Surface waves and their influence on cone cracking in brittle materials

Jason McDonald and Sikhanda Satapathy

US Army Research Laboratory, RDRL-WMP-B, Aberdeen Proving Ground, MD 21005, United States

Corresponding Author:

Jason McDonald

Phone: 00-1-410-278-3707

Fax: 00-1-410-278-2460

Email: jason.r.mcdonald27.civ@mail.mil

Abstract:

Prior experimental studies have shown substantial improvement in the ballistic performance of ceramics due to surface treatment with thin coatings; however, no correlation with material properties and bonding conditions has been found and no mechanism has yet been definitively established. It is expected that the high strain rates present in a ballistic event will produce surface waves and it is plausible that the interaction of these waves with surface flaws may contribute significantly to initial crack growth and the onset of failure in brittle materials. In the present work a Laplace-transform-based computational framework is implemented. This technique can be considered complementary to other computational schemes such as finite element and finite difference techniques for the study of elastic waves in solids. Extension of this framework to multi-layer systems is possible and may provide insight into whether or not the observed surface-treatment effect in ceramics can be attributed to wave modification.

Keywords:

Ceramics

Cracking

Surface waves

1. Introduction

Surface defects and flaws likely play an important role in the impact response of brittle materials, such as ceramics and glass. High speed projectile impact induces a complex evolving stress state in the target. It is conceivable that the interaction of surface flaws with the evolving stress fields can contribute significantly to crack initiation and the onset of failure. Prior experimental studies [1-3] have shown substantial benefit of surface treatments on the ballistic efficiency of ceramics. The correlation between the degree of ballistic improvement and various properties of the layer, including degree of bonding with the substrate, has not definitively been established. Initial attempts using finite element modeling in conjunction with accepted material models for ceramics and alloys has proven unsuccessful in reproducing the experimental results [4]. One possibility for this lack of success could be that surface waves of combined dilatational and vertical shearing motion, often referred to as Rayleigh-type waves, are contributing significantly to the onset of cracking and failure in ceramics. Waves of this type have been studied extensively by seismologists over the years in both single half space and layered systems [5-7]. Some notable findings include: the range of existence of such modes (as a function of material properties) in two-layered systems [8-10], and the dependence of their amplitude on material properties and layer thickness [11, 12]. Extension of these methods to problems of interest in ballistics and impact, using modern computational techniques, may shed light on surface-treatment effects in ceramics.

One drawback in the use of finite element and finite difference methods to model wave propagation are the demands on spatial and temporal discretizations to avoid unphysical numerical effects. It has been well established empirically that (in the case of monochromatic waves) one needs a minimum 10 cells per wavelength (assuming linear interpolation) to prevent

artifacts such as attenuation, anisotropy, dispersion, and polarization from emerging from numerical errors [13]. Inhomogeneous meshes (employing either changes in element size or interpolation order) can also lead to spurious oscillations [13]. As an example, a typical glass with Rayleigh-wave speed ~ 3000 m/s subjected to a loading rate of 10 MHz would require a minimum spatial discretization of 0.03 mm, possibly higher to achieve desired accuracy. It is conceivable that ballistic impact events could produce an initial elastic loading phase of quite high rate prior to the onset of inelastic response since one expects the loading rate to increase with impact velocity [14]. Such high-rate loading may require a spatial discretization of prohibitive computational expense and lead to very small time steps and large accumulated errors in explicit methods. The Laplace-transform-based scheme implemented in the present work does not suffer from this particular drawback since it does not utilize either spatial or temporal discretization: the field computation at any space-time point is independent of the computation at any other point. This property also makes the technique highly parallelizable. The method does, however, have its own drawbacks such as the complication in treating loading conditions in which the spatial and temporal dependences are not readily factorable [14].

The cone-crack system is one of several failure mechanisms (including radial cracks, axial splitting, and comminuted zones) that are observed under dynamic indentation and ballistic loading [15, 16]. Although the details and relative importance of the various failure processes as a function of loading conditions and geometry are not yet well understood, the authors are currently of the belief that reducing the magnitude of tensile waves, near the surface of a brittle solid, may reduce or delay the development of at least some of these failure mechanisms, thereby improving ballistic performance. The goal of the present study is to investigate the wave-flaw interaction near the surface of a brittle solid leading to cone cracks. The reason for focusing on

the cone crack system in this initial study is that a simple model was previously presented by Lawn et al [17] for the growth of cone cracks under quasi-static fields. To this end, we have implemented a computational scheme to study the stress wave interaction with surface flaws. Use of the Laplace transform enables analytical expressions for the various quantities of interest to be obtained in the transformed space. Numerical inversion of these closed-form expressions provides stress and displacement distributions as functions of space and time. The method was first compared to the known analytical solutions for the surface displacement due to the idealized case of a suddenly-applied point load. It was next applied to the problem of cone crack extension from an existing ring-shaped flaw under dynamic frictionless loading of an elastic half space by a rigid cylindrical punch. The computed cone cracks under dynamic loading conditions showed significant differences compared to the case of quasi-static loading conditions as calculated previously [17].

It is possible to extend this scheme to study the effects of indenter elasticity, interface friction, and surface coatings. Including the effects of indenter properties and friction would provide insight into whether the initial stage of failure under dynamic loading is strongly influenced by indenter or projectile type as it is in the static case. The effects of coating properties, layer thickness, and degree of bonding at the interface could also be studied to yield insights into the role of surface constraints in improving ballistic response of ceramics.

