Co-construction of fractions schemes and units coordinating structures

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Abstract

A growing body of research implicates students' ability to coordinate multiple levels of numerical units as an important aspect of their mathematical development. In this paper, we consider relationships between the ways students coordinate units with whole numbers (their multiplicative concepts) and the ways students coordinate units with fractions (their fractions schemes). Interviews with 50 sixth-grade students suggest commonality in the number of levels of units-within-units structures that students construct for the two contexts, consistent with Steffe's (2001) reorganization hypothesis. The results suggest that fractions may be a fertile domain for elementary and middle grades students to continue to develop whole number understandings.

Keywords: Units Coordination, Fractions Schemes, Multiplicative Reasoning, Reorganization Hypothesis

Co-construction of Fractions Schemes and Units Coordinating Structures

1. Introduction

Steffe's (2001) reorganization hypothesis posits that students' ways of thinking about fractions – their fractional schemes – can result from their reorganization of the psychological operations they develop as they construct schemes for working with composite units. However, many students have yet to construct fractions understandings beyond part-whole comparisons by the time they reach middle school. As part of our ongoing efforts to develop ways to support such students, we assessed 50 sixth-grade students' whole number multiplicative concepts and fractions schemes, using items adapted from Hackenberg & Lee (2015) and Norton & Wilkins (2010). Our results indicate that students entering the middle grades are still developing structures for coordinating multiple levels of units, but students generally assimilate the same number of levels of units in whole number contexts as they do in fractions contexts. Analysis of the outlying cases suggests potential for “reverse” or “parallel” reorganizations for some students — fractions may be a fertile domain for middle-grades students to continue to develop whole-number multiplicative concepts.

2. Literature Review

Despite decades of research efforts, students' difficulty understanding fractions persists (Mullis, Martin, Foy, & Arora, 2012). Although it is critical for students to develop ways of thinking about fractions beyond initial part-whole comparisons (Behr, Harel, Post, & Lesh, 1992), students commonly lack experiences from which more powerful meanings for fractions might develop, as do many of their teachers (Izsák, Jacobson, de Araujo, & Orrill, 2012). These other meanings are typically described in terms of measure, operator, ratio, and quotient sub-constructs of fractions (Kieren, 1979).

A robust understanding of fractions includes developing each of the fraction sub-constructs, but developing an understanding of a fraction as a measure, such as a length on a number line, has been identified as particularly important for students' future mathematics achievement (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Siegler, Thompson, & Schneider, 2011). There is growing evidence that students' ability to think of fractions as measurable quantities affords early algebraic reasoning (Hackenberg, 2010; Hackenberg & Lee, 2015; Norton & Wilkins, 2012; Olive & Çağlayan, 2008).

When Lamon (2007) studied the influences of varying the focus of introductory fractions instruction with second-graders, she found that emphasizing the measurement construct was the most effective for helping her students to develop the other interpretations of fractions. But while Lamon's (2007) study suggested benefits to introducing fractions as measures, she found that the most important characteristic affecting individuals' ability to learn fractions concepts was not the sub-construct she emphasized in instruction: "[A] more significant factor in overall success was the development of the central multiplicative structures" (Lamon, 2007, p. 660).

Mathematics education researchers from diverse theoretical frameworks have modeled structural relations between addition, multiplication, exponentiation, fractions, and rational numbers (e.g., Behr, Harel, Post, & Lesh, 1992; Confrey & Smith, 1995; Lamon, 2007; Hackenberg, 2007; Izsák, Orrill, Cohen, & Brown, 2010; Norton & Wilkins, 2012; Piaget, 1970b; Steffe, 1992; 2001; Streefland, 1993; Thompson & Saldanha, 2003; Vergnaud, 1994). The notion of structure serves dual purposes in modeling the development of rational number concepts. There is consideration of the mathematical structure that might *result* from learning (e.g., understanding of the rational number system as a field), and there is consideration of the

psychological structures learners construct *as they* are learning (e.g., the structure of children's counting schemes). In this paper, we focus on the latter to analyze relationships between sixth-grade students' multiplicative concepts and their fractions schemes.

2. Theoretical Framework

We adapt a scheme-theoretic perspective (von Glasersfeld, 1995) to include an interpretation of Piagetian structure (1970b) to consider relationships between students' units coordination with fractions and their units coordination with whole number multiplication.

2.1 Von Glasersfeld's Scheme Theory

A scheme is an individual's established way of experiencing and operating in service of a goal. A scheme consists of three sequential components: a recognition template, mental activity, and an expected result (von Glasersfeld, 1995). When the result of goal-directed activity is unexpected, resolving this *perturbation* is often an impetus for establishing a new scheme.

The recognition template of a scheme is its assimilatory structure, which is obtained from the sensorimotor pattern awareness or mental imagery associated with an activity. Assimilation involves modification of perceptual input so that it fits into an individual's existing conceptual structures (von Glasersfeld, 1995). In the beginning, *participatory* stage of a scheme, the recognition template is closely tied to particular contexts and recent activities (Tzur, 2007).

An *operation* is an interiorized mental action. Operations are abstractions of actions that "can be carried out through representation and not through actually being acted out" (Piaget, 1970a, p. 14). This abstraction is what allows an individual to coordinate operations with other operations. In contrast, the results of mental activity that are someone has merely internalized must still be carried out, at least in his imagination.

Reflective abstraction is the process by which internalized actions previously tied to sensorimotor activity become interiorized (von Glasersfeld, 1995), by which the recognition template becomes less dependent on specific contexts and schemes become more *anticipatory* (Tzur, 2007). When an entire tri-partite scheme structure is interiorized, a scheme is called a *concept* (Hackenberg, 2007). The recognition template and the expected result of a concept are perceived simultaneously, which is what allows concepts to become parts of the assimilatory structure of other schemes.

2.2. Piagetian Structures

Piaget (1970b) defines a *structure* as comprising “three ideas: the idea of wholeness, the idea of transformation, and the idea of self-regulation” (p. 5). *Wholeness* refers to the idea that, although structures have individual elements, the relationships between elements are defined by global rules that apply to a totality. *Transformation* refers to the idea that structures are not static forms; they represent a potential for “intelligible change that always preserves invariance in certain respects” (p. 20). *Self-regulation* refers to the idea that the “transformations inherent in a structure never lead beyond the system, but always engender elements that belong to it and preserve its laws” (p. 14). The versatility of the construct permits flexibility in analyzing both recurrent and nested structural relationships.

A scheme is an example of a structure, for organizing a sequence of operations in service of a goal. “Piaget (1970a/1972) considered a scheme to be a general structure of a set of actions that conserves itself by repetition, consolidates itself by exercise, and applies itself to situations that vary because of contextual modification” (Hunting, Davis, & Pearn, 1996, p. 356). Whereas a person engages in acts of self-regulation while carrying out the activity of a scheme, transformations that are interiorized must be *reversible* because they precede operational activity.

As the result of reflective abstraction, individuals assimilate progressively more complex systems of elements and reversible operations as structures.

3. Conceptual Framework

According to Steffe's (2001) reorganization hypothesis, students' initial fraction schemes can emerge as reorganizations of students' schemes and operations previously organized for whole number counting sequences. We use the construct of Piagetian structure to consider the bi-directionality of such reorganizations for fractions and multiplicative reasoning. We begin with descriptions of multiplicative concepts and fractions schemes.

