SOLITONS IN MAGNETO-OPTIC WAVEGUIDES BY EXTENDED TRIAL FUNCTION SCHEME

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Abstract

This paper obtains soliton solutions to magneto-optic waveguides that appear with Kerr, power and log-law nonlinearities. The extended trial function method is employed to obtain these solutions. Thus, bright, dark and singular soliton solutions are retrieved. In addition, Gaussons are obtained for log-law nonlinear waveguides. All of these solutions appear with constraints that guarantees the existence of solitons and Gaussons.

Keywords: magneto-optic waveguides; solitons; integrability; extended trial function.

1 INTRODUCTION

Optical soliton molecules form the fundamental structure of fiber-optic transmission technology along trans-continental and trans-oceanic distances [1-15]. This technology is modeled by the nonlinear Schrödinger's equation (NLSE). There are various kinds of optical waveguides that make these transmission reality. They are optical fibers, photonic crystal fibers, optical metamaterials and metasurfaces and many others. This paper is going to consider a special kind of optical waveguide that comes with an artificially induced magnetic field. These are called magneto-optic waveguides. This external magnetic field can compell solitons from a state of attraction to a state of isolation which can help to reduce soliton clutter thus salvaging loss from information transfer.

This paper will adopt an integration scheme that is known as external trial function method that will reveal soliton solutions as well as other solutions such as doubly periodic functions which, in the limiting case of the modulus of ellipticity, leads to solitons. Three types of nonlinear media are studied in this paper. They are Kerr law, power law and log law nonlinearity. These solutions all appear with various constraints that are also listed for the existence of solitons and Gaussons. The details are all discussed in the next few sections.

2 MATHEMATICAL MODEL

The governing equation for propagation of solitons through magneto-optic waveguides is given by [1, 2, 8, 14]

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + \{\xi_1 F(|q|^2) + \eta_1 F(|r|^2)\} q = Q_1 r, \quad (1)$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + \{\xi_2 F(|r|^2) + \eta_2 F(|q|^2)\} r = Q_2 q. \quad (2)$$

Eqs. (1) and (2) are coupled nonlinear Schrödinger's equations that govern the propagation of optical solitons through a magneto-optic waveguide. The coefficients of a_l for $l = 1, 2$ represent group velocity dispersion (GVD) while the coefficients of b_l are the spatio-temporal dispersion (STD) terms. It is extremely important to include STD in addition to GVD so that the governing NLSE, given by Eqs. (1) and (2) is well-posed. It was recently proved that NLSE with GVD only, and without STD is an ill-posed problem [3-5]. On the right hand side, Q_l are the magneto-optic parameters. Finally, the functional F represents the form of nonlinearity. Here, $F(|q|^2) q : C \mapsto C$. If complex plane C is a two-dimensional linear space R^2 , $F(|q|^2) q$ is k times continuously differentiable, then [2]

$$F(|q|^2) q \in \bigcup_{m,n=1}^{\infty} C^k((-n, n) \times (-m, m); R^2). \quad (3)$$

In order to solve this coupled NLSE by the extended trial function scheme [3, 7, 10, 11], the following solution structure is selected.

$$q(x, t) = P_1(\xi) \exp[i\phi(x, t)], \quad (4)$$

$$r(x, t) = P_2(\xi) \exp[i\phi(x, t)], \quad (5)$$

where the wave variable ξ is given by

$$\xi = k(x - vt). \quad (6)$$

Here, $P_l(\xi)$ ($l = 1, 2$) represents the amplitude component of the soliton solution and v is the speed of the soliton, while the phase component $\phi(x, t)$ is defined as

$$\phi(x, t) = -\kappa x + \omega t + \theta_0. \quad (7)$$

Here κ is the frequency of the solitons while ω represents the wave number and θ_0 is the phase constant. Substituting Eqs. (4) and (5) into Eqs. (1) and (2) and then decomposing into real and imaginary parts give

$$k^2(a_l - b_lv)P_l'' + (b_l\omega\kappa - \omega - a_l\kappa^2)P_l + \{\xi_l F(P_l^2) + \eta_l F(P_{\bar{l}}^2)\}P_l = Q_l P_{\bar{l}}, \quad (8)$$

and

$$-kv(1 - b_l\kappa)P_l' + k(b_l\omega - 2a_l\kappa)P_l' = 0, \quad (9)$$

respectively. Here, $l = 1, 2$ and $\bar{l} = 3 - l$. From the imaginary part equation it is possible to obtain the speed (v) of the soliton as

$$v = \frac{b_l\omega - 2a_l\kappa}{1 - b_l\kappa}. \quad (10)$$

Now, equating the two values of the soliton speed leads to

$$a_1 = a_2, \quad (11)$$

and

$$b_1 = b_2. \quad (12)$$

The speed of the soliton therefore reduces to

$$v = \frac{b\omega - 2a\kappa}{1 - b\kappa}, \quad (13)$$

where it is assumed that $a_1 = a_2 = a$ and $b_1 = b_2 = b$. The speed of the soliton, given by Eq. (13) stays valid as long as

$$b\kappa \neq 1. \quad (14)$$

Therefore the coupled NLSE for magneto-optic waveguide is re-casted as

$$iq_t + aq_{xx} + bq_{xt} + \{\xi_1 F(|q|^2) + \eta_1 F(|r|^2)\}q = Q_1 r, \quad (15)$$

$$ir_t + ar_{xx} + br_{xt} + \{\xi_2 F(|r|^2) + \eta_2 F(|q|^2)\}r = Q_2 q. \quad (16)$$

Consequently, the real part Eq. (8) modifies to

$$k^2(a - bv)P_l'' + (b\omega\kappa - \omega - a\kappa^2)P_l + \{\xi_l F(P_l^2) + \eta_l F(P_{\bar{l}}^2)\}P_l = Q_l P_{\bar{l}}. \quad (17)$$

In the next three sections, this real part equation (17) will be studied for the three types of nonlinearity for the functions F .

