Dylan's Units Coordinating Across Contexts

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Abstract

Units coordination has emerged as an important construct for understanding students' mathematical thinking, particularly their concepts of multiplication and fractions. To explore students' units coordination development, we conducted an eleven-session constructivist teaching experiment with a pair of sixth-grade students, investigating how they coordinated whole number and fractional units in discrete and continuous settings. In this paper we focus on one student, Dylan, who reasoned with whole number units but not fractional units at the beginning of the teaching experiment. We describe Dylan's development of units coordination as he continued to reason with whole number units in fractional situations, and we discuss implications for instruction.

Keywords: Rational Numbers, Instructional Activities, Units Coordination, Teaching Experiment, Fractions

Dylan's Coordinating Units Across Contexts

Many elementary curricular standards now include an early focus on conceptions of fractions as measures (Lamon, 2007) alongside conceptual understanding of whole number arithmetic. For instance, the United States' *Common Core State Standards for School Mathematics* includes both “represent and solve problems involving multiplication and division” and “understanding fractions as numbers [that can be plotted on a number line]” as third-grade objectives (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2012, p. 21). Researchers have identified a shared cognitive necessity for understanding fractions and for conceptualizing multiplication (and division) – an ability to *coordinate multiple levels of units* (Hackenberg & Tillema, 2009; Steffe, 1992; Steffe & Olive, 2010).

Units coordination has emerged as an important construct for understanding mathematical thinking after elementary school as well. These domains include students' writing of linear equations involving unknown quantities (Hackenberg & Lee, 2015; Olive & Çağalan, 2008), students' ways of operating additively with signed quantities (such as integers; Ulrich, 2012), students' combinatorial reasoning (Tillema, 2015), and teachers' interpretations of fractions representations (Izsák, Jacobsen, de Araujo, & Orhill, 2012). Meaningful attainment of middle and secondary school learning goals is likely to continue to present a challenge to teachers and students, as research suggests that many students enter sixth-grade yet to coordinate multiple levels of units (Boyce & Norton, 2016; Hackenberg, 2013; Hackenberg & Lee, 2015; Norton & Boyce, 2013).

In this paper, we report on an investigation of how engagement in different mathematical situations might foster middle-grades students' development of units coordination. More

specifically, we consider an integrated development of whole number concepts and fractions concepts, with students' development of structures that apply to both situations as an underlying objective. We focus our analysis on a single sixth-grade student, Dylan, who participated in paired-student constructivist teaching experiment (Steffe & Thompson, 2000). We analyze Dylan's units coordination as he engaged with fractions and whole number tasks in discrete and continuous settings. We describe how Dylan progressed in his units coordination development even as he consistently approached fractions tasks by reasoning with whole number units instead of fractional units.

Theoretical and Conceptual Framing

We adopted a scheme theoretic perspective for our study (von Glasersfeld, 1995). In scheme theory, mental activity in service of a goal begins with the recognition of a salient situation; i.e., an individual's fitting of a situational goal to a *scheme* for which a sequence of available mental actions (*operations*) are expected to yield a satisfactory result (von Glasersfeld, 1995). Steffe (2001) describes units coordination as the "mental operation of distributing a composite unit across the elements of another composite unit" (p. 279). He describes how this operation is involved in a child's scheme for counting.

The result of the units coordination permits the child to experience a unit containing five units of two that is not simply a sensory-motor experience. This organized experience is what I regard as a situation of the scheme. The activity of the scheme in the case of the example is to count by two five times to specify the numerosity of the individual units contained in five units of two, and the results of the scheme is the experience of the immediate past counting activity along with its result. (Steffe, 2001, p. 279)

Modifications to a scheme can affect one or more of its three constituent parts: the recognition template, the operations, or the expected result. For instance, students who are *perturbed* by a novel situational goal or surprising outcome may form an accommodation to

successfully fit the situation within an existing scheme or abduct novel activity in an attempt to form a new way of operating in that situation (Norton, 2008; von Glasersfeld, 1995). It is via the process of *reflective abstraction* that a student's scheme transitions from relying on mental or physical activities with sensory material to becoming *interiorized* and anticipatory (Piaget, 2001; Simon, Placa, & Avitzur, 2016).

Consider the following Nickels task: You have three nickels (5-cent pieces); how many more nickels would you need to have 35 cents? Upon hearing the words “three nickels,” a student might immediately know (*assimilate*) that three 5s is also 15 1s. This is an example of *interiorized* coordination of two *levels of units*. The coordination may be *reversible* in the sense that a student might anticipate computing other quantities in terms of *either* 5s or 1s before learning the rest of the task. Another student might arrive at the correct response to the Nickels task, but require actions such as tallying or counting—units coordinating *in activity*—to coordinate the three 5s as fifteen 1s.

We incorporate aspects of Piaget's structuralism (1970) to characterize *organizations* of interiorized, reversible operations. Figure 1 represents the form of a structure for coordinating three levels of units — for knowing 15 as a result of composing fifteen 1s, three 5s, or one 15. A units coordinating structure is a way of operating within such a form (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015).

Figure 1. Representation of a structure for coordinating units.

Stages of units coordinating activity

As students' develop more powerful schemes, they progress through stages characterized by the types units coordinating activity in which they engage (Norton, Boyce, Ulrich, & Phillips, 2015; Steffe & Cobb, 1988). Along with decreased reliance on physical activity or perceptual material, students develop flexibility in their ability to form composite units and an ability to reflect on their reasoning with inverse relationships as they progress from Stage 1 (in which children require activity to form a numerical composite) to Stage 2 (in which children can assimilate a unit composed of other units and further compose or decompose units in activity) to Stage 3 (in which children can assimilate with units within units within units) (Hackenberg, 2010; Norton, Boyce, Ulrich, & Phillips, 2015).

Units coordination with fractional units

The three stages of units coordinating activity describe differences in students' abilities to coordinate units, but an individual's propensity for coordinating units can also vary with the task setting (Boyce, 2014). Different schemes may be invoked if tasks involve continuous physical quantities such as length instead of an abstract and discrete context of money (Norton & Boyce, 2015). For instance, the wording of the Nickels task invokes units coordination with whole

number units familiar to American students. Instead, one might pose the task as, “You have three $\frac{1}{20}$ dollars. How many more $\frac{1}{20}$ dollars is needed to have $\frac{35}{100}$ dollars?” Though the two tasks involve the same amount of money, the latter task may invoke substantively different mental activity.

The *reorganization hypothesis* posits that individuals can reorganize mental operations they develop with counting to construct schemes involving *fractional* units (Hackenberg, 2007; Olive, 1999; Steffe, 2001; Steffe & Olive, 2010). Operations involved in coordinating whole number units include *unitizing*, *disembedding*, *iterating*, and *partitioning*. Individuals can coordinate such operations to *equi-partition*, which is requisite for constructing fractional units (Hackenberg, 2010; Lamon, 2007; Norton & Wilkins, 2010; Steffe & Olive, 2010). Equi-partitioning involves partitioning a whole to form equivalent parts, removing a part without modifying the whole (*disembedding*), forming a unit size (*unitizing*), and then *iterating* that unit to re-form the size of the original whole (Steffe, 2001).