3.1 Multiplicative Concepts

Hackenberg (2010) describes *multiplicative concepts* in terms of the number of levels of units-within-units that students assimilate in situations for which multiplicative reasoning with whole numbers would be seen as appropriate from an adult perspective. The first multiplicative concept (MC1) involves coordination of two levels of units in activity. This is considered the most basic units coordination for multiplication, and involves "inserting a composite unit into the units of another composite unit" (Hackenberg & Tillema, 2009, p. 3). Coordinating units *in activity* means that the student must actually carry out the composition (i.e., by counting or marking with or without visual aids). For example, suppose a student forms a goal of finding out how many 3s are in 12. If the student verbally counted by 1s to 12, double-counting the first count of three as 1, the count from 4-7 as 2, etc., leading to a correct response, this would support an inference that the student had constructed the first multiplicative concept.

The second multiplicative concept (MC2) permits the composition of units described above, but with counting actions interiorized. Students who have constructed the second multiplicative concept have a greater propensity for strategic reasoning, as they can operate on

the units within a composite unit immediately (without having to “make” them). If an MC2 student forms a goal of finding out how many 3s are in 24, she might reason that since there are four 3s in 12 and two 12s in 24, then there would be 4 times 2 = eight 3s in 24.

With the third multiplicative concept (MC3), a student can assimilate composite units of composite units. An MC3 student might operate in the same way as an MC2 student to find the number of 3s in 24, but more quickly and without observable activity. Furthermore, the result of operating would retain a three-level structure of 24, as eight units of three units of one unit (see Figure 1). If the task sequence above was followed by a request to figure out the number of 3s in $24 + 12$, an MC3 student would likely recognize that 36 is twelve 3s, because it is just adding on another four 3s. An MC2 student would be more likely to conflate units or re-build a three-level structure for 36 in activity.

Figure 1. Form of a three-level structure of 24

3.2. Fractions Schemes

Steffe and Olive (2010) describe a hierarchy of fractions schemes in terms of the mental operations inferred from analyzing students’ engagement with fractions tasks within constructivist teaching experiments (e.g., Hackenberg, 2007; Norton, 2008; Olive, 1999; Sáenz-Ludlow, 1994; Steffe, 2001; Tzur, 1999). We focus on describing five schemes that develop in

133 sequence: the part-whole scheme¹, partitive unit fraction scheme, partitive fraction scheme,
134 reversible partitive fraction scheme, and iterative fraction scheme.

135 The operations of the part-whole scheme include partitioning a whole into equally sized
136 parts and disembedding (removing from the whole without modifying the whole) one or more
137 parts from the whole. For example, to understand $\frac{3}{5}$ as 3 parts out of a whole of 5 parts means
138 to understand 3 and 5 as three 1s and five 1s, respectively, as well as understanding that the three
139 1s are contained within the five 1s. Though the part-whole scheme involves fractional language,
140 it does not involve unitizing a fraction. It is the ratio of the number of equal parts in one
141 collection to the number of equal parts in a larger collection (the whole) that determines a
142 fraction name with a part-whole scheme.

143 The partitive unit fraction scheme (PUFS) is the first fraction scheme for which a student
144 unitizes a fraction, and for this reason students who have not yet constructed the PUFS are
145 considered pre-fractional (Steffe & Olive, 2010). Like the part-whole scheme, the PUFS involves
146 partitioning and disembedding operations, but it also includes iteration. A student with a PUFS
147 understands a unit fraction, such as $\frac{1}{5}$, as a part within a whole that, if disembedded from the
148 whole and iterated 5 times, would result in the whole. The sequential operations of partitioning,
149 disembedding, and iterating are collectively referred to as *equipartitioning* (Steffe & Olive,
150 2010). A child with a PUFS can conceive a whole as a composite unit of 5 units, any one of
151 which could be iterated to form the whole (see Figure 2). Thus, the PUFS is reversible, as a
152 student with a PUFS can both form a whole unit from a unit fraction and form a unit fraction
153 from a whole.

¹ The part-whole scheme we describe is sometimes referred to as a “parts-out-of-wholes fractional scheme” to contrast it with “parts-within-wholes fractional schemes” that do not include the disembedding operation (Hackenberg, 2013).

Figure 2. Form of the PUFS two-level units structure for $1/5$

A generalization of the partitive unit fraction scheme to accommodate non-unit proper fractions results in the emergence of a *partitive fraction scheme* (PFS). A student with a PFS understands a proper fraction such as $3/5$ to mean the size of 3 parts out of a whole equipartitioned into 5 equal pieces. The operations of the PFS are a melding of the partitive unit fraction scheme and the part-whole scheme. On the one hand, there is coordination between a unit fraction and a unit 1. On the other, there is coordination between the number of parts in the whole and the number of parts in the proper fraction.

The partitive fraction scheme is limited by its non-reversibility for non-unit fractions. Students must construct a *reversible partitive fraction scheme* (RPFS) in order to form a whole from a non-unit proper fraction. The RPFS permits the “reversing” of the operations of the partitive fraction scheme, so that a student understands that to form a whole from an amount such as $3/5$, she must first find the amount $1/5$ by splitting $3/5$ into 3 equal parts. Although the RPFS involves anticipation of the inverse relationship between iterating and partitioning (the construction of the *splitting* operation, Norton & Wilkins, 2012) to reverse the sequence of

operations in the partitive fractions scheme, the RPFS does not yet involve a third level of units. Like the partitive fraction scheme, it involves units of $1/n$ within 1 and m units of 1 within n units of 1 (see Figure 3).

Figure 3. Form of the RPFS two-level units structure for $3/5$

Although students' partitive unit fraction schemes can become both generalized and reversible to make sense of proper fractions like $3/5$, the relationships between $1/5$ and $3/5$ and between $1/5$ and 1 are not fully integrated because iterations of unit fractions are constrained to being contained within 1. Unlike the partitive fraction schemes, the iterative fraction scheme (IFS) involves interiorization of three levels of units: units of $1/n$, units of m $1/n$ units, and units of 1 (see Figure 4). Upon construction of the IFS, a fractional unit becomes *iterable*, so that numbers like $3/5$ and numbers like $7/5$ each result from iterating the fractional unit $1/5$ (Steffe & Olive, 2010; Tzur, 1999). The construction of the IFS thus allows for a conception of both proper and improper fractions as “numbers in their own right” (Hackenberg, 2007, p. 207).

Figure 4. Form of the IFS three-level units structure for $7/5$.

3.3 Levels of Units

The units coordination structures for fractions schemes and multiplicative concepts share the notion of n levels of units (see Table 1), but the units coordinating structures are not always isomorphic. In the case of whole number multiplication, assimilating two levels of units means to assimilate a whole number as both a composite of units of 1 and a (non-composite) unit within a composite unit. For example, the number 4 can be iterated three times to form 12. An isomorphic multiplicative relation holds for coordinations with the partitive (unit) fractions scheme: 1 is simultaneously a unit that can be iterated and a composite of n units (of $1/n$). The three-level structure of units coordinations with the iterative fraction scheme (IFS) is not isomorphic to the three-level structure for whole numbers. The IFS includes a notion of the (improper) fraction m/n as m of the amount that, when iterated n times, would make both n/n and 1 (for $m > n$). There is no such notion for multiplicative concepts.

Table 1

Units coordination structure relationships by number of levels of units

Levels	Fractions Schemes	Multiplicative Concepts
1	Pre-fractional	MC1 or below
2	Partitive Schemes	MC2
3	Iterative Fraction Scheme	MC3

4. Research Questions

The reorganization hypothesis posits that students can reorganize operations originating with counting to construct fractions schemes. The reorganization hypothesis emerged as a result of teacher-researchers interacting with elementary grades students in fractions contexts, prior to their experiencing fractions instruction in school. We sought to investigate the directionality of the relationship between the construction of units coordinating structures for whole number multiplication contexts and units coordinating structures for fractions contexts. For example, while both the second multiplicative concept (MC2) and the partitive unit fraction scheme (PUFS) require iterating a unit composed within another unit, it is unknown whether students must construct MC2 prior to constructing a PUFS or whether they might instead be constructed simultaneously.

