a function of the pressure dependence parameter with $\mathbf{L}$ set for isotropic plasticity behavior in both tensile and compressive loading conditions and a non-quadratic effective yield function with $k = 15$ and $c = 0.85$.

The final case is that which utilizes both the plastic anisotropy of the material in combination with the pressure dependence. Again, the pressure dependence of the model ranges from $\alpha = 0.0$ to $\alpha = 0.2$ in increments of 0.05 and the loading direction ranges from 0° to 90° . The results from this case study are shown in Figures 8 and 9, which correspond to loading in tension and compression, respectively. The primary observation that can be made from the results

shown in the figures is that the material response now clearly depends on both the pressure and the loading direction, which of course is the expected result. This result can be seen as a combination of the results shown in Figures 6 and 7. The highest allowable equivalent von Mises stress corresponds to loading 0° from the rolling direction and loaded in tension while the lowest allowable equivalent von Mises stress corresponds to loading 90° from the rolling direction and loaded in compression.

Figure 8: Normalized von Mises equivalent stress as a function of the pressure dependence parameter with L set for anisotropic plasticity behavior. Each curve represents a linear interpolation of the solution at different loading directions between 0° and 90° loaded in tension.

4.2. Comparison of the Modified Drucker-Prager Model with Experiments

In previous work by Wilson (2002), it was determined that the classical metal plasticity assumptions, von Mises plasticity in particular, are not adequate for predicting the hardening behavior in certain aluminum alloys. It was found that

Figure 9: Normalized von Mises equivalent stress as a function of the pressure dependence parameter with $\mathbf{L}$ set for anisotropic plasticity behavior. Each curve represents a linear interpolation of the solution at different loading directions between 0° and 90° loaded in compression.

for Al2024-T351 a pressure-dependent model, specifically the Drucker-Prager model, could better predict the overall stress-strain response of the material in tensile loading conditions; von Mises plasticity had a tendency to over-predict the hardening behavior. Work by Gilat et al. showed a strong anisotropic response in rolled Al2024-T351 plate stock. The present case study is aimed at comparing the modified Drucker-Prager model with the experimental data observed in Gilat et al..

The hardening law chosen for the comparison is that of Ludwik,

$$\bar{\sigma} = \sigma_{y0} + A \bar{\varepsilon}^p{}^n \quad (31)$$

where σ_{y0} is the initial yield stress, A is an amplification factor that controls the contribution of $\bar{\varepsilon}^p$, the effective plastic strain, to the hardening of the material,

and n is a material dependent parameter that controls the nonlinear response of the hardening behavior. The elastic and plastic material properties and the parameters corresponding to the KB93 model can be found in Tables 3 and 4, respectively. The FEA model used for this study was that of a dogbone tensile specimen (corresponding to the dimensions used in Gilat et al.) comprised of 50,664 linear hexahedral elements with 59,094 nodes, as seen in Figure 10. There were 9 elements used through the thickness of the specimen. A symmetry boundary condition was imposed on the left boundary while the right boundary was subjected to an applied constant velocity.

Table 3: Elastic and plastic material properties used in the modified Drucker-Prager experimental comparison case study.

E (MPa)	ν	σ_{y0} (MPa)	α	A (MPa)	n
71705	0.33	324	0.03	720	0.625

The comparison of the simulated and experimentally derived (Gilat et al.) stress-strain responses for the Al2024-T351 dogbone tensile specimen can be seen in Figure 11. Engineering stress and strain were used because the reference data (Gilat et al.) was obtained in this form. The strain from the simulation was calculated using the gauge length of the specimen (5.08mm, see Figure 10) as the initial length (also for calculation of the change in length throughout the simulation) while the stress was calculated by extracting the reaction forces at the right face of the specimen and dividing by the original cross-sectional area. This method for calculating the material response is identical to that used to obtain the experimental data. The figure shows that the initial yield and hardening behavior of the material is captured well using the modified Drucker-

Table 4: KB93 model parameters used in the modified Drucker-Prager experimental comparison case study.

k	c	C	α_1	α_2	β_1	β_2	β_3	γ_1	γ_2	γ_3
6	0.3	0.64599	1.0	1.3	-0.35	-0.65	-0.65	1.5	1.5	0.73

Figure 10: FEA model of the tensile dogbone specimen used for the modified Drucker-Prager experimental comparison case study.

