Instability analysis of Shear Bands using the instantaneous growth-rate method

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Abstract

Shear banding, as an unstable process of localization, is a common precursor to fracture in materials under high strain rate loadings, making the detection of the instability point after which localization will occur of significant importance. Stability analysis based on the perturbation method or the acoustic tensor have been employed. However, these methodologies are limited to certain class of problems and are difficult to generalize.

In this work we propose an alternative for identifying the instability point by employing the concept of generalized stability analysis. In this framework, a stability measure is obtained by computing the instantaneous growth rate of the vector tangent to the solution. Such an approach is more appropriate for non-orthogonal problems and is easier to generalize to difficult dynamic fracture problems.

Under conditions where the local instability triggers the non-homogeneous solution growth, i.e. problems that are homogeneous until the appearance of local instability, the generalized stability analysis and the modal stability analysis will closely match. Therefore, the non-homogeneous growth can be approximated by the Rayleigh Quotient of the vector tangent to the solution, which is easier to compute.

We show that for a particular class of problems that respect the aforementioned

Pre-proof
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25 conditions, in 1D and 2D examples, both quantities successfully find the instabil-
26 ity point predicted analytically and validated experimentally in past literature re-
27 sults. This methodology is general and can be applied to a wide array of dynamic
28 fracture problems, for which instability that leads to localization is important.

29 *Keywords:* Shearband, Instantaneous growth-rate, Instability, Failure, Rayleigh
30 Quotient

31 **1. Introduction**

32 Shear bands are material instabilities associated with highly localized intense plas-
33 tic deformation zones. In materials such as metals they typically form under high
34 strain rate loadings and are a strong precursor to fracture[57]. These localized
35 bands have also been observed in materials with non-associative flow laws like
36 porous or granular materials, polymers, rocks and soils [17, 27].

37 The flow law of many structural materials show strain rate dependency. In a ma-
38 terial like steel a high strain rate loading hardens the material. On the other hand,
39 the heat produced by inelastic deformation generates thermal softening that will
40 lower the material's flow stress potentially causing instability. These two effects
41 added to the strain hardening of the material lead to an intricate equilibrium that
42 is highly sensitive to changes in material parameters and loading conditions.

43 Given the very short time intervals associated with the formation of shear bands,
44 adiabatic conditions might be adopted to simplify the resolution of the problem.
45 However, this assumption has been shown to make the numerical solution mesh-
46 dependent [14, 37, 60, 61]. By considering the effect of thermal diffusion, the
47 problem is given an intrinsic length scale that functions as a localization limiter,
48 which leads to mesh insensitive results [9, 10, 56]. Other regularization techniques
49 reported in the literature include strain gradient theories [2–4], which have been
50 used in the context of shear bands in [1, 33, 53] and nonlocal methods[7, 23, 45].

51 Mesh alignment is another form of sensitivity that can be improved using mesh-
52 free formulations[35, 36] or Isogeometric analysis [16].

53 Local stability analysis of shear bands has been an active area of research as it
54 provides valuable information that can be used to model shear bands through phe-
55 nomenological models. Some examples of the importance of stability detection
56 can be found in the work of Rabczuk et al. [46] that set strong displacement dis-
57 continuities with cohesive surfaces where local instability is detected. Song et al.
58 [49] introduce additional tangential degrees of freedom at the discontinuity using
59 phantom nodes. Belytschko et al. [15] inject coarse scale discontinuities based on
60 isolating the unstable regions at the fine scale into a localization domain. Tabar-
61 raei et al. [51] use material instability based on the perturbation method to inject
62 a phantom node discontinuity on a macro level model based on the cohesive laws
63 obtained from a microstructural model. Finally, Rabczuk and Samaniego [47] use
64 instability in an adaptive mesh-free method for discontinuous modeling of shear
65 bands.

66 Among others, two main approaches on the study of stability involving shear
67 bands are the linear perturbation method [5, 8, 10, 12, 13, 22, 26, 31, 38, 44, 55]
68 and the eigenvalue analysis of the acoustic tensor [18, 19, 50, 59]. This paper fo-
69 cuses on the linear perturbation method and not on the loss of ellipticity detected
70 by the eigenvalues of the acoustic tensor since for a rate-dependent material the
71 equations remain elliptical as was shown by Lemonds and Needleman [34].

72 The stability of a system is traditionally studied from the perspective of the modal
73 spectrum of that system. The underlying reasoning is that whichever mode has
74 the largest growth rate corresponding to an exponential growth will dominate the
75 solution. However, for non-normal systems (i.e. systems whose eigenvalues are
76 non-orthogonal) the short time growth-rate of the solution might not correspond
77 to the growth rate of the most unstable mode. On the other hand, non-autonomous
78 systems do not have a constant spectrum because of the time dependency of their
79 operators. Since the problem being studied is neither normal nor autonomous, a

80 generalized stability analysis is of interest [24, 25]. In this work we use the Finite
81 Element Method to obtain a monolithic semi-discrete Jacobian for the shear band
82 problem following the derivation in [41], on which a generalized stability analysis
83 is performed.

84 The local stability criteria proposed by Bai [8], Ling and Belytschko [38] and
85 others, are usually derived from a perturbation analysis, which under the same
86 assumptions is equivalent to a local eigenvalue analysis [6]. These works have
87 shown that this instability condition predicts well the localized region and, for a
88 certain special class of problems, it will mark the point after which the solution
89 will rapidly localize. We focus on these problems and compare the results of the
90 proposed generalized stability analysis with this local instability criterion.

91 There are several different concepts that are important for a generalized stability
92 analysis as shown in Farrell and Ioannou [24, 25]. The study of the growth rate of
93 a perturbation tangent to the solution is of particular interest since it is computa-
94 tionally cheap and can easily be implemented on complex multiphysics systems.
95 Additionally, an analysis based on the concept of the Rayleigh Quotient of an
96 eigenvector is employed and is shown to also produce a good approximation of
97 the instability point.

98 It has been shown that the local spectral analysis based on element eigenvalues
99 recover the analytical local instability criterion[6]. Albeit considerably more ex-
100 pensive, the local spectral analysis posses the same implementation simplicity
101 that the instantaneous growth rate and the Rayleigh Quotient have, i.e. they only
102 require the knowledge of the tangent stiffness matrix. However, a global modal
103 analysis based on the eigenvalues of the full system is computationally prohibitive
104 as the problem grows larger. This fact makes the proposed approach attractive for
105 the study of the global instability behavior since the computational cost is of the
106 same magnitude of the analytical instability criterion.

107 This work is structured as follows. Section 1 contains the current introduction.
108 In Section 2 the shear band problem is described mathematically, followed by its

109 implementation using the Finite Element Method. Section 3 describes the method
 110 of local instability analysis that has been employed in the past and compared with
 111 experimental results and the Generalized stability method used in this work. In
 112 Section 4 the application of the proposed methodology on numerical examples
 113 is showed for both 1D and 2D cases. Finally, the conclusions are presented in
 114 Section 5 after which the Appendixes and the References are located.

115 **2. Problem Statement**

116 *2.1. Shear Bands*

117 The shear band problem is described by the equations of conservation of mo-
 118 mentum and energy, and also the equations for elastic and inelastic constitutive
 119 relations. All of these and boundary conditions are formalized as a set of coupled
 120 PDEs as follows. The Momentum Equation, which includes inertial effects but
 121 ignores body forces, is given by

$$\rho \ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} \quad (1)$$

122 with $\mathbf{u}$ as the displacement field, t as the time, $\boldsymbol{\sigma}$ as the stress tensor and ρ as
 123 the material density. A super imposed dot corresponds to a time derivative. The
 124 Elastic Constitutive Relation, in total form, is

$$\boldsymbol{\sigma} = \mathbf{C}^{elas} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \quad (2)$$

125 Where $\mathbf{C}^{elas}$ is the fourth-order elastic constitutive tensor. The strain tensor has
 126 been additively decomposed into elastic and inelastic parts [48] as

$$\boldsymbol{\varepsilon} = \frac{1}{2} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \quad (3)$$

127 with ε^e and ε^p as the elastic and inelastic parts of the strain, respectively. The
 128 energy equation is written as

$$\rho c_p \dot{T} = \lambda \nabla^2 T + \chi \boldsymbol{\sigma} : \mathbf{D}^p \quad (4)$$

129 where T is the temperature, and λ , c_p , and χ are the conductivity, specific heat
 130 and Taylor-Quinney coefficient, respectively. This equation accounts for diffusion
 131 as well as heat production in proportion to the plastic work, $\boldsymbol{\sigma} : \mathbf{D}^p$ [52], where
 132 $\mathbf{D}^p = \dot{\varepsilon}^p$ is the rate of plastic deformation.