Our conjecture entering the study was that many sixth-grade students who lack a PUFS assimilate two levels of whole number units, but they have not had the opportunity to construct a scheme to coordinate fractional units. School-based instruction in the United States currently emphasizes figural interpretations of fractions with primary focus on part-whole relationships; the resulting meaning of fractions is that of ordered pairs of numbers that are used to describe figural arrangements (Norton & Wilkins, 2010; Pitkethly, & Hunting, 1996). Students commonly lack experiences from which more powerful structures for fractions might develop, as do many of their teachers (Izsák, Jacobson, de Araujo, & Orrill, 2012). Accordingly, we formed the following hypotheses about the relationship between the units coordinating structures for whole numbers and the units coordinating structures for fractions:

Hypothesis 1: There is positive association between the number of levels of units in students' multiplication concepts and students' construction of fractions schemes.

Hypothesis 2: Students construct n -level structures for whole number multiplication
before constructing n -level structures for fractions.

Evidence supporting the veracity of Hypothesis 1 would corroborate *levels of units* as an important construct for analyzing the organizations within a developing multiplicative conceptual field (Vergnaud, 1994). The purpose of Hypothesis 2 is to parse out the nature of the organizations, which could provide guidance for the types of support that might be provided to students. Evidence of the veracity of both Hypothesis 1 and Hypothesis 2 would indicate that coordinating n levels of units with fractions is prefaced by the construction of an n -level structure for coordinating whole number units. Evidence supporting the veracity of Hypothesis 1, but not Hypothesis 2, would suggest that students may co-construct fractions schemes involving n levels of units coordination as they are interiorizing n levels of whole number units.

5. Methods

To explore the relationship between units coordination with whole-number multiplication and fraction schemes, we conducted one-on-one clinical interviews, which were video-recorded (Clement, 2000). We developed a coding scheme by which to quantify (on an ordinal scale) students' ways of operating within the domains of fractions and whole number multiplication. We then statistically analyzed the relationship between students' units coordination with whole number multiplication (as described by their multiplicative concepts) and students' units coordination with fractions (as described by their fraction schemes) using non-parametric measures of association (both symmetric and asymmetric). We then re-analyzed the recordings of the clinical interviews of students who were outliers from the overall pattern. Here our aim was to explain aspects of the quantitative findings and hypothesize other relationships that might explain the students' deviance from the general pattern.

We conducted interviews with students from a rural middle school in the Southeastern region of the United States. We interviewed fifty students (all of the sixth-grade students in the school who returned a signed parent permission form) during a three-day period in the second week of the school year, in August. Each interview was conducted by one of the two authors during the normal class time. Due to space and time constraints, we set up two interviewing stations, each with a video camera and microphone, in opposite corners of an otherwise empty classroom and conducted individual interviews simultaneously.

Our clinical interview protocol consisted of two parts. The first section included tasks from a written assessment of students' fractions schemes (Norton & Wilkins, 2010; Wilkins & Norton, 2011; Wilkins, Norton, & Boyce, 2013). The second section consisted of tasks designed to assess multiplicative concepts, which were based on personal communication with Hackenberg (May 9, 2012) and from Hackenberg & Lee (2015).

5.1. Approach for Fractions Interview Assessment

Our approach in the fractions assessment was to engage students in pairs of tasks from least difficult to most difficult, according to Steffe & Olive's (2010) hierarchy of fractions schemes. Two tasks per scheme were each designed to allow the interviewer to make and confirm inferences of students' construction of a PUFS, PFS, RPFS, and IFS. We posed the tasks in sequence, and after each task, we made notes pertaining to a preliminary assessment of (counter)-indication of the associated scheme. Although the interviewers had flexibility to ask follow-up questions as desired, if there was counter-indication of the associated scheme on two consecutive tasks, we ended the fractions portion of the interview and concluded that the student had not constructed schemes pertaining to the unposed tasks. This approach is justified because

the hierarchical nature of the fractions schemes has been statistically validated with a similar population of students (Wilkins & Norton, 2010).

Tasks 1 and 2 (see Figure 5) were used to assess students' construction of a partitive unit fraction scheme (PUFS). A student's response to Task 1 provides an opportunity to infer her assimilation of fractional language, in her interpretation of "1/7 as long." Indication of PUFS might include observation of a sequential left-to-right marking of the whole into seven approximately equally sized parts. Task 2 provides an opportunity to better understand the operations of a student whose marks are imprecise. By asking follow-up questions about how the student decided to make marks, one can infer whether his fraction conception includes the 1- n size-relation instead of merely a 1- n parts relation. Counter-indication of the PUFS in Task 1 includes drawing a bar the same length as the given bar or segmenting the bar into pieces that are not (approximately) of uniform size. Counter-indication of the PUFS in Task 2 includes naming the fraction as something other than $1/n$, where n is the number of parts the student attributes to the longer stick, and making partition marks on the larger stick that do not appear to incorporate the size of the shorter stick.

1. Your stick is $\frac{1}{7}$ as long as the stick shown below. Draw your stick.

2. What fraction is the smaller stick out of the longer stick?

Figure 5. Tasks used for assessing a PUFS

Tasks 3 and 4 (see Figure 6) were used to assess students' construction of a PFS. Tasks 3 and 4 mirror the first two tasks' hypothetical assimilation requirements, but with composite

287 fractions. If imprecise marks are made in response to Task 3, follow-up questions to explore
288 whether the student treats composite fractions as relative sizes of the whole can be used to
289 discern between inferences of a PWS and a PFS. Follow-up questions to Task 4 are useful for
290 distinguishing whether a student assimilates only *unit* fractions as relative sizes. If a student
291 initially responds to the task with a unit fraction such as one-half or one-third, one might ask the
292 student to consider whether another (composite) fraction might be more appropriate.

3. Your stick is $\frac{3}{5}$ as long as the stick shown below. Draw your stick.

4. What fraction is the smaller stick out of the longer stick?

Figure 6. Tasks used for assessing a PFS

295 Tasks 5 and 6 are designed to assess the student's reversible partitive fraction scheme
296 (RPFS). For both tasks, one counter-indication of the scheme is the assimilation of the task as a
297 request to *create* a fraction of the whole. In Task 5 especially, students might be able to create a
298 correctly sized whole via assimilation within a part-whole scheme. Students might partition the
299 given bar into 4 pieces, and make a whole with 5 pieces below the given bar. A follow-up
300 question about the name of one of the pieces in the original bar can be used to infer whether the
301 student was conceiving of $\frac{4}{5}$ as 4 (out of 5) of $\frac{1}{5}$. Responses of "one" or "one-fourth" would
302 counter-indicate that the student had assimilated the situation within a RPFS.

5. The stick shown below is $\frac{4}{5}$ as long as a whole candy bar. Draw the whole candy bar.

6. The stick shown below is $\frac{3}{7}$ as long as a whole candy bar. Draw the whole candy bar.

Figure 7. Tasks used for assessing a RPFS

In Tasks 7 and 8, students without an iterative fraction scheme (IFS) might immediately indicate a lack of an assimilatory structure for the task. Such counter-indications include expressions of confusion or responses indicating assimilation to a reciprocal task of naming a proper fraction. If a student names the fraction after an inference of iterating activity, then a follow-up question asking for explanation of whether the student was iterating a unit fraction is useful. For example, if a student responded to Task 8 with, “ $\frac{7}{3}$, because the shorter bar is three-thirds and seven of those thirds fit into the longer bar,” this would be strong indication of an IFS.

