☆ Implicit-explicit schemes of finite element method for the non-stationary thermal convection problems with temperature-dependent coefficients

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Abstract

For the time-dependent thermal convection problems with temperature-dependent coefficients, the implicit-explicit scheme is presented, in which mixed finite element method is applied for the spatial approximation of the velocity, pressure and temperature while the time discretization is based on the high-order backward difference scheme. Linear terms are dealt with the implicit scheme while the nonlinear terms are treated by the semi-implicit scheme. The advantages for this scheme are unconditionally stable, decoupled computational and second order accuracy. Finally, numerical tests illustrate the theoretical results of the presented schemes, and display that the highly efficient method conserves the property of divergence free of the original problems.

Key words: Non-stationary thermal convection problems, Temperature-dependent coefficients, Implicit-explicit scheme, Finite element method, Decoupled computation

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1. Introduction

The Boussinesq model is one of the most useful models in fluid and geophysical fluid dynamics. This model also constitutes an important system of equations in atmospheric dynamics

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and a dissipative nonlinear system of equations. Since this system does not contain only the velocity and pressure fields but also the temperature field. Note that the convection is a nonlinear transport process in this model. Hence, it is difficult to achieve long time numerical simulations for this model. Thus, development of an efficient computational method for investigating this model has drawn the attentions of many researchers because of many real applications. There are so far numerous works devoted to the development of efficient numerical schemes for the conduction-convection equations with constant coefficients [1–8]. For example, the finite element method, the penalty method, the variational multiscale method, etc.

Recently, a lot of experiments have shown that the variation of viscosity with the temperature is an important factor for details of the flow, for example, the process of glass production and the mantle convection inside. In particular, the viscosity depends strongly on the temperature of the mantle flow. So the nonlinear natural convection model with temperature-dependent coefficients has drawn the attention. From the mathematical point of view, Lorca and Boldrini [14] have first used the spectral Galerkin method and proved the existence of global weak solutions in 3D and local strong solution in 2D to the initial value problem for a generalized Boussinesq model under mild assumptions on the temperature dependency of the viscosity and thermal conductivity and no-slip boundary condition. In 2009, Gunzburger et al. [16] established the well-posedness of the infinite Prandtl number model for convection with temperature-dependent viscosity under the free-slip boundary condition and zero horizontal fluxes. As far as we know, there are few researches on the fully discrete methods for the nonstationary thermal convection problems with variable coefficients, since the convection term is a nonlinear transport process in our model. Based on the energy method, Tabata and Tagami [15] in 2005 derived the error estimates of finite element methods for this model with temperature-dependent coefficients by using the first-order backward Euler method in time and the conforming finite elements in space without numerical tests. They didn't consider the high-order time discretization schemes and the influence of the Prandtl number and the Rayleigh number on the solution of the nonlinear natural convection model.

In general, there exist three classical schemes to deal with the time-dependent problems, which are the fully implicit, implicit-explicit, and explicit schemes. Among them, high-order

schemes are of more interest because the first-order schemes are not sufficiently accurate for long time approximations. Meanwhile, the stability condition of schemes is also a key issue. Usually an explicit scheme is easy in computation, but it suffers a severely restricted time step size from stability requirement. Although the fully implicit schemes are (almost) unconditionally stable, one has to solve a system of nonlinear equations at each time step. Hence, we often use the implicit-explicit schemes to solve the nonlinear problems. Namely, we adopt an implicit scheme for the linear terms and an implicit-explicit scheme for the nonlinear terms. Moreover, the well-known high-order backward difference formula is one of this approach. Recently, Su et al. [5] proposed a second-order accuracy scheme based on Crank-Nicolson extrapolation for the two-dimensional time-dependent conduction-convection equations with constant coefficients, which is almost unconditionally stable. However the stability of this scheme is not good enough compared with other second-order implicit-explicit schemes for long time numerical simulations.

In this paper, we are concerned with the implicit-explicit decoupled finite element scheme for the nonstationary conduction-convection equations with temperature-dependent coefficients. Based on the high-order backward difference scheme, the first-order and second-order implicit-explicit fully discretized schemes are proposed respectively, in which we use an implicit scheme for the linear terms and the semi-implicit schemes for the nonlinear terms. Mixed finite element method (e.g. P_2 - P_1 - P_2 , P_1b - P_1 - P_1 , etc.) is applied for the spatial approximation of the velocity, pressure and temperature. Then the numerical results illustrate the high efficiency of the proposed schemes. Note that these schemes guarantee the property of solenoidal vector field for the original problem in a way.