3 KERR LAW NONLINEARITY

For Kerr law nonlinearity, $F(s) = s$ [8, 14]. Therefore, the model Eqs. (15) and (16), change to

$$iq_t + aq_{xx} + bq_{xt} + (\xi_1 |q|^2 + \eta_1 |r|^2)q = Q_1 r, \quad (18)$$

$$ir_t + ar_{xx} + br_{xt} + (\xi_2|r|^2 + \eta_2|q|^2) r = Q_2q, \quad (19)$$

and naturally therefore the real part equation from (17) is now given by

$$k^2(a - bv)P_1'' + (b\omega\kappa - \omega - a\kappa^2)P_1 + (\xi_1P_1^2 + \eta_1P_2^2)P_1 = Q_1P_2, \quad (20)$$

$$k^2(a - bv)P_2'' + (b\omega\kappa - \omega - a\kappa^2)P_2 + (\xi_2P_2^2 + \eta_2P_1^2)P_2 = Q_2P_1. \quad (21)$$

The extended trial function scheme will be applied to Eqs. (20) and (21) to retrieve bright, dark and singular solitons.

To start with the extraction of solutions to (20) and (21), the following assumption for the soliton structure is made

$$P_1 = \sum_{i=0}^{\varsigma} \tau_i \Psi^i, \quad (22)$$

$$P_2 = \sum_{i=0}^{\xi} \tilde{\tau}_i \Psi^i, \quad (23)$$

where

$$(\Psi')^2 = \Lambda(\Psi) = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} = \frac{\mu_\sigma \Psi^\sigma + \dots + \mu_1 \Psi + \mu_0}{\chi_\rho \Psi^\rho + \dots + \chi_1 \Psi + \chi_0}. \quad (24)$$

Here $\tau_0, \dots, \tau_\varsigma$; $\tilde{\tau}_0, \dots, \tilde{\tau}_\xi$; $\mu_0, \dots, \mu_\sigma$ and $\chi_0, \dots, \chi_\rho$ are constants to be determined later. Using the relations (22)-(24), we can derive the terms $(P_1')^2$, $(P_2')^2$, P_1'' and P_2'' as below:

$$(P_1')^2 = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\varsigma} i \tau_i \Psi^{i-1} \right)^2, \quad (25)$$

$$(P_2')^2 = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\xi} i \tilde{\tau}_i \Psi^{i-1} \right)^2, \quad (26)$$

and

$$P_1'' = \frac{\Phi'(\Psi)\Upsilon(\Psi) - \Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)} \left(\sum_{i=0}^{\varsigma} i \tau_i \Psi^{i-1} \right) + \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\varsigma} i(i-1) \tau_i \Psi^{i-2} \right), \quad (27)$$

$$P_2'' = \frac{\Phi'(\Psi)\Upsilon(\Psi) - \Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)} \left(\sum_{i=0}^{\xi} i \tilde{\tau}_i \Psi^{i-1} \right) + \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\xi} i(i-1) \tilde{\tau}_i \Psi^{i-2} \right), \quad (28)$$

where $\Phi(\Psi)$ and $\Upsilon(\Psi)$ are polynomials of Ψ . We can reduce Eq. (24) to the elementary integral form as follows:

$$\pm(\xi - \xi_0) = \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}} = \int \sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}} d\Psi. \quad (29)$$

According to the balance principle, one determines a relation of σ , ρ , ς and $\tilde{\varsigma}$ is given by

$$\varsigma = \tilde{\varsigma} = \frac{\sigma - \rho - 2}{2}. \quad (30)$$

When $\sigma = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (30), we then assume that Eqs. (20) and (21) have the following formal solutions:

$$P_1 = \tau_0 + \tau_1 \Psi, \quad (31)$$

$$P_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \quad (32)$$

where τ_i , $\tilde{\tau}_i$ ($i = 0, 1$) are constants to be determined later and Ψ satisfies Eq. (24). Substituting these formal solutions and their necessary derivatives into Eqs. (20) and (21), and solving the resulting system of algebraic equations one recovers

$$\begin{aligned} \mu_1 &= \frac{2\chi_0 [-\xi_1 \tau_0 \tau_1^4 \tilde{\tau}_0^2 + \xi_2 \tau_0^3 \tilde{\tau}_1^4 - \tau_1^2 \tilde{\tau}_1^2 (\xi_1 \tau_0^3 + \tilde{\tau}_0 (\xi_2 \tau_0 \tilde{\tau}_0 - Q_1)) + \tau_0 \tau_1 \tilde{\tau}_1^3 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0)]}{k^2 \tau_1^3 \tilde{\tau}_1^2 (a - bv)}, \\ \mu_2 &= \frac{\chi_0 [-\xi_1 \tau_1^4 \tilde{\tau}_0^2 - 2\xi_1 \tau_0 \tau_1^3 \tilde{\tau}_0 \tilde{\tau}_1 + \xi_2 \tau_0^2 \tilde{\tau}_1^4 - \tau_1^2 \tilde{\tau}_1^2 (\xi_2 \tilde{\tau}_0^2 + 3\xi_1 \tau_0^2) + 2\tau_1 \tilde{\tau}_1^3 (Q_1 - 3\xi_2 \tau_0 \tilde{\tau}_0)]}{k^2 \tau_1^2 \tilde{\tau}_1^2 (a - bv)}, \\ \mu_3 &= -\frac{4\chi_0 [\xi_1 \tau_1^2 (\tau_1 \tilde{\tau}_0 + 2\tau_0 \tilde{\tau}_1) + \xi_2 \tilde{\tau}_1^2 (2\tau_1 \tilde{\tau}_0 + \tau_0 \tilde{\tau}_1)]}{3k^2 \tau_1 \tilde{\tau}_1 (a - bv)}, \quad \mu_4 = -\frac{\chi_0 (\xi_1 \tau_1^2 + \xi_2 \tilde{\tau}_1^2)}{k^2 (a - bv)}, \\ \omega &= \frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)}, \quad \eta_1 = \frac{\xi_1 \tau_1^2 + 2\xi_2 \tilde{\tau}_1^2}{\tilde{\tau}_1^2}, \\ \eta_2 &= \frac{2\xi_1 \tau_1^2 + \xi_2 \tilde{\tau}_1^2}{\tau_1^2}, \quad Q_2 = \frac{-\xi_1 \tau_1^2 (\tau_1 \tilde{\tau}_0 - \tau_0 \tilde{\tau}_1)^2 + \tilde{\tau}_1^2 [Q_1 \tau_1 \tilde{\tau}_1 + \xi_2 (\tau_1 \tilde{\tau}_0 - \tau_0 \tilde{\tau}_1)^2]}{\tau_1^3 \tilde{\tau}_1}, \\ \mu_0 &= \mu_0, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1. \end{aligned} \quad (33)$$