The reorganization of operations to construct schemes for coordinating fractional units does not culminate with equi-partitioning. The sequential partitioning and iterating within the equi-partitioning operation can later become interiorized as inverse operations, resulting in a single *splitting* operation (Norton & Wilkins, 2012; Steffe, & Olive, 2010). Consider the following Candy Bar task (Norton, 2008): “Your candy bar is 5 times as big as my candy bar. This is your candy bar [Shown a rectangle]. How big is my candy bar?” A student who has not constructed splitting is likely to assimilate the given size as a unit of ‘1’ within a composite unit ‘5’. This is because the wording “5 times as big” suggests iteration. For the splitter, these words simultaneously invoke both partitioning and iterating (Norton & Wilkins, 2012).

The construction of the splitting operation allows for more flexibility in assimilating with two levels of units, which fosters coordination of a third level of units (in activity) in more situations. For instance, the construction of splitting is requisite for students' forming the size of a whole from a non-unit proper fraction such as $5/7$ (Hackenberg, 2007; Norton & Wilkins, 2012; Steffe & Olive, 2010). To form the size of the (larger) whole consisting of 7 sevenths, the student must anticipate partitioning and disembedding (see Figure 2).

Figure 2. Forming size of a whole using a reversible partitive fractions scheme

Middle school students' fractional reasoning

The coordination of fractional units is fundamental to understanding fractions as measures, which is critical for students' development of proportional reasoning (Lamon, 2007), the rational number system (Norton & Wilkins, 2012) and the ability to reason with unknown quantities (Hackenberg & Lee, 2015). By the time they reach middle school, most students assimilate with the same number of levels of whole number units as levels of fractional units, but unfortunately many students remain at Stage 1, in which they require activity to coordinate even two levels of units (Boyce & Norton, 2016; Hackenberg, 2013; Hackenberg & Lee, 2015; Steffe, 2007).

Some Stage 1 students' schemes for fractions situations instead involve partitioning without disembedding, resulting in a *parts-within-wholes* conception of a fraction as a ratio (Hackenberg, 2013). Stage 2 students sometimes construct *parts-out-of-wholes* ratio conceptions that incorporate disembedding but not equi-partitioning (Boyce & Norton, 2016; Hackenberg, 2013; Norton & Wilkins, 2010). Though these students mentally "set apart" a part from a whole without modifying the whole, they unitize the part as a discrete unit of '1' prior to counting activity instead of a unitizing a size to be iterated. Such students (including the focus of this paper, Dylan) coordinate whole number units in fractional situations without coordinating fractional units. These Stage 2 students have not (yet) constructed fractions as *measures* (Lamon, 2007).

Supporting middle grades students' development of units coordination

We consider three intertwined learning goals for middle school students who currently assimilate with n (< 3) levels of units. Students may begin to assimilate with n levels of units in

new ways; e.g., by assimilating fractional situations with fractional units rather than whole number units. Students may also begin to coordinate $n + 1$ units in activity in new ways; e.g., in continuous contexts as well as discrete contexts or with decreased reliance on perceptual materials. The third goal is for students to begin to assimilate with an additional level of units and thus enter the next stage. How might middle-grades students be supported in these learning goals?

One way of supporting units coordinating development is for teachers to engage students in situations in which they experience their schemes involving physical manipulation as appropriate yet inadequate. Steffe (1992) provided for such goal formation using tasks involving missing units. For example, after a student forms an array of three rows of five chips each and establishes that there are 15 chips total, the teacher might cover the array, say that there are now 35 chips, and ask the student to figure how many rows there would be, constraining the student to count mental representations.

Teachers might also provide tasks for which coordination of units would allow for greater efficiency. Norton and Boyce (2015) provided for such goal formation. After a student established relations between four levels of physical units (chips, cups, boxes, and crates), the teacher posed tasks in quick succession pertaining to permutations of the relations. For example, after establishing that there were two chips in a cup, three cups in a box, and two boxes in a crate, the teacher might ask, “How many chips are in a cup? How many cups are in a box? How many boxes are in a crate? How many cups are in a crate? How many chips are in a crate?” The student could choose to look at or touch the chips, cups, etc., if he wanted. The goal was for the student to anticipate maintaining relationships between units across tasks so that referring to the physical materials was not always required.

Purpose

In a previous teaching experiment (Boyce & Norton, 2014) focused on supporting sixth-grade students' fraction scheme development, we noticed growth in one student's units coordinating activity in the teaching session immediately following his successful coordination of (two levels of) units with whole numbers. He could confidently and correctly respond to fraction tasks that had engendered perturbation in previous sessions by coordinating two levels of fractional units. Though he did not express awareness of the similarity in his ways of operating with fractions and whole numbers, we thought that switching the task context might have encouraged such reflection, perhaps outside of the teaching sessions. We conjectured that, in addition to the previously mentioned approaches to support students' interiorization of units coordinating activity, alternating between whole number and fractions tasks might lead to reflective abstraction resulting in students' development of more powerful units coordinating structures. We set out to explore this conjecture in another teaching experiment with sixth-grade students.

Methods

We report on the results of a constructivist teaching experiment (Steffe & Thompson, 2000) with a pair of sixth-grade students from the Southeastern United States. The two participants – Dylan and Maddie – were selected because they were available to meet after school and because they each were assessed as having two levels of (whole number) units interiorized during clinical interviews conducted by the authors as part of another study (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015). Unlike Dylan, Maddie had also constructed schemes for coordinating two levels of fractional units at the onset of the teaching experiment. The purpose of the teaching experiment was to understand the students' (growth in) units

coordinating activity in multiple contexts (discrete/continuous, fractions/whole numbers, involving various perceptual material, see Table 1). Motivation for activities within and across sessions came from previous teaching experiments (e.g., Norton & Boyce, 2015; Steffe, 1992) and the conjecture that alternating fractions and whole number tasks could support the students' units coordination development. We focus on Dylan's units coordinating across contexts to consider the learning opportunities for middle-school students who have established parts-out-of-wholes schemes.

There were 11 teaching sessions, taking place in Autumn 2013. We met with the two students in an otherwise empty classroom for approximately 35 minutes, once or twice per week. The second author taught the teaching sessions, and the first author observed, filmed and transcribed each session. The teacher and witness initially completed ongoing and retrospective analyses of the students' reasoning independently, with the witness focusing particularly on Dylan's units coordinating activity. This approach enabled the witness to consider how the sequencing of interactions between students and the teacher affected Dylan's units coordination across the contexts without influencing the teacher's goals and decision-making (see Boyce, 2014 for details). In the Results section, we present synthesis of our retrospective analyses of Dylan's activity. We include discussion of Maddie's activity when it serves to qualify Dylan's activity.

Table 1.