Prager model. This result supports the findings by Wilson (2002) that suggest the Drucker-Prager model can be an adequate basis for structural analysis of aluminum alloys loaded in tension. It should be noted that while this study shows a successful application of the present work to Al2024-T351, using this model in compression dominated load states for this particular material would be a poor choice for reasons already stated in Section 3.1.

4.3. Application of the Anisotropic Modified Cam-Clay Model in 3D Finite Element Analysis

A 3D finite element implementation of the anisotropic, non-quadratic, modified Cam-Clay model described in Section 3.2 was developed for use in the finite element analysis software package, Abaqus. As with the modified Drucker-Prager model, a fully implicit stress update algorithm for the model was developed and used to create a VUMAT applicable within Abaqus/Explicit. The

Figure 11: Comparison of the simulated and experimentally derived stress-strain responses for the Al2024-T351 dogbone tensile specimen.

numerical scheme utilizes an orthogonal return to the yield surface in the 3D stress space. The material is assumed to be either isotropic or orthotropic for the following studies (see Table 2 for the linear transformation tensor parameters). The elastic and plastic properties used for the study are shown in Table

530 5.

Table 5: Elastic and plastic material properties used in the anisotropic modified Cam-Clay case study.

E (MPa)	ν	p_c (MPa)	M
4000	0.21	300	1.0

In order to showcase the applicability of the anisotropic modified Cam-Clay yield criterion in arbitrary 3D loading conditions within finite element analysis, a simple indentation test, described in Figure 12, was analyzed. The problem setup

includes a rigid hemispherical indenter penetrating a deformable material. The deformable material is resting in a rigid container and its behavior is modeled using the anisotropic modified Cam-Clay yield criterion. The rigid indenter and container are modeled as discrete rigid surfaces. This problem setup was chosen due to the large deformation, both in shear and dilatational modes, and also because the modified Cam-Clay yield criterion has an inherent pressure dependence allowing for plastic strain evolution in compression.

Figure 12: Schematic for the large deformation indentation simulation of the anisotropic modified Cam-Clay model.

The finite element model for the deformable material shown in Figure 12 incorporates 16400 nodes and 14560 C3D8R reduced integration hexahedral elements with hourglass control. Due to planar symmetries, the model incorporates only one quarter of the setup shown in Figure 12. Symmetry boundary conditions are applied on the zx and zy planes (coordinate system is shown in Figures 13 and 14). Surface-to-surface contact is implemented between the deformable material and the rigid indenter (traveling with a constant velocity in the z -direction) and the rigid container (fixed in space), respectively. Due to the large deformation, adaptive remeshing was used at a rate of 5 time steps per remesh with 5 mesh sweeps at each remesh cycle.

The final penetration depth of the rigid indenter into the deformable material was constant for each result shown in this section. The results from the finite element study of the anisotropic modified Cam-Clay model are shown in Figures 13 and 14. Figure 13 shows a comparison of the maximum principal stress for the anisotropic modified Cam-Clay model and the quadratic isotropic modified Cam-Clay model. Figure 14 shows a comparison of the predicted equivalent plastic strain for the anisotropic and isotropic modified Cam-Clay yield criteria. The anisotropic behavior of the anisotropic modified Cam-Clay model is clearly seen in Figures 13(a) and 14(a). On the contrary, the standard modified Cam-Clay model (Figures 13(b) and 14(b)) yields a solution that is approximately axisymmetric.