133 The rate of change of the equivalent plastic strain is defined as

$$\dot{\gamma}_p = \sqrt{\frac{2}{3} \mathbf{D}^p : \mathbf{D}^p} \quad (5)$$

134 The inelastic part of the rate of deformation tensor, assuming J2 plasticity, is de-
 135 fined as

$$\mathbf{D}^p = \frac{3}{2} \frac{\dot{\gamma}_p}{\bar{\sigma}} \mathbf{S} \quad (6)$$

136 where the deviatoric stress tensor $\mathbf{S}$ is

$$\mathbf{S} = \boldsymbol{\sigma} - \frac{1}{3} \text{tr}(\boldsymbol{\sigma}) \mathbf{I} \quad (7)$$

137 and the effective stress is given by

$$\bar{\sigma} = \sqrt{\frac{3}{2} \mathbf{S} : \mathbf{S}} \quad (8)$$

138 The inelastic constitutive relation, given in terms of the effective stress, may be
 139 written as

$$\bar{\sigma} = P(T) Q(\bar{\gamma}_p) R(\dot{\gamma}_p) \quad (9)$$

140 where $\bar{\gamma}_p$ is the equivalent plastic strain, which serves as a strain hardening param-
 141 eter and the functions $P(T)$, $Q(\bar{\gamma}_p)$ and $R(\dot{\gamma}_p)$ depend on specific material models.

Any flow law that can be decomposed as in (9) is valid[29] and hereafter the Johnson-Cook material model [30] will be used.

The four equations (1), (2), (4), (9) along with the definition of the flow law describe the evolution of the four unknown fields of displacement, stress, temperature and plastic strain.

Lastly, the boundary conditions are

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \Gamma^u \quad (10)$$

$$T = \bar{T} \quad \text{on} \quad \Gamma^T \quad (11)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on} \quad \Gamma^t \quad (12)$$

$$\mathbf{n} \cdot \mathbf{q} = \bar{q} \quad \text{on} \quad \Gamma^q \quad (13)$$

where $\bar{\mathbf{t}}$ and $\bar{q}$ are the prescribed traction and flux, respectively. This model considers small strains, and additionally neglects thermal strains and the thermoelastic contribution to the system energy.

At $t = 0$ the system is considered to be unstressed, undeformed and at room temperature.

2.2. Finite Element formulation

A mixed finite element formulation is used to discretize the governing equations for the shear band problem. The matrices that result from that discretization are used for both solving the PDEs and analyzing the stability of the problem. The system of equations (1)-(9) is first reformulated in the following form

$$\rho \dot{\mathbf{v}} = \nabla \cdot \boldsymbol{\sigma} \quad (14)$$

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{elas} : \left(\nabla^s \mathbf{v} - \frac{3}{2} \frac{g(\bar{\boldsymbol{\sigma}}, T, \bar{\gamma}_p)}{\bar{\boldsymbol{\sigma}}} \mathbf{S} \right) \quad (15)$$

$$\rho c_p \dot{T} = \lambda \nabla^2 T + \chi \bar{\boldsymbol{\sigma}} g(\bar{\boldsymbol{\sigma}}, T, \bar{\gamma}_p) \quad (16)$$

$$\dot{\bar{\gamma}}_p = g(\bar{\boldsymbol{\sigma}}, T, \bar{\gamma}_p) \quad (17)$$

157 with the rate of deformation tensor given by the symmetric part of the velocity
158 gradient, or directly defined by taking the time derivative of Eq. (3), and reads

$$\nabla^s \mathbf{v} = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \quad (18)$$

159 To obtain the weak form in mixed formulation, each equation is pre-multiplied by
160 an appropriate shape function w_v , w_σ , w_T and $w_{\bar{\gamma}_p}$ for the momentum equation,
161 the elastic constitutive equation, the energy equation and the inelastic constitu-
162 tive equation, respectively. Then, integrating over the domain and applying the
163 divergence theorem where appropriate, the weak form of the residual equations is
164 obtained.

165 Specifically, the residual of the momentum equation (14) reads

$$\mathbf{r}_v = \int_{\Omega} w_v \rho \dot{\mathbf{v}} \, d\Omega + \int_{\Omega} \nabla w_v \boldsymbol{\sigma} \, d\Omega - \int_{\Gamma} w_v \boldsymbol{\sigma} \cdot \mathbf{n} \, d\Gamma \quad (19)$$

166 The weak form of the residual of the elastic constitutive equation (15) is

$$\mathbf{r}_\sigma = \int_{\Omega} w_\sigma \dot{\boldsymbol{\sigma}} \, d\Omega - \int_{\Omega} w_\sigma \mathbf{C}^{elas} : \left(\nabla^s \mathbf{v} - \frac{3}{2} \frac{g(\bar{\boldsymbol{\sigma}}, T, \bar{\gamma}_p)}{\bar{\boldsymbol{\sigma}}} \mathbf{S} \right) \, d\Omega \quad (20)$$

167 The residual of the energy equation (16) reads

$$\mathbf{r}_T = \int_{\Omega} w_T \rho c_p \dot{T} \, d\Omega + \int_{\Omega} \nabla w_T \lambda \nabla T \, d\Omega - \int_{\Omega} w_T \chi^* \eta \bar{\tau}_g \, d\Omega - \int_{\Gamma} w_T \lambda \nabla T \cdot \mathbf{n} \, d\Gamma \quad (21)$$

168 and finally, the weak form of the residual of the inelastic constitutive equation (17)

169 is

$$\mathbf{r}_{\tilde{\gamma}_p} = \int_{\Omega} w_{\tilde{\gamma}_p} \dot{\tilde{\gamma}}_p \, d\Omega - \int_{\Omega} w_{\tilde{\gamma}_p} g \, d\Omega \quad (22)$$

170 These equations can be grouped into a residual vector $\mathbf{r}$ and a solution vector $\mathbf{x}$

171 with displacements, temperature, stress and plastic strain fields as

$$\mathbf{x} = \begin{bmatrix} \mathbf{v} \\ T \\ \boldsymbol{\sigma} \\ \tilde{\gamma}_p \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}_v \\ \mathbf{r}_T \\ \mathbf{r}_\sigma \\ \mathbf{r}_{\tilde{\gamma}_p} \end{bmatrix} \quad (23)$$

172 The coupled nonlinear problem can then be stated as

$$\mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{u}_{ibv}) = 0 \quad (24)$$

173 where $\mathbf{u}_{ibv}$, dropped henceforth for simplification, represents the initial and the

174 boundary conditions. To solve (24), the Newton method is applied, assuming a

175 Taylor series expansion around $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$, such that

$$\mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{r}(\mathbf{x}_0, \dot{\mathbf{x}}_0) + d_{\delta \mathbf{x}} \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) \Big|_{\mathbf{x}_0, \dot{\mathbf{x}}_0} + d_{\delta \dot{\mathbf{x}}} \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) \Big|_{\mathbf{x}_0, \dot{\mathbf{x}}_0} \quad (25)$$

176 where $d_{\delta \mathbf{x}}(\cdot)$ and $d_{\delta \dot{\mathbf{x}}}(\cdot)$ represent the Gâteaux differentials in the $\delta \mathbf{x}$ and $\delta \dot{\mathbf{x}}$ di-

reactions, respectively, which are defined as

$$\begin{aligned} d_{\delta \mathbf{x}} \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) &= \left. \frac{d}{d\varepsilon} \mathbf{r}(\mathbf{x} + \varepsilon \delta \mathbf{x}, \dot{\mathbf{x}}) \right|_{\varepsilon=0} = -\mathbf{K} \cdot \delta \mathbf{x} \\ d_{\delta \dot{\mathbf{x}}} \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) &= \left. \frac{d}{d\varepsilon} \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}} + \varepsilon \delta \dot{\mathbf{x}}) \right|_{\varepsilon=0} = \mathbf{M} \cdot \delta \dot{\mathbf{x}} \end{aligned} \quad (26)$$

Considering that $\mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) = 0$, gives

$$-\mathbf{r}(\mathbf{x}_0, \dot{\mathbf{x}}_0) = \mathbf{M} \cdot \delta \dot{\mathbf{x}} - \mathbf{K} \cdot \delta \mathbf{x} \quad (27)$$

It is interesting to note that (27) represents both the equation for spectral instability analysis by the Indirect method of Lyapunov (explained later) and the equations for Newton's method for solving the nonlinear problem with an implicit scheme. To emphasize this point, note that if the initial point $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$ is a solution, then its residual will vanish and (27) will translate into the Lyapunov stability equation in (40). On the other hand, if the residual **does not** vanish, then a typical Newton correction is obtained, as follows

$$\mathbf{J} \delta \mathbf{x} = -\mathbf{r} \quad (28)$$

where $\mathbf{J}$ corresponds to the Jacobian matrix (often called Tangent Stiffness matrix) and is computed by applying the Gâteaux derivative to the appropriate residual. Depending on the time discretization chosen, the Jacobian matrix can be expressed as a linear combination of the matrices $\mathbf{M}$ and $\mathbf{K}$. If the process is assumed to be quasi-static, the Mass matrix $\mathbf{M}$ will simply become zero. Furthermore, for a Backward Euler discretization in time, the Jacobian matrix is given by

$$\mathbf{J} = \frac{1}{\Delta t} \mathbf{M} - \mathbf{K} \quad (29)$$

where Δt is the time step used to advance the solution. Note that the time integration scheme will affect the Jacobian matrix used to obtain a solution at a particular

time step but not the matrices $\mathbf{M}$ and $\mathbf{K}$ used for the stability analysis. This happens because $\mathbf{M}$ and $\mathbf{K}$ are taken after a converged solution at a given time is reached and are therefore independent from the integration scheme.