Teaching session descriptions

Session	Materials	Mathematics Domains
1	Cuisenaire Rods	Whole numbers, proper fractions

2	Cuisenaire Rods	Proper fractions, improper fractions
3	Cuisenaire Rods	Proper fractions, improper fractions
4	Paper Strips (linear)	Whole numbers, proper and improper fractions
5	<i>CandyDepot</i>	Whole numbers, improper fractions
6	Chips, Cups, Boxes	Whole numbers, proper and improper fractions
7	Cents, Chips, Cups, Boxes	Whole numbers, proper and improper fractions
8	Paper Sheets (area/linear)	Whole numbers, proper and improper fractions
9	<i>CandyDepot</i>	Whole numbers, improper fractions
10	Liquid Volume Measures	Whole numbers, proper fractions
11	Multiple	Whole numbers, proper and improper fractions

Results

Session 1: October 3, 2013

The first session began with a series of exploratory tasks with Cuisenaire rods (see Figure 3). In this free play (Steffe, 1991), Dylan and Maddie suggested several relationships between the Cuisenaire rods' sizes. Dylan reasoned that since five red rods fit into an orange rod and two yellow rods are the same length as five red rods, two yellow rods must fit into an orange rod. Dylan thus coordinated two levels of units: the orange as a unit of five red units.

<i>Color</i>	Tan	Red	Lime	Purple	Yellow
<i>Length</i>	1 cm	2 cm	3 cm	4 cm	5 cm
<i>Color</i>	Green	Black	Brown	Blue	Orange
<i>Length</i>	6 cm	7 cm	8 cm	9 cm	10 cm

Figure 3. Cuisenaire rods' lengths by color

Dylan's initial meanings of fraction

Following the teacher's request for a relationship between different rods involving fractions, Dylan first suggested that he could measure the number of square feet in the orange rod if the tan rods were 1 square foot each – he did not use fractional language. Dylan introduced fractional language when he suggested thinking about money.

D: Suppose you were trying to see how many quarters would fit in a dollar bill.

[Aligns a lime (3-rod) and a tan (1-rod) below a purple (4-rod)] Okay, pretend this purple rod is a dollar.

T: Okay.

D: [Aligns 4 tans below the purple.] Okay, this purple is a dollar. Kind of like the square foot thing. This [tan] is one quarter. So you put that up there [aligns a tan above the purple], okay, I've got more room. I've got another quarter. [Places another tan] That's 50 cent. I've got another quarter [places another tan.] That's 75 cent. Because you're three-fourths of the way there.

T: Okay.

D: You add another quarter, and that gives you your dollar. So I could conclude that four fourths equals 1.

T: Where did the fourths come from?

D: Since there [are] four of these little [tan] blocks; they all fit. Pretend that this [purple] is a fourth. Then four little blocks fit into that [purple], so *it would be four-fourths*.

T: What fraction is a quarter of a dollar?

D: 50 cent. Oh – a fraction?

T: Yeah, what fraction is a quarter of a dollar?

D: A half.

T: A quarter is half of a dollar?

D: Yeah.

T: Okay.

D: No - a quarter! It's [inaudible] of a dollar. One-fourth.

Dylan was reasoning with two levels of units (the number of dollars and the number of quarter-dollars). Though he seems to be describing iteration of one-fourth, two-fourths, three-fourths, etc., this was *after* he had established that four tan rods were equivalent to the purple rod. His explanation suggests his reasoning was based on the perceptual materials. He described the purple as both “a fourth” and “four-fourths,” and he did not retain a fractional representation of a quarter of a dollar.

Dylan and Maddie's interaction

The teacher posed a follow-up task: “How many quarters are in 5 dollars?” While Dylan was building the length of 5 copies of the purple 4-rod [he aligned 4 purples, a lime 3-rod, and a tan 1-rod], Maddie replied, “One-quarter? It would be $\frac{1}{20}$. You would do 4 times 5.” Dylan proceeded to count the number of tan 1-rods in the string of blocks he had appended. Dylan claimed, “it would be $\frac{1}{18}$.” After Maddie repeated, “it would be 20 quarters,” Dylan revised his count. “No, it would be one out of 19. [slight pause] Quarters.”

Dylan response suggested he might be removing the part from the whole instead of disembedding in his naming the fraction, but Dylan concluded he had counted incorrectly and agreed with Maddie after replacing the lime 3-rod and tan 1-rod with a purple 4-rod. “Okay, if there's 5 of these [purple rods] lined up, there's 4 in here [the purple], and there's 5 [purples]. 4

times 5 is 20. Okay, like Maddie said.” In his language, Dylan did not suggest counting “quarters” or “fourths”. Before the composite units were represented identically as purple units, he had not coordinated the tan 1-rod as a unit within the purple 4-rod unit within the whole.

The previous exchanges exemplify and foreshadow Maddie and Dylan’s interactions throughout the teaching experiment. Dylan seldom acquiesced to Maddie’s reasoning, and instead coordinated units in activity to (attempt to) make sense of her explanations. The next exchange exemplifies another quality of their interactions. When encountering situations involving fractional language, Dylan would often respond quickly and correctly, based on his assimilation with two levels of whole number units. Maddie’s propensity to build a third level of units in activity would sometimes result in her not arriving at a (correct) solution to tasks as quickly as Dylan.

The teacher asked Maddie and Dylan to “suppose” that the single orange 10-rod represented $\frac{5}{7}$ and find a rod that would represent $\frac{1}{7}$. Dylan aligned 5 red rods below the orange. “This is $\frac{5}{7}$, and there’s five of those [reds], so take 4 away and you’ve got $\frac{1}{7}$.” As he spoke, Dylan slid the four right-most red pieces away from the orange (see Figure 4). Maddie disagreed at first, claiming that the red 2-rod would be one-*fifth* of the orange 10-rod. It was not until she had iterated the red piece 7 times to form the size of a whole and compared that size with the orange ($\frac{5}{7}$) that she agreed that the red piece would be indeed be 1-seventh. Our interpretation is that Dylan was reasoning about forming the size of ‘1’ from a size of ‘5’; he was not attending to the units’ meanings as one-sevenths.

Figure 4. Dylan's forming the size of one-seventh, given the orange was five-sevenths

Session 2: October 8, 2013

Dylan's identification of unit fractions

To help the students begin to attend to fractional units, the teacher introduced a Missing Unit Game using the Cuisenaire rods. One person would create a unit fraction, whole, and composite unit fraction while the other two averted their eyes. Given an amount named as a fraction, the goal was to produce the other two amounts, which were hidden in envelopes. The teacher and students took turns posing tasks in the Missing Unit Game. The teacher posed initial tasks of finding the unit fraction and whole given a piece named $\frac{2}{3}$ and subsequently given a piece named $\frac{3}{4}$. In each situation the unit fraction was the smallest Cuisenaire rod, the tan 1-rod, and the students successfully completed the tasks without indication of perturbation.

When Maddie posed a task to find the unit and the whole, telling the teacher and Dylan that the purple 4-rod was $\frac{2}{4}$, Dylan correctly suggested that the brown 8-rod would be the whole, but he suggested the purple 4-rod would be the unit. Maddie had hidden the red 2-rod as the unit fraction instead, and she told him he was incorrect – “I gave you that [purple] piece! How could it be the unit?” Dylan replied, “You can actually have three units. One of these

[purple 4-rods] equals two of these [red 2-rods]. Two of these [reds] equals four of these [tan 1-rods].” The teacher next asked Dylan, “If this [purple] was the unit fraction, what would we call it?” After Dylan responded, “two-fourths,” the teacher suggested to Dylan that a unit fraction is “one over something, right?” Dylan then named the purple 4-rod as one-half, but when asked what the name for the red 2-rod would be, he said it would also be one-half. Dylan’s naming of unit fractions suggests that he was just beginning to understand a proper fraction as a composite of unit fractions.