2. Theoretical Background

Consider a homogeneous, isotropic, linearly-elastic half space subjected to an axi-symmetric surface source of compressive waves (the technique presented in the present work is not actually restricted to axi-symmetric problems but the implementation in the three-dimensional case is more complicated and will not be considered here [18]). Let the radial coordinate be denoted by r and the depth coordinate by z . The displacement at a given point can be written in terms of potentials ϕ and ψ :

$$\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} \\ u_z &= \frac{\partial \phi}{\partial z} - \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} \\ u_\theta &= 0 \end{aligned} \quad (1)$$

The advantage of using potentials is that they satisfy the simple wave equations [19, 20]:

$$\begin{aligned} \bar{\nabla}^2 \phi &= \frac{1}{c_d^2} \frac{\partial^2 \phi}{\partial t^2} & \bar{\nabla}^2 \psi &= \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2} \\ c_d^2 &= \frac{\lambda + 2\mu}{\rho} & c_s^2 &= \frac{\mu}{\rho} \\ \bar{\nabla}^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \end{aligned} \quad (2)$$

In Equation (2) λ and μ are the usual elastic constants and ρ is the mass density. The approach taken in the present work is to use the Laplace transform to eliminate the time derivatives in Equation (2) and obtain Helmholtz equations for both potentials:

$$\begin{aligned} \bar{f}(r, z, s) &\equiv \int_0^\infty f(r, z, t) e^{-st} dt \\ \bar{\nabla}^2 \bar{\phi} &= k_d^2 \bar{\phi} & \bar{\nabla}^2 \bar{\psi} &= k_s^2 \bar{\psi} \\ k_d^2 &= \frac{s^2}{c_d^2} & k_s^2 &= \frac{s^2}{c_s^2} \end{aligned} \quad (3)$$

Using the separation-of-variables method the general solutions can be found:

$$\begin{aligned}
108 \quad \bar{\phi}(r, s) &= \int_0^\infty A(k, s) e^{-\kappa_d z} J_0(kr) dk & \bar{\psi}(r, s) &= \int_0^\infty B(k, s) e^{-\kappa_s z} J_0(kr) dk \\
&\kappa_d^2 = k_d^2 + k^2 & \kappa_s^2 &= k_s^2 + k^2
\end{aligned} \tag{4}$$

109 In Equation (4) J_0 is the zeroth-order Bessel function of the first kind. Inserting the expressions
110 in Equation (4) into the expressions in Equation (1) gives the general expressions for the
111 displacements:

$$\begin{aligned}
&\bar{u}_r = \int_0^\infty \left(-A(k) e^{-\kappa_d z} + \kappa_s B(k) e^{-\kappa_s z} \right) k J_1(kr) dk \\
112 \quad \bar{u}_z &= \int_0^\infty \left(-\kappa_d A(k) e^{-\kappa_d z} + k^2 B(k) e^{-\kappa_s z} \right) J_0(kr) dk \\
&\bar{u}_\theta = 0
\end{aligned} \tag{5}$$

113 General expressions for the stresses can be determined using the following relations:

$$\begin{aligned}
&\bar{\sigma}_r = \lambda \left(\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r} + \frac{\partial \bar{u}_z}{\partial z} \right) + 2\mu \frac{\partial \bar{u}_r}{\partial r} \\
&\bar{\sigma}_z = \lambda \left(\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r} + \frac{\partial \bar{u}_z}{\partial z} \right) + 2\mu \frac{\partial \bar{u}_z}{\partial z} \\
114 \quad \bar{\sigma}_\theta &= \lambda \left(\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r} + \frac{\partial \bar{u}_z}{\partial z} \right) + 2\mu \frac{\bar{u}_r}{r} \\
&\bar{\sigma}_{rz} = \mu \left(\frac{\partial \bar{u}_r}{\partial r} + \frac{\partial \bar{u}_z}{\partial z} \right)
\end{aligned} \tag{6}$$

115 Inserting Equation (5) into Equation (6) gives:

$$\begin{aligned}
\bar{\sigma}_r &= \int_0^\infty \left((\lambda k_d^2 - 2\mu k^2) A(k) e^{-\kappa_d z} + 2\mu k^2 \kappa_s B(k) e^{-\kappa_s z} \right) J_0(kr) dk \\
&\quad + 2\mu \int_0^\infty k^2 \left(A(k) e^{-\kappa_d z} - \kappa_s B(k) e^{-\kappa_s z} \right) \frac{J_1(kr)}{kr} dk \\
116 \quad \bar{\sigma}_z &= \int_0^\infty \left((\lambda k_d^2 + 2\mu \kappa_d^2) A(k) e^{-\kappa_d z} - 2\mu k^2 \kappa_s B(k) e^{-\kappa_s z} \right) J_0(kr) dk \\
\bar{\sigma}_\theta &= \int_0^\infty \lambda k_d^2 A(k) e^{-\kappa_d z} J_0(kr) dk + 2\mu \int_0^\infty k^2 \left(-A(k) e^{-\kappa_d z} + \kappa_s B(k) e^{-\kappa_s z} \right) \frac{J_1(kr)}{kr} dk \\
\bar{\sigma}_{rz} &= \int_0^\infty \mu k \left(2\kappa_d A(k) e^{-\kappa_d z} - (\kappa_s^2 + k^2) B(k) e^{-\kappa_s z} \right) J_1(kr) dk
\end{aligned} \tag{7}$$

117 The coefficients A and B are determined from the boundary conditions:

$$\begin{aligned}
118 \quad \bar{\sigma}_z(r, 0, s) &= -F(r) \bar{G}(s) \\
\bar{\sigma}_{rz}(r, 0, s) &= 0
\end{aligned} \tag{8}$$

119 In the present work we only consider loading conditions in which the spatial and temporal
120 responses can be separated. The technique for treating more general cases has been presented by
121 Tsai [14] and may be incorporated in future work. Expressions for the coefficients for various
122 cases are given in the Appendix.

