Ariel's Cycles of Problem Solving: An Adolescent Acquires the Mathematics Register¹

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RUNNING HEAD

ARIEL'S CYCLES OF PROBLEM SOLVING, ACQUIRING THE MATHEMATICS REGISTER

1 Introduction and Overview

Students use multiple modes of expression, such as everyday, conversational language, drawings, as well as other modes such as graphical displays and models with manipulatives to convey their understanding of mathematical concepts, principles, and problem-solving processes (Halliday, 1978). Eventually, for the purposes of both instruction and assessment, students are expected to move to the more developed, precise linguistic encoding and expressions of those understandings---that is, the mathematics register.

The analysis presented in this paper illustrates the interdependency between developing mathematical understanding and using the specialized language unique to the discipline to express those understandings employing the mathematics register. Our focus is on one student, Ariel, as he develops knowledge of both mathematics and of the language (both oral and written) required for his successful problem solving and communication of the problem-solving process (Silliman & Wilkinson, 2015; Sigley et al., 2013).

We analyze the problem solving of Ariel, a 7th grade bilingual student from an urban school district, over an 18-month period. Three issues are addressed by our analysis:

- (1) How does Ariel display his mathematical understanding during problem solving?
- (2) How does Ariel use the mathematics register while solving the problem?
- (3) What systems of representation, including language (oral and written), does Ariel bring to solving the problem?

Initially, Ariel displays his mathematical understanding during his attempt to solve a specific algebraic problem; in this exploration of understanding algebraic ideas, he seeks to identify patterns. We follow his mathematical and language learning from this initial conjecture, to his reformulation and generalization of a rule. Eighteen months later, after studying algebra, Ariel attends to the systems of representation, including both the oral and written expressions of the mathematics register.

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The analysis of one case, Ariel, is considered in the current policy context. Teachers are now expected to engage their students in active conceptual understanding of the mathematics that they are expected to learn; consequently, the significance of this work is broad. The interdependency between mathematical knowledge and competence in using the mathematics register is deemed to be an essential component of the new *Common Core Content Standards (CCCS) and Standards for Mathematics Practice* (National Governors Association Center for Best Practices & Council of Chief State School Officers, CCSSO & NGO, 2010.) Thus, this interdependency is a driving force behind both major standardized assessments about to be implemented nationally, via the *PARCC (Partnership for Assessment for Readiness for College and Careers)* and the *SBAC (Smarter Balanced Assessment Consortium)*. These new assessments and educational standards arose from state-led initiatives to increase educational expectations from kindergarten to grade 12, so that all students would be college-and career-ready no later than the end of high school.

The national call for new content standards explicitly requires students to use discipline-specific language to discuss and create mathematical understandings, to use mathematics to make sense of the mathematical problems, to offer arguments, to be able to understand others' reasoning and share their critiques, and finally, to use tools to build models of those understandings. Thus, both oral and written language is central to mathematical teaching and learning. Working collaboratively is more than traditional cooperative learning. For collaborative learning to thrive, students must accept that the solution is jointly built and requires exchange of ideas, including talking together. Further, the input of each person is important and the ideas of others must be heard and reviewed.

This paper consists of six sections. Following this overview, in the second section we present the perspective that frames the paper and references the current policy context for learning and assessing students' mathematical knowledge. The third section introduces key concepts and features, including the mathematics register. The fourth section focuses on the methods of data collection and analysis, and the fifth section provides the analysis. The final section offers a summary, interpretation, and discussion of implications for teaching, learning, and assessment particularly for students whose first language is not English and who are learning mathematics at the same time they are learning the language of instruction---the mathematics register in English.

2 Common Core State Standards and The Language of Mathematics

The new mathematical standards emphasize “general, cross-disciplinary literacy expectations that must be met for students to be prepared to enter college and workforce training programs ready to succeed” (CCSSO & NGO, 2010, p. 4) by the end of high school. The anchor standards emphasize the *integration of communication processes* into the entire curriculum. These processes are reading, writing, speaking and listening, and *language*. However, language is not defined as the superordinate process, but rather it is narrowly defined as the acquisition of more general academic and domain-specific vocabulary knowledge and effective use of language conventions across modes of expression. These integrated processes function as the foundation for the grade-specific standards, which reflect “end-of-year expectations” in content areas such as mathematics

(CCSSO & NGO, 2010, p. 4.) Recent research (Herbel-Eisenmann et al., this volume) revealed that often, teachers reported they did not know enough about classroom discourse to enact the Standards for Mathematical Practice.

To illustrate the complex language and cultural background knowledge required for assessing mathematical knowledge among school students, we consider the following sample test item taken from the *PARCC* (grade 6-8) (*Partnership for Assessment of Readiness for College and Careers*, *PARCC*, 2013; see <http://parconline.org/samples/item-task-prototypes>). One aspect of the mathematics register is that certain mathematical terms may take on different or more precise meanings in the mathematics register in comparison with other usage contexts. The following grade 6 sample item that draws on ratio concepts and the relationship of unit rates to ratios exemplifies how meanings specific to their mathematics register must become part of students' lexicons. This two-part sample is a prototype of practical applications in the mathematics assessment items developed by the *PARCC*:

Mr. Ruiz is starting a marching band at his school. He first does research and finds the following data about other local marching bands.