Substituting these results into Eqs. (24) and (29), we find that

$$\pm(\xi - \xi_0) = W \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \quad (34)$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4} \Psi^3 + \frac{\mu_2}{\mu_4} \Psi^2 + \frac{\mu_1}{\mu_4} \Psi + \frac{\mu_0}{\mu_4}, \quad W = \sqrt{\frac{\zeta_0}{\mu_4}}. \quad (35)$$

Consequently, we obtain the traveling wave solutions to Eqs. (18) and (19) below:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$,

$$\begin{aligned} q(x, t) &= \left\{ \tau_0 + \tau_1 \lambda_1 \pm \frac{\tau_1 W}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0} \right\} \\ &\times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \end{aligned} \quad (36)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 \pm \frac{\tilde{\tau}_1 W}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (37)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_1 + \frac{4\tau_1 W^2 (\lambda_2 - \lambda_1)}{4W^2 - \left[(\lambda_1 - \lambda_2) \left(k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right) \right]^2} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (38)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 + \frac{4\tilde{\tau}_1 W^2 (\lambda_2 - \lambda_1)}{4W^2 - \left[(\lambda_1 - \lambda_2) \left(k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right) \right]^2} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (39)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$,

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_2 + \frac{\tau_1 (\lambda_2 - \lambda_1)}{\exp \left[\frac{\lambda_1 - \lambda_2}{W} \left(k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right) \right] - 1} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (40)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_2 + \frac{\tilde{\tau}_1 (\lambda_2 - \lambda_1)}{\exp \left[\frac{\lambda_1 - \lambda_2}{W} \left(k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right) \right] - 1} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (41)$$

and

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_1 + \frac{\tau_1 (\lambda_1 - \lambda_2)}{\exp \left[\frac{\lambda_1 - \lambda_2}{W} \left(k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right) \right] - 1} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (42)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 + \frac{\tilde{\tau}_1 (\lambda_1 - \lambda_2)}{\exp \left[\frac{\lambda_1 - \lambda_2}{W} \left(k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right) \right] - 1} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (43)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_1 - \frac{2\tau_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2) \cosh \left[\frac{k\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (44)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 - \frac{2\tilde{\tau}_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2) \cosh \left[\frac{k\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (45)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$,

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_2 + \frac{\tau_1 (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \operatorname{sn}^2 \left[\pm \frac{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right], l \right]} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (46)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_2 + \frac{\tilde{\tau}_1 (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \operatorname{sn}^2 \left[\pm \frac{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right], l \right]} \right\} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (47)$$

where

$$l^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \quad (48)$$

Note that λ_i ($i = 1, \dots, 4$) are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \quad (49)$$

When $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\xi_0 = 0$, the solutions (36)-(45) can be reduced to plane wave solutions

$$q(x, t) = \left\{ \pm \frac{\tau_1 W}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (50)$$

$$r(x, t) = \left\{ \pm \frac{\tilde{\tau}_1 W}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (51)$$

$$q(x, t) = \left\{ \frac{4\tau_1 W^2 (\lambda_2 - \lambda_1)}{4W^2 - \left[k(\lambda_1 - \lambda_2) \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]^2} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (52)$$

$$r(x, t) = \left\{ \frac{4\tilde{\tau}_1 W^2 (\lambda_2 - \lambda_1)}{4W^2 - \left[k(\lambda_1 - \lambda_2) \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]^2} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (53)$$

singular soliton solutions

$$q(x, t) = \frac{\tau_1 (\lambda_2 - \lambda_1)}{2} \left\{ 1 \mp \coth \left[\frac{k(\lambda_1 - \lambda_2)}{2W} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (54)$$

$$r(x, t) = \frac{\tilde{\tau}_1 (\lambda_2 - \lambda_1)}{2} \left\{ 1 \mp \coth \left[\frac{k(\lambda_1 - \lambda_2)}{2W} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (55)$$

and bright soliton solutions

$$q(x, t) = \left\{ \frac{A}{C + \cosh \left[B \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (56)$$

$$r(x, t) = \left\{ \frac{\tilde{A}}{C + \cosh \left[B \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (57)$$

where

$$A = \frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}, \quad \tilde{A} = \frac{2\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}, \quad B = \frac{k\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W}, \quad C = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}. \quad (58)$$

Here, A and $\tilde{A}$ are the amplitudes of the solitons, while B is the inverse width of the solitons. These solitons are valid for $\tau_1 < 0$ and $\tilde{\tau}_1 < 0$. Furthermore, when $\tau_0 = -\tau_1 \lambda_2$, $\tilde{\tau}_0 = -\tilde{\tau}_1 \lambda_2$ and $\xi_0 = 0$, Jacobi elliptic function solutions (46) and (47) can be reduced to

$$q(x, t) = \left\{ \frac{A_1}{C_1 + \text{sn}^2 \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)} \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (59)$$

$$r(x, t) = \left\{ \frac{\tilde{A}_1}{C_1 + \text{sn}^2 \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)} \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (60)$$

where

$$A_1 = \frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}, \quad \tilde{A}_1 = \frac{\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}, \quad C_1 = \frac{\lambda_4 - \lambda_2}{\lambda_1 - \lambda_4}, \quad (61)$$

$$B_j = \frac{(-1)^j k \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W} \quad \text{for } j = 1, 2.$$

Remark-1: When the modulus $l \rightarrow 1$, a second form of singular optical soliton solutions emerge

$$q(x, t) = \left\{ \frac{A_1}{C_1 + \tanh^2 \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (62)$$

$$r(x, t) = \left\{ \frac{\tilde{A}_1}{C_1 + \tanh^2 \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (63)$$

where $\lambda_3 = \lambda_4$.