The block structure of the matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_v & 0 & 0 & 0 \\ 0 & \mathbf{M}_\sigma & 0 & 0 \\ 0 & 0 & \mathbf{M}_T & 0 \\ 0 & 0 & 0 & \mathbf{M}_{\tilde{\gamma}_p} \end{bmatrix} \quad (30)$$

and the matrix $\mathbf{K}$ by

$$\mathbf{K} = \begin{bmatrix} 0 & \mathbf{K}_{L v \sigma} & 0 & 0 \\ \mathbf{K}_{L \sigma v} & \mathbf{K}_{L \sigma \sigma} & \mathbf{K}_{NL \sigma T} & \mathbf{K}_{NL \sigma \tilde{\gamma}_p} \\ 0 & \mathbf{K}_{NL T \sigma} & \mathbf{K}_{L T T} + \mathbf{K}_{NL T T} & \mathbf{K}_{NL T \tilde{\gamma}_p} \\ 0 & \mathbf{K}_{NL \tilde{\gamma}_p \sigma} & \mathbf{K}_{NL \tilde{\gamma}_p T} & \mathbf{K}_{NL \tilde{\gamma}_p \tilde{\gamma}_p} \end{bmatrix} \quad (31)$$

Here $\mathbf{M}$ is the mass matrix and $\mathbf{K} = \mathbf{K}_L + \mathbf{K}_{NL}$ represents the sum of the stiffness matrices associated with linear material behavior such as elasticity and thermal diffusion ($\mathbf{K}_L$) and the tangent stiffness matrices associated with material non-linear behavior ($\mathbf{K}_{NL}$).

The inner blocks in $\mathbf{M}$ and $\mathbf{K}$ are presented in Arriaga et al. [6]. The subscripts on each block indicate the fields which are coupled, thus the pair (i, j) represents the variation in the j direction of the residual associated with field i .

3. Theoretical background

This section focus on the stability analysis of the problem.

In section 3.1 the local analytical instability criterion is recalled, evidentiating the correspondence between this result and experimental observations. It is also

210 shown why this analytical condition, commonly used in the literature, is equiva-
211 lent to the spectral stability analysis in Arriaga et al. [6]. The presentation of these
212 topics is essential for validation of the proposed criterion.

213 In section 3.2 the method of generalized stability analysis is used to derive a set
214 of additional conditions for which instability is expected. There, the relationship
215 between this method and the spectral stability analysis is shown, motivating the
216 prediction of the results presented in section 4.

217 3.1. Local Instability

218 According to Marchand and Duffy [40], three stages in a shear band deformation
219 process can be observed in experiments and are identified in Figure 1. These
220 stages are

- 221 • Stage I: Homogenous plastic deformation
- 222 • Stage II: Inhomogenous plastic deformation
- 223 • Stage III: Highly localized deformation and stress collapse

224 It is worth noting that the concept of homogeneity in this context refers to a ho-
225 mogeneous (as opposed to heterogeneous) distribution of the physical quantities
226 that describe the problem (strain, stress, temperature, etc...) where changes occur
227 smoothly and gradually over the domain.

Figure 1: Stages of Shear band deformation. I-homogeneous solution, II-onset of localization, III-stress collapse.[40]

228 Since it is at stage II that inhomogeneous plastic deformation begins, it is expected
 229 that the instability criterion will predict the initiation of this stage. In fact, as
 230 shown by Bai [8], the analytical local instability condition for this problem is
 231 given by:

$$1 - \frac{\chi \bar{\sigma}_0 P_0}{\rho c_p Q_0} < 0 \quad (32)$$

232 where the subscript 0 indicates the value of that quantity at the current time and

$$P_0 = - \frac{\partial \bar{\sigma}}{\partial T} \Big|_{\mathbf{x}=\mathbf{x}_0} \quad ; \quad Q_0 = \frac{\partial \bar{\sigma}}{\partial \bar{\gamma}_p} \Big|_{\mathbf{x}=\mathbf{x}_0} \quad (33)$$

233 It has been shown in Arriaga et al. [6] that the local instability can be computed
 234 by doing an eigenvalue analysis of the element stiffness matrix. This was more
 235 expensive than the computation of the analytical local instability criterion but was
 236 much easier to implement in more complex situations. Also, this result shows that
 237 the local criterion is in fact a spectral stability analysis.

238 The technique used to prove this result is the indirect method of Lyapunov[43]
 239 which is equivalent to the linear perturbation method. This technique is briefly
 240 explained here to provide insight on the assumptions and simplifications being
 241 made and to show the correspondence with spectral stability analysis.

242 Firstly, consider the system of equations expressed in its autonomous form as

$$\mathcal{L}(\dot{\mathbf{x}}) = F(\mathbf{x}) \quad (34)$$

243 where $\mathbf{x}$ represents all non-constant variables of the problem, $\mathcal{L}$ represents a lin-
 244 ear operator and $F(\mathbf{x})$ is a non-linear function of $\mathbf{x}$.

245 The variables $\mathbf{x}$ can be expressed as a sum of the equilibrium solution with a
 246 perturbation $\delta \mathbf{x}$ as:

$$\mathbf{x} = \mathbf{x}_0 + \delta \mathbf{x} \quad (35)$$

247 Linearizing $F(\mathbf{x})$ by a Taylor Series Expansion around $\mathbf{x}_0$ and considering only
 248 the first order term gives

$$F(\mathbf{x}) = F(\mathbf{x}_0) + F'(\mathbf{x}_0)\delta\mathbf{x} \quad (36)$$

249 Thus, (34) can be written as

$$\mathcal{L}(\dot{\mathbf{x}}_0 + \delta\dot{\mathbf{x}}) = F(\mathbf{x}_0) + F'(\mathbf{x}_0)\delta\mathbf{x} \quad (37)$$

250 Since $\mathbf{x}_0$ is a solution point, $\mathcal{L}(\dot{\mathbf{x}}_0) = F(\mathbf{x}_0)$, which means that

$$\mathcal{L}(\delta\dot{\mathbf{x}}) = \mathbf{K} \cdot \delta\mathbf{x} \quad (38)$$

251 where $\mathbf{K} = F'(\mathbf{x}_0)$ is the linearized Jacobian matrix of $F(\mathbf{x})$ at $\mathbf{x}_0$. In our model,
 252 this matrix will be obtained from mixed the finite element formulation described
 253 earlier. This process centers the stability analysis problem on the equilibrium point
 254 such that it becomes the origin, permitting the use of a Lyapunov indirect stability
 255 analysis[43]. Representing the operator $\mathcal{L}(\cdot)$ by a matrix $\mathbf{M}$ and considering a
 256 linear perturbation of the form

$$\delta\mathbf{x} = \hat{\mathbf{x}}e^{\omega t} \quad (39)$$

257 where $\omega \in \mathbb{C}$ is an eigenvalue at the complex space $\mathbb{C}$ and $\hat{\mathbf{x}}$ its corresponding
 258 eigenvector representing the mode shape of the perturbation, gives the following
 259 generalized eigenvalue problem

$$\omega\mathbf{M}\delta\mathbf{x} = \mathbf{K}\delta\mathbf{x} \quad (40)$$

260 This method of stability analysis was proposed by Lyapunov[39] and has been
 261 referred to as the first method of Lyapunov or the indirect method of Lyapunov[20,
 262 43]. Here we showed that the linear perturbation method leads to an eigenvalue
 263 problem. On the other hand, by applying the linear perturbation method[8] the
 264 local instability criterion is obtained. Therefore, it is clear that the local instability

265 criterion methodology (perturbation) is equivalent to a spectral stability analysis
266 (eigenvalues).