	Band 1	Band 2	Band 3
Number of Brass Instrument Players	123	42	150
Number of Percussion Instrument Players	41	14	50

Part A

Enter your answer in the box.

Mr. Ruiz realizes that there are brass instrument player(s) per percussion player.

Part B

Mr. Ruiz has 210 students who are interested in joining the marching band. He decides to have 80% of the band be made up of percussion and brass instruments. Use the unit rate you found in Part A to determine how many students should play brass instruments.

Show or explain all your steps.

Figure 1: Example Test Item. *Note: From Partnership for Assessment of Readiness for College and Careers (PARCC) Computer-based Sample Test Items.*

This example includes precise technical vocabulary (e.g., *data*, *unit rate*), and complex clauses (e.g., *who are interested in joining the marching band*). Regarding vocabulary: although most dictionary meanings of *rate* refer to mathematical notions associated with *estimate* (e.g., How do you *rate* your chances of winning the lottery?) or *amount* (What's your average *rate* for the mile run?), this sample item uses specialized vocabulary particular to formulating tables of equivalent ratios (Silliman & Wilkinson, 2015.) In this test item, *rate* must be understood relationally as a unit for determining equivalence linkages between two quantities. Furthermore, students must understand the arithmetic distinction between *percent* of a quantity as a *rate per 100* (i.e., *80% of the band*

be made up of percussion and brass instruments) and unit rate as a quantity realized as a proportion (*Use the unit rate you found in Part A to determine how many students should play brass instruments*).

The larger question raised by this example is whether the language challenges built into the test items themselves attenuate students' opportunities to demonstrate what they do know about the mathematics assessed by these instruments. Do students know the mathematics content, but are unable to formulate/demonstrate their knowledge as required by the test? Perhaps students do not understand what the test item requires because they do not possess the linguistic, cultural, and background knowledge required for test item comprehension. This paper contributes to the discussion of this larger issue by providing a close analysis of one student's cycle of problem solving. We focus on Ariel's mathematical strategies and his language choices in expressing his communications in small group discussion. First, we turn to a consideration of the specialized language of learning/teaching/assessing mathematics in schools.

3 Conceptual Background: The Mathematics Register

Each discipline, whether mathematics, history, geography, or biology, employs its own specialized language---a register---that is quite complex and differs from everyday conversation. Each defines the unique way of cultivating reading, writing, speaking, and ways of reasoning that students must master, if they are to be deemed proficient in the discipline and in multiple ways (e.g., tests, texts) of demonstrating that proficiency (Wilkinson & Silliman, 2000). The academic language register is defined as the "specialized language, both oral and written, of academic settings that facilitate communication and thinking about disciplinary content" (Nagy & Townsend, 2012, p. 92.) Disciplinary language registers can be both oral and written and share lexical and syntactic features including: (1) Increased conciseness in word selection to avoid redundancy; (2) a higher frequency of informational words as the means to achieve more concise expression; and (3) syntactic processes embedding complex ideas into fewer words (Snow, 2010).

Mathematics is constructed in a blending of natural language, mathematics symbolism, and visual display (O'Halloran, 2005.) Consequently, for mathematics learning, students must be able not only to articulate the natural language that is highly technical, dense and precise, but also make connections among the three semiotic systems. O'Halloran (2005) provided a detailed description of the mathematics register by focusing on processes, representations, and symbolism. She highlighted the major role that this range of meaning systems plays in constructing and sharing mathematical understandings. Gibbons (2009) pointed to the relationship between varied classroom activity structures (e.g. dyads, small groups, whole-class work, individual students presentation and written solutions) and the texts that students create in those particular contexts. For example, students in small groups may point to mathematical representations on their papers in lieu of stating them orally.

Specific characteristics of the mathematics register provide insights into what may make it potentially difficult for all students, but in particular for English learners (those for whom English is not the first language), and consequently may interfere with their developing mathematical knowledge. For example, there are characteristic phrases and

certain modes that are acceptable for mathematical processes such as argumentation. According to Forman et al., arguments, for example, should be precise, brief, and mathematically logical (Forman et al., 1997).

3.1 Mathematics Lexicon.

The mathematics register requires a highly technical and precise language that is densely structured. Mathematics employs the use of technical vocabulary that is uniquely mathematical (e.g., *trigonometric ratios*, *hypotenuse*, *sine*, *cosine*, *quadrilateral*); further, there are “everyday words” that are used to convey mathematical concepts (e.g., *length*, *prove*, *greater*, *value*). These terms do not occur in isolation, but often are elements of elaborated noun phrases (e.g., *ratios of the lengths of the sides of a right triangle*; *two trigonometric ratios that involve the hypotenuse*; *one pair of opposite sides of an inscribed quadrilateral*). All of these serve the processes of defining, identifying, attributing, and other semantic relationships (Fang, Schleppegrell, & Moore, 2014). In addition, verbs may act differently in the mathematics register in comparison with other domains of usage, such as those involving *doing* or *thinking* verbs. Relational verbs including *is*, *be*, and *have* are often used to signify relationships between objects and processes in the mathematics register (O’Halloran, 1998).