The remainder of this paper is organized as follows. In Section 2, we introduce the notations, an abstract functional setting of the time-dependent thermal convection problems with temperature dependent coefficients. Mixed finite element strategy is recalled and some well-known results used throughout this paper in Section 3. Fully discrete method based on the implicit-explicit decoupled scheme is given in Section 4. In Section 5, numerical experiments are given to verify the theoretical results completely. Finally, we end with a short conclusion in Section 6.

2. Preliminaries

Let Ω be a bounded, convex and open subset of $\mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz continuous boundary $\partial\Omega$. We consider the following time-dependent thermal convection problems with temperature-dependent coefficients

$$\begin{cases} Pr^{-1}(u_t + (u \cdot \nabla)u) - \nabla \cdot (\nu(\theta)D(u)) + \nabla p - Ra\beta(\theta)\theta = f, & \text{in } \Omega \times (0, T], \\ \nabla \cdot u = 0, & \text{in } \Omega \times (0, T], \\ \theta_t - \nabla \cdot (\kappa(\theta)\nabla\theta) + u \cdot \nabla\theta = g, & \text{in } \Omega \times (0, T], \\ u(x, 0) = u_0, \quad \theta(x, 0) = \theta_0, & \text{on } \Omega \times \{0\}, \\ u = u_D, \quad \theta = \theta_D & \text{in } \partial\Omega \times (0, T], \end{cases} \quad (1)$$

where $u \in \mathbb{R}^d$ represents the velocity vector, $p = p(x, t)$ the pressure, $\theta = \theta(x, t)$ the temperature, Pr the Prandtl number, which is the ratio of the representative value of the kinematic viscosity to the representative value of the thermal diffusivity, and Ra the Rayleigh number, which denotes the ratio of relative heating to the overall dissipation. $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized deformation tensor. $\nu, \kappa \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^d$ denote the generalized viscosity, thermal conductivity, and thermal expansion coefficients depending on the temperature respectively. Such dependence could be of great importance in cases of the large temperature in some certain applications. (u_0, θ_0) and (u_D, θ_D) denote the initial value and boundary value respectively. f and g denote the external force and heat source respectively. T is the given final time and $u_t = \partial u / \partial t$, $\theta_t = \partial \theta / \partial t$. Furthermore, assume the given functions $\nu, \kappa \in C(\overline{\Omega} \times [0, T])$ and $\beta \in C(\overline{\Omega} \times [0, T])$.

With the standard Sobolev spaces

$$X = (H^1(\Omega))^d, \quad X_0 = (H_0^1(\Omega))^d, \quad W = H^1(\Omega), \quad W_0 = H_0^1(\Omega),$$

$$M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q dx = 0\},$$

a weak formulation of (1) reads: find $(u, p, \theta) \in (X, M, W)$ for all $t \in (0, T]$ such that for all $(v, q, s) \in (X_0, M, W_0)$ and $(u, \theta)|_{\partial\Omega} = (u_D, \theta_D)$,

$$\begin{cases} Pr^{-1}((u_t, v) + b(u; u, v)) + B_\nu((u, p); (v, q)) - a_\beta(\theta, v) = (f, v), \\ (\theta_t, s) + a_\kappa(\theta, s) + \bar{b}(u; \theta, s) = (g, s), \\ u(x, 0) = u_0, \quad \theta(x, 0) = \theta_0, \end{cases} \quad (2)$$

with

$$\begin{aligned}
a_v(u, v) &= (\nu(\theta)\nabla u, \nabla v), \quad d(v, q) = (q, \operatorname{div} v), \quad a_\kappa(\theta, s) = (\kappa(\theta)\nabla \theta, \nabla s), \\
b(u; v, w) &= ((u \cdot \nabla)v, w) + \frac{1}{2}(w(\operatorname{div} u), v) = \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v), \\
\bar{b}(u; \theta, s) &= ((u \cdot \nabla)\theta, s) + \frac{1}{2}(\theta(\operatorname{div} u), s) = \frac{1}{2}((u \cdot \nabla)\theta, s) - \frac{1}{2}((u \cdot \nabla)s, \theta), \\
B_v((u, p); (v, q)) &= a_v(u, v) - d(v, p) + d(u, q), \quad a_\beta(\theta, v) = (Ra\beta(\theta)\theta, v).
\end{aligned}$$