When it was Dylan’s turn to choose the task, he presented the orange 10-rod as “one-third,” hid three oranges in the “whole” envelope, and hid a tan 1-rod in the unit fraction envelope. The witness (having seen the hidden pieces) asked if Dylan could give “another name for the one-third that would help them figure out what the unit would be.” Dylan’s suggestions, “two-sixths ... or three-eighths,” suggested that he had not utilized the tan to establish the relationship between the fraction and the whole. For Dylan, the “unit” was first treated as something superfluous and arbitrary, except that it needed to fit into another fraction rod some whole number of times.

Session 3: October 10, 2013

In the next session, the teacher posed Missing Units Game tasks in a continuous setting, with rectangular strips of construction paper. Whereas in the discrete setting of Cuisenaire rods there is a smallest piece (the tan 1-rod), in the continuous setting there is no smallest-size piece to try to name as a unit. The teacher gave the students a strip named $\frac{3}{5}$ and asked them to find the whole and the unit. Dylan’s responses in this session show how he first created a piece to serve as the unit without equipartitioning.

Dylan strategy centered on producing the complement of $\frac{3}{5}$ in the whole—two fifths, because, as he later explained, “it was five-fifths and there was two fifths leftover in the whole.” It was unclear how Dylan found the amount he named as two-fifths, as he may have fortuitously formed the (nearly) correct size from a sheet of paper available on his desk. Dylan iterated the three-fifths strip twice along the width of the sheet of paper, and he made a mark on the strip where the second iteration extended beyond the sheet (see Figure 5). He then made a copy of the strip (naming it as three-fifths), and he removed the excess from original strip (naming the result as two-fifths).

Figure 5. Dylan's measuring

When asked to explain how he found two-fifths, Dylan shrugged his shoulders to indicate that he wasn't sure. “I measured it to pretend that this was like 1 – like 2 of the fifths. It came out to that, so I cut it.” When the teacher asked the students to find one-fifth, Dylan started to iterate the “cut-off” piece from left to right on top of his nearly square two-fifths strip. When finding that the cut-off piece did not fit into the two-fifths piece exactly twice, he rotated the two-fifths strip ninety degrees and began to iterate the piece across the $\frac{2}{5}$ strip.

The teacher interrupted Dylan and told the students that they were correct about the size of the whole and asked them to find the unit from just the $\frac{3}{5}$ strip. Dylan made two equally-spaced marks on the $\frac{3}{5}$ strip, and when the teacher asked, “where is one-fifth?” Dylan shaded in the left-most piece. Though that activity by itself might suggest that Dylan had *split* $\frac{3}{5}$ into 3 one-fifths, Dylan may have instead placed the marks by re-presenting his immediately prior activities with the “cut-off” piece (which was fortuitously the right size). In the next session, the teacher continued to explore Dylan’s reasoning with paper strips.

Commonalities in Dylan’s strategies with multiplicative and fractional tasks

Session 4: October 15, 2013

In the fourth session, the teacher presented a sequence of structurally similar tasks with the paper strips, first with whole numbers and then fractions. At the beginning of the episode, the teacher asked the students to make a 7 inch strip from a given 4 inch strip, with no ruler available. Dylan operated much as he had at end of the previous episode, but this time his reasoning was more explicit. He copied the given strip, cut the copy in half, and then cut that part in half and put the three pieces of size 4 inches, 2 inches, and 1 inch together to form the correct size. When the teacher next gave each student a 12-inch strip and asked them to make a 5-inch strip, Dylan’s strategy was different. Dylan estimated a piece he thought to be 2 inches and put together two and a half of those pieces to form the size 5 inches. It is unclear how formed an estimate of the size of the 2-inch strip – he did not appear to reasoning with the size of the 12-inch strip.

The teacher followed up with fractions tasks. The students were given a $\frac{4}{9}$ strip and asked to find the unit and the whole. Dylan began by forming the whole. He adjoined two copies of the $\frac{4}{9}$ strip and then made a half of a half of a third copy. He seemed to be reasoning with

whole numbers— $4+4+(\text{half of half of } 4)$ —in the same way he had previously found the length of 7 inches given the length of 4 inches. He subsequently named the smallest piece as “the unit” and appended it to the others to form “the whole.”

When the teacher next gave the students a strip labeled “ $8/5$ ” and the teacher asked them to find the unit and the whole, Dylan expressed surprise, “Are you serious! What if we can’t figure this out?” He made a copy of the given strip, and then began to partition the copy. He produced six partitions before nearly reaching the end of the strip, and then he adjusted marks to fit in two more partitions. He cut off the last piece, which he named as the unit. He then made a single cut to remove the first five of the pieces and named it as the whole. Dylan referred to the whole he had made as “5 fives” and named the fraction as “eighths”, suggesting that he was not coordinating fractional units.

Looking across the four tasks of Session 4 reveals commonalities in Dylan’s strategies across multiplicative and fractional tasks. When Dylan was trying to make a larger size from a given amount, he began by iterating the given amount. Once the remainder was larger than the original unit, he would form a new unit (e.g., by partitioning and disembedding half) and repeat this process until he reached the goal. When Dylan was trying to make something smaller than what he started with, he would first create an even smaller unit. This new unit was not necessarily associated with the starting amount (e.g., he did not compare his estimate of 2 inches with the extent of the 12 inch strip). Having formed this smaller unit, he would apply the aforementioned strategy to make the intermediate size.

Dylan’s coordinating units in CandyDepot

Session 5: October 17, 2013

The fifth session marked the introduction of the *CandyDepot* educational game (LTRG, 2013) to the teaching experiment. Units coordination is the explicit focus of *CandyDepot* (Norton, Boyce, & Hatch, 2015). Students first choose unit relationships: the number of bars in a bundle and the number of bundles in a box. The software then produces a (pseudo)random sequence of customer orders in terms of bars, bundles, or boxes that the students can satisfy by dragging objects to a truck (see Figure 6). The goal of the game is to satisfy orders correctly and efficiently, by quickly dragging a minimal number of items. For example, if the customer order were 38 bars, with 2 bars per bundle and 2 bundles per box, the most efficient way of satisfying the order would be to drag 9 boxes and 1 bundle.

The students took turns solving the tasks, some of which involved whole number orders and some of which involved fractional orders of bundles or boxes. When there were 2 bars per bundle and 2 bundles per box, Dylan and Maddie easily satisfied whole number orders. When Dylan first encountered a fractional order ($9/2$ of a box) he seemed to presume that half a box was a bar (rather than a bundle) and focused on efficiently packing nine bars. After the game indicated that he was incorrect, Maddie explained to him that he should have packed 9 bundles.

Figure 6. Screenshot of CandyDepot Level 1 (LTRG, 2013)

When there were later three bars per bundle and two bundles per box, Dylan got an order for 12 bundles. He again did not consider the unit, assimilating the task as a goal to produce 12, with bars as the default unit. He shipped two boxes (two 6's). Even after Maddie alerted him that he needed 12 bundles and the teacher pointed out that he had shipped 12 bars, Dylan exclaimed, "No, I did 12 boxes - or 2 boxes." Maddie: "Which is 12 bars, not 12 bundles."