7. What fraction is the longer stick out of the shorter stick?

8. What fraction is the longer stick out of the shorter stick?

Figure 8. Tasks used for assessing a IFS

5.2. Approach for Multiplicative Concepts Interview Assessment

There were five tasks used for assessing students' multiplicative concepts. The first involved rectangular bars made of colored construction paper. The interviewer put a red bar that

was 9 inches long and a yellow bar that was 2.25 inches long on the student's desk and asked, "How many times does the yellow bar fit into the red bar?" The student could manipulate the bars in response, and if "4" was not the result (e.g., due to estimation or counting errors) then the teacher would ask the student to repeat the process in order to establish that the yellow bar fit into the red bar 4 times. The red bar was then removed from view, and a blue bar that was .75 inches long was placed below the yellow bar, with the same line of questioning. Upon a similar establishment that the blue bar fit into the yellow bar 3 times, the student was asked, "Without using the bars, can you decide how many times the blue bar fits into the red bar?" Upon a response, the interviewer encouraged the student to explain her response by referring to the actual bars.

In this initial bars task, the interviewer's focus was on how a student's responses related to her assimilation of composite units and her use of multiplication facts. If a student responds that the blue bar fit in the red bar "about 12 times" and then, after the bars were visible, she indicates counting increments of 1 coordinated with measuring the length of the red bar using the blue bar as a unit, this would suggest she had constructed the first multiplicative concept. To make an inference regarding the second multiplicative concept, the interviewer might follow-up with, "Is there a way you could use the yellow bars to find the answer?"

For other students, the interviewer encouraged explanations to discern whether seemingly facile and confident responses masked less-sophisticated units coordinations. If a student quickly responds "12" and explains that the reason is that "4 times 3 is 12", this is moderate indication that he thinks of 12 as the composition of 4 units of 3. By asking him to justify his use of multiplication using the bars, the interviewer could infer whether his response was based on rote fact. If his justification were restricted to demonstrating that the blue bar fit in the red bar 12

340 times, this would suggest that he had constructed the first multiplicative concept but not the
341 second.

342 The next three tasks were similar to the first task, except that students were asked to
343 determine size relations for the bars that differed from their physical relations. The original bars
344 were removed from the table, and a red bar of length 25 cm, a yellow bar of length 7 cm, and a
345 blue bar of length 3 cm were placed on the desk. Three indeterminate bars tasks involving these
346 physical bars were posed to the student in sequence (see Figure 9). Between each task, the
347 interviewer asked the student to explain her response.

1. If the Yellow bar fit into the Red bar 5 times and the Blue bar fit into the Yellow bar 2 times, how many times would the Blue bar fit into the Red bar?
2. If the Yellow bar fit into the Red bar 5 times and the Blue bar fit into the Red bar 15 times, how many times would the Blue bar fit into the Yellow bar?
3. If the Blue bar fit into the Red bar 8 times and the Blue bar fit into the Yellow bar 4 times, how many times would the Yellow bar fit into the Red bar?

348

349 *Figure 9. Indeterminate Bars Tasks*

350 The interviewers attended closely to similarities and differences in students' activities
351 across these three tasks. If students assimilate the first indeterminate bars task as the same type
352 of task as the initial bars task and quickly respond and subsequently explain the result is 5 times
353 $2 = 10$, this would suggest that they were not relying on their perception of the figurative
354 materials. The second indeterminate bars task involves reversing the relationship of the earlier
355 tasks. If a student assimilates the task as the same as the previous tasks, and responds that the
356 Blue bar would fit into the Yellow bar 15 times $5 = 75$ times, this suggests his composition of
357 units within units may not be reversible. His assimilation of the task as having the same form but

reversing his reasoning to respond $15 \div 5 = 3$ times would indicate at least MC2. The final bars task involves reversibility of a different form, as the previous tasks each request a response in terms of iterations of Blue bars. If students were able to quickly and confidently reason that the Yellow bar would fit into the Red Bar two times in the final bar task, the flexibility in the students' units coordination would indicate MC3.

The bars tasks were followed by two tasks that pertained to students' assimilation of experientially real problem situations. The first was about school field trips. "There are 96 students going on the field trip and 6 buses. The same number of students is to ride on each bus. Draw a picture to show how you will determine how many students will ride on each bus. How many students will ride on each bus?" After drawing the 6 buses, if a student began to distribute 1s across the 6 buses, the interviewer would stop the student and ask her to explain her strategy and whether she could think of a faster way. Indication of the second multiplicative concept would include distributing composite units across the six buses. If a student suggested dividing 96 by 6, the interviewer would ask the student to justify the choice of division.

The final task was about desks in classrooms. "In a classroom, there are 6 rows with 4 seats in each row. The teacher adds 12 more seats to the classroom. How many total rows can be made with four seats in each row?" The interviewer would repeat the task if the student requested. If the student gave an incorrect response, the interviewer would ask the student to explain his thinking. If the student gave a correct response of 9 rows, the interviewer immediately followed up with, "How many seats will be in the classroom?" and a subsequent request for explanation. If a student could immediately reason that there would be 36, because there would be nine rows of four seats in each row, this would be strong indication that he had constructed the third multiplicative concept (MC3).

5.3. Coding of Fractions Schemes and Multiplicative Concepts

For assessing fractions schemes, we constructed the following rubric, *a priori*, for coding the videos of the clinical interviews. We took for granted the theoretical hierarchy of fractions schemes—supported by several empirical studies (e.g., Norton & Wilkins, 2010; Wilkins & Norton, 2011; Wilkins, Norton, & Boyce, 2013)—as we assessed each of the four progressions: PUFS, PFS, RPFS, and IFS. For each scheme, we looked for evidence of each listed aspect and identified evidence explicitly, with the understanding that there should be stronger indication than counter-indication for each of these aspects in order to code that the student had constructed the scheme. For each of the aspects described in Figure 10, we used the following scoring:

0: If we found stronger counter-indication than indication for an aspect of the scheme, or we could identify no evidence for any aspect of the scheme;

1: If we found explicit evidence for each aspect of the scheme;

?: If we could find stronger indication than counter-indication for each aspect of the scheme, or found evidence for some but not all aspects.

Scheme	Aspect(s)
PUFS	Iterating a unit fractional part Appropriate use of fractional language Understanding of 1-to- n relationship between a unit fractional part and a whole
PFS	Treats a proper fractional part as iterations of a unit fractional part Understands proper fractional part as a relative size to the whole
RPFS	Coordinates partitioning and iterating to produce the unit fractional part and reproduce the whole
IFS	Treats improper fractions as iterations of a unit fractional part while maintaining the 1-to- n relationship between the unit fractional part and the whole

Figure 10. Aspects of Fractions Schemes for Coding Evidence

We created a spreadsheet listing each student's name by row and each of the aspects in the columns. For each student and aspect, we listed the time stamp on the video and a description of the evidence by which we were making a claim of indication or counter-indication. We each

coded the results of our own interviews in a spreadsheet. We then conferred to review videos of students for whom a ‘?’ was coded and came to a consensus for a final coding of ‘0’ or a ‘1’ for each student’s fractions scheme assessment.

Coding multiplicative concepts was similar. We coded 1 if either no multiplicative activity occurred (i.e., if 1 was the only available unit) or if students built a two-level structure in activity. In such cases, students might know multiplication facts and procedures, but they did not build a three-level unit structure in activity. We coded 2 if there was evidence that students built a three-levels-of-units structure, i.e., that 15 was five 3s. We coded 3 if three levels of units were interiorized, as evidenced by facile, confident, and creative uses of a three-level structure. Again, evidence for the coding was listed by timestamp and a description of the observation.