As mentioned in Section 3.2, the effect of including anisotropy in the plastic behavior is that it changes the apparent slope of the critical state line, M . As the load state and direction changes, the effective slope of the critical state line changes, relative to the effective stress measure, $\bar{\phi}$, in such a way that the yield surface will swell or shrink in the deviatoric direction. Thus, the parameter M for the anisotropic modified Cam-Clay model is a reference slope that is fixed, however, if measured through simulation, M would appear to increase or decrease relative to the value for the standard quadratic isotropic modified Cam-Clay model depending on the loading direction.

The preconsolidation pressure, p_c , has no apparent change. This is a distinction between models of this type and models of the Drucker-Prager-type in that for the modified Drucker-Prager model the matrix flow stress appears to change with the anisotropic behavior. For Cam-Clay-type models, the matrix flow stress is closely related to the value p_c , which does not change with the present hardening assumptions such that perfect plasticity is observed. Regardless of the anisotropic properties used for the anisotropic modified Cam-Clay model, the actual and perceived values of p_c will remain identical because the anisotropy acts only on the deviatoric part of the stress state. In the event that the general yield criterion development framework were used to create a model incorporating hardening behavior in p_c , the lack of sensitivity of p_c to anisotropy

Figure 13: Comparison of the maximum principal stress contour for the (a) non-quadratic ($k = 6, c = 0.86$), anisotropic (see Table 2), modified Cam-Clay model and the (b) quadratic ($k = 1, c = 0.0$), isotropic (see Table 2), modified Cam-Clay model.

would still occur, provided that the hardening behavior is not sensitive to deviatoric stress states (i.e., a function of dilatational modes of deformation only).

Figure 14: Comparison of the equivalent plastic strain contour for the (a) non-quadratic ($k = 6$, $c = 0.86$), anisotropic (see Table 2), modified Cam-Clay model and the (b) quadratic ($k = 1$, $c = 0.0$), isotropic (see Table 2), modified Cam-Clay model.

5. Conclusion and Future Work

A general yield criterion has been developed in order to enable the ability to investigate pressure-dependent materials exhibiting anisotropic plastic effects. The developed general form for the pressure-dependent anisotropic yield criteria can be applied to materials either additively, multiplicatively, or as a combination of the two. Using this form of the general yield criterion the authors have shown the applicability to a simple additive pressure-dependent model that converges to the standard Drucker-Prager model when the yield surface is selected to be quadratic and the components of the linear transformation tensor set for isotropic plasticity. This model is applicable for, but is not necessarily limited to, many geomechanical materials in which the yield limit in tension is much lower than in compression. As discussed in Section 4.2, the modified Drucker-Prager model shows excellent agreement with experiments for some metallic materials loaded in tension. The framework has also been applied to the modified Cam-Clay model; the resulting model reduces to the standard modified Cam-Clay model when the yield surface is quadratic and the linear transformation tensor

set for isotropic plasticity. The anisotropic modified Cam-Clay model is a general application model for both geomechanical and metallic materials in that it forms a closed yield surface.

It should be noted that the general yield criterion can be extended to an infinite number of variations depending on the form of the embedded pressure-dependent functions described previously. This is particularly useful for developing new yield criteria that incorporate shear and hydrostatic stress states. The impact of this work is in the applicability to existing models in that the effective stress, $\bar{\phi}$, defined based on the KB93 yield criterion allows one to use an infinite number of variations in terms of shear sensitivity, thereby allowing one to insert this effective stress in place of the original effective stress value of a given model; this is enhanced by allowing one to include many variations of pressure sensitivity functions. The two of these properties combined yield an ideal general framework that can accurately model the plastic behavior of various material systems, e.g., pressure sensitive or initial anisotropic behavior, using the same general formulation, largely reducing the complexity in yield surface description for different classes of materials. Moreover, the nature of the developed general flow rule reduces implementation time for newly developed models in that the contribution to plastic evolution of the pressure dependence functions are simply additive amendments to the existing anisotropic model implementation.

