Distribution of incompressible flow within interdigitated channels and porous electrodes

Robert J. Kee^{a,*}, Huayang Zhu^a

^a*Mechanical Engineering, Colorado School of Mines, Golden, CO 80401, USA*

Abstract

This paper develops a general model with which to evaluate flow uniformity and pressure drop within interdigitated-channel structures, especially in the context of redox flow batteries. The governing equations are cast in dimensionless variables, leading to a set of characteristic dimensionless parameter groups. The systems of governing equations are solved computationally, with the results presented graphically. Because the results are general, the underlying model itself is not needed to apply the quantitative design guidelines. However, the paper presents and discusses all the information required to recreate the model as needed.

Keywords:

Interdigitated channels, Flow battery, Fuel cell, Porous media, Design guidelines

*Corresponding author. Tel: +1-303-273-3379.

Email address: rjkee@mines.edu (Robert J. Kee)

1. Introduction

Redox flow batteries, polymer-electrolyte fuel cells (PEMFC), and potentially other technologies can be designed with interdigitated channels that supply flow through underlying porous electrode structures. Device performance often depends on tradeoffs between flow uniformity over large areas and overall pressure losses.

The objective of the present paper is to develop quantitative guidelines that assist the design of interdigitated-channel configurations and operating conditions. The approach is based upon a combination of models that are formulated in terms of characteristic dimensionless groups. The results are presented graphically, revealing quantitative tradeoffs between flow uniformity and pressure drop. The model results are reduced to a general set of design guidelines that do not require running the model for particular geometric configurations or operating conditions.

Figure 1 illustrates some of the salient features of an interdigitated redox flow battery layout. Small rectangular flow channels are situated within the bipolar plates. The “feed” channels are completely closed at the exit end of the structure, and “exhaust” channels are completely closed at the entrance end of the structure. Thus, all the fluid (electrolyte) that enters the feed channels must flow through the porous electrodes and leave via the exhaust channels. Electrochemical reactions proceed within the porous electrodes, with ion mobility across the membrane that separates the anode and cathode.

A variety of channel layouts, including interdigitated channels, are used in redox flow batteries [1–5] and fuel cells [6–16]. Much of the modeling literature is based on various approaches for two- and three-dimensional compu-

26 tational fluid dynamics. Some investigations specifically consider under-rib
27 convection, especially for PEMFC serpentine designs [17–19]. Most models
28 also incorporate electrochemistry as well as ion mobility across the mem-
29 brane that separates anode and cathode. In essentially all cases, the physical
30 problem is solved directly, without an attempt to generalize in dimensionless
31 terms.

32 The present paper is devoted entirely to flow distribution; it does not
33 consider chemistry or electrochemistry within the porous structures. Flow
34 batteries are often designed with relatively low single-pass conversion within
35 the porous electrode structures. Thus, it is reasonable to neglect the effects of
36 electrode chemistry and membrane transport when considering the flow dis-
37 tribution. The important contribution of the present approach is to develop
38 generalized, but quantitative, guidelines for the flow distribution.

39 Although the results of the present analysis are general, there are also
40 limitations based upon physical assumptions. The model is certainly compro-
41 mised if the mass transport across the ion-transport membrane (cf., Fig. 1),
42 is comparable to the mass transport between channels. The model assumes
43 fully developed flow within the channels (i.e., entry-length effects are ne-
44 glected). Laminar flow is typical in flow-battery channels, but the model
45 is not restricted to laminar flow. For gas-phase flows (e.g., as in PEMFC),
46 the incompressibility assumption can be compromised. Although including
47 an equation of state in the model itself is easily done, doing so limits the
48 generality of the dimensionless results. Finally, molecular diffusion is ne-
49 glected. In applications such as PEMFC fuel cells with gaseous fuels, axial
50 diffusive transport within the channels and molecular diffusion within the

51 gas-diffusion structures can play important roles.

52 The model focuses specifically on two interdigitated channels and the
53 underlying porous structure. Thus, there is an implicit assumption that each
54 channel pair within a full battery layout behaves the same as all others. In
55 other words, the manifold or header structures are designed to feed all the
56 interdigitated channels equally.

57 The theoretical and computational approach developed here has its foun-
58 dation in earlier models for flow distribution in parallel-flow header-channel
59 configurations [20]. The new elements of the present paper are related to
60 the generalization of flow through porous electrode structures that connect
61 the interdigitated channels. The present model and the dimensionless results
62 provide easy-to-use design guidance. Nevertheless, it is also clear that com-
63 plete design process benefits greatly from much more detailed models that
64 incorporate chemical, electrochemical, and thermal behaviors.