3.2 Mathematics Syntax.

The grammatical patterns unique to the mathematics register include dense noun phrases, nominalizations, logical connectors, and verbs, which are often employed in mathematical arguments, justifications, and constructions of mathematics. All of these elements present challenges for students learning mathematics. In the case of mathematics: Elaborated noun phrases, and short written and spoken phrases, may represent key ideas and understanding with very few words. Mathematical tasks, including tests, often employ nominalizations, which express mathematical processes using complex language, for example, verbs or adjectives converted into nouns and/or noun phrases, (e.g. *permutation*.) In addition, mnemonics are not uncommon, as for example, in the case of the word *Foil*, which references a procedure for finding the product of two binomials: F, the product of the first terms; O, the product of the outer terms; I, the product of the inner terms; and L, the product of the last terms (Steege & Bailey (1997). In this case, the idea of distributing multiplication over addition, an important mathematical idea, is replaced by a mechanical procedure, devoid of meaning.

Finally, mathematics also uses subordinators marking clausal dependency (e.g., *if...then*) in specific ways and that are different from everyday conversation. In mathematical word problems, and in developing theorems and proofs, conjunctions such as *if*, *when*, and *therefore* are used in precise technical ways, and constructions such as *given* take on new meanings in these contexts. Furthermore, mathematical operations that are used to construct mathematical reasoning often are unstated. O’Halloran (1998) showed how solving mathematical problems involves “long chains of reasoning that provide little or no indication of the results, definitions, axioms, operation properties or laws that have been used” (p. 377).

3.3 Mathematics Discourse Features.

Teaching and learning mathematics in schools both rely on the discourse structures of description, sequencing, procedural iteration, and justifications---all of which are manifest in both teachers' and students' oral and written discourse. In addition, assessment items included for the PARCC often require students show their answers and include the explanation and/or justification. For direct explanations, teachers makes explicit statements tailored to assist students in understanding the underlying concept, the relevance of applying the concept in particular situations, or how concepts are used. For justifications, teachers'/students' responses provide the operation necessary for completing the problem and the reason that operation was appropriate.

These linguistic features—technical vocabulary, semi-technical terms, elaborated noun phrases, nominalizations, subordinators with precise meanings (a class of conjunctions), and implicit logical relationships, explanations and justifications—present formidable challenges to students working to comprehend and solve mathematical problems. Finally, much of the extant literature on the mathematics register has focused on students for whom English is not their first or only language (e.g. Moschkovich, 2007; Setati & Adler, 2000).

4 Methods

4.1 Context.

The data set analyzed for this paper is taken from the corpus of data collected for Rutgers University under an NSF funded grant entitled *Research on Informal Mathematical Learning (IML, REC-0309062)* program as archived in the *Rutgers Video Mosaic Collaborative* (<http://www.videomosaic.org>). This federally funded project provided an innovative after school mathematics enrichment program to one middle school in central New Jersey. At the time of the project, the district was identified as an Abbott District by New Jersey. Abbott Districts are public school districts in New Jersey covered by a New Jersey Supreme Court ruling that found that the education provided to urban school children was inadequate and unconstitutional. The Court ordered the State to assure that these children were to receive an adequate and constitutional education through implementation of a comprehensive set of programs and reforms, including standards-based education supported by parity funding, supplemental programs, preschool education, and school facilities improvements.

4.2 Participants.

Ariel, a 13-year old bilingual student, was one of the participants of the *Informal Mathematical Learning (IML)* program. The *IML* included two cohorts of 20 middle-school students attending one urban school in the central New Jersey; Ariel was a participant of the second cohort. Both Spanish and English were Ariel's languages at home. However, Ariel is not an English learner (an EL, those who do not have English as their first language), in that he was not designated as an EL by the school district. Consequently, Ariel did not receive EL services in school. Perhaps the assessment of Ariel's English proficiency was insufficiently sensitive to his English proficiency. Similar to many children in the U.S. who struggle with formal school learning, Ariel did not have immediate success in the mathematics tasks initially presented in the *IML* program.

The *IML* program provided an enrichment experience for students in that they

worked on challenging mathematical tasks. The assumption underlying the design was that mathematical tasks must provide multiple opportunities for students to engage in the range of mathematical practices and language. The tasks offered the opportunity for students to build mathematical concepts and required more than using numbers, computation or symbol manipulation. Students reveal their conceptual understanding thorough explaining and justifications. The *IML* program was designed to provide a nurturing environment in which students were invited to work together on challenging, open-ended tasks, free of the school constraints and stressors of grading and testing. The mathematical topics that were explored in these sessions were not a component of the grade-level curriculum; consequently, students' work in the program was not directly influenced by regular classroom instruction.