267 3.2. Generalized stability of a perturbation tangent to the solution

268 3.2.1. Tangent perturbation

269 An important quantity of interest for a general stability analysis is the growth rate
270 of a perturbation tangent to the current solution.

271 To understand the importance of this quantity **it is** useful to notice that a typical
272 spectral analysis defines an instability by the existence of a mode with positive
273 exponential growth rate in the system, in which case that mode will completely
274 dominate the solution at $t \rightarrow \infty$. However, this rationale assumes that the operators
275 of the system do not change, or change very slowly in time.

276 If the operators change very quickly, as in the shear band problem, then the growth
277 rate of the unstable perturbation may not be significantly larger than the actual rate
278 of change of the solution. Therefore, the growth of the perturbation might not be
279 fast enough to dominate the solution before the system changes considerably.

280 On the other hand, if the solution is already progressing in an unstable direction,
281 it is expected that this unstable mode will dominate the progress of the solution.

282 One way to obtain the vector that is tangent to the solution is by taking the incre-
283 ment of the solution on a first order integration scheme given by

$$\mathbf{x}^{n+1} = \mathbf{x}^n + \mathbf{x}_s \quad (41)$$

284 where $\mathbf{x}^n$ is the solution at step n and $\mathbf{x}_s$ is the increment of the solution step.

285 The Backward Euler integration scheme assumes that the rate of the increment is
286 the same as the rate at the converged time step (i.e. the tangent of the solution at

287 the converged time step). Hence, the incremental step is given by

$$\mathbf{x}_s = \Delta t \dot{\mathbf{x}}^{n+1} \quad (42)$$

288 with Δt is the time increment and $\dot{\mathbf{x}}^{n+1}$ is the rate of change of the solution at step
289 $n + 1$.

290 This means that the vector $\mathbf{x}_s$ that represents the step from $\mathbf{x}^n$ to the converged
291 solution $\mathbf{x}^{n+1}$ obtained with the Newton Method (where $\mathbf{K}$ and $\mathbf{M}$ were also com-
292 puted for that time step) is in fact the tangent to the solution at $\mathbf{x}^{n+1}$ and the desired
293 vector for our study.

294 One can immediately recognize that no extra effort was put into the calculation
295 of this vector for the Backward Euler integration scheme. If other integration
296 schemes were used, then the rate vector would have to be computed explicitly
297 at the converged time step. Naturally, the discrete solution step might still be an
298 acceptable approximation of the tangent if the time step is not excessively large.
299 On the other hand, even in the Backward Euler scheme, if the time step is large
300 then the operators might change significantly between time steps, rendering the
301 tangent direction less accurate.

302 3.2.2. *Instantaneous growth rate*

303 We define the instantaneous growth rate of a general perturbation vector. For
304 consistency with the literature, let us first define the linearized dynamic operator
305 as

$$\mathbf{A} = \mathbf{M}^{-1} \mathbf{K} \quad (43)$$

306 Therefore, the perturbed problem for a converged solution, as seen in (27), may
307 be expressed as

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} \quad (44)$$

308 Note that, generally speaking, $\mathbf{A}$ is not constant and is only valid for small pertur-

309 bations with respect to the current solution.

310 Assuming that $\mathbf{A}$ is autonomous (linearity is already implied when looking at the
311 perturbed system), the solution at time t is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}_0 \quad (45)$$

312 with t being the time in reference to the current time and $\mathbf{x}_0 = \mathbf{x}(0)$ the perturbation
313 vector which we want to study.

314 The growth of that perturbation vector is therefore given by:

$$\sigma^2(\mathbf{x}_0, t) = \frac{\mathbf{x}(t)^T \mathbf{x}(t)}{\mathbf{x}_0^T \mathbf{x}_0} \quad (46)$$

315 plugging in (45)

$$\sigma^2(\mathbf{x}_0, t) = \frac{\mathbf{x}_0^T e^{\mathbf{A}^T t} e^{\mathbf{A}t} \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{x}_0} \quad (47)$$

316 The instantaneous growth rate will therefore be given by

$$G_I(\mathbf{x}_0) = \lim_{t \rightarrow 0} \frac{\partial \sigma^2(\mathbf{x}_0, t)}{\partial t} \quad (48)$$

317 Since $\mathbf{x}_0$ is not a function of t we can expand the matrix $e^{\mathbf{A}^T t} e^{\mathbf{A}t}$ in its Taylor Series
318 and compute its limit when $t \rightarrow 0$, which leads to:

$$G_I(\mathbf{x}_0) = \frac{\mathbf{x}_0^T H(\mathbf{A}) \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{x}_0} = \frac{\mathbf{x}_0^T \mathbf{A} \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{x}_0} \quad (49)$$

319 where $H(\mathbf{A}) = (\mathbf{A}^T + \mathbf{A})/2$ is the Hermitian part of $\mathbf{A}$.

320 If we revert to using $\mathbf{K}$ and $\mathbf{M}$, then

$$G_I(\mathbf{x}_0) = \frac{\mathbf{x}_0^T \mathbf{M}^{-1} \mathbf{K} \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{x}_0} \quad (50)$$

321 Notice that, if $\mathbf{x}_0$ is an eigenvector, then it is clear that $G_I(\mathbf{x}_0)$ will be the eigen-
 322 value of that eigenvector.

323 This is an important result because it shows why the spectral analysis is usually
 324 considered a good way of detecting instability. The idea is that, if any mode has
 325 a positive real part, then that mode will grow exponentially at a rate given by
 326 the eigenvalue of that mode. If that is the case, then the solution will be mostly
 327 changing in that direction and therefore the tangent of the solution should be very
 328 close to an eigenvalue.

329 As can be observed in Appendix A, in a system with orthogonal operators, the
 330 instantaneous growth rate behaves as a weighted average of the eigenvalues scaled
 331 by the modal participation factors. Therefore, in this type of problem $G_I(\mathbf{x}_0) \leq$
 332 $\max(\omega)$ with ω being the eigenvalues of the system. However, it is known that
 333 non-normal systems might have growth rates larger than the eigenvalues of that
 334 system. Regardless, if the tangent to the solution is close to an eigenvector, then
 335 the difference between $G_I(\mathbf{x}_0)$ and $\max(\omega)$ should be small.

336 3.2.3. Rayleigh Quotient

337 The previous procedure is used to find the instantaneous growth rate of any vector.
 338 However, as mentioned before, the tangent direction is expected to be close to
 339 an eigenvector with an unstable eigenvalue. Therefore, an approximate way of
 340 computing the growth rate is by using the well known Rayleigh Quotient given
 341 as:

$$G_{RQ}(\mathbf{x}_0) = \frac{\mathbf{x}_0^T \mathbf{K} \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{M} \mathbf{x}_0} \quad (51)$$

342 Once again, if $\mathbf{x}_0$ is an eigenvector, then $G_{RQ}(\mathbf{x}_0)$ will be the eigenvalue of that
 343 eigenvector. As can be observed in Appendix B, in a system with orthogonal oper-
 344 ators, the Rayleigh Quotient is also a weighted average of the eigenvalues scaled
 345 by the modal participation factors, but now also scaled by the modal masses.

346 The immediate advantage of the Rayleigh Quotient over the Instantaneous Growth
347 rate is the fact that it **does not** require the inversion of $\mathbf{M}$. Since $\mathbf{M}$ is positive def-
348 inite and well conditioned, this operation will be cheap using iterative methods.
349 However, for large problems and many time steps, this cost might become rele-
350 vant.

351 Another advantage is that static modes¹ will make $\mathbf{M}$ singular and the inversion of
352 $\mathbf{M}$ impossible or very complex (pseudo inverse). This of course should not hap-
353 pen if the Mass matrix is positive definite. Nevertheless, referring to the example
354 for a system with orthogonal operators presented in Appendix B, we notice that if
355 we have a mode that is very close to being a static mode, i.e. with a small modal
356 mass, it is likely that its eigenvalue will be very large. In the instantaneous growth
357 grate of a perturbation formulation in equation 50 nothing is done to counter the
358 influence of this mode, but the Rayleigh Quotient scales down this mode based on
359 its low mass. This becomes a significant concern in non-normal systems since the
360 eigenmodes of the operators are not orthogonal, so that almost insignificant sub-
361 spaces might imply very large modal components (see Appendix C for a pictorial
362 example).