The trilinear forms $b(\cdot; \cdot, \cdot)$ and $\bar{b}(\cdot; \cdot, \cdot)$ satisfy

$$\begin{aligned}
|b(u; v, w)| &\leq N \|\nabla u\|_0 \|\nabla v\|_0 \|\nabla w\|_0, \quad \forall u, v, w \in X, \\
|\bar{b}(u; \theta, s)| &\leq \bar{N} \|\nabla u\|_0 \|\nabla \theta\|_0 \|\nabla s\|_0, \quad \forall (u, \theta, s) \in (X, W, W),
\end{aligned} \tag{3}$$

where

$$N = \sup_{u, v, w \in X} \frac{|b(u; v, w)|}{\|\nabla u\|_0 \|\nabla v\|_0 \|\nabla w\|_0}, \quad \bar{N} = \sup_{u \in X, \theta, s \in W} \frac{|\bar{b}(u; \theta, s)|}{\|\nabla u\|_0 \|\nabla \theta\|_0 \|\nabla s\|_0}.$$

Using the similar techniques in [14–16], if the spaces X and M satisfy the continuous inf-sup condition, the existence and uniqueness of solution (u, p, θ) for problem (2) can be obtained under some reasonable propositions on the given functions f, g, ν, κ and β , the initial boundary conditions (u_0, θ_0) and (u_D, θ_D) , and the smoothness of boundary. And the maximum principle of this problem can be also obtained. Interested readers can refer to [14, 15] for details.

3. Mixed finite element method

Now we introduce the standard finite-dimensional subspaces $(X_h, M_h, W_h) \subset (X, M, W)$ which are characterized by K_h , a partitioning of Ω into triangles K with the mesh size $h \in (0, 1)$, assumed to be uniformly regular in the usual sense. Set $X_{0h} = X_h \cap X_0$ and $W_{0h} = W_h \cap W_0$. For further details, readers can refer to [12].

The standard Galerkin finite element approximation of (2) based on (X_h, M_h, W_h) reads as follows: find $(u_h, p_h, \theta_h) \in (X_h, M_h, W_h)$ such that, for all $0 \leq t \leq T$, $(u_h, \theta_h)|_{\partial\Omega} = (u_{Dh}, \theta_{Dh})$ (u_{Dh} and θ_{Dh} are the interpolation of u_D and θ_D respectively) and $(v_h, q_h, s_h) \in (X_{0h}, M_h, W_{0h})$,

$$\begin{cases} Pr^{-1}((u_{ht}, v_h) + b(u_h; u_h, v_h)) + B_{v_h}((u_h, p_h); (v_h, q_h)) - a_{\beta_h}(\theta_h, v_h) = (f, v_h), \\ (\theta_{ht}, s_h) + a_{\kappa_h}(\theta_h, s_h) + \bar{b}(u_h; \theta_h, s_h) = (g, s_h), \\ u_h(x, 0) = u_{0h}, \quad \theta_h(x, 0) = \theta_{0h}. \end{cases} \tag{4}$$

Here, $a_{v_h}(u, v) = (v(\theta_h)\nabla u, \nabla v)$, $a_{\kappa_h}(\theta, s) = (\kappa(\theta_h)\nabla\theta, \nabla s)$, $a_{\beta_h}(\theta, v) = (Ra\beta(\theta_h)\theta, v)$, $(u_{0h}, v_h) = (u_0, v_h)$ and $(\theta_{0h}, s_h) = (\theta_0, s_h)$. Note that we can also use the interpolation finite element method to deal with the nonlinear term $v(\theta_h)$, $\kappa(\theta_h)$ and $\beta(\theta_h)$ respectively. For further details, readers can refer to [18].

Then, we define the subspace V_h of X_h given by

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0, \forall q_h \in M_h\}.$$

We know from [3] that the pair (X_h, M_h) and V_h satisfy the following approximation properties:

Lemma 3.1. *Let $I_h : (L^2(\Omega))^d \rightarrow V_h$ be the standard L^2 -projection. Then*

$$\|v - I_h v\|_{0,\Omega} + h\|\nabla(v - I_h v)\|_{0,\Omega} \leq ch^i \|v\|_{i,\Omega}, \quad \forall v \in (H^i(\Omega))^d \cap V_0, \quad (5)$$

for $i = 1, 2, 3$, with $V_0 = \{v \in (H_0^i(\Omega))^d; \nabla \cdot v = 0\}$.