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124 3. Numerical Solution Method

125 Numerical evaluation of the expressions in Equations (5) and (7) presents several difficulties
126 which must be addressed to obtain an accurate and stable computational scheme. One
127 complication involves the oscillatory nature of the Bessel functions appearing in the integrands;
128 if not properly addressed this behavior will cause prohibitively slow convergence in any
129 numerical quadrature scheme. The solution is to subdivide the integral into a summation of sub
130 integrals between the zeros of the Bessel function $J_n(x)$:

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$$\int_0^{\infty} f(x) J_n(x) dx = \sum_{j=0}^{\infty} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f(x) J_n(x) dx \quad (9)$$

$$x_0^{(n)} = 0, J_n(x_{i=1,2,\dots}^{(n)}) = 0$$

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The convergence of the resulting series is then accelerated by applying the epsilon algorithm of Wynn [21]. The algorithm transforms, using a non-linear iteration, the sequence of partial sums of the infinite series in Equation (9), into a new sequence (the so-called diagonal Padé approximants) having the same limit as the original sequence but typically better convergence properties. In the present work we used the *Quadpack* [22] routines QAGP, for evaluating the sub-integrals in Equation (9), and QEXT for the epsilon algorithm. For the QAGP routine we chose a 61-point Kronrad integration rule, as described in chapter 2 of reference [22], to ensure high accuracy in the evaluation of the sub-integrals. Both routines require a user-specified relative tolerance to determine the stopping criteria. For the present work we used a relative tolerance of 10^{-8} .

A second difficulty in evaluating the integrals involves the branch points associated with the dilatational (P) and shear (S) waves:

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$$k_b^{(d)} = \frac{\text{Im}(s)}{c_d}$$

$$k_b^{(s)} = \frac{\text{Im}(s)}{c_s} \quad (10)$$

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and the pseudo pole associated with the Rayleigh (R) wave $k_p^{(r)}$, which is given by the real root of the Rayleigh function:

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$$\left(2k_p^{(r)^2} + k_s^2\right)^2 - 4\kappa_d \kappa_s k_p^{(r)^2} = 0 \quad (11)$$

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In the present work Equation (11) is solved by iteration using the secant method described in [23]. Special care must be taken to isolate these points for special treatment in order to ensure

accuracy and stability of the method [20]. For the routine QAGP, used in the present work, the user can specify a set of internal points for further subdivision of the original interval (between successive zeros of the Bessel function). This allows problem points (where changes of slope or large gradients occur) to be identified for use as endpoints. The two branch points and the pseudo pole must all be specified as internal points; in addition, for the pseudo pole it was found necessary to specify four extra points on either side of it in order to achieve sufficient accuracy to capture its effect.

In order to obtain the final time-domain solutions an inverse Laplace transform routine must be used. The routine chosen for the present study was a revision of routine INVLTF which is described in detail in [24]. It utilizes a Fourier-series representation of the Bromwich integral and the QD algorithm (which calculates successive Padé approximations in the form of continued fractions) to accelerate the convergence of the resulting series. The formally infinite series, which results from discretization of the Bromwich integral, is truncated once a user-specified relative accuracy is obtained. In the present work we used a relative accuracy of 10^{-3} and default values for all other parameters described in [24].

4. Results

4.1 Verification of the implementation

In order to verify the numerical solution method we utilized the result of Pekeris [25] for the case of a suddenly-applied compressive point load located at the origin (Figure 1). He obtained closed-form solutions for both the vertical and horizontal displacements at the surface in terms of rational functions and elliptic integrals.

$$F(r) = F_0 \delta(r) / 2\pi r = \lim_{a \rightarrow 0} ((F_0 / \pi a^2) \exp(-r^2 / a^2))$$

Figure 1: A compressive point load of strength F_0 is applied to the surface of an elastic half space. The spatial dependence can also be viewed as the limiting case of a Gaussian pulse. The temporal behavior of the load is given by the Heavyside function ($H(t < 0) = 0, H(t > 0) = 1$).

Since the analytical solution contains singularities it is impossible for any numerical technique to exactly reproduce it, therefore, the verification method employed was to show that the numerical solution obtained for a suddenly-applied load, with a Gaussian spatial dependence, approaches the closed-form solution of Pekeris in the far field region ($r \gg a$). The loading function assumed is given by:

$$\begin{aligned} \sigma_z(r, 0, t) &= -\frac{F_0}{\pi a^2} \exp(-r^2 / a^2) H(t) \\ \sigma_{rz}(r, 0, t) &= 0 \end{aligned} \quad (12)$$

A value of $a = 1$ mm was chosen since projectiles in ballistics applications typically have lateral dimensions measured in mm. Computed results for the vertical and horizontal surface displacements, for three values of the relative field point ($r / a = 10, 100$, and 1000), are shown in figures 2 and 3 respectively, along with the closed-form solutions for a point source. The figures demonstrate how the computed solutions approach the point-source results for $r \gg a$. It was found that as the value of (r / a) becomes large the convergence of the summation resulting

from the discretization of the integrals over k becomes much slower. The origin of this behavior is not well understood but it may be due to the difficulty of the numerical method to capture the behavior of the solution in the vicinity of the pseudo discontinuity/singularity associated with the Rayleigh-wave arrival. Fortunately this behavior is not a serious issue for the application of interest in the present work since we want to model the material response in the near and intermediate field regions ($r/a < 50$) where this convergence issue does not manifest itself.