The aforementioned Drucker-Prager-like anisotropic yield criterion was implemented in the finite element analysis software, Abaqus. A single element model was developed in order to validate the general yield criterion created herein for multiple cases including anisotropy, pressure dependence, and combinations of the two. The modified Drucker-Prager model was then calibrated for a specific aluminum alloy and compared with experimental data that showed a good agreement. Likewise, the anisotropic modified Cam-Clay model was implemented in Abaqus and applied to a 3D structure under large deformation with combined shear and pressure load states. The implementation of the general yield surface into a stress update algorithm for use in finite element

analysis will be discussed in future work. The ability to implement the general dimensional yield surface presented in this work within a finite element analysis framework is of ample importance for large-scale analysis of complex processes. The future work of the authors will also include specific forms for anisotropic pressure-dependent models for materials that are prone to damage, as discussed in the introduction.

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Appendix A. Karafillis and Boyce Anisotropic Yield Criterion

The general anisotropic yield criterion developed by Karafillis and Boyce (1993) (KB93) will be discussed in this section. This model incorporates a linear stress transformation to transform a general 3D anisotropic yield criterion into a simpler 3D isotropic yield criterion. A specific form for the linear stress transformation tensor will be discussed in more detail in Section 4.1 but for a comprehensive treatment on the topic one should refer to the original paper by Karafillis and Boyce as we will only describe the portions of the model that are relevant for the current theoretical development and case study. As stated in the introduction, this model utilizes two separate yield criterion via a mixing coefficient that allows for adaptation of the overall yield surface for a vast number of materials.

The first of the two yield criteria for the model, neglecting any anisotropic behavior for the moment, is,

$$\phi_1 = (S_1 - S_2)^{2k} + (S_2 - S_3)^{2k} + (S_3 - S_1)^{2k} = 2\bar{\sigma}^{2k} \quad (\text{A.1})$$

where S_i are the principal values of the deviator of the Cauchy stress, $\bar{\sigma}$ is the matrix flow stress, and k is the integer shaping exponent that is used to augment the order of the yield surface. For example, if $k = 1$, then the yield surface is quadratic. It should be noted that when $k = 1$ the von Mises (J_2) yield criterion is recovered and when $k \rightarrow \infty$ the Tresca yield criterion is recovered. Thus, the lower bound for ϕ_1 is the Tresca yield criterion and the upper bound corresponds to J_2 plasticity. It should also be noted that for the present development we will assume the material behaves as perfectly plastic and therefore the matrix flow stress will remain constant but any hardening law (see Smith et al. (2014)) can be implemented to govern the behavior of the material during evolution of the state of plasticity. The second yield criterion is,

$$\phi_2 = S_1^{2k} + S_2^{2k} + S_3^{2k} = \frac{2^{2k} + 2}{3^{2k}} \bar{\sigma}^{2k} \quad (\text{A.2})$$

When $k = 1$ in equation A.2, the von Mises yield criterion is again recovered.

When $k \rightarrow \infty$ in equation A.2, the upper bound of the KB93 model is achieved.

The general yield criterion for the KB93 model then becomes,

$$\phi = (1 - c)\phi_1 + cm\phi_2 = 2\bar{\sigma}^{2k} \quad (\text{A.3})$$

where,

$$m = \frac{3^{2k}}{2^{2k-1} + 1}$$

and c is the mixing coefficient. When $c = 0$, ϕ_1 is recovered, and when $c = 1$, ϕ_2 is recovered. Although we have not given any consideration to the anisotropy of a material up to this point, the generality of the above model should be apparent. The transferability of this model from one material to another is one of the primary attractive properties that have driven the current research.

The yield loci corresponding to $k = 1$ and $c = 0$ for the KB93 yield criterion, as well as for both of the two comprising yield functions, correspond to J_2 plasticity. The yield loci corresponding to $k = 15$ and $c = 0.85$ for the KB93 yield criterion and the two comprising yield functions in plane stress space and

in the π -plane are shown in Figures A.1 and A.2, respectively. It can be seen in these figures that the two mixing yield surfaces, ϕ_1 and ϕ_2 , now closely correspond to the lower and upper bound yield surfaces of the KB93 model, respectively. The effective KB93 yield surface, ϕ , a mixture of ϕ_1 and ϕ_2 , lies in between, as represented by the solid lines shown in Figure A.1 (plane stress representation) and Figure A.2 (π -plane representation).