In the urban middle school for the *IML* program, 98 percent of the students were African American or Latino. All of the student volunteers for the project were African American or Latino. The *IML* program was a longitudinal study, spanning three years. The first cohort started the program in the 6th grade and continued until the 8th grade; the second cohort started a year later in 6th grade and completed the program in 7th grade. During their participation in the program, students met after school twice a week for 60 to 90 minutes during the academic year. Over the year, there were approximately 24 such meetings. In addition, there were two-week all day summer institutes.

The forthcoming analysis reveals Ariel's long journey to achieve success in mathematical understanding and in sharing/representing his work at least with partial success via the mathematics register. For the entire duration of his participation in *IML* program, Ariel attended all classes at the middle school in English only classes. No special efforts were made to address his English proficiency. This paper presents the analysis of Ariel's problem solving during one session and a follow-up interview with him where he worked on the same problem 18 months later.

4.3 Researcher Participation and Data Collection.

There were three mathematical content strands for the project: Combinatorics, probability, and algebraic thinking. Students were encouraged to work collaboratively and were expected to justify their mathematical reasoning to their classmates by solving a problem; they worked in small groups or pairs. Students were arbiters of whether or not a solution was correct and a solution was deemed as correct when all students agreed that *the solution made sense*. Students were given extended time to work on mathematical tasks and were provided with tasks that concerned similar mathematical ideas, thus providing opportunities to revisit previous work and refine mathematical thinking. The researcher's role in the program was to: (1) Provide students with interesting and motivating tasks, (2) encourage students' collaborative work where they justified their solutions to their classmates, (3) encourage students to make their reasoning public, and (4) attend to the reasoning of other students so that all could make sense of the variety of mathematical representations and ways of reasoning. The researcher moved throughout the classroom during the task sessions, asking questions and providing prompts to the students as they worked together. The sessions were recorded with three cameras with separate audio recording equipment and a videographer. The tapes were later transcribed. Members of the research team (e.g.,

the graduate student and teacher observers) who were not facilitating the sessions took field notes. The field notes included descriptions of ongoing activities and discussions, and any notable occurrences, such as speaker announcements. The research team collected the artifacts generated.

4.4 Data Set.

The data for this study consisted of the analysis of a rich data set of two videotaped sessions separated by 18 months. During the first one-hour problem-solving session, Ariel participated for 45 minutes. The second session consisted of a 60-minute task-based interview. During the first session, Ariel was introduced to a task that challenged him to find a rule that could be expressed as a linear function. The task is referred to as the *Ladders Problem*, which is described below. In the first session, one researcher facilitated the session. Ariel worked with another classmate, James. The final interview was conducted by a different member of the research team. The primary data sources included the video-recordings, their transcripts, student produced work, and researcher notes. All of these materials are archived on Rutgers University *Video Mosaic Collaborative* (VMC) (<http://www.videomosaic.org>). These data consist of six segments taken from the 45-minute session with Ariel, and a final session in which he revisits the earlier *Ladders Problem*.

4.5 Data Analysis.

After review of the video for all of the sessions, the most relevant excerpts, that is the critical events of Ariel's problem solving, were identified to determine which activities were likely to contain a significant mathematical idea, and/or a shift in the discussion indicating a resolution of the problem (see Powell, Francisco & Maher, 2003). These critical events were highlighted for further analysis. The entire data set is referenced as *Ariel's Cycles of Problem Solving*. This identification process included initial review of the tapes, transcripts, and field notes taken during the small-group sessions. Each excerpt of a key sequence was bounded by the introduction of new mathematical idea and/or a significant shift indicating new, deeper understanding of the mathematical challenge of the *Ladders Problem* (given below).

Each of the excerpts was analyzed for evidence of the salient characteristics of the mathematics register (e.g., technical vocabulary, semi-technical terms, elaborated noun phrases, complex clausal embedding, nominalizations, subordinators with precise meanings, and implicit logical relationships). Data analysis included identification of recurring themes in Ariel's discourse, and the language he used to discuss his problem solving. Shifts in the content related to the mathematics register were noted, as were particular features of the mathematics register. In the segments to follow, turns at talk are enumerated within each segment, followed by analysis/interpretation both mathematically and linguistically.

5 Findings: The Cycles of Ariel's Problem Solving

The analysis shows the progression of Ariel's mathematics understanding of the *Ladders Problem*, and the increasing sophistication of Ariel's use of the mathematics register.

BUILDING LADDERS

A company makes ladders of different heights, from very short ones to very tall ones. The shortest ladder has only one rung, and looks like this (We could build a model of it with 5 light green Cuisenaire rods.)

A two-rung ladder could be modeled using 8 light green rods, and looks like this.

Build a rod model to represent a 3-rung ladder.
How many rods did you use?

Figure 2: The Ladders Problem

Participants:

Ariel: The focal student

James: A fellow student, who is present in one session, partially

Researchers (R): Ask Ariel questions to stimulate problem solving

5.1 Researcher Establishes Terminology and Task Rules.

At the beginning of the problem solving session (Event #1 in Sigley and Wilkinson, 2015), Ariel is exploring ideas about functions. The Researcher begins the session by arranging five Cuisenaire rods of the same color and length to form the shape of a ladder.