65 2. Channel flow

66 Assuming steady-state, single-component, isothermal, incompressible, plug
67 flow, the mass-continuity equations within the channels can be written as

$$\rho \frac{du_f}{dz} = -\frac{K(p_f - p_e)}{H_c}, \quad (1)$$

68

$$\rho \frac{du_e}{dz} = +\frac{K(p_f - p_e)}{H_c}, \quad (2)$$

69 where u_f and u_e are mean velocities in the feed and exhaust channels, respec-
70 tively. The rectangular channels are assumed to be W_c wide, H_c high, and
71 L_c long. The independent variable z is the axial position within the chan-
72 nels. The mass flux between adjacent channels (i.e., through the underlying

porous media) can be represented as

$$m'' = K (p_f - p_e), \quad (3)$$

where p_f and p_e are the local pressures within the feed and exhaust channels, respectively. The “flux” is referenced to the surface area on the floor of the channels. The parameter K is a function of the porous-media properties and physical dimensions, but is assumed to be a constant for any particular system.

The axial momentum equations within the channels can be written as

$$\frac{d(\rho u_f u_f)}{dz} = -\frac{dp_f}{dz} - \tau_w \frac{P_c}{A_c}, \quad (4)$$

$$\frac{d(\rho u_e u_e)}{dz} = -\frac{dp_e}{dz} - \tau_w \frac{P_c}{A_c}, \quad (5)$$

where τ_w is the local shear stress at the channel walls, $P_c = 2(W_c + H_c)$ is the channel’s wetted perimeter, and $A_c = W_c H_c$ is the channel’s cross-sectional area. Because the mass flux between the channels is assumed to be normal to the axial direction, a mass-flux term does not directly appear in the axial momentum equations. However, the mass flux is indirectly involved via the velocity coupling with the continuity equations.

The wall shear stress can be represent in terms of a friction factor as

$$f = \frac{\tau_w}{\rho u^2/2}. \quad (6)$$

For fully developed laminar flow in a rectangular channel, the friction factor can be correlated in terms of a Reynolds number as

$$\text{Re}f = 13.74 + 10.38 \exp\left(-\frac{3.4}{\alpha}\right), \quad (7)$$

where $\alpha = H_c/W_c$ ($0 \leq \alpha \leq 1$) is the channel aspect ratio [20, 21]. The Reynolds number $Re = \rho u D_h / \mu$ is evaluated based upon the mean velocity and the channel hydraulic diameter $D_h = 4A_c/P_c$. The fluid's dynamic viscosity is represented as μ . Although Eq. 7 is a laminar friction-factor correlation, the dimensionless analysis does not depend upon the particular correlation. Any other representation of the Ref (e.g., representing turbulent flow) is fully accommodated in the analysis.

The two momentum equations can be summarized as

$$2\rho u_f \frac{du_f}{dz} = -\frac{dp}{dz} - \left(\frac{P_c}{2A_c D_h} \mu Ref \right) u_f. \quad (8)$$

$$2\rho u_e \frac{du_e}{dz} = -\frac{dp}{dz} - \left(\frac{P_c}{2A_c D_h} \mu Ref \right) u_e. \quad (9)$$

The interdigitated-channel problem can be generalized by writing the equations in dimensionless variables. A set of dimensionless variables can be defined as

$$\hat{z} = \frac{z}{L_c}, \quad \hat{u}_f = \frac{u_f}{U_{in}}, \quad \hat{u}_e = \frac{u_e}{U_{in}}, \quad (10)$$

$$\hat{p}_f = \frac{p_f}{\rho U_{in}^2}, \quad \hat{p}_e = \frac{p_e}{\rho U_{in}^2}. \quad (11)$$

The inlet velocity for the feed channel is specified as U_{in} .

Following transformation to the dimensionless form, the system of equations can be summarized as

$$\frac{d\hat{u}_f}{d\hat{z}} = -\Omega (\hat{p}_f - \hat{p}_e), \quad (12)$$

$$\frac{d\hat{u}_e}{d\hat{z}} = +\Omega (\hat{p}_f - \hat{p}_e). \quad (13)$$

$$2\hat{u}_f \frac{d\hat{u}_f}{d\hat{z}} = -\frac{d\hat{p}}{d\hat{z}} - \Gamma \hat{u}_f, \quad (14)$$

108

$$2\hat{u}_e \frac{d\hat{u}_e}{d\hat{z}} = -\frac{d\hat{p}}{d\hat{z}} - \Gamma\hat{u}_e, \quad (15)$$

109 The dimensionless equations contain two dimensionless groups of physical
110 and fluid parameters:

$$\Omega \equiv \frac{L_c}{H_c} K U_{\text{in}}, \quad (16)$$

111

$$\Gamma \equiv \frac{P_c}{2A_c D_h} \frac{L_c}{U_{\text{in}}} \frac{\mu}{\rho} \text{Ref}. \quad (17)$$