Figure A.1: Plane stress yield surfaces for ϕ , ϕ_1 , and ϕ_2 with $k = 15$ and $c = 0.85$ in the KB93 model.

Figure A.2: Yield surfaces for ϕ , ϕ_1 , and ϕ_2 in the π -plane with $k = 15$, $c = 0.85$ in the KB93 model.

In order to consider the plastic anisotropy of the material we will need to develop the aforementioned linear transformation tensor. If we assume that there exists some $\mathbf{L}$ that is a fourth-order linear transformation tensor such that,

$$\tilde{\mathbf{S}} = \mathbf{L} : \boldsymbol{\sigma} \quad (\text{A.4})$$

where $\tilde{\mathbf{S}}$ is the so-called Isotropic Plasticity Equivalent “IPE” deviatoric stress tensor and $\boldsymbol{\sigma}$ is the actual Cauchy stress corresponding to the anisotropic yield criterion. The components of $\mathbf{L}$ must then be related to the anisotropy of the material. The exact form of the tensor $\mathbf{L}$ is not important for the moment so for now we will just assume that it does exist and has the transformation property shown above.

The plastic anisotropy is taken into account through transformations of the actual Cauchy stress and the rate of plastic deformation tensor conjugate to the IPE stress via the linear transformation tensor. The transformation of the actual Cauchy stress is used to evaluate the yield surface in equation A.3 above using the principal components of the IPE stress tensor as opposed to the deviator of

the actual Cauchy stress,

$$\phi = (1 - c)\phi_1(\tilde{S}_i) + cm\phi_2(\tilde{S}_i) = 2\bar{\sigma}^{2k} \quad (\text{A.5})$$

The evolution equation of the plastic dissipation for this model can be obtained from the following equality,

$$\dot{w}^p = \boldsymbol{\sigma} : \mathbf{D}^p = \tilde{\mathbf{S}} : \mathbf{d}^p = \bar{\sigma} \dot{\epsilon}^p \quad (\text{A.6})$$

where $\mathbf{D}^p$ is the standard rate of plastic deformation tensor and $\mathbf{d}^p$ is the rate plastic deformation tensor, which is conjugate to the IPE stress tensor. If an associative flow rule is used,

$$\mathbf{d}^p = \dot{\lambda} \frac{\partial \phi}{\partial \tilde{\mathbf{S}}} = \dot{\lambda} \mathbf{Q}^T \cdot \frac{\partial \phi}{\partial \boldsymbol{\Lambda}} \cdot \mathbf{Q} \quad (\text{A.7})$$

where $\mathbf{Q}$ is a rotation matrix whose rows are the eigenvectors of the IPE stress tensor and $\boldsymbol{\Lambda}$ is a diagonal matrix whose components are the principal values of the IPE stress tensor. Assuming that $\mathbf{d}^p$ has already been determined, one can find the anisotropic counterpart using equation A.6,

$$\mathbf{D}^p = \mathbf{L} : \mathbf{d}^p \quad (\text{A.8})$$

Therefore, this model can be implemented by performing a transformation of the actual Cauchy stress into the IPE stress, followed by determining and updating the principal values of the IPE stress as would be done for any isotropic plasticity problem, and then transforming the variables back to the anisotropic Cauchy stress via the rate of plastic deformation tensor. Kinematic hardening can be incorporated by including a backstress with a corresponding evolution equation such that,

$$\tilde{\mathbf{S}} = \mathbf{L} : (\boldsymbol{\sigma} - \mathbf{B}) \quad (\text{A.9})$$

where $\mathbf{B}$ is the backstress tensor. For simplicity, the backstress will not be discussed in any detail for the present study but the theoretical development

presented in the following section is still applicable to kinematic behavior by replacing the actual Cauchy stress by the tensor,

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma} - \boldsymbol{B} \quad (\text{A.10})$$

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