Dylan successfully modified his approach to coordinate three levels of units in activity when he later encountered an order for $20/3$ bundles. He immediately wrote out multiples of 3 on paper until he reached 18; he then began circling the numbers in pairs, in reference to the two bundles per box (see Figure 7). He was thus coordinating pairs of threes as pairs of bundles

within boxes. His activity relied on perceptual material, as he was drawing and circling numbers, but he treated a box as a unit of two bundles, each of which was a unit of three (bars).

Figure 7. Dylan's coordination of pairs of 3s as pairs of bundles

Session 6: October 23, 2013

Dylan's coordinating units with nested discrete manipulatives

In the next session, the teacher introduced tasks involving physical manipulatives: chips within cups within boxes (Norton & Boyce, 2015). The teacher invited Dylan to choose the relationships. He placed 4 chips in each of 12 cups, and he then placed them in a box in a 4 by 3 array. Before the teacher could ask a question, Dylan connected the activity to the Missing Units Game, assuming the teacher would ask for the unit and the whole. Dylan offered that the whole would be 48 and the unit would be “one-forty-eighth”. After Maddie volunteered that 1 chip would be $1/48^{\text{th}}$ of the box, the teacher asked the students for the number of chips in one-third of the box. Both students divided 48 by 3, with Dylan responding “16” first. Dylan's explanation shows how he coordinated units *in activity* by distributing 15s and 1s.

Dylan: Because 45 [looks down at his hand underneath table] how many 15s can fit into that? That's 3. And then – wait. There was 48, so I knew 15 could go into it. That was 3. Then I saw how many 1s could fit into ... [2 second pause. Shrugs shoulders].

When the teacher subsequently asked for the number of chips in $\frac{2}{3}$ of a box, Dylan immediately answered, 32, “because I knew $\frac{1}{3}$ was 16, and $16 + 16 = 32$.” His actions were consistent with his ways of operating in Session 4, especially his reference to adding two $\frac{1}{3}$ s rather than use of a multiplicative relation. He seemed to be coordinating two levels of units at a time: $\frac{2}{3}$ as $\frac{1}{3}$ plus $\frac{1}{3}$, and (previously) 48 as a unit of three 16s.

Session 7: October 29, 2013

Dylan's coordinating units with(out) physical referents

The seventh session again involved the chips within cups within boxes (2 chips within 10 cups within a box), but this time a new quantity was added: 3 cents per chip. The teacher asked Maddie, “If each chip is worth 3 cents, how much would the box be worth?” After a bit of confusion from Dylan, both students agreed the answer was 60 cents, but Dylan solved the task by multiplying 6×10 and did not understand Maddie's solution of 20×3 .

Maddie: “He said 6 times 10, but I did 20 times 3.”

Dylan: “Why would you do 20 times 3 [furrows brow] ... I don't get why she did that [looks at box] because, there's no 3 in there.”

Dylan eventually resolved the issue: “Yeah, because if each chip was 3 cents [raises 3 fingers] and there's 20 chips [staring ahead] 3 times 20 is 60”. He thus relied on figurative material to stand in for the cents, which had no visible referent. In contrast, Maddie could move flexibly within the three level structure, immediately equating 10 6s and 20 3s, not just arithmetically, but with appropriate reference to the units.

When the teacher posed the next task, directing it to Maddie, “What if you had 40 cups; how many chips would there be?” Dylan surprisingly responded first. He reasoned that 40 cups was 4 boxes, and each box was 20 chips, so there would be four 20s (80). He referred to “80 dollars” at first, but he generated his response in about 10 seconds. He explained that he had “counted to 4” and indicated that he counted by 10s four times to determine that there were 4 boxes. He later said he had counted by 20s four times after determining that there would be 4 boxes. So, he seemed to have performed two separate units coordinations: 40 as 4 units of 10; and then 80 as 4 units of 20. In each case, he was iterating a composite unit in activity and transitioning from one units coordination to the other, possibly relying on the available perceptual material.

Dylan's coordination of three levels of units in activity with whole numbers

When the series of tasks were directed to Dylan, his responses indicate that he consistently built on previous responses to generate the next result with a multiplicative relation (e.g., for each of the x units, there are y units, so $x * y$). Recall that there were 3 cents per chip, 2 chips within each cup, and 10 cups within a box.

T: Ready? Let's suppose that you have 5 boxes. How many cups would you have?

D: [Looks at box] Cups. 50.

T: How many chips?

D: [5 sec, looking straight ahead] A hundred.

T: How many cents would it be worth, the 5 boxes?

D: [Immediately] 300.

T: How'd you get that?

D: Because if there's a hundred chips [glances at box] and then they're each 3 cents, that would make it 300.

T: If you have 30 cups, how many chips would you have?

D: [Looks at box] 30 cups? [Subvocally '2'] 60.

T: How many cents would they be worth?

D: [Looking straight ahead] If there were 60 chips [2 sec] A hundred 80.

T: How many boxes would you have?

D: [Taps out 9 fingers in succession on desk] 9.

T: How'd you get that?

D: [Looks at box] Because there's 20 chips in the thing. We're trying to get, how many?

T: I said 30 cups.

D: [Mouths number words] 30 cups in 1 box?

In response to the final question, Dylan was required to reverse his way of operating, which he was able to do, except he conflated the units of chips and cents. His actions suggest he was not responding with rote multiplication, but his conflation of chips and cents suggests that he was not coordinating intermediary units (chips being the intermediary between cents and boxes). He seemed to be engaging in reversible multiplicative reasoning (Hackenberg, 2010), which involves coordinating three levels of units in activity.

Dylan's coordination of three levels of units in activity with fractions

The teacher next switched to the fractions context, encouraging the students to work together to solve the series of tasks.

T: What would $\frac{1}{5}$ of a box be?

D: [Reaches into box and grabs 2 cups, holds up. Puts cups back.] 4 chips, 2 cups.

T: What would 4 fifths of a box be?

D: 5 chips?

M: 5 chips... 20 divided by 4.

D: It would be 8 chips.

...

M: I thought you said 1 fourth of a box at first, or something.

Maddie assimilated $\frac{4}{5}$ as the unit fraction $\frac{1}{4}$, rather than assimilating it as 4 units of a $\frac{1}{5}$ unit. In this case, the $\frac{1}{5}$ unit was itself a composite unit. Meanwhile, Dylan had again conflated units (chips and cups). After Dylan looked in the box and counted the chips, he explained other ways to correctly reason that $\frac{4}{5}$ of the box would be 16 chips.

D: So this [2 cups] was 1 fifth, and we're trying to find four. So, 1 fifths, 2 fifths, 3 fifths, 4 fifths [forming a 2×4 array of cups]. There's 2 in each [pointing to top left cup], so 2, 4, 6, 8, 10, 12, 15 [corrected to] 14, 16. I know two ways [Maddie] could have got what I got. Because one-fifth [indicates leftmost 2 cups] is 4 chips. If there's 4 of those, 4 times 4.

M: That's how I did it!

D: Or, she could have noticed that there's like 2 rows. 1 row equals 8, so 2 times 8 = 16.

Dylan used the established relation of one-fifth as 2 cups to coordinate whole number units without further attending to his understanding of $\frac{1}{5}$ as one of 5 parts within a whole. This is further illustrated in the follow-up segment, in which the teacher asked for the size of $\frac{7}{5}$ of a box.