5.4. Symmetric and Asymmetric Tests of Association

We developed an ordinal ranking for each of the fractions schemes from our coding, ranging from 1 for pre-fractional, or lack of a PUFS, 2 for the PUFS but not PFS, etc., to 5 for construction of all the schemes including the IFS. Similarly, multiplicative concepts were scored 1, 2, or 3 for construction of the first (or below), second, or third multiplicative concepts, respectively. Though they each have a rank ordering, it is not an interval scale that equates differences between rankings. For example, although the RPFS is one rank above the PFS, and the IFS is one rank above the RPFS, the difference between a RPFS and a PFS and the difference between a RPFS and an IFS are not assumed to be equivalent.

To test the association between two ordinal level measures, we consider the amount of *concordance* in the data. For each student, an ordered pair (x, y) is assigned so that x is in $\{1, 2, 3, 4, 5\}$ and y is in $\{1, 2, 3\}$. A pair of such ordered pairs (a *dyad*) is said to be concordant if the sign of the difference in the x coordinates matches the sign of the difference in the y coordinates.

For example, if Bobby was assessed a PUFS and MC2, and Suzy was assessed an IFS and MC3, then the dyads would be concordant, because the sign of the difference in both x and y is the same (positive if considering the change Suzy - Bobby). If the differences between x and y were non-zero but different signs, the dyads would be discordant (see Table 2).

Note that in all calculations of association with ordinal variables, the magnitude of the differences does not have a bearing on the calculation (this requires interval level variables). In general, tests of association involving concordance involve computing all possible pairwise concordance relations, and then comparing the resulting number of concordant dyads and discordant dyads. In the simplest case (the gamma statistic), a test statistic consisting of the ratio of the difference in the number of concordant and discordant dyads to the total number of dyads is computed. If the absolute value of this ratio is near one, there is a strong association between the variables (Gonzales & Nelson, 1996).

Table 2
Concordance Examples

Student 1 (x_1, y_1)	Student 2 (x_2, y_2)	(Student 2 – Student 1) ($x_2 - x_1, y_2 - y_1$)	Sign of product $\text{sign}(x_2 - x_1)(y_2 - y_1)$	Description
(2, 3)	(5, 4)	(3, 1)	+	concordant
(2, 3)	(3, 5)	(1, 2)	+	concordant
(2, 3)	(2, 4)	(0, 1)	0	weak tie
(2, 3)	(2, 3)	(0, 0)	0	strong tie
(2, 3)	(1, 3)	(-1, 0)	0	weak tie
(2, 3)	(1, 4)	(-1, 1)	–	discordant

There are several ways to use concordance to test for association between two ordinal-level variables; the main differences are the handling of situations in which one or both of the differences in x and y are zero. When both differences are zero, that is termed a “strong tie,” and when one of the differences is not zero, that is termed a “weak tie.” In symmetric measures of association, ties in the x variable are treated the same as ties in the y variable. In asymmetric

measures of association, weak ties are excluded if the tie is in the independent variable and included if the tie is in the dependent variable.

To test our two hypotheses, we considered both symmetric and asymmetric measures and strong and weak ties. Two symmetric measures of association we used are the gamma statistic, in which all ties are excluded from the calculation, and Kendall's tau-b, in which all weak ties are included. We used the Somers' d statistic (Somers, 1962) as an asymmetric measure of association.

6. Results

6.1. Test of Hypothesis One

We used PASW (SPSS version 18) to compute the cross tabulations for multiplicative concepts and fractions schemes listed in Table 3. Notably, there was only one student who was assessed with an IFS at the beginning of his sixth grade school year, and half of the students we interviewed were pre-fractional (NF).

Table 3
Cross-Tabulation of Multiplicative Concepts vs. Fractions Schemes

Multiplicative Concepts	Highest Constructed Fractions Scheme					Total
	NF	PUFS	PFS	RPFS	IFS	
1	25	5	0	0	0	30
2	2	13	0	2	0	17
3	0	0	0	2	1	3
Total	27	18	0	4	1	50

To test Hypothesis 1, we consider whether there is an association between multiplicative conceptions and fractions schemes. We first tested an asymmetric association, treating Multiplicative Concepts as the independent variable and Fractions Schemes as the dependent

variable. The value of the Somers' d statistic, as computed in SPSS, is .786, with (exact) $p < .001$. We thus reject this null hypothesis and conclude that there is an asymmetric association between the two variables. However, under the null hypothesis that there is not a symmetric association, significance of both the gamma statistic (.962; exact $p < .001$) and Kendall's tau-b (.750; exact $p < .001$) suggest the two constructs are symmetrically positively associated. Note that a statistically significant symmetric association implies a statistically significant asymmetric association in both directions. To summarize the differences between the three statistics (Somers' d, gamma, and Kendall's tau-b), it does not matter whether ties are handled symmetrically or asymmetrically, coarsely or finely – the two variables are strongly positively associated in each case. With regard to Hypothesis 1, we conclude that there is a symmetric association between the number of levels of units in students' multiplication concepts and students' construction of fractions schemes.

6.2. Test of Hypothesis Two

Whereas the presence of an asymmetric association without a symmetric association would be consistent with temporal dependency, the presence of a symmetric association does not imply contemporaneous development. In Table 4, we collapse the cross-tabulation of multiplicative concepts and fractions schemes in order to test Hypothesis 2, that an additional level of multiplicative units is interiorized before an additional level of units is coordinated with fractions. For Hypothesis 2, the null hypothesis is that the frequency of students coordinating $n+1$ levels of units with fractions but only assimilating n (or fewer) levels of multiplicative units will be lesser or equal to the frequency of students assimilating $n+1$ levels of multiplicative units but only coordinating n (or fewer) levels of units with fractions. Under the alternative hypothesis that the whole numbers units coordinating structures are constructed first, we would reject the

null hypothesis if, for those students whose number of levels of units differed across the two constructs, there were significantly more students constructing MC- n without coordinating n levels of units with fractions than vice versa.

Table 4

Cross-Tabulation of Multiplicative and Fractional Levels of Units

<i>MC-n</i>	<i>n</i> -levels of Fractions Units		Total
	With <i>MC-n</i>	Without <i>MC-n</i>	
With n -level Fractions Units	41	5 [^]	46
Without n -level Fractions Units	4 [*]	0	4
Total	45	5	50

* 2 students assessed with MC2 and NF; 2 students assessed with MC3 and RPFS.

[^] 5 students assessed with MC1 and PUFS.

We failed to reject the null hypothesis using a one-tailed binomial test ($p = .74$), as there were 4 students in one group and 5 in the other (see Table 4). Thus, we cannot make a statistical claim about the temporal precedence of students' construction of n -level whole number units coordinating structures and students' construction of n -level fractions schemes. Considering that multiplication instruction precedes fractions instruction in our participants' school curriculum, we were particularly intrigued by the 5 cases in which students were assessed with a PUFS but only MC1. We returned to the videos of the clinical interviews in order to investigate whether there were commonalities between them that might explicate our findings.

6.3 Investigation of Special Cases

In examining the five cases in which we assessed students as MC1 with a PUFS, we recognized that measurement error could be a factor. We also know that many middle school students will be in the process of constructing PUFS and/or interiorizing two levels of units, rendering inferences about their current ways of operating especially difficult. Indeed, a related measurement error seemed to play a role in at least one of the five cases. Upon re-examination of

his clinical interview, we now understand one student, Jacob, as an MC2 student who had constructed a PUFS (consistent with Hypothesis 2) and not as an MC1 student who had constructed a PUFS (contradicting the model). However, the remaining four cases raise questions about the theoretical model and/or our assessments of students' ways of operating.

We begin by sharing evidence affirming that Jacob had constructed a PUFS. We then share evidence that calls into question our initial assessment of Jacob as an MC1 student. The analysis we share will provide the reader with an example of how we made inferences from video data, and will also provide some indication for how difficult it can be to assess students' ways of operating. Finally, we consider the remaining four cases in which we assessed MC1 students as constructing a PUFS and offer some explanations for why these cases might occur.