Figure 3: Five light green Cuisenaire Rods arranged as a ladder

The Researcher added one more rod to each side with a third rod between both Ariel and James; he asked the students what they thought this structure might represent. Ariel agreed that the shape could represent a ladder and that the parallel center rods could be called *steps*. Note that Ariel uses language minimally in this first segment of the session.

1.1	Researcher	Watch what I am doing. [arranges rods in a ladder] Alright. So, what does this look like?
1.2	Ariel	An H.
1.3	Researcher	Maybe, it's possible. How many rods do I have here?
1.4	James and Ariel	Five.

1.5	Researcher	Five right? If I do like this? What does this now look like for you?
1.6	Ariel	A chain or a necklace or something?
1.7	Researcher	Suppose it stands up?
1.8	Ariel	A building?
1.9	James	Like a tower.
1.10	Researcher	If it stands what would you call it?
1.11	Ariel	A building.
1.12	Researcher	Very good. Well, you can call it so many different names. I am going to call it a ladder. Does it look like one? In this particular case, what would you call this if we called this a ladder?
[pointing to a step]
1.13	James	A step?
1.14	Researcher	Step. Ok so how many steps do we have?
1.15	James and Ariel	Two.
	Researcher	Ok, so going back to the previous one that you called eight. Could that also be a ladder?
1.16	James	Yes.
1.17	Researcher	How many steps?
1.18	James	One.
1.19	Researcher	So this one is a ladder with one step and how many blocks? Rather rods we call it.
1.20	James and Ariel	Five.
1.21	Researcher	And now this is a ladder with two, right? And how many?
1.22	James	Eight.
1.23	Researcher	Eight, so my question, so you can build it as high as you want. So my question to you is that how many would that be if we had rods, how many would we need to build a ladder with ten steps?
1.24	Ariel	Wow, three steps. Ah, I got the idea.
1.25	James	I got it, 34.

In this segment, the Researcher establishes the terminology for Ariel to use to communicate with his partner James and with the Researcher. Ariel and James show through their actions and oral and written language that they can do what the task requires. While establishing the procedure for the session, and the terminology for the task, the Researcher illustrates the solution for a ladder with one and two rungs.

5.2 Ariel Solves the Problem for a Small Number of Steps by Building a Model.

There is no oral/written language for this segment of the session (Event #2 in Sigley and Wilkinson, 2015), as Ariel is physically building the ladder with varying number of rungs; he states that has an idea. Mathematically, he is building a physical model that relates to the problem he is attempting to solve.

Figure 4: Ariel builds a ladder with several rungs

5.3 Ariel's Conjecture for 10 Steps Based on His Rule: "Solve for $n/2$, Multiply by 2."

In this segment (Event #2 in Sigley and Wilkinson, 2015), Ariel offers a conjecture about how to solve the problem, and is asked by the Researcher to explain his reasoning orally.

- 3.1 Ariel Five. I counted how many rods it is and then I just multiplied by two because five times two is ten. It was 17 rods, so 17 times two is 34.
- 3.2 Researcher Again, again, slowly. You are so fast for me.
- 3.3 Ariel OK, I did five steps, and then I counted how many rods that was. It was 17 and since five is half of 10, five times two is 10, so I just multiplied the number of rods by two and I got 34 for 17 rods.

Ariel states that to solve the problem for 10 steps, he will look at a ladder for five steps and multiply it by two because five times two is ten. When asked to go *slowly* by the Researcher, Ariel clarifies what he is doing by specifically referencing he *counted how many rods that was* and that he was multiplying *the number of rods by two*. Mathematically, he is solving the problem by: $f(n) = f(n/2) * 2$ and is treating both answers as proportional to each other. Ariel's language is neither precise nor complex, although he uses the technical word *multiplied*.

5.4 Ariel Supports His Conjecture.

In this segment (Event #4 in Sigley and Wilkinson, 2015), Ariel offers support for his conjecture about how to solve the problem when he is asked by the Researcher to explain his reasoning orally for the task of eight steps.

- 4.1 Researcher So, how about for eight? So you're saying, so how would you do for eight?
- 4.2 Ariel For eight?
- 4.3 Researcher [points to paper] So you're saying, you're taking. So why did you pick 34?
- 4.4 Ariel Because ten steps is 34.
- 4.5 Researcher So out of this 100, you choose 10 right? You fix 10, which has 34, right? How would you do for four, how would you do for eight?
- 4.6 Ariel For eight?

- 4.7 Researcher Using the same idea.
- 4.8 Ariel [constructs a ladder with 4 steps and counts the number of rods in that ladder] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26. [Ariel builds a ladder using eight steps and counts the rods in his ladder by tapping the rods with his pen and counts out 26 rods. Ariel counts the number of steps in his ladder.] 1, 2, 3, 4, 5, 6, 7, 8. How the heck? [Ariel counts the rods in his four-step ladder and ends with 14]

The Researcher now invites Ariel to solve the *Ladders Problem* for eight steps using the same idea as he did to solve the problem for 10 steps (using his $f(n) = f(n/2)*2$ formula). Initially, Ariel builds a ladder with four steps and produces a correct solution of 14 by counting the number of rods. He then uses his strategy of doubling the number of rods and conjectures that there would be 28 rods for a ladder with eight steps. To check his answer, he builds a ladder with eight steps and counts out 26, not 28 rods. After checking the number of steps in his ladder he utters: *How the heck?* and checks the number of rods in the four-step ladder. Ariel's linguistically complex language is minimal; primarily he references via his counting, by which he explains and justifies his strategy and answer. Ariel uses neither nominalizations nor elaborated noun phrases.