Figure 2: Comparison of results computed using the approach outlined in the present work versus the closed-form expressions (black solid line) obtained by Pekeris [25]. The load was taken to be a compressive Gaussian pulse of strength F_0 ($F(r) = F_0 / \pi a^2 \exp(-r^2 / a^2)$), with Heavyside temporal response ($G(t) = H(t)$). It was assumed that $\lambda = \mu$, $c_d = 5700$ m/s, and $a = 1$ mm. The results in this figure are for the normalized vertical displacement ($= u_z \pi \mu r / F_0$) versus normalized time ($= t c_s / r$), at location r on the surface ($z = 0$). The result has a singularity at the arrival of the Rayleigh wave ($t c_s / r \approx 1.088$). A small bump is observed at the arrival of the P wave ($t c_s / r \approx 0.577$) and a sudden change in slope at the arrival of the S wave ($t c_s / r = 1.0$).

Figure 3: Comparison of results computed using the approach outlined in the present work versus the closed-form expressions obtained by Pekeris [25]. The load was taken to be a compressive Gaussian pulse of strength F_0 ($F(r) = F_0 / \pi a^2 \exp(-r^2 / a^2)$), with Heavyside temporal response ($G(t) = H(t)$). It was assumed that $\lambda = \mu$, $c_d = 5700$ m/s, and $a = 1$ mm. The results in this figure are for the normalized horizontal displacement ($= u_r \pi \mu r / F_0$) versus normalized time ($= tc_s / r$), at location r on the surface ($z = 0$). The closed-form result has a singularity at the arrival of the R wave ($tc_s / r \approx 1.088$). For large values of r / a the numerical technique has trouble capturing the pseudo discontinuity/singularity at the arrival of the R wave. A small bump is observed at the arrival of the P wave ($tc_s / r \approx 0.577$) and a change in sign of the slope at the arrival of the S wave ($tc_s / r = 1.0$).

4.2. Cylindrical punch indenter

In the literature the point-source solution is often employed, as the kernel in a convolution integral, to construct solutions for arbitrary source distributions. For our purposes this is not a practical approach because of the difficulty in computing the point-source solution. It is far more efficient to find the numerical solution to the system defined in (5), (7) and (8) for the particular source of interest. The sphere and the cylinder are often employed in dynamic indentation and

ballistic experiments. For the method employed in the present work it is far simpler to consider the cylindrical flat-punch (Figure 4) because the radius of the contact circle stays fixed as the load is increased which allows for a factorization of spatial and temporal dependences consistent with equation (8). The procedure for treating the sphere has been described by Tsai [14] but it is considerably more difficult and will not be considered in the present work. The actual loading conditions for an impact event have only been derived for the quasi-static case in which wave effects can be neglected (the Hertzian theory). Tsai [14] presented an approach for deriving the first correction term to Hertzian theory for a sphere but the mathematics is quite involved. In the present work we will use an ansatz for the loading condition on an elastic half space by a cylindrical indenter in the dynamic regime. We assume that the cylinder is rigid (infinite elastic moduli) and that there is no friction between the contacting surfaces. We will also neglect inelastic and dissipative effects. In the case of static loading the distribution of pressure on the surface is given by the well-known Hertzian solution [26]:

$$F(r) = \begin{cases} F_0 / (2\pi a(a^2 - r^2)^{1/2}) & r < a \\ 0 & r > a \end{cases} \quad (13)$$

We will assume that we can use the Hertzian result in Equation (13) for the spatial dependence of our loading condition and that the time dependence is given by a simple ramp function:

$$G(t) = \begin{cases} 0 & t \leq 0 \\ t/\tau & 0 < t < \tau \\ 1 & t \geq \tau \end{cases} \quad (14)$$

It is unlikely that this loading condition represents the actual dynamic loading condition that occurs during impact but it is a convenient way of producing elastic waves whose effects can be studied.

Figure 4: Cylindrical flat-punch indenter of radius a and total load strength F_0 . The spatial distribution of the normal stress on the surface is given by:

$(F(r < a) = (F_0 / 2\pi a) / (a^2 - r^2)^{1/2}, F(r > a) = 0)$. In the case of static loading the radial and hoop stresses on the surface, outside the contact circle, are given by:
 $\sigma_r(r > a, z = 0) = F_0(1 - 2\nu) / 2\pi r^2$ and $\sigma_\theta(r > a, z = 0) = -F_0(1 - 2\nu) / 2\pi r^2$ respectively.

Figures 5 and 6 show the vertical and horizontal displacement of the surface at a radial value $r = 5a$ for various values of the ramp time τ . The limit $\tau \rightarrow 0$ corresponds to the extreme dynamic case (suddenly-applied load) while in the limit $\tau \rightarrow \infty$ the material should instantaneously assume the static solution at any given load value. It is apparent from the figures that the solutions approach the static-solution values in the limit $tc_s / a \gg 1$ as would be expected. The solutions also develop sharper features as τ decreases.

Vertical Displacement

Figure 5: Computed normalized vertical displacement ($=u_z \pi \mu a / F_0$) versus normalized time ($=tc_s / a$), at location $r/a = 5$ on the surface ($z = 0$). The load was taken to be a cylindrical-punch indenter of strength F_0 and radius $a = 500 \mu\text{m}$ ($F(r < a) = (F_0 / 2\pi a) / (a^2 - r^2)^{1/2}$, $F(r > a) = 0$). A ramped temporal response ($G(t < 0) = 0$, $G(0 < t < \tau) = t/\tau$, $G(t > \tau) = 1$) was assumed. The features become less sharp as the ramp time τ is increased. The case $\tau = 0$ corresponds to the Heavyside function ($G(t) = H(t)$).