6.3.1. Jacob's Case

Jacob responded to the first PUFS task (see Task 1 in Figure 5) by partitioning the given stick into fourths, using three equally spaced partitions. He then erased the partitions and made six new partitions, producing seven equal parts in the given stick. Finally, he drew a small stick below the leftmost part, with the same length. When the interviewer asked him why he had done this, Jacob replied, "because I split this up into seven pieces and copied it [referring to the small bar he had drawn below]." When asked how he knew the piece would be $\frac{1}{7}$, Jacob explained, "It would fit in there exactly seven times." Jacob's actions provide strong indication of a PUFS because he had disembedded a part from the whole and established its size relation with the whole by mentally iterating the part seven times. His response to PUFS Task 2 (see Figure 5) provides further indication of a PUFS: He used the length of the small stick to measure off the length of the long stick with his fingers; "it's about one fifth."

521 For the units coordinating tasks, indications were not as clear or consistent and, thus,
522 inference became more challenging. Recall that the first task asks the student to determine how
523 many times a small blue bar fits into a long red bar, given that the small blue bar fits into a
524 medium yellow bar three times, and that the medium yellow bar fits into the long red bar four
525 times. Jacob responded “12... 4 times 3”, explaining, “the yellow bar would be three blue ones
526 and if the yellow bar fits in here four times, you would have to do four times three to figure it
527 out.” Jacob’s explanation would seem to provide strong indication that he was an MC2 student,
528 treating the yellow bar as a unit of three blue bars, and he responded similarly to the second units
529 coordinating task (see Task 1 in Figure 9). However, his response to the third units coordinating
530 task (Task 2 in Figure 9) called this into question:

531 Interviewer: This time, the yellow bar fits into the red bar five times. The blue bar fits
532 into the red bar 15 times. How many times does the blue bar fit into the yellow bar?

533 Jacob: How many times does the yellow bar...

534 Interviewer: [repeats question]

535 Jacob: [looks aside for 10 seconds] Seventy-five?

536 Interviewer: How did you get seventy-five?

537 Jacob: Fifteen times five.

538 Interviewer: Okay. And why did you multiply those?

539 Jacob: Because the yellow bar fits into the red bar five times and the blue bar fits into the
540 red bar 15 times.

541 Interviewer: Can you draw a picture to show?

542 Jacob: [With some coaching, Jacob draws a long bar draws five equal parts under it,
543 exhausting the length of the long bar. He then draws 15 equal parts above the long bar,
544 exhausting the length of the long bar]

545 Interviewer: How many times does it look like the blues fit into the yellow in this
546 picture? [referring to the picture Jacob had drawn]

547 Jacob: Three or four times.

548 Interviewer: Three or four? Okay, I'm going to give you the question one more time,
549 from the beginning. Here's the question... [repeats the original question].

550 Jacob: [looks aside for 2 seconds] Three.

551 Interviewer: Why is it three?

552 Jacob: Because I had to divide 5 by... 15 by 5.

553 Interviewer: Why? Why do you have to divide?

554 Jacob: [after about 10 seconds] because... [another 5 seconds] I don't know.

555

556 Initially, Jacob's response suggested to us that he had been operating procedurally in each
557 of the three units coordinating tasks up to this point, without actually organizing the unit
558 structures involved in the tasks. Although it was clear that Jacob had understood what the task
559 was asking (as the interviewer had him repeat the task before responding), Jacob did not see any
560 trouble with his response until, upon the interviewer's suggestion, Jacob drew a representation of
561 the situation, at which point he decided the yellow bar would fit into the red bar about four times.
562 Finally, after the interviewer repeated the original question, Jacob decided it would fit three
563 times, explaining that he needed to divide 15 and 5. Still, Jacob could not explain why dividing
564 would give the right answer. On the other hand, Jacob recovered in responding to the fourth units
565 coordinating task (see Task 3 in Figure 9), immediately responding "2" because "4 times 2."

566 In hindsight, we might have read too much into Jacob's confusion about the third task.
567 Although he multiplied the given values at first and could not explain why one should divide the
568 two numbers instead, this should not negate his strong responses in the first two tasks. After all,
569 he was able to provide clear explanations for those solutions, indicating that he had assimilated
570 two levels of units, especially in the first task where he had iterated the yellow bar four times

while treating it as a unit of four blue bars. Moreover, his response to the fourth task indicates that he could make appropriate adjustments in the way the units were related; rather than multiplying the given numbers, 4 and 8, he multiplied a hypothetical unit, 2, by 4 to produce 8.

6.3.2 Other MC1 students assessed with PUFS

Unlike Jacob's case, our reexamination of the other four MC1/PUFS students (Adam, Cody, Devin, and Katy) affirmed our initial assessments that they had not yet interiorized two levels of units. For example, consider Adam's response to the first units coordination task. Like Jacob, Adam responded "12," explaining that he got this answer by multiplying 3 and 4. But when the interviewer asked him why he multiplied, Adam replied, "if you add them it will be too less." This indicates that multiplication was a convenient available numerical operation that justified the result of mentally iterating the blue bar across the red bar, rather than an operation that represented a necessary relationship between the units. Each of the other three students responded similarly to each of the units coordination tasks, visualizing the number of times the smaller piece would fit into the larger piece, rather than relying on multiplicative relationships that would necessarily determine this number.

With regard to PUFS, all four students may have successfully completed the tasks without interiorizing that scheme as an anticipatory structure. Consider, for example, Adam's response to the first PUFS task (Task 1 in Figure 5). Fifty seconds lapsed from the time Adam first read the task to the time he first responded, with the interviewer having to read the task in between. This alone indicates that Adam lacked an assimilatory structure with which to anticipate an appropriate response. When he finally responded, the following interaction ensued:

Adam: [draws a stick the same length as the given stick] So it's like if this is like broken

593 into seven pieces and then draw one seventh of those?

594 Interviewer: I think that sounds right, what you said.

595 Adam: [partitions his drawn stick into seven equal pieces and shades in one of them]

596 Interviewer: Nice. Let me ask you a question. How did you decide where to make this
597 first mark right here? [pointing to the first partitioning mark Adam had made, near the
598 left end of the drawn stick]

599 Adam: First I kept the measurement of this stick in my head. Then I like divide it in my
600 head and draw six lines to see if that would work. If not, I'd put like either less space or
601 more space in between.

602

603 Adam's actions of shading one of seven parts, rather than simply drawing a one-seventh
604 stick, or even disembedding one of the seven parts, indicates that he might have been using a
605 parts-within-wholes scheme, by which fractions are understood as numerical relationships
606 between a pair of whole numbers; e.g. 1 within 7 equal parts (Hackenberg, 2013). On the other
607 hand, Adam's response to the second task (see Task 2 in Figure 5) indicates that he had
608 constructed a PUFS, at least as a participatory scheme.

609 Adam: [Reads the task and then uses his index fingers to span the length of the smaller
610 given stick. He then translates that span from the left end of the longer stick and makes a
611 mark by his right index finger. He continues this process, working from left to right, until
612 he has partitioned the longer stick into seven parts, though parts at the right end seem to
613 be progressively smaller] So it's one-seventh of this one [pointing to the longer stick].

614 Interviewer: Okay. So how could you check that it's one seventh?

615 Adam: 'Cause I could measure this [spanning smaller stick again] and that would be one
616 block [lining up with left end of longer stick again] out of these seven pieces [scanning
617 length of longer stick].