Remark-2: However, if $l \rightarrow 0$, periodic singular wave solutions are listed as

$$q(x, t) = \left\{ \frac{A_1}{C_1 + \sin^2 \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (64)$$

$$r(x, t) = \left\{ \frac{\tilde{A}_1}{C_1 + \sin^2 \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1^2 (a\kappa^2 - \xi_2 \tilde{\tau}_0^2) - \tau_1 \tilde{\tau}_1 (Q_1 - 2\xi_2 \tau_0 \tilde{\tau}_0) - \xi_2 \tau_0^2 \tilde{\tau}_1^2}{\tau_1^2 (b\kappa - 1)} \right) t + \theta \right\} \right], \quad (65)$$

where $\lambda_2 = \lambda_3$.

4 POWER LAW NONLINEARITY

For power law nonlinearity, $F(s) = s^n$, where n represents the power law nonlinearity parameter. Here stability issue dictates $0 < n < 2$ and also $n \neq 2$ for self-focusing singularity [2]. The model equations (15) and (16) therefore change to

$$iq_t + aq_{xx} + bq_{xt} + (\xi_1 |q|^{2n} + \eta_1 |r|^{2n}) q = Q_1 r, \quad (66)$$

$$ir_t + ar_{xx} + br_{xt} + (\xi_2 |r|^{2n} + \eta_2 |q|^{2n}) r = Q_2 q, \quad (67)$$

and naturally therefore the real part equation from (17) is now given by

$$k^2(a - bv)P_1'' + (b\omega\kappa - \omega - a\kappa^2)P_1 + (\xi_1 P_1^{2n} + \eta_1 P_2^{2n})P_1 = Q_1 P_2, \quad (68)$$

$$k^2(a - bv)P_2'' + (b\omega\kappa - \omega - a\kappa^2)P_2 + (\xi_2 P_2^{2n} + \eta_2 P_1^{2n})P_2 = Q_2 P_1. \quad (69)$$

Using the balancing procedure between P_l'' and P_l^{2n+1} in Eqs. (68) and (69), we have

$$N = \frac{1}{n}. \quad (70)$$

To obtain an analytic solution, we use the transformation

$$P_1 = U_1^{\frac{1}{2n}} = U_2^{\frac{1}{2n}} = P_2, \quad (71)$$

in Eqs. (68) and (69) to find

$$k^2(a - bv) \{ (1 - 2n)(U_1')^2 + 2nU_1 U_1'' \} + 4n^2 (b\omega\kappa - \omega - a\kappa^2 - Q_1) U_1^2 + 4n^2 (\xi_1 + \eta_1) U_1^3 = 0, \quad (72)$$

$$k^2(a - bv) \{ (1 - 2n)(U_2')^2 + 2nU_2 U_2'' \} + 4n^2 (b\omega\kappa - \omega - a\kappa^2 - Q_2) U_2^2 + 4n^2 (\xi_2 + \eta_2) U_2^3 = 0. \quad (73)$$

The extended trial function scheme will be applied to Eqs. (72) and (73) to retrieve bright, dark and singular solitons.

Balancing the order of $U_l U_l''$ and U_l^3 in Eqs. (72) and (73), we determine a relation of σ , ρ , ς and $\tilde{\varsigma}$ as

$$\varsigma = \tilde{\varsigma} = \sigma - \rho - 2. \quad (74)$$

Case-1: When $\sigma = 3$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (74), Eqs. (72) and (73) have the solutions in the form

$$U_1 = \tau_0 + \tau_1 \Psi, \quad (75)$$

$$U_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \quad (76)$$

where τ_i , $\tilde{\tau}_i$ ($i = 0, 1$) are constants to be determined later and Ψ satisfies Eq. (24). Substituting these solutions and their necessary derivatives into Eqs. (72) and (73), and solving the resulting system of algebraic equations one recovers the following solution set:

$$\begin{aligned} \mu_0 &= \mu_0, \quad \tau_1 = \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1, \quad Q_1 = Q_2, \\ \mu_1 &= \frac{2[-2n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 2n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + k^2\mu_0\tilde{\tau}_1^3(1+n)(a-bv)]}{k^2\tilde{\tau}_0\tilde{\tau}_1^2(1+n)(a-bv)}, \\ \mu_2 &= \frac{-8n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 8n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + k^2\mu_0\tilde{\tau}_1^3(1+n)(a-bv)}{k^2\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(a-bv)}, \\ \mu_3 &= -\frac{4n^2\tau_1\chi_0(\eta_1 + \xi_1)}{k^2(1+n)(a-bv)}, \quad \xi_2 = \frac{\eta_1\tau_1 + \xi_1\tau_1 - \eta_2\tilde{\tau}_1}{\tilde{\tau}_1}, \quad \tau_0 = \frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1}, \\ \omega &= \frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n)[k^2\mu_0\tilde{\tau}_1^2(bv-a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa-1)}. \end{aligned} \quad (77)$$

Substituting these results into Eqs. (24) and (29), we have

$$\pm(\xi - \xi_0) = \sqrt{W_1} \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \quad (78)$$

where

$$\Lambda(\Psi) = \Psi^3 + \frac{\mu_2}{\mu_3}\Psi^2 + \frac{\mu_1}{\mu_3}\Psi + \frac{\mu_0}{\mu_3}, \quad W_1 = \frac{\zeta_0}{\mu_3}. \quad (79)$$

Consequently, we obtain the traveling wave solutions to Eqs. (66) and (67) below:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3$, rational function solutions are:

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_1 + \frac{4\tau_1 W_1}{\left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right]^2} \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (80)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 + \frac{4\tilde{\tau}_1 W_1}{\left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right]^2} \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (81)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, dark soliton solutions are:

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_2 + \tau_1 (\lambda_1 - \lambda_2) \tanh^2 \left(\frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_1}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right] \right) \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (82)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_2 + \tilde{\tau}_1 (\lambda_1 - \lambda_2) \tanh^2 \left(\frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_1}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right] \right) \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (83)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)^2$ and $\lambda_1 > \lambda_2$, singular soliton solutions are:

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_1 + \tau_1 (\lambda_1 - \lambda_2) \operatorname{cosech}^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_1}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (84)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 + \tilde{\tau}_1 (\lambda_1 - \lambda_2) \operatorname{cosech}^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_1}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (85)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, Jacobi elliptic function solutions are:

$$q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_3 + \tau_1 (\lambda_2 - \lambda_3) \operatorname{sn}^2 \left(\mp \frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_1}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right], l \right) \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (86)$$

$$r(x, t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1 \lambda_3 + \tilde{\tau}_1 (\lambda_2 - \lambda_3) \operatorname{sn}^2 \left(\mp \frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_1}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right], l \right) \right\}^{\frac{1}{2n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (87)$$

where

$$l^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}. \quad (88)$$

Note that λ_i ($i = 1, 2, 3$) are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \quad (89)$$

When $\tau_0 = -\tau_1 \lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1 \lambda_1$ and $\xi_0 = 0$, the solutions (80)-(85) are reduced to plane wave solutions

$$q(x, t) = \left\{ \frac{A_2}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)} \right\}^{\frac{1}{n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (90)$$

$$r(x, t) = \left\{ \frac{\tilde{A}_2}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)} \right\}^{\frac{1}{n}} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2 \eta_1 \tau_1 \chi_0 \tilde{\tau}_0^3 - 4n^2 \xi_1 \tau_1 \chi_0 \tilde{\tau}_0^3 + \tilde{\tau}_1 (1+n) [k^2 \mu_0 \tilde{\tau}_1^2 (bv - a) + 4n^2 \chi_0 \tilde{\tau}_0^2 (a\kappa^2 + Q_2)]}{4n^2 \chi_0 \tilde{\tau}_0^2 \tilde{\tau}_1 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (91)$$

bright 1-soliton solutions

$$q(x, t) = \left\{ \frac{A_3}{\cosh^{\frac{1}{n}} \left[B_3 \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (92)$$

$$r(x, t) = \left\{ \frac{\tilde{A}_3}{\cosh^{\frac{1}{n}} \left[B_3 \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (93)$$

and singular soliton solutions

$$q(x, t) = \left\{ \frac{A_4}{\sinh^{\frac{1}{n}} \left[B_3 \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (94)$$

$$r(x, t) = \left\{ \frac{\tilde{A}_4}{\sinh^{\frac{1}{n}} \left[B_3 \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right\} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (95)$$

where

$$A_2 = 2\sqrt{\tau_1 W_1}, \quad A_3 = [\tau_1(\lambda_2 - \lambda_1)]^{\frac{1}{2n}}, \quad A_4 = [\tau_1(\lambda_1 - \lambda_2)]^{\frac{1}{2n}}, \\ \tilde{A}_2 = 2\sqrt{\tilde{\tau}_1 W_1}, \quad \tilde{A}_3 = [\tilde{\tau}_1(\lambda_2 - \lambda_1)]^{\frac{1}{2n}}, \quad \tilde{A}_4 = [\tilde{\tau}_1(\lambda_1 - \lambda_2)]^{\frac{1}{2n}}, \quad B_3 = \frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}. \quad (96)$$

Here A_3 , $\tilde{A}_3$ and A_4 , $\tilde{A}_4$ are respectively the amplitudes of 1-solitons and singular solitons, while B_3 is the inverse width of the solitons. So, the solitons exist for $\tau_1 > 0$ and $\tilde{\tau}_1 > 0$. Furthermore, when $\tau_0 = -\tau_1\lambda_3$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_3$ and

$\xi_0 = 0$, the solutions (86) and (87) are simplified as

$$q(x, t) = A_5 \operatorname{sn}^{\frac{1}{n}} \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (97)$$

$$r(x, t) = \tilde{A}_5 \operatorname{sn}^{\frac{1}{n}} \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (98)$$

where

$$A_5 = [\tau_1(\lambda_2 - \lambda_3)]^{\frac{1}{2n}}, \quad \tilde{A}_5 = [\tilde{\tau}_1(\lambda_2 - \lambda_3)]^{\frac{1}{2n}}, \quad B_j = \frac{(-1)^j k}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_1}} \quad \text{for } j = 4, 5. \quad (99)$$

Remark-3: When the modulus $l \rightarrow 1$, dark soliton solutions fall out:

$$q(x, t) = A_5 \tanh^{\frac{1}{n}} \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (100)$$

$$r(x, t) = \tilde{A}_5 \tanh^{\frac{1}{n}} \left[B_j \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4n^2\eta_1\tau_1\chi_0\tilde{\tau}_0^3 - 4n^2\xi_1\tau_1\chi_0\tilde{\tau}_0^3 + \tilde{\tau}_1(1+n) [k^2\mu_0\tilde{\tau}_1^2(bv - a) + 4n^2\chi_0\tilde{\tau}_0^2(a\kappa^2 + Q_2)]}{4n^2\chi_0\tilde{\tau}_0^2\tilde{\tau}_1(1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (101)$$

where $\lambda_1 = \lambda_2$.