D: A box and 3 cup - 4 cups.

M: Huh? [frowns] A box and... [1 sec] 8 chips.

D: [Looks into the box] It's a box and 12 chips.

...

D: Because you - there's 3 fifths leftover. 1 [moves 2 cups from the box to his desk] because that was 1 fifth, 2 fifths [second pair of cups] 3 [forms 2 by 3 array of cups]. Because it was 7 fifths, so a whole would be 5, so there's 1 and there's going to be [squints] 2 fifths leftover. So it would be 1 and [2 sec] 8 chips. Is that what you said Maddie? 8 chips?

T: 1 and 8 chips.

D: Yeah.

T: So what do you have on your desk?

D: [Picks up leftmost 2 cups from the array]. This is 8 chips.

T: Okay, good.

D: I had it mixed up, like my numbers – I thought there was 3 fifths leftover. I thought it was eight. Eight-fifths.

Dylan's response indicates he assimilated the improper fraction as a whole plus a remainder. In contrast to his similar confluences when working with $\frac{8}{5}$ with paper strips in Session 4, the availability to manipulate an established discrete referent (two cups = $\frac{1}{5}$) enabled him to reflect on his activity. We infer that Dylan conflated units when he tried to use the relationship of $\frac{4}{5}$ of a box as 8 cups to solve the task. Though Dylan had named the whole as "1 box", it seems he initially associated the visible array of 8 cups with the whole instead, as his goal became to find the number of chips in the excess of 5 cups within 8 cups. It was not until he re-formed the array to represent the remainder of $\frac{3}{5}$ that he realized that the whole should be "5" instead of "8".

*Dylan's reasoning with units of area***Session 8: October 31, 2013**

In the eighth session, Dylan and Maddie were given a sequence of tasks similar to those encountered in Session 4, but they began with area models (sheets of paper) instead of linear models (strips of paper). The session began with the following task: "This [square] is 9 times as big as the one I'd like you to make. How could you make yours?" Dylan partitioned the sheet vertically, but he exhausted the whole after only 8 parts and so partitioned one of them in half to make 9 parts and then cut out the right-most part. Maddie partitioned into thirds, vertically and horizontally.

Dylan was subsequently able to meaningfully adopt Maddie's strategy. The students each successfully made one-third of a square by partitioning vertically to form a strip. The teacher then asked, "just using that one-third, can you make a fifteenth?" Maddie responded by partitioning the one-third strip into 5 parts: "Because 15 divided by 3 is 5." Dylan observed and indicated he understood her reasoning. When the teacher next provided a different square, and said, "this square is 12 times as big as the square I want you to make", both students *split* the square into 12 equal parts, taking advantage of two-dimensionality to coordinate 12 as four 3s by partitioning vertically and horizontally. Dylan's responses indicate his assimilation of both linear and area models of fractions with two levels of whole units.

Session 9: November 6, 2013

In the ninth session, the students returned to *CandyDepot*. Dylan's activity began by his successful completion of several whole number orders of bars. For example, when there were 5 bundles per box and 2 bars per bundle, he quickly satisfied an order for 29 bars by shipping 2 bars, 4 bundles, and 1 bar. This was an impressive response indicating that he may have been

coordinating three levels of units. However, he may have equated boxes as 10s and bundles as 2s within 29, which would involve coordinating only two levels of whole number units (omitting the intermediate relation of the box as 5 bundles).

Dylan's identification of two strategies for completing fractional orders in Candy Depot

Dylan was focused on producing a whole number of bars in fractions situations, as specified by the numerator of the given fraction. When the setting switched to 2 bars per bundle and 2 bundles per box, Dylan's order was for $19/2$ bundles. He filled the order with 9 boxes and a bar, over Maddie's objections that 9 boxes would be 18 bundles. He seemed to be reasoning that there would be 9 wholes and one half, but in the process of dividing by 2, he conflated the referent units of bundles and boxes. When he later encountered an order of $11/2$ boxes, he immediately placed 3 boxes, a bundle, and a bar, apparently assimilating the $11/2$ as 11 bars.

When there were 2 bars per bundle and 6 bundles per box, Dylan proficiently satisfied orders for 61 and 55 bars, presumably employing the previously described strategy involving counting the number of 12s and 2s. This time when he received an order for 11 bundles, he appropriately adjusted for the new unit by converting the bundles into bars first: "11 times 2 is 22." It was clear to the teacher at this point that Dylan was reasoning with bars, but it was unclear whether he was attending to or conflating the referent units when encountering fractional orders. Dylan suggested that he could "switch back and forth" between his bar method and a method focusing on the referent unit. He could consider the number of bars, or he could consider the number of bundles and treat the number of bars as a remainder.

[Order is $17/3$ bundles, 3 bars per bundle and 3 bundles per box]

D: [Drags a box. Hesitates momentarily over a box, then drags 2 bundles and then 2 bars]

T: Wow, that was a tough one. What were you thinking there?

D: [Inaudible] it's right, but if you ask me I don't remember exactly how I did it.

M: A third means bundles.

D: One-third?

T: Well, this says 17 thirds of a bundle. A third of what is a bundle?

M: A third of a box is a bundle.

...

D: I saw there's three bars, and I saw how many could go into 17, and that came out to be 3. No, 5. 5 full bars. Or bundles. Then, only 3 can go in there [points to box]. So, 1, and then 2 in there [points to the bundle], and then there's two leftover.

Dylan reasoned with $17/3$ bundles as a mixed number: 5 wholes and 2 leftovers. He explained how he could form 5 as a composite of one 3 and two 1s, but he did not demonstrate thinking of the bars as "one-thirds." When he later received an order of $10/3$ boxes, he packed 3 boxes and a bundle. If he had used his bar strategy and focused on making 10 bars, he would have packed a box and a bar. Instead, it seemed he could coordinate 10 bundles as 3 units of 3 whole boxes with a unit of 1 bundle left over, which would involve coordinating three levels of (whole number) units in activity.

Session 10: November 7, 2013

Dylan's reasoning with liquid measures

In the penultimate session the context shifted to a continuous setting, as the students worked with relationships of measuring spoons and cups. The setup was similar to the initial chips/cups/boxes tasks, except that the students found relationships between the liquid measures by pouring water and recorded the relations on paper. For instance, Dylan wrote $1 \frac{1}{2}$ teaspoon = $1 + \frac{1}{2}$ teaspoon, referring to the relation between the teaspoon and the half tablespoon (or $1 \frac{1}{2}$

teaspoon). The teacher encouraged the students to pour water to determine size relations, but because the teacher was under the mistaken assumption the students were working with a tablespoon rather than a half-tablespoon, he initially asked students to pretend the relations held.

As Dylan began pouring water from the large spoon into the cup, to determine their relationship, Maddie interrupted him, "You know what you should have done? Figure out how many go into [1/2 cup] and then times it by 2. That would be a lot easier." Dylan seemed confused at first, until Maddie explained further: "The half a cup. See how many of those go into here, instead of having to fill up that whole thing. Then times whatever you get by 2." Dylan then realized that he could just double the previously recorded Tbsp/cup relation (8) to get 16. Assuming 8 represented the correct relation between the half-tablespoon and cup (rather than one imposed by the teacher who had assumed they were using a full tablespoon), Dylan adjusted: "Yeah - that should be 4 [pointing to the relation between the Tbsp and half-cup] and another should be 8 [taps again]."