363 4. Numerical Results

364 In this section we choose numerical examples where the local instability triggers
365 the solution to start evolving mainly in the direction of localization. In other
366 words, when local instability happens, the direction of the solution change will al-
367 most immediately be dominated by the localization direction. These type of prob-
368 lems usually model very small scales where the elastic wave propagation plays
369 an insignificant role. Problems with larger scales will have other effects included
370 in the direction of the solution change, like elastic wave propagation and reflec-
371 tion, stress redistribution, shear band propagation, etc. These effects counter the

¹Static modes: Modes with modal mass equal to zero.

property of the small scale examples where the local instability triggers the global localization.

We study the behavior on 1D and 2D shearband benchmark problems. The material considered in both examples is a 4340 Steel with a Johnson-Cook constitutive law [30] given in (52). The parameters associated with this material are presented in Tables 1 and 2.

$$\bar{\sigma} = \left[A + B(\bar{\gamma}_p)^N \right] \left[1 - \left(\frac{T - T_0}{T_m - T_0} \right)^m \right] \left[1 + c \ln \left(\frac{\dot{\gamma}_p}{\dot{\gamma}_{pr}} \right) \right] \quad (52)$$

Table 1: Material properties for 4340 Steel

Property Name	Symbol	Value	Unit
Mass density	ρ	7830	kg/m ³
Specific heat	c_p	477	J/kgK
Thermal conductivity	λ	38	W/mK
Young's modulus	E	200	GPa
Poisson's ratio	μ	0.29	-
Shear modulus	G	77.5	GPa
Taylor-Quinney coefficient	χ	0.9	-

Table 2: Johnson-Cook parameters for 4340 Steel

Property Name	Symbol	Value	Unit
Yield shear stress	A	792	MPa
Shear stress hardening parameter	B	510	MPa
Strain hardening parameter	N	0.26	-
Reference Temperature	T_0	298	K
Melting Temperature	T_m	1793	K
Thermal softening exponent	m	1.03	-
Strain-rate hardening parameter	c	0.014	-
Reference strain-rate	$\dot{\gamma}_r^p$	1.0	1/s

Neumann boundary conditions of zero flux for the Temperature field are applied on all edges and plane strain conditions are assumed when applicable.

380 The small strain simplification is used without loss of generality, since the focus
 381 of this work is on the behavior of the instability measures and not the quantitative
 382 results obtained from the modeling.

383 An hyperbolic secant type of imperfection is used in order to trigger the shearband.
 384 Hence, in the analysis we scale by $\eta_{imp}(r_n)$ the parameters A and B , with $\eta_{imp}(r_n)$
 385 given by

$$\eta_{imp}(r_n) = 1 - \alpha_{red} \left(\frac{2}{e^{r_n} + e^{-r_n}} \right) \quad (53)$$

386 where α_{red} is the reduction of the material parameters in percentage at the center
 387 of the imperfection and r_n is the normalized distance given by

$$r_n(x, y) = \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{r_0} \quad (54)$$

388 with x_0 and y_0 representing the center of the imperfection and r_0 its radius.

389 A velocity-type loading is applied as dirichlet boundary conditions. Velocity con-
 390 trol is defined as shown in figure 2 with v_r being the steady velocity and t_r the
 391 time to reach that velocity. The S-curve is a polynomial that guarantees that the
 392 derivative of the Jerk² is continuous. We choose this to avoid discontinuities and
 393 kinks in the profile of the Jerk since the thermo-mechanical coupled system is sen-
 394 sitive to the applied velocity and may generate jumps in the instantaneous stability
 395 measure.

396 4.1. 1D rod under uniform shear

397 4.1.1. Problem description

398 A steel rod is studied using a 1D formulation often employed as an idealized ver-
 399 sion of the Torsional Kolsky Bar experiment [21, 28, 32, 54], which is commonly

²The Jerk is the derivative of the Acceleration with respect to time and is the quantity respon-
 sible for generating elastic waves

Figure 2: Velocity profile used as a loading function to generate the shearband.

used to study a pure shear deformation without necking effects [11, 58].

To capture the shear band localization more accurately, the rod is discretized with a non-constant element size. Considering symmetry, the relative position of the elements (p) from the center to one edge (half-rod) can be defined by the following function

$$p(\bar{x}) = \begin{cases} S_0 \bar{x} & \text{for } \bar{x} \leq \beta \\ S_0 \bar{x} + (1 - S_0) \left(\frac{\bar{x} - \beta}{1 - \beta} \right)^P & \text{for } \bar{x} > \beta \end{cases} \quad (55)$$

where $\bar{x}$ the relative node number, P is the exponent used for varying the element size, β is the percentage of nodes concentrated at the center of the rod with constant element size and S_0 is the slope for the central elements, defined as:

$$S_0 = \frac{d_e}{L} N \quad (56)$$

where L and N are respectively the length and the number of nodes of the half-rod and d_e defines the constant element size in the center of the rod.

In the following sections, the results are presented for $N = 201$, $\beta = 0.25$, $P = 5$, $\frac{d_e}{L} = 1 \times 10^{-3}$ and $L = 0.5 \times 10^{-3}$ m. All values refer to the half-rod, which means that the final length is 10^{-3} m. The total number of nodes is 401 since the central node is repeated. Figure 3a shows the relative position of the nodes in the half-rod as a function of the relative node number and figure 3b shows the relative element size also as a function of the relative node number.

Figure 3: Representation of a 1D right half-rod. The black dot marks β

As defined before, the material considered for the 1D problem is a 4340 Steel with a Johnson-Cook constitutive law [30] and an hyperbolic secant type of imperfection is considered at the center of the rod with $\alpha_{red} = 0.01$, $r_0 = 10\mu\text{m}$ and $x_0 = L/2 = 500\mu\text{m}$.

The analysis is performed on a range of nominal strain rates between 10^1s^{-1} and 10^4s^{-1} by applying the velocity profile shown in figure 2 to both edges of the rod. Our goal is to detect the onset of shearband localization using the proposed methods: Instantaneous growth rate method and Rayleigh Quotient method and compare those with the analytical criteria proposed by Bai [8].

4.1.2. Results

Consider Figure 4 that represents the shear stress at the edge of the rod as a function of the nominal strain (Fig.4a) and the equivalent plastic strain along the rod at different moments in time (Fig. 4b). The various deformation states are illustrated by the different colors. It can clearly be observed that the plastic strain grows uniformly along the rod during the initial stage of the deformation (disregarding the slight inhomogeneity due to the imperfection). This is consistent with the experimental results of Marchand and Duffy [40] as it corresponds to Stage

433 I of Figure 1. Only after the onset of instability and the initiation of Stage II do
 434 we start to observe an inhomogeneous distribution of the solution with increasing
 435 growth rate.

436 The special characteristic of this problem is precisely the fact that plastic deformation starts to develop homogeneously and that localization is globally initiated
 437 when Stage II starts. These types of problems are only realistic when considering
 438 a small portion of material at the vicinity of the shear band, and therefore are
 439 not big enough to model shear band propagation. However, they still provide a
 440 good way to model the region of the shear band and its immediate vicinity as a
 441 comparison with the experimental work of Marchand and Duffy [40] suggests.
 442

Figure 4: Behavior of the rod with progression of time. 4a shows the stress-strain behaviour of a rod being sheared at a strain-rate of $10^4 s^{-1}$. Several points are chosen for which the entire distribution of the plastic strain along the rod is represented in 4b. The red indicator marks the onset of instability.

443 Figure 5 shows the comparison in the detection of onset of shear band localization
 444 between the analytical stability criteria for the central point of the 1D rod (Eq.
 445 32), the Rayleigh Quotient (Eq. 51) and the instantaneous growth rate (Eq. 50).
 446 Five different nominal strain rates are tested. The gray line represents the stress at
 447 the center of the rod and the blue, green and red lines correspond to the analytical
 448 criterion, the Rayleigh Quotient and the instantaneous growth rate, respectively. It

449 can be seen that all three measures agree and approximate well the local instability
 450 point.

Figure 5: Stability analysis of 1D rod (center point) at different strain rates. The local instability condition given by Eq. 32 (where a negative value corresponds to instability) is compared to the Rayleigh Quotient as given in (51) and the Instantaneous Growth rate as given in (50).