112 Each of the governing equations is a first-order ordinary differential equa-
113 tion. Together, the system of equations is a fourth-order ordinary-differential-
114 equation boundary-value problem. As such, four boundary conditions are
115 needed for solution. At the inlet end (i.e., $\hat{z} = 0$), the dimensionless mean
116 velocities are written as

$$\hat{u}_f(0) = 1, \quad \hat{u}_e(0) = 0. \quad (18)$$

117 Note that the exhaust channel is blocked at the inlet end. At the exit end
118 (i.e., $\hat{z} = 1$),

$$\hat{u}_f(1) = 0, \quad \hat{p}_e(1) = 0. \quad (19)$$

119 The pressure at the end of the exhaust channel is specified as a reference
120 pressure of zero. The feed channel is blocked at the exit end (i.e., $\hat{u}_f(1) = 0$).
121 Note that three boundary conditions are specified for velocity, but only one
122 for pressure. In other words, the pressure at the inlet of the feed channel is
123 not specified directly, but is determined from the solution. Such an implicit
124 boundary condition affects the computational solution algorithm.

125 The dimensionless parameters include physical dimensions, fluid proper-
126 ties (viscosity and mass density), operating parameters (velocity), a friction-

127 factor correlation (*Ref*), and a porous-media parameter K . Typical op-
 128 erating conditions, channel dimensions, and electrolyte fluid properties are
 129 available in the literature (e.g., Darling and Perry [4]). As discussed subse-
 130 quently, the K parameter depends on the porous-media permeability. Perme-
 131 ability, which depends on the electrode material itself and compression load,
 132 is available from manufacturer's specifications (e.g., SIGRACELL[®] battery
 133 felt, SGL Group, Wiesbaden, Germany).

134 3. Computational solution

135 As illustrated by the stencil shown in Fig. 2, the differential equations
 136 are discretized using a finite-volume mesh network. Both the boundary con-
 137 ditions and the flow directions play central roles in the finite-difference dis-
 138 cretization. The continuity equations are differenced in opposite directions,
 139 thus enabling the propagation of the velocity boundary conditions into the
 140 channel interiors. That is,

$$\left(\frac{\hat{u}_{f,j} - \hat{u}_{f,j-1}}{\Delta \hat{z}} \right) + \Omega (\hat{p}_{f,j} - \hat{p}_{e,j}) = 0, \quad (2 \leq j \leq J), \quad (20)$$

$$\left(\frac{\hat{u}_{e,j+1} - \hat{u}_{e,j}}{\Delta \hat{z}} \right) - \Omega (\hat{p}_{f,j} - \hat{p}_{e,j}) = 0, \quad (1 \leq j \leq J-1), \quad (21)$$

$$2\hat{u}_{f,j} \left(\frac{\hat{u}_{f,j} - \hat{u}_{f,j-1}}{\Delta \hat{z}} \right) + \left(\frac{\hat{p}_{f,j} - \hat{p}_{f,j-1}}{\Delta \hat{z}} \right) + \Gamma \hat{u}_{f,j} = 0, \quad (2 \leq j \leq J), \quad (22)$$

$$2\hat{u}_{e,j} \left(\frac{\hat{u}_{e,j} - \hat{u}_{e,j-1}}{\Delta \hat{z}} \right) + \left(\frac{\hat{p}_{e,j+1} - \hat{p}_{e,j}}{\Delta \hat{z}} \right) + \Gamma \hat{u}_{e,j} = 0, \quad (1 \leq j \leq J-1). \quad (23)$$

144 In the momentum equations, note that the convective derivatives are both
 145 “upwinded” [21]. The formulation assumes that there is no flow reversal.
 146 That is, both $\hat{u}_f$ and $\hat{u}_e$ are always positive (i.e., proceeding from the feed end
 147 toward the exit end). Although in principle there could be local flow reversals,
 148 any flow reversal would indicate a very poor design. The pressure derivatives
 149 are differenced in the opposite direction from the velocity derivatives. The
 150 sense of the exhaust-channel pressure derivative propagates information from
 151 the specified exhaust pressure into the interior. Note, however, that the
 152 entrance pressure is not directly specified. Note also that the exhaust-channel
 153 convective derivative involves $\hat{u}_{e,j-1}$, which is not defined at $j = 1$. However,
 154 because $\hat{u}_{e,1} = 0$, the entire term vanishes at the boundary.