5.5 Ariel Tests His Conjecture by Building a Ladder for 10 Rungs.

In this segment (Event #4 in Sigley and Wilkinson, 2015), Ariel provides reasoning for solving the problem.

- 5.1 James So these two must be wrong [points to ladders]
- 5.2 Ariel Which one? The 10 one? [Ariel construct a ladder with 10 rungs and counts the numbers of steps in his ladder]
- 5.3 Researcher Which one are you doing now?
- 5.4 Ariel [counting the steps in his ladder] 30: I give up man.
- 5.5 Researcher 30, what?
- 5.6 Ariel 32.
- 5.7 Researcher 32 for what?
- 5.8 Ariel For this one, the 10 steps.
- 5.9 Researcher What did you do to get a ladder for 10 and then 100?
- 5.10 Ariel Oh, a ladder with 10? I just did for five and then I multiplied by two and we have 17 rods and then I did that again, and it was 34. So then I went over here and 34 times 10.
- 5.11 Researcher Where does this 34 come from?
- 5.12 Ariel The rods. 18, it is wrong. It is 32.
- 5.13 Researcher So how are we going to know? We have to find a way to be able to tell how many rods no matter how many steps.
- 5.14 Ariel I don't know now. I just noticed like I could just multiply by two and subtract two.

With prompting from James, Ariel checks his answer for a ladder with 10 steps by building it. Ariel counts out the number of rods in his ladder with 10 steps and gets 32 rods which is different than his initial answer of 34. When the Researcher asks Ariel about the

origin of his answer 34, Ariel states that it is wrong and that the correct answer is 32. Ariel notices another pattern: That if he subtracts two from his doubling strategy he will get the correct answer, but Ariel fails to indicate that he knows the reason for this action. Ariel's oral language is minimal, although he uses the technical terms of *multiplied*, *multiply*, and *subtract*. Ariel is reflecting on his work about what he understands. He qualifies this further in his oral expression, but there is no use of dense noun phrases.

5.6 Ariel Formulates a Composite Rule.

In this segment (Event #4 in Sigley and Wilkinson, 2015), Ariel constructs separate rules for determining the number of rods in a ladder that is dependent on whether or not the number of steps is even or odd; he writes down these rules.

For odd numbers I go to the nearest even number take $\frac{1}{2}$ of that even #, count the rods for a ladder with that many steps multiply it by 2 subtract 2 and add 3

For even numbers I take $\frac{1}{2}$ of that number and make a ladder with that many steps. Then I multiply the # of rods of that ladder by 2 then I subtract 2.

Figure: Ariel's written work expressing his odd and even rules

- 6.1 Ariel (Odd rule) For odd numbers, I go to the nearest even number and take one-half of that even number, count the rods for a ladder with that many steps, multiply by 2, subtract 2 and add 3.

Because for every new thingy it is three rods and it will give me 29.

- 6.2 Ariel (Even rule) For even numbers, I take one-half of that number and make a ladder with that many steps. Then I multiply the number of rods of that ladder by two then I subtract two.

Ariel develops a composite recursive function that depends on whether the number of steps is odd or even. For an odd number of rods $f(n) = f((n-1)/2)*2-2+3$; for an even number of rods he uses $f(n) = f(n/2)*2-2$; and then he tests his rule by building ladders with eight and nine steps. Ariel tests his rule for a ladder with 80 steps. Instead of building a ladder with 40 steps, he uses his answer for 10 steps and multiplies it by four and then uses his rule that produces an incorrect answer. When Ariel is unable to construct a model for $f(n/2)$ he uses his proportional reasoning strategy. In his formulation of the rules, Ariel uses the technical terms *multiply*, *even*, *odd*, *subtract*, and *number*, revealing his adoption of some elements of the standard features of the mathematical register. In 6.1 there is a coordinating construction (*and*), where he is linking steps sequentially. Several aspects of his language are more complex than earlier in this session, with longer sentences, due to his frequent use of elaborated (dense) noun phrases such as *nearest even number*, *half of that number*, and *the number of rods of that ladder*. He uses a listing strategy with these phrases. After Ariel states the odd number rule and writes it down, he justifies his work: *Because for every new thingy it is three rods and it will give me 29*.

5.7 Ariel Explains His Problem Solving (18 months later).

Eighteen months later (Event #5-8 in Sigley and Wilkinson, 2015), after enrolling in an algebra class, Ariel is presented exactly the same problem with the Cuisenaire rods available for him to use. Initially, he works on solving the *Ladders Problem* for 10 steps, and progresses to the more general solution.