451 The results in Figure 5 can be summarized in a single plot as shown in Figure 6.
 452 There, the nominal strain at the points where the local stability criterion, the
 453 Rayleigh Quotient and the instantaneous growth rate change sign and the peak
 454 stress at the center of the rod are represented as a function of the strain-rate. As
 455 expected, the instability has a trend similar to the peak stress and all measures

456 for instability agree. For a simple problem like this, the peak stress is occurring
 457 roughly at a given critical value of accumulated inelastic work since plastic strain
 458 is the single source of the softening and the solution is spatially homogeneous up
 459 to the peak (ignoring the effect of the imperfection). Given that diffusion **does not**
 460 play a role when the solution is homogeneous, we would expect that the integral of
 461 the stress-strain curve from zero to the peak stress to be independent of the strain
 462 rate. Therefore, since the yield stress increases with strain rate, the strain for the
 463 peak stress would have to reduce slightly to maintain the value of accumulated
 464 inelastic work. This can be observed in Figure 5 from the downward slope of the
 465 maximum stress curve with respect to the strain rate.

Figure 6: Instability points as a function of the strain rate.

466 It becomes clear that both methods proposed in this paper for detection of the
 467 instability produce a good approximation of the instability point and agree well
 468 with the analytical criteria of Bai [8]. Consequently, it is possible to use these
 469 methods to provide a cheap and convenient alternative to detect the instability
 470 point when it is difficult to formulate the analytical criterion, which is the case in

471 general general shear band problems.

472 4.2. 45° shear band under uniform compression

473 4.2.1. Problem description

474 In this section a square plate is compressed under uniform impact loading gener-
 475 ating a shear band oriented at a 45° angle. The geometry of the square plate is
 476 given in Figure 7 with the edge dimension of $D=100\mu\text{m}$.

Figure 7: A square plate under uniform impact loading. The colors represent the factor η_{imp} defined in equation (53). Also indicated are two corner points that are used for plotting purposes: the bottom-left corner (BL) and the top-right corner (TR).

477 The mesh consists of 40×40 quad elements, modeled by an irreducible mixed-
 478 finite element formulation with $\bar{\mathbf{B}}$ -bar to reduce shear-locking effects. The veloc-
 479 ity and temperature fields are discretized with bilinear C^0 continuous functions
 480 and the stress and equivalent plastic strain are piecewise C^{-1} continuous. Details
 481 on the numerical implementation can be found in Arriaga et al. [6], Berger-Vergiat
 482 et al. [16], McAuliffe and Waisman [41, 42].

483 As defined before, the material considered is a 4340 Steel with a Johnson-Cook
 484 constitutive law [30] and an hyperbolic secant type of imperfection (defined earlier
 485 in equation (53) and graphically represented in Figure 7) is considered with $\alpha_{red} =$
 486 0.1 , $r_0 = D/10$ and $x_0 = y_0 = 0\mu\text{m}$. Note that this location (indicated in figure 7

487 as BL) will be the first to become unstable and for that reason it is used as the
488 reference location for the local instability behavior.

489 The plate is compressed uniaxially with a velocity control loading following the
490 profile shown in Figure 2 with $v_r=10\text{m/s}$ and $t_r=0.4\mu\text{s}$, which corresponds to a
491 nominal strain rate of 10^5s^{-1} for this plate.

492 4.2.2. Results

493 Similarly to the 1D case, this model is part of a particular set of examples where
494 the inhomogeneous deformation triggered locally develops almost simultaneously
495 in the entire plate. This effect comes as a consequence of the small size of the plate
496 width and is realistic only for studying the shear band region and its immediate
497 vicinity.

498 Figure 8 shows the comparison between the Von-Mises stress at the bottom left
499 corner of plate where the imperfection is maximum (BL), the stress at top right
500 corner where the shear band passes but the imperfection is practically zero (TR)
501 and the average Von-Mises stress on the plate. This comparison shows that the
502 response in the early stage of yielding is uniform (apart from the imperfection)
503 and that the nature of the differences between stresses are only geometrical and
504 not temporal. This illustrates well that both the elastic wave propagation speed
505 and the shear band propagation speed are too large to be causing any significant
506 effect on a plate of the size being modeled. Once again, this is only realistic
507 when studying the vicinity of the shear band since for larger scales elastic wave
508 propagation and shear band propagation play significant roles.

509 Figure 9 shows the values of the analytical stability criterion (Eq. 32), the Rayleigh
510 Quotient (Eq. 51) and the instantaneous growth rate (Eq. 50) for the reference po-
511 sition (BL). The gray line represents the Von-Mises stress and the blue, green and
512 red lines correspond to the analytical criterion, the Rayleigh Quotient and the in-
513 stantaneous growth rate, respectively. It is shown that both quantities approximate
514 well the local instability point. Additionally, notice that the growth rate stabilizes

Figure 8: Evolution of Von-Mises stress at the maximum imperfection point (BL-bottom left corner), the shear band imperfection free point (TR - top right corner) and the average Von-Mises stress on the plate as function of the nominal strain.

515 in the long term since the solution is now evolving in a steady-state like path,
 516 where the heat generated at the shear band is balanced by the thermal diffusion.

517 In Figure 10 the equivalent plastic strain field is represented at four different mo-
 518 ments in time showing the three stages mentioned earlier in section 3.1. Fig-
 519 ure 10b shows a moment before the instability point, where the deformation, even
 520 though it has considerable plasticity, is still homogeneous (besides the imperfec-
 521 tion). The second figure (Fig. 10c) shows a moment immediately after the insta-
 522 bility point and once again, apart from the effect of the imperfection, the velocity
 523 field looks homogeneous. The next figure (Fig. 10d) shows an intermediate time
 524 in stage II and mild localization can be observed. Finally, the last figure (Fig. 10e)
 525 shows a time well into stage III where extreme localization is visible. Notice that
 526 the shear band appears to be relatively wide in this representation due to the small
 527 size of the model. However, taking the dimensions of the plate into consideration,
 528 the shear band is of the order of $10\text{-}30\mu\text{m}$. As expected, the non-homogeneous de-
 529 formation associated with a shear band only occurs after the instability point that

Figure 9: Stability analysis of the 2D plate at the reference position (BL) for a strain rate of 10^5 s^{-1} . The local instability condition given by Eq. 32 (where a negative value corresponds to instability) is compared to the Rayleigh Quotient as given in (51) and the Instantaneous Growth rate as given in (50).

initiates stage II. This confirms the predictive capability of the proposed method when compared to experimental results.

5. Concluding Remarks

In this work we have shown that the instantaneous growth rate of the vector tangent to the solution, derived from general stability analysis, predicts reasonably well the onset of shear band localization. This technique can easily be implemented within existing implicit FEM codes since the matrices and vector required to obtain the growth rates are the same as those computed for the Newton iterations. Moreover, we have also demonstrated that the instantaneous growth rate approach can be simplified into a special choice of the Rayleigh Quotient approximation which requires a few matrix vector products and therefore leads to a

(a) Identification of selected points. The black line is the stress at the reference point (BL) and the gray line the value of G_{RQ} .

(b) Stage I: $t=2.0 \mu s$

(c) Instability point: $t=3.4 \mu s$

(d) Stage II: $t=5.0 \mu s$

(e) Stage III: $t=10.0 \mu s$

Figure 10: Representation of the equivalent plastic strain field for the three different stages mentioned in section 3.1.

541 more efficient implementation. The Rayleigh Quotient approximation of the in-
542 stantaneous growth rate works well when the global localization of the model is
543 triggered by the appearance of local instability.

544 Additionally, the proposed methodology is particularly useful to study problems
545 for which an analytical criterion depends on functions that are fitted to experimen-
546 tal results, for example when different constitutive laws are used, thus making the
547 analytical criterion case specific and hence more cumbersome to derive.

548 Another advantage of this methodology is the possibility to modify the equations
549 to include additional phenomena like non-local techniques, phase field or non-
550 constant parameters like the Taylor Quinney coefficient, thus avoiding having to
551 derive the analytical criterion for it. This is of significant importance because,
552 even though the analytical criterion might convey more information about the
553 location of the unstable regions of the material, it is usually difficult to derive
554 and simplifying assumptions might be necessary to obtain it.

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558 **Appendix A. Instantaneous Growth rate on an system with orthogonal oper-**
 559 **ators**

560 The instantaneous growth rate of a vector $\mathbf{x}$ is given by

$$G_I = \frac{\mathbf{x}^T \mathbf{M}^{-1} \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \quad (\text{A.1})$$

561 a modal decomposition of $\mathbf{x}$ yields

$$\mathbf{x} = \mathbf{Q} \mathbf{v} \quad (\text{A.2})$$

562 where $\mathbf{Q}$ is the matrix that defines the eigenspace of the operators and $\mathbf{v}$ the com-
 563 ponents on the eigenspace resulting from the decomposition of $\mathbf{x}$.