Horizontal Displacement

Figure 6: Computed normalized horizontal displacement $(= u_r \pi \mu a / F_0)$ versus normalized time $(= t c_s / a)$, at location $r / a = 5$ on the surface $(z = 0)$. The load was taken to be a cylindrical-punch indenter of strength F_0 and radius $a = 500 \mu \text{m}$ $(F(r < a) = (F_0 / 2\pi a) / (a^2 - r^2)^{1/2}, F(r > a) = 0)$. A ramped temporal response $(G(t < 0) = 0, G(0 < t < \tau) = t / \tau, G(t > \tau) = 1)$ was assumed. The features become less sharp as the ramp time τ is increased. The case $\tau = 0$ corresponds to the Heavyside function $(G(t) = H(t))$.

Figures 7 and 8 show the largest r - z plane principal stress σ_1 and the principal stress in the azimuthal direction σ_2 for the same simulations. It is clear that the in-plane stress develops a very significant tensile peak as τ decreases which appears to sharpen into a singularity in the limit $\tau \rightarrow 0$. This tensile peak can be very important with regard to cone cracking under dynamic loading as discussed in the next section. It is also interesting to note that the hoop stress develops a tensile peak in the dynamic regime whereas in the static regime it is compressive. This would tend to imply that it is possible to open radial cracks in the dynamic case but not in the static one. This is in agreement with dynamic indentation experiments which show that below a certain impact velocity only the cone-crack system appears while above it the radial-crack system starts to appear [16, 27].

Figure 7: Computed normalized in-plane tensile stress ($= \pi a^2 \sigma_1 / F_0$) versus normalized time ($= tc_s / a$), at location $r/a = 5$ on the surface ($z = 0$). The load was taken to be a cylindrical-punch indenter of strength F_0 and radius $a = 500 \mu\text{m}$:
 $(F(r < a) = (F_0 / 2\pi a) / (a^2 - r^2)^{1/2}, F(r > a) = 0)$. A ramped temporal response
 $(G(t < 0) = 0, G(0 < t < \tau) = t / \tau, G(t > \tau) = 1)$ was assumed. The features become less sharp as the ramp time τ is increased. The case $\tau = 0$ corresponds to the Heavyside function ($G(t) = H(t)$). It is clear that for sufficiently small τ (dynamic loading) there is a substantial increase in the tensile stress as compared to static loading.

Principal Stress in Azimuthal Direction

Figure 8: Computed normalized hoop stress ($= \pi a^2 \sigma_2 / F_0$) versus normalized time ($= tc_s / a$), at location $r/a = 5$ on the surface ($z = 0$). The load was taken to be a cylindrical-punch indenter of strength F_0 and radius $a = 500 \mu\text{m}$:
 $(F(r < a) = (F_0 / 2\pi a) / (a^2 - r^2)^{1/2}, F(r > a) = 0)$. A ramped temporal response ($G(t < 0) = 0, G(0 < t < \tau) = t / \tau, G(t > \tau) = 1$) was assumed. The features become less sharp as the ramp time τ is increased. The case $\tau = 0$ corresponds to the Heavyside function ($G(t) = H(t)$). It is interesting that in the dynamic case the hoop stress has a significant tensile portion whereas it is compressive in the static case.

It follows from dimensional analysis that the shapes of the computed curves are determined by the value of the dimensionless number $c_d \tau / a$ with $c_d \tau / a \gg 1$ corresponding to quasi-static behavior and $c_d \tau / a \ll 1$ to highly-dynamic behavior. The results of the present work indicate that even for $c_d \tau / a \sim 1$ the behavior is already fairly dynamic in nature.

It was mentioned previously that in the dynamic case the peak tensile stress becomes much larger than in the static case. In addition, the peak tensile stress in the dynamic case falls off much less rapidly with distance from the source than in the static case. Figure 9 shows that for

319 $c_d \tau / a = 1.14$ the peak tensile stress falls off approximately as $(r/a)^{-0.5}$ in contrast to $(r/a)^{-2}$
 320 for the static case $c_d \tau / a \rightarrow \infty$. One can also observe the separation of the different components
 321 of the net waveform as the observation point moves further from the source (a smaller hump
 322 arrives at earlier times followed by a larger spike). This is due to the different components
 323 (associated with the P, S, and R modes) having different velocities. The portion of the net
 324 waveform associated with the P and S modes falls off as $(r/a)^{-2}$ and becomes broader as it
 325 propagates; in contrast, the portion associated with the R mode displays $(r/a)^{-0.5}$ asymptotic
 326 dependence. These results for the asymptotic behavior of the various components is well-known
 327 and it is reassuring that the present approach reproduces them. In the next section it will be
 328 shown that these difference in asymptotic behavior lead to profound differences in the cone
 329 cracking behavior under dynamic and quasi-static loading.

330
 331 Figure 9: Computed normalized in-plane tensile stress ($= \pi a^2 \sigma_1 / F_0$) versus normalized
 332 time ($= tc_s / a$), at various locations (r/a) on the surface ($z = 0$). The load was taken to
 333 be a cylindrical-punch indenter of strength F_0 and radius $a = 500 \mu\text{m}$:

334 $(F(r < a) = (F_0 / 2\pi a) / (a^2 - r^2)^{1/2}, F(r > a) = 0)$. The temporal response was assumed

to be ramped ($G(t < 0) = 0, G(0 < t < \tau) = t/\tau, G(t > \tau) = 1$) with ramp length $\tau = 10^{-7}$ sec. The dilatational wave speed was assumed to be $c_d = 5700$ m/s and the Poisson ratio $\nu = 0.25$. The characteristic dimensionless number in this case is $c_d \tau / a = 1.14$.

4.3. Cone cracks: static versus dynamic loading

One application of the computational method presented in the present work is to model cone cracks formed in brittle materials under dynamic loading by an indenter. Studies under quasi-static conditions are numerous and date all the way back to Hertz's investigations in the 1880's [28]. A detailed computational study of quasi-static cone cracking in soda-lime glass due to a spherical indenter was presented by Lawn et al [17]. The goal in the present study was to perform a similar calculation under dynamic loading conditions so that the predicted cracks can be compared to those in the quasi-static case. The basic idea is illustrated in Figure 10.