- 7.1 Ariel How many rods would you use to build a 10-rung ladder? Well, I'm not going to build it, I'll just do a x-y table. That the difference between the y, the y variables was from five to eight it increased by three, from eight to 11 it increased by three, so you would just keep on increasing by three, so until and then, it'll keep on going. And then it shows that it's linear obviously because in a linear equation the first difference is always the same. So, it's linear.

x	y
1	5
2	8
3	11
4	14
5	17
6	20
7	23
8	26
9	29
10	32

Figure 5: Ariel's x-y Table for The Ladders Problem

In solving the problem, Ariel constructs an x-y table and examines the first differences between the entries in the table. He notices that the first difference is always the same (3), and concludes that the equation has to be a linear. Ariel then continues to solve the problem for any number of rungs.

- 7.2 Ariel How could you represent the number of rods needed to build a ladder with any number of rungs? That would be – oh that's easy. Y equals 3x plus 2.
- 7.3 Researcher Wait, where'd that come from?
- 7.4 Ariel Cause, how could you represent the number of rods needed to build a ladder with any number of rungs. So, you'd get the number of rungs from multiplying the ladder which it is, like if it was the first ladder, second ladder, third ladder, multiply by three like on this one, it would be nine and plus two is eleven. So, substitute the number, for, in each x, it would be three times three plus two it would be nine plus two it's 11. And it works out, for every one.

The Researcher asks Ariel to explain why the plus 2 is included in the equation.

- 7.5 Ariel Because, I just looked at it and if you multiply each by 3, it's gonna be, m plus the y intercept, which is gonna be 2, cause if it's adding 3 each time, if you reverse this to when it was at 0, it would be a 2 right there. Wait, yeah, it'd be a 2 right there. And then, this [indicating 3] would be your slope of 3, and your y intercept of 2 [indicating the value of (0,2)]. And

then it's a linear equation.

Ariel provides a general solution of $y = 3x + 2$, where x is the number of steps in the ladder, and also he explains the meaning behind the 3 and 2 in the equation. He efficiently identifies the solution and expresses it elegantly. Ariel explains that when building a ladder, you are adding three each time. The *plus two* is present because when there are no steps, there are two rods on the bottom of the ladder. In his explanation, Ariel uses mathematical language (e.g. *difference between the y variables*) as he justifies why the equation representing the solution is linear. In this session, Ariel not only solves the problem, demonstrating his understanding, but also his explanations reveal his adoption of some key elements of the mathematics register such as precise technical vocabulary terms in all of these five excerpts (7.1, 7.2, 7.3; 7.4, and 7.5). Specifically, he uses the following terms: *Multiplying, equals, intercept, plus, times, slope, table, represent*); he uses semi-technical phrases (*substitute, reverse, number*) dense elaborated noun phrases (*y variables, linear equation, x y table*), nominalizations (*the first difference*), subordinations with precise meanings (*if*). Ariel express complexity is varied ways, including elaborated noun phrases (*the number of rods needed to build a ladder with any number of rungs*;) and also the use double embedding (*if you reverse this to when*). Ariel references implicit logical relationships (*would be*)---a modal to express some degree of uncertainty. Ariel uses appositives (*like if it was the first ladder, second ladder, third ladder*). His statement combines both oral and more literate forms. This analysis reveals that Ariel knows and uses how to express linguistic complexity, but the analysis does not reveal his full realization of the mathematics register.

Ariel's explanations/justifications from seventh grade forward, show increasing sophistication moving from the particular to the general, and from using *everyday* language to the more precise language of the mathematics register. Further, his mathematical solutions are capable of being expressed in general form, because he has more precise technical language. As his understanding of the complex mathematical concepts emerges and he appears to be more comfortable with his understanding, Ariel's oral and written language becomes more complex and is synchronized with his increasing understanding. As Ariel explains the underlying mathematical principles in the task, he uses greater precision in vocabulary and incorporates some of the highly technical aspects of the mathematics register. In the final session, 18 months later, Ariel demonstrates his command of both the underlying mathematical knowledge, representing his knowledge with a function table. He uses the mathematics register, employing many of the complex elements, as he conveys that understanding in a precise manner.

6 Discussion and Implications

6.1 Mathematical Understanding Comes First, Then Adoption of the Mathematics Register.

The analysis shows the progression of Ariel's mathematics understanding of the *Ladders Problem*, and the increasing sophistication of Ariel's use of the mathematics register, even though he has not completely adopted this form. He uses a mix of oral and literate language in the final session. Once he has grasped the problem to be solved, Ariel's first step is to build his mathematical ideas about how to solve the problem, and then eventually map that

process onto linguistic expressions, eventually conforming to the standard expressions of the mathematics register. In the initial cycles, Ariel relies primarily on the strategies of counting orally, written numerical symbolic representation, and manipulating the Cuisenaire rods. He provides some oral explanations and commentary, although they are not complex linguistically; generally, they do not reveal his adoption of the mathematics register. Throughout the first six segments, it is clear that Ariel does not know how to solve the general problem and generates incorrect problem solving strategies and rules, resulting in incorrect answers. In the final session—18 months later---Ariel has abandoned the oral counting and manipulatives (rods) strategies. He solves the general problem because he understands it. His goal is to formulate an explanation of the problem- solving process resulting in the correct answer and communicating his discoveries in the well-formed conventional manner that is the standard, precise verbal encoding of the mathematics register. The analysis of Ariel's problem solving alerts us to recognize that students may not be in full command---for production and comprehension---of the mathematics register until they understand the mathematics. Ariel presents his various attempts to solve the problem, making full use of all semiotic systems (symbolic representation, manipulatives, oral counting, pointing,) but he only fully relies on the mathematics register in the final session, after he has constructed his understanding of the algebraic concept.