Case-2: If we take $\sigma = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 2$ in Eq. (74), we assume that Eqs. (72) and (73) have the following formal solutions:

$$U_1 = \tau_0 + \tau_1\Psi + \tau_2\Psi^2, \quad (102)$$

$$U_2 = \tilde{\tau}_0 + \tilde{\tau}_1\Psi + \tilde{\tau}_2\Psi^2, \quad (103)$$

where $\tau_i, \tilde{\tau}_i$ ($i = 0, 1, 2$) are constants to be determined later and Ψ satisfies Eq. (24). Substituting these formal solutions and their necessary derivatives into Eqs. (72) and (73), and solving the resulting system of algebraic

equations one recovers the following solution set:

$$\begin{aligned}
\mu_0 &= -\frac{n^2\tau_2\chi_0\tilde{\tau}_0^2(\eta_1+\xi_1)}{k^2\tilde{\tau}_2^2(1+n)(a-bv)}, & \mu_1 &= -\frac{2n^2\tau_1\chi_0\tilde{\tau}_0(\eta_1+\xi_1)}{k^2\tilde{\tau}_2(1+n)(a-bv)}, \\
\mu_2 &= -\frac{n^2\chi_0(\eta_1+\xi_1)(\tau_1^2\tilde{\tau}_2+2\tau_2^2\tilde{\tau}_0)}{k^2\tau_2\tilde{\tau}_2(1+n)(a-bv)}, & \mu_3 &= -\frac{2n^2\tau_1\chi_0(\eta_1+\xi_1)}{k^2(1+n)(a-bv)}, \\
\mu_4 &= -\frac{n^2\tau_2\chi_0(\eta_1+\xi_1)}{k^2(1+n)(a-bv)}, & \xi_2 &= \frac{\tau_2(\eta_1+\xi_1)-\eta_2\tilde{\tau}_2}{\tilde{\tau}_2}, & \tau_0 &= \frac{\tau_2\tilde{\tau}_0}{\tilde{\tau}_2}, & \tilde{\tau}_1 &= \frac{\tau_1\tilde{\tau}_2}{\tau_2}, & \tau_1 &= \tau_1, \\
\tau_2 &= \tau_2, & \tilde{\tau}_0 &= \tilde{\tau}_0, & \tilde{\tau}_2 &= \tilde{\tau}_2, & Q_1 &= Q_2, & \omega &= \frac{-4\tau_2^2\tilde{\tau}_0(\eta_1+\xi_1)+\tilde{\tau}_2[\tau_1^2(\eta_1+\xi_1)+4\tau_2(1+n)(a\kappa^2+Q_2)]}{4\tau_2\tilde{\tau}_2(1+n)(b\kappa-1)}.
\end{aligned} \tag{104}$$

Substituting these results into Eqs. (24) and (29), we observe that

$$\pm(\xi-\xi_0) = W_2 \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{105}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad W_2 = \sqrt{\frac{\xi_0}{\mu_4}}. \tag{106}$$

Integrating Eq. (105) and taking $\xi_0 = 0$, we obtain the traveling wave solutions to Eqs. (66) and (67) in the forms:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, we obtain solutions as

$$\begin{aligned}
q(x, t) &= \left[\sum_{j=0}^2 \tau_j \left(\lambda_1 \pm \frac{W_2}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)} \right) \right]^{\frac{1}{2n}} \\
&\times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2\tilde{\tau}_0(\eta_1+\xi_1) + \tilde{\tau}_2[\tau_1^2(\eta_1+\xi_1) + 4\tau_2(1+n)(a\kappa^2+Q_2)]}{4\tau_2\tilde{\tau}_2(1+n)(b\kappa-1)} \right) t + \theta \right\} \right], \tag{107}
\end{aligned}$$

$$\begin{aligned}
r(x, t) &= \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_1 \pm \frac{W_2}{k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)} \right) \right]^{\frac{1}{2n}} \\
&\times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2\tilde{\tau}_0(\eta_1+\xi_1) + \tilde{\tau}_2[\tau_1^2(\eta_1+\xi_1) + 4\tau_2(1+n)(a\kappa^2+Q_2)]}{4\tau_2\tilde{\tau}_2(1+n)(b\kappa-1)} \right) t + \theta \right\} \right]. \tag{108}
\end{aligned}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, we get solutions as

$$\begin{aligned}
q(x, t) &= \left[\sum_{j=0}^2 \tau_j \left(\lambda_1 + \frac{4W_2^2(\lambda_2 - \lambda_1)}{4W_2^2 - \left[k(\lambda_1 - \lambda_2) \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]^2} \right) \right]^{\frac{1}{2n}} \\
&\times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2\tilde{\tau}_0(\eta_1+\xi_1) + \tilde{\tau}_2[\tau_1^2(\eta_1+\xi_1) + 4\tau_2(1+n)(a\kappa^2+Q_2)]}{4\tau_2\tilde{\tau}_2(1+n)(b\kappa-1)} \right) t + \theta \right\} \right], \tag{109}
\end{aligned}$$

$$r(x, t) = \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_1 + \frac{4W_2^2(\lambda_2 - \lambda_1)}{4W_2^2 - \left[k(\lambda_1 - \lambda_2) \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]^2} \right) \right]^j^{\frac{1}{2n}} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (110)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, we have solutions as

$$q(x, t) = \left[\sum_{j=0}^2 \tau_j \left(\lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp \left[\frac{k(\lambda_1 - \lambda_2)}{W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] - 1} \right) \right]^j^{\frac{1}{2n}} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (111)$$

$$r(x, t) = \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp \left[\frac{k(\lambda_1 - \lambda_2)}{W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] - 1} \right) \right]^j^{\frac{1}{2n}} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (112)$$

and

$$q(x, t) = \left[\sum_{j=0}^2 \tau_j \left(\lambda_1 + \frac{\lambda_1 - \lambda_2}{\exp \left[\frac{k(\lambda_1 - \lambda_2)}{W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] - 1} \right) \right]^j^{\frac{1}{2n}} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (113)$$

$$r(x, t) = \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_1 + \frac{\lambda_1 - \lambda_2}{\exp \left[\frac{k(\lambda_1 - \lambda_2)}{W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right] - 1} \right) \right]^j^{\frac{1}{2n}} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (114)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, we recover bright soliton solutions as

$$q(x, t) = \left[\sum_{j=0}^2 \tau_j \left(\lambda_1 - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2) \cosh \left[\frac{k\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right) \right]^j \frac{1}{2n} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (115)$$

$$r(x, t) = \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_1 - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2) \cosh \left[\frac{k\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right) \right]^j \frac{1}{2n} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right]. \quad (116)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, we find solutions as

$$q(x, t) = \left[\sum_{j=0}^2 \tau_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \operatorname{sn}^2 \left[\pm \frac{k\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), l \right]} \right) \right]^j \frac{1}{2n} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (117)$$

$$r(x, t) = \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \operatorname{sn}^2 \left[\pm \frac{k\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), l \right]} \right) \right]^j \frac{1}{2n} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (118)$$

where

$$l^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \quad (119)$$

Note that λ_i ($i = 1, \dots, 4$) are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \quad (120)$$

Remark-4: When the modulus $l \rightarrow 1$, the solutions (117) and (118) are reduced to singular solitons given by

$$q(x, t) = \left[\sum_{j=0}^2 \tau_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \tanh^2 \left[\pm \frac{k \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right) \right]^j \frac{1}{2n} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (121)$$

$$r(x, t) = \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \tanh^2 \left[\pm \frac{k \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right) \right]^j \frac{1}{2n} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (122)$$

where $\lambda_3 = \lambda_4$.