Of course, we will make use of the approximate divergence-free finite element space V_{0h} :

$$V_{0h} = \{v_h \in V_h; (\nabla \cdot v_h, q_h) = 0, \forall q_h \in M_h\}.$$

From the results of [12], the pair (X_h, M_h, W_h) and V_{0h} satisfy the following properties:

P1. For each $v \in (H^i(\Omega))^d \cap X$ with $\nabla \cdot v = 0$ and $q \in H^{i-1}(\Omega) \cap L_0^2(\Omega)$ with $i = 1, 2, 3$, there exist approximations $\pi_h v \in V_{0h}$ and $\rho_h q \in M_h$ such that

$$\|\nabla(v - \pi_h v)\|_0 \leq ch^{i-1} \|v\|_i, \quad \|q - \rho_h q\|_0 \leq ch^{i-1} \|q\|_{i-1}.$$

P2. There exists $r_h : W \rightarrow W_h$, such that for all φ

$$\begin{aligned} (\nabla(\varphi - r_h \varphi), \nabla \varphi_h) &= 0, \quad \forall \varphi_h \in W_h, \\ \int_{\Omega} (\varphi - r_h \varphi) dx &= 0, \quad \|\nabla r_h \varphi\|_0 \leq \|\nabla \varphi\|_0, \end{aligned}$$

and that when $\varphi \in W^{r,q}(\Omega)$ ($1 \leq q \leq \infty$), we have

$$\|\varphi - r_h \varphi\|_{-s,q} \leq ch^{r+s} |\varphi|_{r,q}, \quad -1 \leq s \leq m, 0 \leq r \leq m+1. \quad (6)$$

P3. The inverse inequality:

$$\|\nabla v_h\|_0 \leq ch^{-1} \|v_h\|_0, \quad \forall v_h \in X_h.$$

P4. For each $q_h \in X_h$, there exists $v_h \in X_h$, $v_h \neq 0$ such that

$$d(v_h, q_h) \geq \beta \|q_h\|_0 \|\nabla v_h\|_0,$$

here $\beta > 0$ is a constant depending only on Ω . This means that the spaces X_h and M_h are supposed to satisfy the discrete inf-sup condition. In the following, we consider the Taylor-Hood element and the Mini element to approximate the velocity and pressure.

4. Fully discrete method based on the implicit-explicit scheme

In this section we first consider the time discretization of the finite element approximation (4) based on the backward Euler scheme. Let N and M be two positive integers. Set $\tau = T/N$ be the mesh width in time and $t_n = n\tau$ for $n = 1, 2, \dots, M$. The backward Euler scheme applied to the spatially discrete problem of (4) consists of determining functions $\{u_h^n\}_{n \geq 1} \subset X_h$, $\{p_h^n\}_{n \geq 1} \subset M_h$ and $\{\theta_h^n\}_{n \geq 1} \subset W_h$ as functions of the recursive equations:

$$\begin{cases} Pr^{-1}((\partial_\tau u_h^n, v_h) + b(u_h^n; u_h^n, v_h)) + B_{v_h^n}((u_h, p_h); (v_h, q_h)) - a_{\beta_h^n}(\theta_h^n, v_h) = (f^n, v_h), \\ (\partial_\tau \theta_h^n, s_h) + a_{\kappa_h^n}(\theta_h^n, s_h) + \bar{b}(u_h^n; \theta_h^n, s_h) = (g^n, s_h), \\ u_h^0 = u_{0h}, \quad \theta_h^0 = \theta_{0h}. \end{cases} \quad (7)$$

for all $(v_h, q_h, s_h) \in (X_{0h}, M_h, W_{0h})$ and $(u_h^0, \theta_h^0)|_{\partial\Omega} = (u_{Dh}, \theta_{Dh})$. Here, $\partial_\tau u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau}$, $a_{v_h^n}(u, v) = (v(\theta_h^n) \nabla u, \nabla v)$, $a_{\kappa_h^n}(\theta, s) = (\kappa(\theta_h^n) \nabla \theta, \nabla s)$ and $a_{\beta_h^n}(\theta, v) = (Ra\beta(\theta_h^n)\theta, v)$. Note that the above nonlinear scheme is fully implicit and coupled, so we need use the different iterative method to solve this system.