Teachers might find it beneficial to create situations in which students deploy all of their diverse repertoires of mathematics knowledge and skills to the instructional tasks at hand. Since mathematical proficiency includes a blend of understanding the use of mathematical symbolism, natural language and visual displays, students can make connections among all three semiotic systems, each having its own conventions and poses specific challenges (O'Halloran, 2005; this volume.) During this process, students may use one or more of the semiotic systems simultaneously as Ariel did, mixing and meshing elements of each to express current understandings. This process of mixing registers---the informal everyday conversation and the specific mathematics register--- is similar in some ways to the contemporary linguistic concept of *translanguaging* (Garcia, 2009.) *Translanguaging* refers to the spontaneous, everyday way of making meaning, shaping experiences, and communicating by bilinguals. This concept incorporates the notion that bilingual students employ fluid, and strategic mobilization of their linguistic resources from both languages across communicative contexts. Some have argued, persuasively, that bilingual students should be encouraged to mix and mesh elements of both languages, thus utilizing all of their linguistic and cultural resources as tools to support their academic language development and school learning (Garcia, 2009). We consider a similar claim that, while constructing their mathematical understanding, students should be encouraged to communicate those understandings by employing all of their resources---linguistic, symbolic, and gestural. A related point is that assessments of mathematics knowledge, both formative and summative, are valid when they allow students to demonstrate what they know about the mathematics and not solely what they know about the mathematics register. It is important for teachers to identify the many mathematical understandings and resources and repertoires of each student, and then determine how best to leverage those resources so that each student can take full advantage of the learning opportunities and the assessments presented in classrooms.

6.2 Mathematics Instruction for English Learners Should Not Be Watered Down for English Learners (ELs.)

The analysis given in this paper has further implications for those students whose first language is not English and who are learning mathematics at the same time they are learning the language of instruction---the mathematics register.

We can only imagine how ELs are challenged both by English and the linguistic complexity of the mathematics register. Does the content complexity generated through embedded dense noun phrases and subordinators stand as a barrier to conceptual understanding? In mathematics, dense noun phrases are relatively common in texts and tests, especially in the higher grades (Fang and Schleppegrell, 2014). Unpacking test and text sentences for ELs assumes that their requisite conceptual and text organization knowledge is accessible to them. If so, they then must apply a flexible repertoire of cognitive and linguistic resources to maintain and efficiently manage the lexical and semantic relationships embedded in the elaborated (dense) noun phrases inherent in language of the mathematics register. Consequently, the following questions arise: Do/should teachers model the mathematics register and its use in both texts and tests? Do teachers provide opportunities for students to be aware of multiple representations of mathematical knowledge, including the formal register?

A primary implication of this work is that EL students should focus first and foremost on grasping and solving the mathematical problem at hand, using all of their resources---language and non-language. An English learner, such as Ariel, would likely begin working on the *Ladders Problem* using manipulatives to build models and then by counting and/or writing symbols in Spanish (his first language or L1) as he/she looks for patterns and eventually a rule that solves the problem. The next step is to provide both authentic opportunities and critical support so that a student such as Ariel would express understandings in English (his second language or L2.) An example in the present study is seen when the Researcher and James engage with Ariel in a discussion about the problem.

The specialized vocabulary of mathematics is another challenge for ELs. Direct and literal translation of terms from English (L2) to Spanish (L1) is not likely to be an effective instructional strategy if the EL does not have the concept at least to some degree. And, even then, translation presents challenges. Recent research has established that EL/bilingual students' vocabulary is distributed across both languages (L1 and L2) and the underlying semantic base is larger than the lexicon of either language (Bedore & Peña, 2011). Furthermore, EL students tend not to have large proportions of overlapping vocabulary, so that we should not expect them to provide the same responses in L1 and L2, at least initially (Bedore & Peña, 2011). During this period, ELs may be resistant to learning the same term in both languages.

The question arises as to whether there is a role for pre-teaching vocabulary in mathematics for ELs. Pre-teaching is a common practice, yet the efficacy of this practice is unclear. Importantly, pre-teaching vocabulary should be presented in meaningful contexts

for ELs, where the relationship of the vocabulary item to the task at hand---solving a particular problem is clear and necessary. In addition, research has established that multiple exposures to key mathematical vocabulary items are critical for learning a concept, so that the EL students can link their concept to their representations in both L1 and L2 (Bedore & Peña, 2011). Requiring students to present their answers only in the fully formed mathematics register will likely be ineffective. The practice of segregating EL students in one location during mathematics instruction, so that the material can be “watered down” and focused primarily on rote memorization and procedures, should be avoided.

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