564 Therefore

$$G_I = \frac{\mathbf{v}^T \mathbf{Q}^T \mathbf{M}^{-1} \mathbf{Q} \mathbf{Q}^{-1} \mathbf{K} \mathbf{Q} \mathbf{v}}{\mathbf{v}^T \mathbf{Q}^T \mathbf{Q} \mathbf{v}} \quad (\text{A.3})$$

565 If the operators are normal, then $\mathbf{Q}^{-1} = \mathbf{Q}^T$, which gives

$$G_I = \frac{\mathbf{v}^T \mathbf{M}^{*-1} \mathbf{K}^* \mathbf{v}}{\mathbf{v}^T \mathbf{v}} \quad (\text{A.4})$$

566 with $\mathbf{M}^* = \mathbf{Q}^T \mathbf{M} \mathbf{Q}$ and $\mathbf{K}^* = \mathbf{Q}^T \mathbf{K} \mathbf{Q}$. Since $\mathbf{M}^*$ and $\mathbf{K}^*$ are diagonal because of
 567 orthogonality:

$$G_I = \frac{\sum_i \mathbf{v}_i^2 \omega_i}{\sum_i \mathbf{v}_i^2} \quad (\text{A.5})$$

568 with $\mathbf{v}_i$ as the i^{th} component of the modal decomposition of $\mathbf{x}$, $\omega_i = \frac{\mathbf{K}_i^*}{\mathbf{M}_i^*}$ as the
 569 eigenvalue associated with the i^{th} mode and $\mathbf{K}_i^*$ and $\mathbf{M}_i^*$ as the i^{th} diagonal entry
 570 of $\mathbf{K}^*$ and $\mathbf{M}^*$, respectively.

571 **Appendix B. Rayleigh Quotient on an system with orthogonal operators**

572 The Rayleigh Quotient of a vector $\mathbf{x}$ is given by

$$G_{RQ} = \frac{\mathbf{x}^T \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} \quad (\text{B.1})$$

573 a modal decomposition of $\mathbf{x}$ yields

$$\mathbf{x} = \mathbf{Q} \mathbf{v} \quad (\text{B.2})$$

574 where $\mathbf{Q}$ is the matrix that defines the eigenspace of the operators and $\mathbf{v}$ the com-
575 ponents on the eigenspace resulting from the decomposition of $\mathbf{x}$.

576 Therefore

$$G_{RQ} = \frac{\mathbf{v}^T \mathbf{Q}^T \mathbf{K} \mathbf{Q} \mathbf{v}}{\mathbf{v}^T \mathbf{Q}^T \mathbf{M} \mathbf{Q} \mathbf{v}} = \frac{\mathbf{v}^T \mathbf{K}^* \mathbf{v}}{\mathbf{v}^T \mathbf{M}^* \mathbf{v}} \quad (\text{B.3})$$

577 with $\mathbf{M}^* = \mathbf{Q}^T \mathbf{M} \mathbf{Q}$ and $\mathbf{K}^* = \mathbf{Q}^T \mathbf{K} \mathbf{Q}$.

578 Since $\mathbf{M}^*$ and $\mathbf{K}^*$ are diagonal because of orthogonality:

$$G_{RQ} = \frac{\sum_i \mathbf{v}_i^2 \mathbf{K}_i^*}{\sum_i \mathbf{v}_i^2 \mathbf{M}_i^*} = \frac{\sum_i \mathbf{v}_i^2 \omega_i \mathbf{M}_i^*}{\sum_i \mathbf{v}_i^2 \mathbf{M}_i^*} \quad (\text{B.4})$$

579 with $\mathbf{v}_i$ as the i^{th} component of the modal decomposition of $\mathbf{x}$, $\omega_i = \frac{\mathbf{K}_i^*}{\mathbf{M}_i^*}$ as the
580 eigenvalue associated with the i^{th} mode and $\mathbf{K}_i^*$ and $\mathbf{M}_i^* > 0$ as the i^{th} diagonal
581 entry of $\mathbf{K}^*$ and $\mathbf{M}^*$, respectively.

582 Appendix C. Non-orthogonality amplification

583 In Figure C.1 is shown the effect of doing a decomposition of a vector in a non-
584 orthogonal basis. We can observe that the components of a vector in this basis
585 might have a disproportionate weights when compared to the actual magnitude
586 that they represent.

Figure C.1: Amplification of the relative weight of the components on a non-orthogonal decomposition

References

- [1] Abu Al-Rub, R. K., Voyiadjis, G. Z., 2006. A Finite Strain Plastic-damage Model for High Velocity Impact using Combined Viscosity and Gradient Localization Limiters: Part I - Theoretical Formulation. *International Journal of Damage Mechanics* 15 (4), 293–334.
- [2] Abu Al-Rub, R. K., Voyiadjis, G. Z., 2006. A physically based gradient plasticity theory. *International Journal of Plasticity* 22 (4), 654–684.
- [3] Aifantis, E. C., 1984. On the Microstructural Origin of Certain Inelastic Models. *Journal of Engineering Materials and Technology* 106 (4), 326–330.
- [4] Aifantis, E. C., 1987. The physics of plastic deformation. *International Journal of Plasticity* 3 (3), 211–247.
- [5] Anand, L., Kim, K., Shawki, T., 1987. Onset of shear localization in viscoplastic solids. *Journal of the Mechanics and Physics of Solids* 35 (4), 407–429.
- [6] Arriaga, M., McAuliffe, C., Waisman, H., 2015. Onset of shear band localization by a local generalized eigenvalue analysis. *Computer Methods in Applied Mechanics and Engineering* 289, 179–208.
- [7] Bažant, Z. P., Jirásek, M., 2002. Nonlocal integral formulations of plasticity and damage: survey of progress. *Journal of Engineering Mechanics* 128 (11), 1119–1149.
- [8] Bai, Y., 1982. Thermo-plastic instability in simple shear. *Journal of the Mechanics and Physics of Solids* 30 (4), 195–207.
- [9] Batra, R., 1987. The initiation and growth of, and the interaction among, adiabatic shear bands in simple and dipolar materials. *International Journal of Plasticity* 3 (1), 75–89.
- [10] Batra, R., Chen, L., 2001. Effect of viscoplastic relations on the instability strain, shear band initiation strain, the strain corresponding to the minimum shear band spacing, and the band width in a thermoviscoplastic material. *International Journal of Plasticity* 17 (11), 1465–1489.
- [11] Batra, R., Kim, C., 1992. Analysis of shear banding in twelve materials. *International Journal of Plasticity* 8 (4), 425–452.
- [12] Batra, R., Wei, Z., 2007. Instability strain and shear band spacing in simple tensile/compressive deformations of thermoviscoplastic materials. *International Journal of Impact Engineering* 34 (3), 448–463.

- 617 [13] Baucom, J. N., Zikry, M. A., 1999. Perturbation analysis of high strain-rate shear localization
618 in B.C.C. crystalline materials. *Acta Mechanica* 137 (1-2), 109–129.
- 619 [14] Belytschko, T., Chiang, H.-Y., Plaskacz, E., 1994. High resolution two-dimensional shear
620 band computations: imperfections and mesh dependence. *Computer Methods in Applied
621 Mechanics and Engineering* 119 (1–2), 1–15.
- 622 [15] Belytschko, T., Loehnert, S., Song, J.-H., 2008. Multiscale aggregating discontinuities: A
623 method for circumventing loss of material stability. *International Journal for Numerical
624 Methods in Engineering* 73 (6), 869–894.
- 625 [16] Berger-Vergiat, L., McAuliffe, C., Waisman, H., 2014. Isogeometric analysis of shear bands.
626 *Computational Mechanics*, 1–19.
- 627 [17] Bigoni, D., 2000. Bifurcation and instability of non-associative elastoplastic solids. In: *Ma-
628 terial Instabilities in Elastic and Plastic Solids*, Petryk H. Springer, pp. 1–52.
- 629 [18] Bigoni, D., Zaccaria, D., 1994. On the eigenvalues of the acoustic tensor in elastoplasticity.
630 *Eur. J. Mech., A/Solids* 13 (5), 621–638.
- 631 [19] Borja, R. I., 2002. Bifurcation of elastoplastic solids to shear band mode at finite strain.
632 *Computer Methods in Applied Mechanics and Engineering* 191 (46), 5287–5314.
- 633 [20] Chen, G., 2001. Stability of Nonlinear Systems. In: *Wiley Encyclopedia of Electrical and
634 Electronics Engineering*. John Wiley & Sons, Inc., pp. 4881–4896.
- 635 [21] Chen, W. W., Song, B., 2010. *Split Hopkinson (Kolsky) Bar: Design, Testing and Applica-
636 tions*. Springer.
- 637 [22] Dai, L., Bai, Y., 2008. Basic mechanical behaviors and mechanics of shear banding in BMGs.
638 *International Journal of Impact Engineering* 35 (8), 704–716.
- 639 [23] Eringen, A. C., 2002. *Nonlocal Continuum Field Theories*. Springer New York.
- 640 [24] Farrell, B. F., Ioannou, P. J., 1996. Generalized Stability Theory. Part I: Autonomous Opera-
641 tors. *Journal of the Atmospheric Sciences* 53 (14), 2025–2040.
- 642 [25] Farrell, B. F., Ioannou, P. J., 1996. Generalized Stability Theory. Part II: Nonautonomous
643 Operators. *Journal of the Atmospheric Sciences* 53 (14), 2041–2053.
- 644 [26] Fressengeas, C., Molinari, A., 1987. Instability and localization of plastic flow in shear at
645 high strain rates. *Journal of the Mechanics and Physics of Solids* 35 (2), 185–211.