Remark-5: However, if $l \rightarrow 0$, the solutions (117) and (118) are reduced to singular periodic solutions:

$$q(x, t) = \left[\sum_{j=0}^2 \tau_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \sin^2 \left[\pm \frac{k \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right) \right]^j \frac{1}{2n} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (123)$$

$$r(x, t) = \left[\sum_{j=0}^2 \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \sin^2 \left[\pm \frac{k \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right]} \right) \right]^j \frac{1}{2n} \times \exp \left[i \left\{ -\kappa x + \left(\frac{-4\tau_2^2 \tilde{\tau}_0 (\eta_1 + \xi_1) + \tilde{\tau}_2 [\tau_1^2 (\eta_1 + \xi_1) + 4\tau_2(1+n)(a\kappa^2 + Q_2)]}{4\tau_2 \tilde{\tau}_2 (1+n)(b\kappa - 1)} \right) t + \theta \right\} \right], \quad (124)$$

where $\lambda_2 = \lambda_3$.

5 LOG-LAW NONLINEARITY

For log law nonlinear media, $F(s) = \ln s$, solitons in magneto-optic waveguides is modeled by [8, 14]

$$iq_t + aq_{xx} + bq_{xt} + (\xi_1 \ln |q|^2 + \eta_1 \ln |r|^2) q = Q_1 r, \quad (125)$$

$$ir_t + ar_{xx} + br_{xt} + (\xi_2 \ln |r|^2 + \eta_2 \ln |q|^2) r = Q_2 q. \quad (126)$$

Thus for the model (125)-(126), the real part equation (17) reduces to

$$k^2(a - bv)P_1'' + (b\omega\kappa - \omega - a\kappa^2)P_1 + 2\xi_1 P_1 \ln P_1 + 2\eta_1 P_1 \ln P_2 - Q_1 P_2 = 0, \quad (127)$$

$$k^2(a - bv)P_2'' + (b\omega\kappa - \omega - a\kappa^2)P_2 + 2\xi_2 P_2 \ln P_2 + 2\eta_2 P_2 \ln P_1 - Q_2 P_1 = 0. \quad (128)$$

To obtain an analytic solution, we use the transformations

$$P_1 = \exp \frac{1}{U_1} = \exp \frac{1}{U_2} = P_2, \quad (129)$$

in Eqs. (127) and (128) to find

$$k^2(a - bv) \{ (U_1')^2 + 2U_1(U_1')^2 - U_1^2 U_1'' \} + 2(\xi_1 + \eta_1)U_1^3 + (b\omega\kappa - \omega - a\kappa^2 - Q_1)U_1^4 = 0, \quad (130)$$

$$k^2(a - bv) \{ (U_2')^2 + 2U_2(U_2')^2 - U_2^2 U_2'' \} + 2(\xi_2 + \eta_2)U_2^3 + (b\omega\kappa - \omega - a\kappa^2 - Q_2)U_2^4 = 0. \quad (131)$$

The extended trial function scheme will be applied to Eqs. (130) and (131) to retrieve bright, dark and singular solitons.

According to the balance principle, we determine a relation of σ , ρ , ς and $\tilde{\varsigma}$ as

$$\varsigma = \tilde{\varsigma} = \sigma - \rho - 2. \quad (132)$$

When $\sigma = 3$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (132), Eqs. (130) and (131) have the solutions in the form

$$U_1 = \tau_0 + \tau_1 \Psi, \quad (133)$$

$$U_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \quad (134)$$

where τ_i , $\tilde{\tau}_i$ ($i = 0, 1$) are constants to be determined later and Ψ satisfies Eq. (24). Substituting these solutions and their necessary derivatives into Eqs. (130) and (131), and solving the resulting system of algebraic equations one recovers the following solution set:

$$\begin{aligned} \mu_0 &= \frac{2\tau_1\chi_0\tilde{\tau}_0^3(Q_2 - Q_1)}{k^2\tilde{\tau}_1^2(a - bv)(\tau_1 - \tilde{\tau}_1)}, & \mu_1 &= \frac{6\tau_1\chi_0\tilde{\tau}_0^2(Q_2 - Q_1)}{k^2\tilde{\tau}_1(a - bv)(\tau_1 - \tilde{\tau}_1)}, \\ \mu_2 &= \frac{6\tau_1\chi_0\tilde{\tau}_0(Q_2 - Q_1)}{k^2(a - bv)(\tau_1 - \tilde{\tau}_1)}, & \mu_3 &= \frac{2\tau_1\chi_0\tilde{\tau}_1(Q_2 - Q_1)}{k^2(a - bv)(\tau_1 - \tilde{\tau}_1)}, \\ \xi_1 &= \frac{-\eta_1\tau_1 + \tilde{\tau}_1(\eta_1 + Q_1 - Q_2)}{\tau_1 - \tilde{\tau}_1}, & \xi_2 &= \frac{-\eta_2(\tau_1 - \tilde{\tau}_1) + \tau_1(Q_1 - Q_2)}{\tau_1 - \tilde{\tau}_1}, \\ \tau_1 &= \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1, \quad \tau_0 = \frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1}, & \omega &= \frac{\tau_1(ak^2 + Q_1) - \tilde{\tau}_1(ak^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}. \end{aligned} \quad (135)$$