4.1. The first-order implicit-explicit scheme

In this subsection, we use the linearization technique to present the first-order implicit-explicit scheme, for all $n \geq 1$, find $(u_h^n, p_h^n, \theta_h^n) \in (X_h, M_h, W_h)$ such that $(v_h, q_h, s_h) \in (X_{0h}, M_h, W_{0h})$,

$$\begin{cases} Pr^{-1}((\partial_\tau u_h^n, v_h) + b(u_h^{n-1}; u_h^n, v_h)) + B_{v_h^{n-1}}((u_h, p_h); (v_h, q_h)) - a_{\beta_h^{n-1}}(\theta_h^n, v_h) = (f^n, v_h), \\ (\partial_\tau \theta_h^n, s_h) + a_{\kappa_h^{n-1}}(\theta_h^n, s_h) + \bar{b}(u_h^{n-1}; \theta_h^n, s_h) = (g^n, s_h), \\ u_h^0 = u_{0h}, \quad \theta_h^0 = \theta_{0h}. \end{cases} \quad (8)$$

The scheme of (8) consists of two subproblems: one is a linearized convection-diffusion equation and the other is a linearized Navier-Stokes equations. First we can solve the second equation of (8) and obtain θ_h^n , and then solve the first equation of (8) to obtain (u_h^n, p_h^n) .

4.2. The second-order implicit-explicit scheme

In this subsection, we use the second-order backward difference formula (BDF) to march in time. Let

$$u^* = 2u_h^n - u_h^{n-1}, \quad \theta^* = 2\theta_h^n - \theta_h^{n-1}, \quad \beta^* = 2\beta_h^n - \beta_h^{n-1},$$

and define

$$a_{v^*}(u, v) = (v(u^*)\nabla u, \nabla v), \quad a_{\kappa^*}(\theta, s) = (\kappa(\theta^*)\nabla \theta, \nabla s), \quad a_{\beta^*}(\theta, v) = (Ra\beta(\theta^*)\theta, v),$$

we get the following second-order implicit-explicit scheme, for all $n \geq 2$, find $(u_h^n, p_h^n, \theta_h^n) \in (X_h, M_h, W_h)$ such that $(v_h, q_h, s_h) \in (X_{0h}, M_h, W_{0h})$,

$$\begin{cases} Pr^{-1}\left(\left(\frac{3u_h^{n+1}-4u_h^n+u_h^{n-1}}{2\tau}, v_h\right) + b(u^*; u_h^{n+1}, v_h)\right) + B_{v^*}((u_h^{n+1}, p_h^{n+1}); (v_h, q_h)) \\ \quad - a_{\beta^*}(\theta_h^{n+1}, v_h) = (f^{n+1}, v_h), \\ \left(\frac{3\theta_h^{n+1}-4\theta_h^n+\theta_h^{n-1}}{2\tau}, s_h\right) + a_{\kappa^*}(\theta_h^{n+1}, s_h) + \bar{b}(u^*; \theta_h^{n+1}, s_h) = (g^{n+1}, s_h), \end{cases} \quad (9)$$

In the first two step processes, we use the first-order implicit-explicit scheme to compute $(u_h^1, p_h^1, \theta_h^1)$ and $(u_h^0, p_h^0, \theta_h^0)$.

And more importantly, the optimal convergence rate will be achieved without imposing any restriction on the time step size τ , under the following reasonable assumptions on the regularities of the exact solution (u, p, θ) :

$$(A) \begin{cases} u \in L^\infty(0, T; (H^2(\Omega))^d) \cap L^2(0, T; (H^3(\Omega))^d), & p \in L^2(0, T; H^2(\Omega)), \\ u_t \in L^\infty(0, T; (H^1(\Omega))^d) \cap L^2(0, T; (H^2(\Omega))^d), & u_{tt} \in L^2(0, T; (H^1(\Omega))^d), \\ \theta \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \\ \theta_t \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), & \theta_{tt} \in L^2(0, T; H^1(\Omega)). \end{cases}$$

Next we state the main results of this paper for above two schemes.