Dylan was able to follow Maddie's reasoning once translated it into whole number unit relations with known quantities: the unknown quantity was 2 units of the known quantity of 8. Furthermore, he seemed to understand that if a half spoon fit into the half cup 8 times, then the full unit (spoon) would fit into the half cup half as many times (4); he subsequently reasoned that the full unit would fit into the full cup twice as many times as that (8). When working with continuous quantities (albeit with presumed discrete relations), Dylan thus demonstrated reversibility in his coordination of two whole units.

Dylan's focus on empirical results

Dylan questioned the half-cup/cup relation, wanting to check it. The teacher was surprised and asked him what it should be, to which he responded, "26." Dylan was treating the

half-cup as a unit of 13 big spoons (as he had previously measured). After checking that two half cups fit into one cup, Dylan began counting the number of large spoonful of water that fit into the cup. Whereas Maddie questioned his pouring, Dylan questioned Maddie's previous reasoning. This suggests something structural (i.e., an operational necessity) in Maddie's reasoning that was not structured in Dylan's reasoning. Dylan's focus on the empirical results was like that of a Stage 1 student, Cody, who also focused on measuring and counting individual cups and spoons rather than using established, previously measured relations when engaging in similar tasks (Norton & Boyce, 2015).

Dylan's attempt to reverse his reasoning with three levels of units

Dylan began trying to determine the relation between the small spoon (teaspoon) and the cup, given that the teaspoon fit into the half-tablespoon $1\frac{1}{2}$ times (which he re-checked) and that the half-tablespoon fit into the cup 32 times. He reasoned as follows: "So, if you do 16 divided by 2 [writing] that equals 8, but you can't really do a full minus or divided by 2 because it would be - You can't really do a full [inaudible] because 2 full cups didn't really fit into the thing. So, you'd have to add half of that back... So, it would go to [1 sec] 12."

Dylan demonstrated that he had not constructed a reversible structure for reasoning with three levels of units in this context. He seemed to be reasoning that if 16 large spoons fit into the cup, and $1\frac{1}{2}$ small spoons fit into a large spoon, then 16 divided by $1\frac{1}{2}$ small spoons would fit into the cup. He had been able to resolve such tasks by coordinating three levels of units in activity when the units were whole numbers. Dylan persisted in this line of reasoning until the teacher's questioning led to a perturbation--that the small spoon would have to go into the cup more than 16 times (the number of times the large spoon went into the cup)—which Dylan only experienced after beginning to physically fit small spoonful of water into the cup.

Dylan made a second error in that he presumed linearity in division: assuming dividing by $1\frac{1}{2}$ would yield a result halfway between dividing by 1 and dividing by 2. Ironically, when Dylan fixed the first issue (reversibility), this second issue was also resolved, because the presumed linearity ($16+8=24$) works in multiplication. Still, this did not represent an effort at multiplicative reasoning, but rather additive reasoning ("I should have added it") suggesting that Dylan was still not coordinating fractional units.

Session 11: November 12, 2013

In the final session, Dylan and Maddie went through a series of activities that repeated sequences they had encountered earlier in the teaching sessions. The purpose of the session was to provide closure as well an opportunity to test for stability (or growth) in relation to our interpretations of the students' activities in the previous sessions. The session began with a return to the water tasks. The teacher asked Dylan how he could find the number of large spoons in a half-cup given that there are 24 small spoons in a half-cup and $1\frac{1}{2}$ small spoons in a large spoon. Dylan recalled his solution of taking half of 24 and then adding back half of what he took away. He continued to be unsure of whether the result should be 18 or 16. When the teacher tried to redirect the students to consider $1\frac{1}{2}$ as three halves, neither student had a way of using the improper fraction to solve the task.

The teacher next gave each student two strips of licorice and asked the students to eat $\frac{3}{5}$ of one of them. Maddie began measuring the licorice by using the width of her finger as an estimate of an inch. Dylan, who was watching Maddie, made 5 iterations of a length spanned by two fingers, exhausting the length of the licorice. He then iterated this length three times and marked the end point with his fingernail. Dylan's actions are consistent with equi-partitioning. He made three-fifths as three iterations of a piece that fit into the whole five times: $\frac{3}{5}$ as three

units of a $\frac{1}{5}$ piece. But when Dylan explained his reasoning, he iterated the remaining $[\frac{2}{5}]$ piece “1, 2, and then half of that” across the whole. He again seemed to be coordinating three levels of whole units in activity: 5 as a unit composed of 2 and 2, and 1, which is half of 2. Dylan's lack of attention to fractional units was evidenced by his responses to the repeated question of how many fifths there would be in the whole, replying, “there would be two and a half”.

The teaching experiment closed with *CandyDepot*. Dylan choose 2 bars per bundle and 2 bundles per box, and he displayed great expediency in solving the first two (simple) orders, one of which was for 30 bars (7 boxes and a bundle). When the next order came up as 15 bundles, he immediately recognized it: “Oh - this is the exact same problem... It was the exact same problem because it was just like, backwards... It was 15 divided by 2, because there's 2 bars in a bundle... So it was backwards - the 15 divided by - or 15 times 2 is 30. Then it's backward.” Dylan's response indicates he had assimilated 15 bundles as both 15 2s and 30 bars.

In the next task—93 bars--which Maddie was solving, Dylan counted by 4s to 20 and then counted 4 20s to 80, reasoning that each 20 was five 4s. As such, once again he was coordinating three levels of units in activity: 80 as 4 units of 20, each of which was 5 units of 4 (boxes). To coordinate all of these units, he must have (at least) assimilated 20 as a unit of 5 units of 4 (boxes). If he did not rely on the boxes as figurative material to collapse the final unit, this would imply he had assimilated three levels of (whole number) units.

Conclusions

In this paper, we have described results of a teaching experiment conducted with two sixth-grade students, Maddie and Dylan. The purpose of this study was to explore how varying the context of tasks involving composite units might support students' development of units

coordinating structures. The contexts included both whole numbers and fractions, as tasks of were introduced in discrete and continuous settings with various manipulatives. We focused analysis on Dylan, who began the teaching experiment assimilating with two levels of (whole number) units.

We had conjectured that varying engagement with whole number and fractions tasks might support Dylan's development of units coordination. We considered three intertwined aspects of this growth: coordinating three levels of units in activity in a novel way (i.e., with fractional units), assimilating with two levels of units in a novel situation (e.g., in a continuous setting), and assimilating with a third level of units. Results of the teaching experiment suggest that Dylan made progress in coordinating three levels of whole number units, but he consistently approached fractions tasks by reasoning with whole number units instead of fractional units. This was most evident when Dylan was solving fractions tasks involving continuous contexts, as he would posit a whole unit of arbitrary size to serve as a unit fraction rather than producing the unit fraction through equipartitioning (cf., Steffe, 2001).