Figure 10: Cone crack formation from a frictionless cylindrical flat-punch indenter. A flaw of initial length c_0 is assumed to exist at a distance r_0 from the center of a

cylindrical flat-punch of radius a . The rate of growth of the crack length c is determined by a function of the stress intensity factor K_I as expressed in equation (15). The intensity factor is calculated from an integral of the local tensile stress weighted by a Green's function as given in equation (16).

A pre-existing ring-shaped flaw is assumed to exist on the surface of a half space. A rigid cylindrical indenter, of radius a , is applied (without friction) to the surface with a given force F_0 and a ramp time τ . The crack extends due to the tensile stress acting along the existing crack path (mode I) and is assumed to follow the unit eigenvector of the principal stress σ_3 (perpendicular to σ_1) at the existing crack tip. The growth rate of the crack is assumed to be given by some function, experimentally determined, of the stress intensity factor K_I :

$$\frac{dc}{dt} = v(K_I) \quad (15)$$

At time t Equation (15) is used to compute an increment of crack extension $\Delta c(t) = v(K_I(t))\Delta t$ which is added to the existing crack path. The direction of crack extension is assumed to be given by the unit eigenvector of the principal stress σ_3 (perpendicular to σ_1) at the crack tip. The value of $K_I(t)$ is determined using an approximation similar to that employed in [17] for the case of static fields. In that work the intensity factor was calculated by integrating, along the existing crack path, the component of the tensile stress normal to the crack surface weighted by the appropriate Green's function. One approximation in the approach is that the Green's function used is for a planar crack in the plane-strain approximation rather than a cone crack which possesses curvature. A second approximation is that the crack is assumed to grow in the stress field of the uncracked solid. In the present case we approximate the effects of wave loading by using the retarded stress to account for the fact that the stress acting at a point along

the crack path will not produce a positive contribution to the intensity factor until the arrival of the corresponding Rayleigh wave at the crack tip [29, 30].:

$$K_I(t) = 2\sqrt{\frac{c(t)}{\pi}} \int_0^{c(t)} \frac{\sigma(b, t - (c(t) - b)/c_R)}{\sqrt{c(t)^2 - b^2}} db \quad (16)$$

In equation (16) $c(t) - b$ is the length of existing crack path between point b and the crack tip $c(t)$ (see Figure 10), and c_R is the Rayleigh-wave speed. In the present work we are also neglecting the reduction in intensity factor during rapid crack growth [29, 30]. The reason for not including this effect in this initial study was the complication in designing a stable non-linear iteration due to the exponential behavior of the $v(K_I)$ curve when changing between the quasi-static and dynamic growth regimes (see figure 12). We are continuing to investigate this and hope to include this effect in future work. In [17] a curve for $v(K_I)$, based on experimental studies of soda-lime glass was presented and is shown in figure 12. We will use this curve for the present study.

Figure 12: Crack speed versus intensity factor function $v(K_I)$ used in the calculations. It is based on experimental work for soda-lime glass [17].

Although it has been pointed out that this approach does not quantitatively predict the correct crack angle [17, 31] it does seem to reproduce much of the qualitative behavior observed in experiment.

A comparison of the predicted crack angles for both the quasi-static $c_d \tau / a = 1.14 \times 10^6$ and dynamic $c_d \tau / a = 1.14$ cases are given in figures 13, 14 and 15.

Cone Cracks for Various Flaw Locations (Quasi-static Loading)

Figure 13: Cone cracks created from existing flaws, at various distances from the edge of a cylindrical flat-punch indenter, by a quasi-static load of strength $F_0 = 20 \text{ N}$. The indenter radius was taken to be $a = 500 \mu\text{m}$, the ramp time $\tau = 0.1 \text{ s}$, and the dilatational wave speed $c_d = 5700 \text{ m/s}$ ($c_d \tau / a = 1.14 \times 10^6$). The Poisson ratio was assumed to be $\nu = 0.21$ which is typical of soda-lime glass.

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Figure 14: Cone cracks created from existing flaws, at various distances from the edge of a cylindrical flat-punch indenter, by a dynamic load of strength $F_0 = 20 \text{ N}$. The indenter radius was taken to be $a = 500 \mu\text{m}$, the ramp time $\tau = 10^{-7} \text{ s}$, and the dilatational wave speed $c_d = 5700 \text{ m/s}$ ($c_d \tau / a = 1.14$). The Poisson ratio was assumed to be $\nu = 0.21$ which is typical of soda-lime glass.

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Figure 15: Direct comparison of cone cracks created from an existing flaw, one millimeter from the center of a cylindrical flat-punch indenter, for both quasi-static ($c_d \tau / a = 1.14 \times 10^6$) and dynamic ($c_d \tau / a = 1.14$) loading cases. The indenter radius was taken to be $a = 500 \mu\text{m}$ and the dilatational wave speed $c_d = 5700 \text{ m/s}$. The Poisson ratio was assumed to be $\nu = 0.21$ which is typical of soda-lime glass. The dynamic-loading case creates a longer crack at a larger angle with respect to the surface than the quasi-static case.

The wave speed $c_d = 5700 \text{ m/s}$ and Poisson ratio $\nu = 0.21$ are typical of soda-lime glass. One observation is that as the radius of the initial flaw (r_0) increases the length of the crack in the quasi-static case decreases more rapidly than in the dynamic case. It can also be seen that the initial portion of the trajectory, in the quasi-static case, tends to be perpendicular to the surface before turning sharply; this does not seem to happen in the dynamic case. Figure 15 illustrates the difference in shape and crack angle for growth under quasi-static and wave fields. In the case of wave fields the crack grows at a significantly larger angle, with respect to the surface, than in the quasi-static case.