Theorem 4.1. *If assumption (A) holds, and (u, p, θ) and (u_h, p_h, θ_h) are the solutions of problem (2) and (8), respectively. Then there hold that*

$$\max_{1 \leq n \leq M} \|u_h^n - u\|_0^2 + \tau \sum_{n=1}^M (\|\nabla(u_h^n - u)\|_0^2 + \|p_h^n - p\|_0^2) \leq c(\tau^2 + h^4),$$

and

$$\max_{1 \leq n \leq M} \|\theta_h^n - \theta\|_0^2 + \tau \sum_{n=1}^M \|\nabla(\theta_h^n - \theta)\|_0^2 \leq c(\tau^2 + h^4).$$

This result shows that the optimal error estimates of the finite element solution $(u_h^n, p_h^n, \theta_h^n)$ to the system of (2), first-order accurate in time and second order accurate in space.

Theorem 4.2. *If assumption (A) holds, (u, p, θ) and (u_h, p_h, θ_h) are the solutions of problems (2) and (9), respectively. Then there hold that*

$$\max_{1 \leq n \leq M} \|u_h^n - u\|_0^2 + \tau \sum_{n=1}^M (\|\nabla(u_h^n - u)\|_0^2 + \|p_h^n - p\|_0^2) \leq c(\tau^4 + h^4),$$

and

$$\max_{1 \leq n \leq M} \|\theta_h^n - \theta\|_0^2 + \tau \sum_{n=1}^M \|\nabla(\theta_h^n - \theta)\|_0^2 \leq c(\tau^4 + h^4).$$

This result shows that the optimal error estimates of the finite element solution $(u_h^n, p_h^n, \theta_h^n)$ to the system of (2), second order accurate both in space and time. For the discrete problem, error estimates of the velocity and the temperature are established without any stability conditions. They are optimal in the sense that they have the same orders as the interpolation errors of the finite elements. Moreover, the optimal estimate of the pressure can be also obtained. The derivation of the error estimates is based on the energy method, which is an extension of that in the nonstationary Navier-Stokes equations [9, 11, 13]. Here, we omit these proof for simplifying the presentation.

5. Numerical experiments

In this section, three numerical experiments are given to illustrate the performances of the first order implicit-explicit scheme of (8) and the second order implicit-explicit scheme of (9) for the unsteady thermal convection problem with temperature-dependent coefficients. The first example is a flow problem with the smooth polynomial solutions. The second example is a physical benchmark case, and the third one is a nonconvex domain modelling problem. The software FreeFEM++, developed by Hecht et al. [17], is used in our numerical experiments and the linear solver UMFPACK is adopted in our program. In this section, the stable P_2 - P_1 - P_2 finite element pair is used to approximate the velocity, pressure and temperature. In order to simplify the computation, we choose the temperature-dependent coefficients $\nu(\theta) = (1 + \theta)^{-2}$, $\kappa(\theta) = 1 + \theta^2$ and $\beta(\theta) = [\theta, 1 - \theta]$ in the thermal convection problem of (1). Other coefficients can also be adopted and similar results can be obtained.

5.1. Analytical solution

In this test, we set the parameters $Pr = Ra = 1$, the final time $T = 1$, and the functions f and g given by the following exact solution

$$\begin{aligned} p(x, y, t) &= 10(2x - 1)(2y - 1) \cos(t), \\ u_1(x, y, t) &= 10x^2(x - 1)^2y(y - 1)(2y - 1) \cos(t), \\ u_2(x, y, t) &= -10x(x - 1)(2x - 1)y^2(y - 1)^2 \cos(t), \\ T(x, y, t) &= 10(x^2(x - 1)^2y(y - 1)(2y - 1) - x(x - 1)(2x - 1)y^2(y - 1)^2) \cos(t). \end{aligned}$$

We investigate the convergence rate of the first order backward Euler scheme and second order BDF scheme for problem (1) with the time step and mesh size satisfy $\tau = h^2$ and $\tau = h$, respectively. The numerical results of two different numerical schemes are shown in Tables 1-2, in which

$$K_{div} = \max_{K \in K_h(\Omega)} \left| \int_K \operatorname{div} u_h dx \right|.$$

The experimental rates of convergence with respect to the mesh size h are calculated by the formula $\frac{\log(E_i/E_{i-1})}{\log(h_i/h_{i-1})}$, where E_i and E_{i-1} are the relative errors corresponding to the mesh sizes h_i and h_{i-1} , respectively. From the numerical results in Tables 1-2, we can see that there is no much difference for the errors of different variables in different norms of two kinds of numerical methods, both two numerical schemes remain the same property (i.e., $\operatorname{div} u = 0$) as the original equations. However, the BDF scheme takes much less CPU time than the backward Euler scheme, which is not surprising since the temporal discretization of BDF scheme is of two order in time step.