- 646 [27] Gajo, A., Bigoni, D., Wood, D., 2004. Multiple shear band development and related in-
647 stabilities in granular materials. *Journal of the Mechanics and Physics of Solids* 52 (12),
648 2683–2724.
- 649 [28] Hopkinson, B., 1914. A Method of Measuring the Pressure Produced in the Detonation of
650 High Explosives or by the Impact of Bullets. *Philosophical Transactions of the Royal Society*
651 of London. Series A, Containing Papers of a Mathematical or Physical Character 213 (497-
652 508), 437–456.
- 653 [29] Hor, A., Morel, F., Lebrun, J.-L., Germain, G., 2013. Modelling, identification and applica-
654 tion of phenomenological constitutive laws over a large strain rate and temperature range.
655 *Mechanics of Materials* 64, 91 – 110.
- 656 [30] Johnson, G. R., Cook, W. H., 1985. Fracture characteristics of three metals subjected to
657 various strains, strain rates, temperatures and pressures. *Engineering Fracture Mechanics*
658 21 (1), 31 – 48.
- 659 [31] Kim, K. H., 1987. Shear Localization in Viscoplastic Solids. Ph.D. Dissertation, Department
660 of Mechanical Engineering, Massachusetts Institute of Technology.
- 661 [32] Kolsky, H., 1949. An Investigation of the Mechanical Properties of Materials at very High
662 Rates of Loading. *Proceedings of the Physical Society. Section B* 62 (11), 676–700.
- 663 [33] Lages, E. N., Paulino, G. H., Menezes, I. F. M., Silva, R. R., 1999. Nonlinear Finite Element
664 Analysis using an Object-Oriented Philosophy – Application to Beam Elements and to the
665 Cosserat Continuum. *Engineering with Computers* 15 (1), 73–89.
- 666 [34] Lemonds, J., Needleman, A., 1986. Finite element analyses of shear localization in rate and
667 temperature dependent solids. *Mechanics of Materials* 5 (4), 339–361.
- 668 [35] Li, S., Hao, W., Liu, W. K., 2000. Mesh-free simulations of shear banding in large deforma-
669 tion. *International Journal of Solids and Structures* 37 (48–50), 7185–7206.
- 670 [36] Li, S., Liu, W. K., 2000. Numerical simulations of strain localization in inelastic solids us-
671 ing mesh-free methods. *International Journal for Numerical Methods in Engineering* 48 (9),
672 1285–1309.
- 673 [37] Li, S., Liu, W. K., Rosakis, A. J., Belytschko, T., Hao, W., 2002. Mesh-free Galerkin simu-
674 lations of dynamic shear band propagation and failure mode transition. *International Journal*
675 *of Solids and Structures* 39 (5), 1213–1240.

- 676 [38] Ling, X., Belytschko, T., 2009. Thermal softening induced plastic instability in rate-
677 dependent materials. *Journal of the Mechanics and Physics of Solids* 57 (4), 788–802.
- 678 [39] Lyapunov, A. M., 1992. *General Problem of the Stability Of Motion*, 1st Edition. CRC Press.
- 679 [40] Marchand, A., Duffy, J., 1988. An experimental study of the formation process of adiabatic
680 shear bands in a structural steel. *Journal of the Mechanics and Physics of Solids* 36 (3),
681 251–283.
- 682 [41] McAuliffe, C., Waisman, H., 2013. Mesh insensitive formulation for initiation and growth
683 of shear bands using mixed finite elements. *Computational Mechanics* 51 (5), 807–823.
- 684 [42] McAuliffe, C., Waisman, H., 2014. A Pian–Sumihara type element for modeling shear bands
685 at finite deformation. *Computational Mechanics* 53 (5), 925–940.
- 686 [43] Murray, R. M., Li, Z., Sastry, S. S., 1994. *A Mathematical Introduction to Robotic Manipu-*
687 *lation*. CRC Press.
- 688 [44] Pan, J., 1983. Perturbation analysis of shear strain localization in rate sensitive materials.
689 *International Journal of Solids and Structures* 19 (2), 153–164.
- 690 [45] Peerlings, R., Geers, M., de Borst, R., Brekelmans, W., 2001. A critical comparison of nonlo-
691 cal and gradient-enhanced softening continua. *International Journal of Solids and Structures*
692 38 (44–45), 7723–7746.
- 693 [46] Rabczuk, T., Areias, P. M. A., Belytschko, T., 2007. A simplified mesh-free method for shear
694 bands with cohesive surfaces. *International Journal for Numerical Methods in Engineering*
695 69 (5), 993–1021.
- 696 [47] Rabczuk, T., Samaniego, E., 2008. Discontinuous modelling of shear bands using adaptive
697 meshfree methods. *Computer Methods in Applied Mechanics and Engineering* 197 (6–8),
698 641–658.
- 699 [48] Simo, J. C., Hughes, T. J. R., 1998. *Computational Inelasticity*. Springer.
- 700 [49] Song, J.-H., Areias, P. M. A., Belytschko, T., 2006. A method for dynamic crack and shear
701 band propagation with phantom nodes. *International Journal for Numerical Methods in En-*
702 *gineering* 67 (6), 868–893.
- 703 [50] Szabó, L., 1997. On the Eigenvalues of the Fourth-Order Constitutive Tensor and Loss of
704 Strong Ellipticity in Elastoplasticity. *International Journal of Plasticity* 13 (10), 809–835.

- 705 [51] Tabarraei, A., Song, J.-H., Waisman, H., 2013. A Two-scale Strong Discontinuity Approach
706 For Evolution Of Shear Bands Under Dynamic Impact Loads. *International Journal for Mul-*
707 *tiscale Computational Engineering* 11 (6), 543–563.
- 708 [52] Taylor, G. I., Quinney, H., 1934. The Latent Energy Remaining in a Metal after Cold Work-
709 ing. *Proceedings of the Royal Society of London. Series A* 143 (849), 307–326.
- 710 [53] Voyiadjis, G. Z., Abu Al-Rub, R. K., 2006. A Finite Strain Plastic-damage Model for High
711 Velocity Impacts using Combined Viscosity and Gradient Localization Limiters: Part II -
712 Numerical Aspects and Simulations. *International Journal of Damage Mechanics* 15 (4),
713 335–373.
- 714 [54] Wright, T., 1990. Approximate analysis for the formation of adiabatic shear bands. *Journal*
715 *of the Mechanics and Physics of Solids* 38 (4), 515–530.
- 716 [55] Wright, T., 1992. Shear band susceptibility: Work hardening materials. *International Journal*
717 *of Plasticity* 8 (5), 583–602.
- 718 [56] Wright, T., Batra, R., 1985. The initiation and growth of adiabatic shear bands. *International*
719 *Journal of Plasticity* 1 (3), 205–212.
- 720 [57] Wright, T. W., 2002. The physics and mathematics of adiabatic shear bands. Cambridge
721 University Press.
- 722 [58] Wright, T. W., Walter, J. W., 1987. On stress collapse in adiabatic shear bands. *Journal of the*
723 *Mechanics and Physics of Solids* 35 (6), 701–720.
- 724 [59] Xue, L., Belytschko, T., 2010. Fast methods for determining instabilities of elastic–plastic
725 damage models through closed-form expressions. *International Journal for Numerical Meth-*
726 *ods in Engineering* 84 (12), 1490–1518.
- 727 [60] Zhou, M., Needleman, A., Clifton, R., 1994. Finite element simulations of shear localization
728 in plate impact. *Journal of the Mechanics and Physics of Solids* 42 (3), 423–458.
- 729 [61] Zhou, M., Rosakis, A., Ravichandran, G., 1996. Dynamically propagating shear bands in
730 impact-loaded prenotched plates—I. Experimental investigations of temperature signatures
731 and propagation speed. *Journal of the Mechanics and Physics of Solids* 44 (6), 981–1006.