618

619 In responding to the second task, Adam seemed to coordinate the two lengths in
620 activity—relying on the physical activity of measuring off the length of the longer stick with the
621 shorter stick. Thus, we find that Adam could begin solving PUFS tasks by coordinating two

622 levels of units in activity even as he was limited to coordinating two levels of units in activity
623 when working with the bars tasks (Figure 5). Indeed, we may have witnessed Adam constructing
624 unit fractions by coordinating the short stick and the long stick as whole number units.
625 Furthermore, this fractional situation might have been a rich context for him to begin to
626 interiorize two levels of units.

627 Each of the remaining MC1/PUFS students also had trouble assimilating the first PUFS
628 task (Task 1 in Figure 5). For example, both Cody and Devin initially responded to the task by
629 drawing a stick with the same length as the given stick. Although we cannot discount the role
630 that reading comprehension might have played in responding to a task that asked for a stick that
631 was “ $\frac{1}{7}$ as long as” the given stick, in their initial responses, neither student did anything to
632 account for the “ $\frac{1}{7}$ ” in the task’s wording. The remaining student, Katy, also had trouble
633 assimilating the task. After looking at it for about 15 seconds, she said she did not know what to
634 do. After the interviewer asked her what she thought the task was asking, Katy responded, as
635 follows:

636 Katy: I have to draw one seventh of that? [pointing to the given stick in Task 1 of Figure
637 1].

638 Interviewer: Yeah. Do you think you can do it?

639 Katy: I don’t know how to measure.

640 Interviewer: If your stick is one-seventh of that [pointing to the given stick], just *about*
641 how big would it be? Can you show me with your fingers or draw a picture of about how
642 long it would be?

643 Katy: [draws two small vertical marks, indicating a spacing that was about one-sixth the
644 length of the given bar]

645 Interviewer: Okay. So what would the stick look like? Like could you draw what the stick
646 would look like?

647 Katy: Like, is it saying longer or shorter [pointing to the words in the task]?

648 Interviewer: It's saying your stick is one-seventh as long as this [pointing to the given
649 stick]. So the length of your stick is one-seventh of this length. So, about how long would
650 that be? I think you were showing me about how long that would be.

651 Katy: [points to the space between the two vertical lines she had drawn]

652 Interviewer: Can you draw the stick?

653 Katy: [draws a stick between the two vertical lines]

654 Interviewer: Okay. How could you check to see if that's a good answer?

655 Katy: Using those things maybe? [pointing to the bars used for the units coordinating
656 tasks]

657 Interviewer: Is there any way you could do it just using the paper [pointing to written
658 task] to figure out if that's [pointing to drawn stick] a good answer?

659 Katy: I could, like, draw it [beginning to make a mark in the given stick].

660 Interviewer: Okay. Why don't you try that?

661 Katy: [draws seven parts, one at a time, along the bottom of the given stick] ...seven of
662 them [then draws a new, smaller stick to the side and begins erasing the seven parts she
663 had just drawn along the given stick]

664 Interviewer: What happened right there?

665 Katy: They were too big.

666 Interviewer: They were too big? How did you know they were too big?

667 Katy: Because I couldn't fit in there [pointing to given stick] ten times.

668 Interviewer: Why did you want to fit *ten* of them?

669 Katy: Because I thought that was the whole.

670

671 On the strength of their responses to the second PUFS task (Task 2 in Figure 1), which
672 were similar to Adam's response to that task, we assessed Cody, Devin, and Katy as students
673 operating with a PUFS. However, Adam's response suggests that some students can carry out the
674 activity of iterating the smaller stick into the larger stick, without anticipating the results of the
675 activity prior to acting. Rather, each student seemed to have constructed a PUFS as a

676 participatory scheme.

677 **6. Conclusions**

678 Steffe's (2001) reorganization hypothesis was developed via reflections on interactions
679 with students during longitudinal teaching experiments. In contrast, the clinical interview
680 methods we used in this study permitted us to make inferences about the ways of operating that a
681 relatively large group of students had constructed prior to their relatively short interactions with
682 us. By considering Piagetian structures (Piaget, 1970b), we were able to analyze commonalities
683 between the fractions schemes students had constructed and their multiplicative concepts, in
684 terms of the number of levels of units (Hackenberg, 2010).

685 Consistent with the reorganization hypothesis, we found a strong positive association
686 between the number of levels of units in entering sixth-grade students' multiplicative concepts
687 and the number of levels of units they coordinated in their fractions schemes. We expected the
688 statistical association between multiplicative concepts and fractions schemes to be asymmetric,
689 with fractions schemes depending on multiplicative concepts, but not vice versa. This is because
690 we hypothesized that students would not be able to operate with n levels of fractional units
691 without having first interiorized n levels of whole number units (Hackenberg, 2007; Norton &
692 Boyce, 2013; Steffe & Olive, 2010). Quantitative results regarding sequentiality in students'
693 construction of fractions schemes and multiplicative concepts were inconclusive. That is, we did
694 not find evidence that students construct n -level multiplicative concepts before they construct n -
695 level fractions schemes.

696 From our theoretical perspective, students' constructions of more powerful structures are
697 accommodations in response to perturbations experienced in operating (von Glasersfeld,

1995). We interpret our findings as evidence that students entering sixth-grade are just as likely to have made accommodations to their units coordinating structures for whole number multiplication as they are to have made accommodations to their fractions schemes. Our qualitative results are limited by the relatively short time we engaged with the students, but they suggest that students may co-construct units coordinating structures for whole numbers as they develop fractions schemes.

The analysis of the cases of Adam, Cody, Devin, and Katy confirm that they were MC1 students and indicate that they were each temporarily perturbed by the PUFS tasks. Though they had not constructed assimilatory structures for anticipating solutions to the tasks, they were able to conjecture and subsequently carry out in activity the operations of a PUFS. Thus, these students could coordinate two levels of fractional units *in activity* without having interiorized two levels of whole number units. Taken together with our quantitative results, this qualitative analysis suggests that fractions may be a fertile context for students to continue developing units coordination.

7. Discussion

Our findings add support to research demonstrating that unitizing and multiplicative reasoning are requisite for more robust understandings of fractions (e.g., Behr, Harel, Post, & Lesh, 1992; Lamon, 2007; Norton & Wilkins, 2010; Steffe & Olive, 2010). Many mathematics educators have focused on how instructional use of particular figural representations, such as number lines or circle models, can mediate students' difficulties with expanding part-whole interpretations of fractions to allow for students' sense-making of fractions arithmetic (e.g., Ball, 1993; Cramer & Wyberg, 2009). The current study suggests that supporting units coordination may be an additional mediator, for which fractions can serve as a context for remediation. Our

results also add support to the notion that the construction of these units coordinating structures is a learning goal that should be supported throughout elementary schooling and beyond (Norton & Boyce, 2013). As construction of three levels of units is crucial for understandings of algebra (Hackenberg, 2013; Hackenberg & Lee, 2015) and integers (Ulrich, 2012), providing opportunities for students to participate in authentic activity involving fractional units may be valuable for supporting student thinking in such domains.

We were disheartened but not surprised to find that many of the students in our study (54%) had yet to construct even the most basic understanding of fractions as measures – the partitive unit fraction scheme. Considering the current group of students was younger, the current results are in line with those previously reported for an otherwise similar rural population of students (Norton & Wilkins, 2009; 2010; Wilkins & Norton, 2011). Future studies with other populations of sixth-graders are warranted to quantitatively test the association of three-level units coordination structures and more advanced fractions schemes involved in fractions arithmetic (Steffe & Olive, 2010).

We were alarmed to find that so few entering sixth-graders (42%) had constructed the second multiplicative concept. Steffe (2007) estimated that approximately 50-70% of students will have interiorized two or more levels of units by the beginning of sixth grade. Though we acknowledge that the relatively short period of engagement with students in this study may have contributed to systematic underassessment of students' abilities, we believe at the school level, lower stages of units coordination, like educational performance overall, may correspond with lower levels of socio-economic-status (Haycock, 2001).