Figures 16 and 17 demonstrate how dramatic the difference is between the quasi-static and dynamic cases when it comes to how quickly the crack length decays with the distance between flaw location and the edge of the indenter.

Figure 16: Cone cracks created from existing flaws, at relatively large distances from the edge of a cylindrical flat-punch indenter, by a dynamic load of strength $F_0 = 20 \text{ N}$. The indenter radius was taken to be $a = 500 \mu \text{ m}$, the ramp time $\tau = 10^{-7} \text{ s}$, and the dilatational wave speed $c_d = 5700 \text{ m/s}$ ($c_d \tau / a = 1.14$). The Poisson ratio was assumed to be $\nu = 0.21$ which is typical of soda-lime glass. It is striking how slowly the crack length falls off with distance from the edge of the indenter.

Figure 17: Comparison of crack length versus initial flaw location for both quasi-static ($c_d \tau / a = 1.14 \times 10^6$) and dynamic ($c_d \tau / a = 1.14$) loading cases for a cylindrical flat-punch indenter with a total load strength $F_0 = 20$ N . The indenter radius was $a = 500 \mu\text{m}$ and the dilatational wave speed $c_d = 5700$ m/s . The Poisson ratio was assumed to be $\nu = 0.21$ which is typical of soda-lime glass. It is striking how in the quasi-static case the cracks become negligibly small above $r_0 \approx 5a$ while in the dynamic case they have only decreased by about 25% at $r_0 = 10a$.

Figure 17 clearly shows that in the quasi-static case the crack length is essentially zero for $r_0/a > 5$ while in the dynamic case the crack length has only decreased by about 25% for $r_0/a \approx 10$.

The reason for the slow decay in the dynamic case is demonstrated in Figure 18. For dynamic crack growth the crack length is essentially determined by the amount of time the stress intensity factor stays above the threshold value for saturation of the crack speed at 1500 m/s

459 ($K_I \approx 9 \times 10^5 \text{ Pa} \cdot \text{m}^{1/2}$). As can be seen in the figure this amount of time decreases only very
 460 slowly as the distance between the flaw location and indenter edge increases.

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 462 Figure 18: Intensity factor K_I versus time for cone cracks initiating from flaws at
 463 various distances from the center of a cylindrical flat-punch indenter. The indenter has a
 464 radius $a = 500 \mu\text{m}$ and a total load $F_0 = 20 \text{ N}$ with ramp time $\tau = 10^{-7} \text{ s}$. The
 465 dilatational wave speed and Poisson ratio were assumed to be $c_d = 5700 \text{ m/s}$ and
 466 $\nu = 0.21$ respectively which are typical values for soda-lime glass. Since we are dealing
 467 with the dynamic regime ($c_d \tau / a = 1.14$) the crack length is essentially determined by
 468 how long the intensity factor remains above the value for crack speed saturation (
 469 $K_I \approx 9 \times 10^5 \text{ Pa} \cdot \text{m}^{1/2}$).

470
 471 Another striking difference between the quasi-static and dynamic cases is illustrated in figures
 472 19-22. It was pointed out previously [17, 31] that in the quasi-static case the crack angle is
 473 strongly dependent on the Poisson ratio ν . This is demonstrated in Figure 19. Interestingly this
 474 strong dependence is not found in the dynamic case as shown in Figure 20. Perhaps even more
 475 dramatic is the difference in how the Poisson ratio affects the crack length in the two cases. As

476 shown in Figure 21 the quasi-static crack length decreases dramatically with increasing ν while
 477 in the dynamic case, as shown in Figure 22, the length actually increases slowly with increasing
 478 ν .

479
 480 Figure 19: Cone cracks created from an existing flaw by a cylindrical flat-punch indenter
 481 of radius $a = 500 \mu\text{m}$ and total load $F_0 = 200 \text{ N}$ with ramp rate $\tau = 0.1 \text{ s}$. The
 482 dilatational wave speed was assumed to be $c_d = 5700 \text{ m/s}$. The initial flaw location was
 483 $r_0 = 4a$. The different curves are for different values of the Poisson ratio ν . In this
 484 quasi-static regime ($c_d \tau / a = 1.14 \times 10^6$) the angle that the crack grows at, with respect to
 485 the surface, is strongly dependent on the value of the Poisson ratio.
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Figure 20: Cone cracks created from an existing flaw by a cylindrical flat-punch indenter of radius $a = 500 \mu\text{m}$ and total load $F_0 = 20 \text{ N}$ with ramp rate $\tau = 10^{-7} \text{ s}$. The dilatational wave speed was assumed to be $c_d = 5700 \text{ m/s}$. The initial flaw location was $r_0 = 4a$. The different curves are for different values of the Poisson ratio ν . In this dynamic regime ($c_d\tau/a = 1.14$) the angle that the crack grows at, with respect to the surface, is essentially independent on the value of the Poisson ratio. This is completely different behavior compared to the quasi-static case.

Effect of Poisson Ratio on Crack Length (Quasi-static)

Figure 21: Crack length versus time for cone cracks generated from an existing flaw by a cylindrical flat-punch indenter of radius $a = 500 \mu\text{m}$ and total load $F_0 = 200 \text{ N}$ with ramp rate $\tau = 0.1 \text{ s}$. The dilatational wave speed was assumed to be $c_d = 5700 \text{ m/s}$. The initial flaw location was $r_0 = 4a$. The different curves are for different values of the Poisson ratio ν . It can be seen that, for this quasi-static case ($c_d \tau / a = 1.14 \times 10^6$), as ν increases the final crack length rapidly decreases.