Dylan's learning during the teaching experiment also included his development of ways to reason with whole number units in fractional situations. Despite not coordinating fractional units, Dylan demonstrated growth in his modification of schemes for proper fractions to accommodate improper fractions. This was most evident in his reasoning in discrete contexts (the chips, cups, and boxes tasks and tasks in *CandyDepot*). It seems that instead of developing understandings of fractions as measures, Dylan was developing understandings of (improper) fractions as ratios (Behr, Harel, Post, & Lesh, 1992). His reasoning in these contexts could potentially be leveraged to support his development of relational thinking and algebraic reasoning (Empson, Levi, & Carpenter, 2011).

Dylan often adopted Maddie's activity to correctly solve fractions tasks by iterating when physical representations were available, but he interpreted the result of such activity using his parts-out-of-wholes scheme. This was particularly apparent at the end of the teaching experiment, when Dylan's actions were consistent with equipartitioning but disconnected from his reasoning. Norton and D'Ambrosio (2008) describe a similar case involving another sixth-grade student, Will, who participated in a paired-student teaching experiment involving linear fraction representations. Like Will, Dylan relied on procedural schemes to produce equivalent fractions, utilized a parts-out-of-wholes scheme to produce a whole from a given proper fraction, and adopted his partner's activities with manipulatives.

Teaching experiments with younger students suggest that students may develop schemes incorporating fractional units as a result of reflecting on their sustained activity with linear fraction representations. For instance, Steffe (2001) describes how Laura, a third-grade student, came to use her iterating operation to develop a measurement understanding of fractions while collaborating with a peer. Cases of sixth-grade students Will (Norton & D'Ambrosio, 2008), Isaac (Norton & Boyce, 2013) and Austin (Boyce, & Norton, 2014) suggest that older students can begin to reason with fractional units as a result of reflecting on their ways of operating with linear fraction bar representations too.

Dylan's case suggests that maintained focus on reasoning with fractions is more likely to support students who have well-established parts-out-of-wholes schemes in constructing fractional units than alternating between multiplication and fractions tasks. Dylan expressed more awareness of the limits of his reasoning when encountering fractions tasks in abstract and continuous settings, and if the teaching experiment had continued longer, sustained activity in these contexts may have led to perturbation whose resolution could result in Dylan's developing

schemes to coordinate fractional units. There may be implications for supporting prospective elementary teachers in beginning to coordinate fractional units. Like Dylan, prospective teachers often have well-established parts-out-of-wholes schemes and procedures for reasoning about fractions (Busi, Lovin, Norton, Siegfried, Stevens, and Wilkins, 2015; Olanoff, Lo, & Tobias, 2014).

A limitation of this study is that the intentional varying of task contexts to consider both Dylan and Maddie's units coordinating structures limited our ability to describe Dylan's learning in operational terms. To further understand how varying contexts influences students' units coordination, we plan to analyze Maddie's reasoning. Alternating contexts may have helped to support students' development of three levels of units with both whole numbers and fractions – but whole numbers for Dylan, and fractions for Maddie.

References

- Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). NY: Macmillan Publishing.
- Boyce, S. (2014). Modeling students' units coordinating activity. (Doctoral dissertation). Retrieved from <http://vtechworks.lib.vt.edu>
- Boyce, S. (in review). Modeling units coordinating activity from the witness' perspective. Submitted to *Educational Studies in Mathematics*.
- Boyce, S., & Norton, A. (2016). Co-construction of fractions schemes and units coordinating structures. *Journal of Mathematical Behavior*. 41, 10 -25

- Boyce, S., & Norton, A. (2014, July). *Modeling Austin's propensity to coordinate units*. Poster presented at the 38th annual meeting of the International Group for the Psychology of Mathematics Education, Vancouver, BC.
- Busi, R., Lovin, L., Norton, A., Siegfried, J., Stevens, A., & Wilkins, J. L. M. (2015). An investigation of PreK-8 preservice teachers' construction of fraction schemes and operations. In Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K. & Dominguez, H. (Eds.) *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 616–623). East Lansing, MI: Michigan State University.
- Empson, S. B., Levi, L., & Carpenter, T. (2011). The algebraic nature of fractions: Developing relational thinking in elementary school. In J. Cai & E. Knuth (Eds.) *Early algebraization: A global dialogue from multiple perspectives* (pp. 409 – 428). NY: Springer.
- Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, 26(1), 27 – 47.
- Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, 28 (4), 383-432.
- Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, 32, 538 -563.
- Hackenberg, A. J., & Lee, M. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, 46(2),

- Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, 28(1), 1 – 18.
- Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C. H. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, 43(4), 391-427.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.) *Second handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics* (pp. 629-667). Charlotte, NC : Information Age Publishing
- Learning Transformation Research Group (2013). *CandyDepot Educational Game for iPad*. Retrieved from <http://ltrg.centers.vt.edu/projects/games/apps/candyDepot.htm>
- Mullis, I. V., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 international results in mathematics*. International Association for the Evaluation of Educational Achievement. Chestnut Hill, MA: TIMMS & PIRLS International Study Center.
- National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors.
- Norton, A. (2008). Josh's conjectural operations: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education*, 39(4), 401-430.
- Norton, A., & Boyce, S. (2015). Coordinating $n+1$ levels of units *Journal of Mathematical Behavior*, 40, 211-242.

- Norton, A., Boyce, S., & Hatch, J. (2015). Coordinating units and constructing fractions at the CandyDepot. *Mathematics Teaching in the Middle School*, 21(5) 280-287.
- Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. M. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education Journal*, 10 (2), 111-136.
- Norton, A., Boyce, S., Ulrich, & Phillips, N., (2015). Students' units coordination activity: A cross-sectional analysis. *Journal of Mathematical Behavior*, 39, 51-66.
- Norton, A., & D'Ambrosio, B. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education*, 39(3), 220-246.
- Norton, A., & Wilkins, J. L. M. (2010). Students' partitive reasoning. *Journal of Mathematical Behavior*, 29, 181-194.
- Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education*, 43(5), 557-583.
- Olanoff, D., Lo, J. J., & Tobias J. M. (2014). Mathematical content knowledge for elementary mathematics: a focus on fractions. *The Mathematics Enthusiast*, 11(2), 267 – 310.
- Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning*, 1(4), 279-314.
- Olive J., & Çağalan, G. (2008). Learners' difficulties with quantitative units in algebraic word problems and the teacher's interpretation of those difficulties. *International Journal of Science and Mathematics Education*, 6, 269-292.
- Piaget, J. (2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). New York: Taylor & Francis. (Original work published 1977)
- Piaget, J. (1970). *Structuralism*. New York: Columbia University Press.

- Simon, M. A. , Placa, N., & Avitzur, A, (2016). Participatory and anticipatory stages of mathematical concept learning: Further empirical and theoretical development. *Journal for Research in Mathematics Education*, 47(1), 63—93.
- Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91–104.
- Steffe, L. P. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 177-194). New York: Kluwer Academic Publishers.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), pp. 259-309.
- Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior*, 20(3), 267 – 307.
- Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: SUNY Press.
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. NY: Springer.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*, 267 – 306. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Thompson, P. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25(3), 165-208.

Tillema, E. S. (2014). Students' coordination of lower and higher dimensional units in the context of constructing and evaluating sums of consecutive numbers. *Journal of Mathematical Behavior*, 36, 51-72.

Ulrich, C. (2012). Additive relationships and signed quantities. (Doctoral dissertation). Retrieved from <http://www.libs.uga.edu/etd>

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C: Falmer Press.