Substituting these results into Eqs. (24) and (29), we observe that

$$\pm(\xi - \xi_0) = \sqrt{W_3} \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \quad (136)$$

where

$$\Lambda(\Psi) = \Psi^3 + \frac{\mu_2}{\mu_3}\Psi^2 + \frac{\mu_1}{\mu_3}\Psi + \frac{\mu_0}{\mu_3}, \quad W_3 = \frac{\chi_0}{\mu_3}. \quad (137)$$

Consequently, we obtain the traveling wave solutions to Eqs. (125) and (126) in the forms:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3$, the following Gaussons solutions are obtained:

$$q(x, t) = \exp \left[\tau_0 + \tau_1 \lambda_1 + \frac{4\tau_1 W_3}{\left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right]^2} \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (138)$$

$$r(x, t) = \exp \left[\tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 + \frac{4\tilde{\tau}_1 W_3}{\left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right]^2} \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right]. \quad (139)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, the following solutions are recovered:

$$q(x, t) = \exp \left[\tau_0 + \tau_1 \lambda_2 + \tau_1 (\lambda_1 - \lambda_2) \tanh^2 \left(\frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right] \right) \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (140)$$

$$r(x, t) = \exp \left[\tilde{\tau}_0 + \tilde{\tau}_1 \lambda_2 + \tilde{\tau}_1 (\lambda_1 - \lambda_2) \tanh^2 \left(\frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right] \right) \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right]. \quad (141)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)^2$ and $\lambda_1 > \lambda_2$, the following solutions fall out:

$$q(x, t) = \exp \left[\tau_0 + \tau_1 \lambda_1 + \tau_1 (\lambda_1 - \lambda_2) \operatorname{cosech}^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (142)$$

$$r(x, t) = \exp \left[\tilde{\tau}_0 + \tilde{\tau}_1 \lambda_1 + \tilde{\tau}_1 (\lambda_1 - \lambda_2) \operatorname{cosech}^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right]. \quad (143)$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, the following solutions emerge:

$$q(x, t) = \exp \left[\tau_0 + \tau_1 \lambda_3 + \tau_1 (\lambda_2 - \lambda_3) \operatorname{sn}^2 \left(\mp \frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_3}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right], l \right) \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (144)$$

$$r(x, t) = \exp \left[\tilde{\tau}_0 + \tilde{\tau}_1 \lambda_3 + \tilde{\tau}_1 (\lambda_2 - \lambda_3) \operatorname{sn}^2 \left(\mp \frac{1}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_3}} \left[k \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) - \xi_0 \right], l \right) \right]^{-1} \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (145)$$

where

$$l^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}. \quad (146)$$

We note that λ_i ($i = 1, 2, 3$) are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \quad (147)$$

When $\tau_0 = -\tau_1 \lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1 \lambda_1$ and $\xi_0 = 0$, the solutions given by (138)-(143) are reduced to

$$q(x, t) = \exp \left[\frac{k^2 \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)^2}{4\tau_1 W_3} \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (148)$$

$$r(x, t) = \exp \left[\frac{k^2 \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right)^2}{4\tilde{\tau}_1 W_3} \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (149)$$

$$q(x, t) = \exp \left[\frac{1}{\tau_1 (\lambda_2 - \lambda_1)} \cosh^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (150)$$

$$r(x, t) = \exp \left[\frac{1}{\tilde{\tau}_1 (\lambda_2 - \lambda_1)} \cosh^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (151)$$

and

$$q(x, t) = \exp \left[\frac{1}{\tau_1(\lambda_1 - \lambda_2)} \sinh^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (152)$$

$$r(x, t) = \exp \left[\frac{1}{\tilde{\tau}_1(\lambda_1 - \lambda_2)} \sinh^2 \left(\frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_2}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right]. \quad (153)$$

Moreover, when $\tau_0 = -\tau_1\lambda_3$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_3$ and $\xi_0 = 0$, the solutions given by (144) and (145) are reduced to

$$q(x, t) = \exp \left[\frac{1}{\tau_1(\lambda_2 - \lambda_3)} \operatorname{ns}^2 \left(\mp \frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (154)$$

$$r(x, t) = \exp \left[\frac{1}{\tilde{\tau}_1(\lambda_2 - \lambda_3)} \operatorname{ns}^2 \left(\mp \frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right), \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right]. \quad (155)$$

Remark-6: When the modulus $l \rightarrow 1$, the following exact solutions fall out:

$$q(x, t) = \exp \left[\frac{1}{\tau_1(\lambda_2 - \lambda_3)} \coth^2 \left(\mp \frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (156)$$

$$r(x, t) = \exp \left[\frac{1}{\tilde{\tau}_1(\lambda_2 - \lambda_3)} \coth^2 \left(\mp \frac{k}{2} \sqrt{\frac{\lambda_1 - \lambda_3}{W_3}} \left(x - \left\{ \frac{b\omega - 2a\kappa}{1 - b\kappa} \right\} t \right) \right) \right] \\ \times \exp \left[i \left\{ -\kappa x + \left(\frac{\tau_1 (a\kappa^2 + Q_1) - \tilde{\tau}_1 (a\kappa^2 + Q_2)}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)} \right) t + \theta \right\} \right], \quad (157)$$

where $\lambda_1 = \lambda_2$.

6 CONCLUSIONS

This paper obtained soliton and Gausson solutions to magneto-optic waveguides having three forms of nonlinear structures, namely Kerr law, power law and log-law. These are bright, dark and singular solitons. In addition,

several other forms of solutions naturally emerged from the integration algorithm. They are cnoidal waves, plane waves and several others. All of these solutions are listed with respective constraint conditions for their existence. The soliton solutions to magneto-optic waveguides will be extremely useful in the technology of fiber-optic transmission. The results of this paper thus carry a lot of future prospects from the research perspective. Other laws of nonlinearity will be addressed later. Additionally, perturbation terms will be included to give a rounded picture to information transmission technology. Such results are awaited at this time and the readers are suggested to stay tuned.

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