Figure 22: Crack length versus time for cone cracks generated from an existing flaw by a cylindrical flat-punch indenter of radius $a = 500 \mu\text{m}$ and total load $F_0 = 20 \text{ N}$ with ramp rate $\tau = 10^{-7} \text{ s}$. The dilatational wave speed was assumed to be $c_d = 5700 \text{ m/s}$. The initial flaw location was $r_0 = 4a$. The different curves are for different values of the Poisson ratio ν . It can be seen that, for this dynamic case ($c_d \tau / a = 1.14$), as ν increases the final crack length slowly increases. This behavior is drastically different than what is found for the quasi-static case.

5. Conclusions

In the present work a computational scheme to study surface wave effects was implemented. Use of the Laplace transform enabled analytical expressions for the various quantities of interest to be obtained in the transformed space. Numerical inversion of these closed-form expressions provides stress and displacement distributions as functions of space and time. The method was first compared to the known analytical solutions for the surface displacement due to the idealized case of a suddenly-applied point load. It was next applied to the problem of cone crack extension from an existing flaw under dynamic loading of a frictionless cylindrical punch. The

computed cone cracks under dynamic loading conditions showed significant differences compared to the case of quasi-static loading conditions. It is possible to extend this scheme to study the effects of indenter elasticity, interface friction, and surface coatings. Including the effects of indenter properties and friction would provide insight into whether the initial stage of failure under dynamic loading is strongly influenced by indenter or projectile type as it is in the static case. The effects of coating properties, layer thickness, and degree of bonding at the interface could also be studied to yield insights into the role of surface constraints in improving ballistic response of ceramics. The incorporation of the above mentioned effects into the method presented below will be the subject of a future publication.

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Appendix (Coefficients for various spatial distributions of the loading function):

Delta-Function Load:

For a point-source (delta function) load, with strength F_0 , the expression for $F(r)$ in equation (8) becomes:

$$F(r) = \frac{F_0}{2\pi} \frac{\delta(r)}{r} = \frac{F_0}{2\pi} \int_0^\infty k J_0(kr) dk \quad (\text{A-1})$$

Inserting this expression along with the expressions in equation (7) into equation (8) gives:

$$\begin{aligned} A(k, s) &= -\frac{F_0}{2\pi\mu} \frac{k(\kappa_s^2 + k^2)}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d\kappa_s k^2} \bar{G}(s) \\ B(k, s) &= -\frac{F_0}{2\pi\mu} \frac{2\kappa_d k}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d\kappa_s k^2} \bar{G}(s) \end{aligned} \quad (\text{A-2})$$

Gaussian Load:

For a load of strength F_0 distributed as a Gaussian pulse of width a , the expression for $F(r)$ is:

$$F(r) = \frac{F_0}{\pi a^2} \exp\left(-\frac{r^2}{a^2}\right) = \frac{F_0}{2\pi} \int_0^\infty k \exp\left(-\frac{k^2 a^2}{4}\right) J_0(kr) dk \quad (\text{A-3})$$

This gives the following expressions for the coefficients:

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603

$$\begin{aligned} A(k, s) &= -\frac{F_0}{2\pi\mu} \frac{k(\kappa_s^2 + k^2)}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d\kappa_s k^2} \exp\left(-\frac{k^2 a^2}{4}\right) \bar{G}(s) \\ B(k, s) &= -\frac{F_0}{2\pi\mu} \frac{2\kappa_d k}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d\kappa_s k^2} \exp\left(-\frac{k^2 a^2}{4}\right) \bar{G}(s) \end{aligned} \quad (\text{A-4})$$

604

605 It is clear from the above expressions that the coefficients in (A-4) reduce to those in (A-2) as

606 $a \rightarrow 0$.

607

608 **Cylindrical Punch:**

609 For a load of strength F_0 distributed over a cylindrical flat-punch indenter of radius a , the

610 expression for $F(r)$ is:

611

612

$$F(r) = \begin{cases} \frac{F_0}{2\pi a^2} \left(1 - \frac{r^2}{a^2}\right)^{-1/2} = \frac{F_0}{2\pi a} \int_0^\infty \sin(ka) J_0(kr) dk & r < a \\ 0 & r > a \end{cases} \quad (\text{A-5})$$

613 which gives:

614

$$\begin{aligned} A(k, s) &= -\frac{F_0}{2\pi\mu} \frac{k(\kappa_s^2 + k^2)}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d\kappa_s k^2} \frac{\sin(ka)}{ka} \bar{G}(s) \\ B(k, s) &= -\frac{F_0}{2\pi\mu} \frac{2\kappa_d k}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d\kappa_s k^2} \frac{\sin(ka)}{ka} \bar{G}(s) \end{aligned} \quad (\text{A-6})$$

615

616 **Constant load over circular area:**

617 For a load of strength F_0 distributed uniformly over a circular area of radius a , the expression

618 for $F(r)$ is:

619

$$620 \quad F(r) = \begin{cases} \frac{F_0}{\pi a^2} = \frac{F_0}{\pi a} \int_0^\infty J_1(ka) J_0(kr) dk & r < a \\ 0 & r > a \end{cases} \quad (\text{A-7})$$

621 which gives:

$$622 \quad \begin{aligned} A(k, s) &= -\frac{F_0}{\pi \mu} \frac{k(\kappa_s^2 + k^2)}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d \kappa_s k^2} \frac{J_1(ka)}{ka} \bar{G}(s) \\ B(k, s) &= -\frac{F_0}{\pi \mu} \frac{2\kappa_d k}{(2k^2 + \kappa_s^2)^2 - 4\kappa_d \kappa_s k^2} \frac{J_1(ka)}{ka} \bar{G}(s) \end{aligned} \quad (\text{A-8})$$

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