We cannot know the classroom experiences of our students prior to our experiences with them, but it may be that our students were not provided with enough opportunities and

experiences to reflect and reason on the results of their own ways of operating in numerical situations in elementary school (Wright, Stanger, Stafford, & Martland, 2006). Fortunately, based on the clinical interviews with Adam, Cody, Devin, and Katy, we hypothesize that such students might eventually interiorize two-level units coordination structures via reflection on their conjectural activities with unit fractions if they are provided with these opportunities in middle school. To encourage this development, we suggest teachers repeatedly engage students in tasks that involve whole numbers and tasks that involve fractions within the same situations, so that students can attend to the commonalities and differences in their ways of reasoning.

The *CandyDepot Educational Game for iPad* (Learning Transformation Research Group, 2013) provides a context for such activity. Students choose whole number relations between the number of bars per bundle and bundles per box, and then are tasked to efficiently pack both whole number and improper fractional orders in terms of boxes, bundles, or bars. For example, a student might pack 2 bars per bundle and 3 bundles per box and then be asked to find an efficient way to pack 59 bars. A student might reason solely with whole numbers of bars, thinking of bundles as units of 3 and boxes as units of 6, or a student could also reason with fractions of a box, thinking of bundles as units of $\frac{1}{3}$ and bars as units of $\frac{1}{6}$. Recent research suggests that repeatedly engaging middle-grade students in reflection on their reasoning in such situations (varying whether the quantity to be packed is prescribed as a whole number or a fraction of bars, bundles, or boxes) can support students' development of units coordination (Norton, Boyce, & Hatch, in press).

To build on these findings, our future plans include conducting teaching experiments with sixth-grade students to further investigate how students' activities with fractions and whole numbers may contribute to their continued development of units coordinating structures in

767 middle school. There is also a need for studies with other populations of students to understand
768 how to support those students' development. Studies with older students or students from a high-
769 achieving school may allow for more substantive analysis of the associations of multiplicative
770 concepts and fractions schemes involving two or more levels of units. Interviews with
771 elementary school students encountering fractions instruction might also be used to investigate
772 whether there are differences in the ways middle school students' and younger students'
773 multiplicative concepts and fractions schemes are related.

References

- Ball, D. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: an integration of research* (pp. 157-195). Hillsdale, NJ : Lawrence Erlbaum Associates.
- Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics* (pp. 296 – 333). New York: MacMillan.
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547 – 589). Mahwah, NJ: Lawrence Erlbaum Associates.
- Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, 26(1), 66-86.
- Cramer, K., & Wyberg, T. (2009). Efficacy of different concrete models for teaching the part-whole construct for fractions. *Mathematical Thinking and Learning*, 11(4), 226-257.
- Gonzalez, R., & Nelson, T. O. (1996). Measuring ordinal association in situations that contain tied scores. *Psychological Bulletin*, 119(1), 159.
- Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, 26(1), 27 – 47.
- Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction* 28(4), 383-432.

- Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, 28(1), 1-18.
- Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, 32, 538 -563.
- Hackenberg, A. J., & Lee, M. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, 46(2),
- Haycock, K. (2001). Closing the achievement gap. *Educational Leadership*, 58(6) 6 – 11.
- Hunting, R. P., Davis, G., & Pearn, C. A. (1996). Engaging whole-number knowledge for rational-number learning using a computer-based tool. *Journal for Research in Mathematics Education*, 27(3), 354-379.
- Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C. H. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, 43(4), 391-427.
- Izsák, A., Orrill, C. H., Cohen, A. S., & Brown, R. E. (2010). Measuring middle grades teachers' understanding of rational numbers with the mixture Rasch model. *The Elementary School Journal*, 110(3), 279-300.
- Kieren, T. E (1979). The development in children and adolescents of the construct of rational numbers as operators. *Alberta Journal of Educational Research*, 25(4), 234–47.
- Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, 27(2), 170-193.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on*

- mathematics teaching and learning*, (pp. 629–667). Charlotte, NC: Information Age Publishing.
- Learning Transformation Research Group (2013). *CandyDepot Educational Game for iPad*. Retrieved from <http://ltrg.centers.vt.edu/projects/games/apps/candyDepot.htm>
- Mack, N. (1995). Confounding whole-number and fractions concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, 26(5), 422-441.
- Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College
- National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors.
- Norton, A. (2008). Josh's operational conjectures: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education*, 39(4), 401-430.
- Norton, A., & Boyce, S. (2013). A cognitive core for the Common State Standards. *The Journal of Mathematical Behavior*, 32(2), 266-279.
- Norton, A., Boyce, S., & Hatch, J. (in press). Coordinating units and constructing fractions at the CandyDepot. *Mathematics Teaching in the Middle School*.
- Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fractions schemes. *The Journal of Mathematical Behavior*, 28(2), 150-161.
- Norton, A., & Wilkins, J. L. M. (2010). Students' partitive reasoning. *The Journal of Mathematical Behavior*, 29(4), 181–194.

- Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education*, 43(5), 557-583.
- Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning*, 1(4), 279-314.
- Olive, J., & Çağlayan, G. (2008). Learners' difficulties with quantitative units in algebraic word problems and the teacher's interpretation of those difficulties. *International Journal of Science and Mathematics Education*, 6(2)
- Piaget, J. (1970a). *Genetic epistemology*. New York: Columbia University Press.
- Piaget, J. (1970b). *Structuralism*. New York: Basic Books.
- Piaget, J. (1972). *The principles of genetic epistemology*. London: Routledge & Kegan Paul.
- Pitkethly, A., & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies of Mathematics*, 30(1), 5-38.
- Sáenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education*, 25(1), 50-85.
- Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, 62, 273-296.
- Somers, R. H. (1962). A new asymmetric measure of association for ordinal variables. *American Sociological Review*, 27(6), 799-811.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), pp. 259-309.
- Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior*, 20(3), 267 - 307.
- Steffe, L. P. (2007, April). Problems in mathematics education. Paper presented for the Senior

Scholar Award of the Special Interest Group for Research in Mathematics Education
(SIG/RME)

at the Annual Conference of the American Educational Research Association, Chicago,
IL.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. NY: Springer.

Streefland, L. (1993). *Fractions in realistic mathematics education: A paradigm of
developmental research*. Boston: Kluwer Academic Publishers.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J.
Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and
standards for school mathematics* (pp. 95–113). Reston, VA: National Council of
Teachers of Mathematics.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the
teacher's role in promoting that learning. *Journal for Research in Mathematics
Education*, 30(4), 390-416.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory
and anticipatory stages in learning a new mathematical conception. *Educational Studies
in Mathematics*, 66, 273–291.

Ulrich, C. (2012). Additive relationships and signed quantities. (Doctoral dissertation).
Retrieved from <http://www.libs.uga.edu/etd>

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey
(Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp.
41-59). Albany, NY: SUNY Press.

- Von Glasersfeld, E. (1995). *Radical constructivism: a way of knowing and learning*. Washington, D.C: Falmer Press.
- Wilkins, J. L. M., & Norton, A. (2010). Students' partitive reasoning. *Journal of Mathematical Behavior*, 29, 181-194.
- Wilkins, J. L. M., & Norton, A. (2011). The splitting loop. *Journal for Research in Mathematics Education*, 42(4), 386-416.
- Wilkins, J. L. M., Norton, A., & Boyce, S. J. (2013). Validating a written instrument for assessing students' fractions schemes and operations. *The Mathematics Educator*, 22(2), 31-54.
- Wright, R. J., Stanger, G., Stafford, A. K., & Martland, J. (2006). *Teaching number: Advancing children's skills and strategies* (2nd ed.). London: Sage.