5.2. Thermal lid-driven modeling

Next, we take the thermal lid-driven flow which has no exact solution for testing the efficiency of the second order implicit-explicit scheme for problem (1), this modelling is a standard benchmark problem. In this case, the boundary conditions are $\theta = 0$ on part of the left boundary, $\theta = 5(y - 0.5)(0.9 - y)$ on part of the right boundary and $\frac{\partial \theta}{\partial n} = 0$ on the rest boundaries, zero Dirichlet conditions on velocity are imposed, see Figure 1 for detail. The unstructured

triangulation of Ω into triangles is used with mesh size $h = \frac{1}{40}$. In this example, we just confirm the performance of backward difference formula.

Firstly, we present the isobars, the isotherms and the streamlines of BDF scheme with different $Pr = 1, 10$ and 100 at the same final time $T = 0.1$ and $Ra = 1$, see Figure 2 for detail. From Figure 2, we can see that there are no difference for the isotherms (both in patterns and values), which means that the varies of Pr have small influence for the isotherms. On the other hand, some differences appear in the isobars and streamlines. The patterns of isobars vary, and the circle of the streamlines goes down as Pr increases. Secondly, we provide the isobars, the isotherms and the streamlines at $T = 1$ and $T = 10$ in Figures 3-4, respectively. As pointed out before, the variation of Pr has small influence for the isotherms, while there exist some isolines of the pressure near the corners of the left side, and the streamlines become a circle as Pr increases. Finally, from Figures 1-3 we see that the parallel behaviour of pressure isolines is distorted in the central part of the region as the increase of Prandtl numbers, the temperature isolines cover the whole domain and the highest isoline appears in the upper and right corner and the lowest isoline lies in the lower and left corner. These phenomenons verify our the physics phenomenon very well.

Next, we compare the numerical isobars, the isotherms and the streamlines of backward difference formula with different Ra at the same final time and $Pr = 1$, see Figures 5-7 for detail. From these figures, we can see that there are no difference for the isotherms at same final time, but the pressure isolines are different, there about 10 times between two successive isobars with different Ra (the values of isobars in two successive patterns are 10 time as the values of Ra are also 10 times). The differences between the streamlines with different Ra at same final time are not big. As have point out in above, the temperature isolines also confirm the generally physic phenomenon very well.

5.3. *The isolated island*

Finally, we test the efficiency of the BDF scheme for the time-dependent thermal convection problem with temperature-dependent coefficients in a nonconvex domain. The geometrical information, the boundary conditions and initial data are presented in Figure 8 (a), nonconform

triangles partition is used and adaptive mesh is also used in the process of the computation, see Figure 8 (b)-(c).

We present the numerical results of this benchmark problem in Figure 9 at different final time $T = 0.1, 1$ and 10 with Rayleigh number $Ra = 1$ and Prandtl number $Pr = 1$. Due to the nonlinear term, the fluid flows from right to left, and a cycle appears in the corner of left. As time increases, the cycle becomes small. On other hand, another streamline circle goes to the middle of the nonconvex domain as time increases. Furthermore, the streamline circles go upper to the center of the domain. The pressure isolines also vary as time increases. The isotherms have an obviously variations. All the temperature isolines are symmetric about the center of domain at different times. The isotherms arrive at the upper as time increase and the isolines are parallel at the top of the domain.

6. Conclusions

In this work, we present the backward Euler formula and backward difference formula to treat the time-dependent thermal convection problem with temperature-dependent coefficients. The implicit scheme is used to deal with the linear terms and the semi-implicit scheme to the nonlinear terms. Following the classical mixed finite element theoretical results, the optimal error estimates are established. From the view of point of numerical tests, we find that two kinds of numerical schemes have excellent stability and convergence. Moreover, two numerical schemes conserve the divergence free of the original problem. Furthermore, the numerical results confirm the comment knowledge very well.

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Table 1: The errors and convergence orders of the backward Euler scheme with $\tau = h^2$.

$1/h$	$\frac{\ u-u_h\ _0}{\ u\ _0}$	Rate	$\frac{\ \nabla(u-u_h)\ _0}{\ \nabla u\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate
4	4.0355e-2	-	1.6622e-1	-	4.8575e-2	-
8	5.0011e-3	3.01	4.4626e-2	1.90	1.2110e-2	2.00
16	6.2468e-4	3.00	1.1422e-2	1.97	3.0260e-3	2.00
32	8.2344e-5	2.92	2.8753e-3	1.99	7.5645e-4	2.00
64	1.2451e-5	2.73	7.2018e-4	2.00	1.8911e-4	2.00
$1/h$	$\frac{\ \nabla(T-T_h)\ _0}{\ \nabla T\ _0}$	Rate	$\frac{\ T-T_h\ _0}{\ T\ _0}$	Rate	K_{div}	CPU(s)
4	9.6251e-2	-	2.3659e-2	-	1.6970e-4	2.15
8	2.6603e-2	1.86	3.0886e-3	2.94	1.0519e-5	20.86
16	6.8263e-3	1.96	3.8696e-4	3.00	4.3210e-7	300.09
32	1.7184e-3	1.99	4.9262e-5	2.97	1.5548e-8	4611.44
64	4.3037e-4	2.00	6.6327e-6	2.89	5.7268e-10	> 20 hours

Table 2: The errors and convergence orders of the BDF scheme with $\tau = h$.

$1/h$	$\frac{\ u-u_h\ _0}{\ u\ _0}$	Rate	$\frac{\ \nabla(u-u_h)\ _0}{\ \nabla u\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate
4	3.9883e-2	-	1.6619e-1	-	4.8575e-2	-
8	4.9108e-3	3.02	4.4620e-2	1.90	1.2110e-2	2.00
16	6.0374e-4	3.02	1.1420e-2	1.97	3.0260e-3	2.00
32	7.5368e-5	3.00	2.8749e-3	1.99	7.5645e-4	2.00
64	9.7143e-6	2.96	7.2010e-4	2.00	1.8911e-4	2.00
$1/h$	$\frac{\ \nabla(T-T_h)\ _0}{\ \nabla T\ _0}$	Rate	$\frac{\ T-T_h\ _0}{\ T\ _0}$	Rate	K_{div}	CPU(s)
4	9.6248e-2	-	2.3641e-2	-	1.6901e-4	1.22
8	2.6602e-2	1.86	3.0868e-3	2.94	1.0512e-5	6.02
16	6.8262e-3	1.96	3.8499e-4	3.00	4.3205e-7	42.82
32	1.7184e-3	1.99	4.8073e-5	3.00	1.5547e-8	334.25
64	4.3036e-4	2.00	6.0340e-6	2.99	5.7269e-10	2729.08

Figure 1: The boundary conditions and initial data of example 2.

Figure 2: Numerical isobars, the isotherms and the streamlines of BDF scheme with $Ra = 1$ at $T = 0.1$. The first line with $Pr = 1$, the second line with $Pr = 10$, the third line with $Pr = 100$.

Figure 3: Numerical isobars, the isotherms and the streamlines of BDF scheme with $Ra = 1$ at $T = 1$. The first line with $Pr = 1$, the second line with $Pr = 10$, the third line with $Pr = 100$.

Figure 4: Numerical isobars, the isotherms and the streamlines of BDF scheme with $Ra = 1$ at $T = 10$. The first line with $Pr = 1$, the second line with $Pr = 10$, the third line with $Pr = 100$.

Figure 5: Numerical isobars, the isotherms and the streamlines of BDF scheme with $Pr = 1$ at $T = 0.1$. The first line with $Ra = 10$, the second line with $Ra = 100$.

Figure 6: Numerical isobars, the isotherms and the streamlines of BDF scheme with $Pr = 1$ at $T = 1$. The first line $Ra = 10$, the second line with $Ra = 100$.

Figure 7: Numerical isobars, the isotherms and the streamlines of BDF scheme with $Pr = 1$ at $T = 10$. The first line with $Ra = 10$, the second line with $Ra = 100$.

(a)

(b)

(c)

Figure 8: (a) The boundary conditions and initial data of example 2, (b) Initial mesh, (3) Adaptive mesh.

Figure 9: Numerical isobars, the isotherms and the streamlines of BDF scheme with $Pr = 1$, $Ra = 1$. The first column at $T = 0.1$, the second column at $T = 1$, the third column at $T = 10$.