155 Three of the boundary conditions are straightforward to implement. That
 156 is,

$$\hat{u}_{f,J} = 0, \quad \hat{u}_{e,1} = 0, \quad \hat{p}_{e,J} = 0. \quad (24)$$

157 The velocity boundary conditions are nominally associated with the continu-
 158 ity equations and the pressure boundary conditions are nominally associated
 159 with the momentum equations. This is a coupled boundary-value problem,
 160 where there is no need to directly associate the boundary conditions with
 161 governing equations. If the feed-channel inlet velocity is specified, then an
 162 implicit boundary condition is needed for the pressure. That is, the bound-
 163 ary condition that is nominally associated with pressure (i.e., the momentum
 164 equation) at the inlet is

$$\hat{u}_{f,1} = 1.0. \quad (25)$$

165 Following discretization, the computational problem is one of solving a
 166 system of nonlinear algebraic equations. The solution is represented as the

167 values for velocity and pressure at each of the finite-volume mesh cells. A
 168 range of numerical-solution approaches can be used. In essentially all cases,
 169 the discrete equations are written in residual form. That is, for example,
 170 consider the feed-channel continuity equation

$$\left(\frac{\hat{u}_{f,j} - \hat{u}_{f,j-1}}{\Delta \hat{z}} \right) + \Omega (\hat{p}_{f,j} - \hat{p}_{e,j}) = R_{f,j}. \quad (26)$$

171 Each equation has such a residual form. The solution algorithm iterates on
 172 the values of the velocities and pressures until some norm of the residual
 173 vector is reduced to a sufficiently small value. At this point, the solution is
 174 found. The present study uses the MATLAB function “fsolve” to accomplish
 175 the solution using a Levenberg-Marquardt algorithm.

176 Given values of Ω and Γ , the solution to the dimensionless channel prob-
 177 lem produces profiles of $\hat{u}_f(\hat{z})$, $\hat{u}_e(\hat{z})$, $\hat{p}_f(\hat{z})$, and $\hat{p}_e(\hat{z})$. From these solutions,
 178 two important summary metrics can be easily extracted. The first is the net
 179 pressure drop between the feed-channel inlet and the exhaust-channel outlet,

$$\Delta \hat{p} = \hat{p}_{f,1} - \hat{p}_{e,J}. \quad (27)$$

180 The net physical pressure drop follows as

$$\Delta p = p_f(0) - p_e(L) = \rho U_{\text{in}}^2 \Delta \hat{p}. \quad (28)$$

181 The other important performance metric is the uniformity of the porous-
 182 media mass flux along the length of the channel pair. The local dimensionless
 183 flux is written as

$$\hat{m}'' = \Omega (\hat{p}_f - \hat{p}_e). \quad (29)$$

184 The relative uniformity along the length of the channel pair can be repre-
 185 sented as

$$\Delta\hat{m} = \frac{\max[\hat{m}''] - \min[\hat{m}'']}{\max[\hat{m}'']}. \quad (30)$$

186 Usually, the desired result is $\Delta\hat{m} \ll 1$.

187 Figure 3 is a composite contour plot that maps $\Delta\hat{p}$ and $\Delta\hat{m}$ as functions
 188 of Ω and Γ . The map shows that reducing the values of Ω and Γ leads to
 189 good uniformity, but at the cost of higher pressure drop. The practical oper-
 190 ating space is usually in the lower regions of the diagonal contours showing
 191 good flow uniformity. Most of the parameters that make up the dimension-
 192 less groups Ω and Γ are physical parameters that are easily understood and
 193 evaluated. However, the parameter K (Eq. 3) must be derived from the
 194 flow through the porous media. The generalization and evaluation of K is
 195 discussed in the following sections.

196 Although the dimensionless map (Fig. 3) shows a great deal of summary
 197 information, it is also informative to consider a few profiles along the chan-
 198 nel length. Figure 4a shows profiles of dimensionless velocities, pressures,
 199 and mass fluxes along the dimensionless channel length with $\Gamma = 1000$ and
 200 $\Omega = 10^{-2}$. As is always the case, the feed-channel velocity decreases mono-
 201 tonically and the exhaust-channel velocity increases monotonically. Under
 202 these conditions, the velocity profiles are found to be significantly nonlinear,
 203 with both upward and downward curvatures. The pressure drop is relatively
 204 large and also nonlinear. The mass flux between the channels varies greatly
 205 along the channel length, with a minimum near the mid-length of the chan-
 206 nels. As seen from the black dot on the inset map (reproduction of Fig. 3),
 207 the flux nonuniformity is high and pressure drop is also high. Because the

208 profiles and the characteristic parameters (Γ and Ω) are dimensionless, the
209 profiles represent a wide range of profiles in physical (dimensional) terms.

210 The nonlinear behavior of the velocity and pressure profiles is caused by
211 nonlinear physical interactions. The channel velocities are affected directly
212 by mass exchange between the channels and the continuity equations (Eqs. 1
213 and 2). Pressure varies along the channel length owing to the momentum
214 balance and viscous drag at the channel walls (Eqs. 8 and 9). The nonlinear
215 interactions between velocity and pressure lead to possibly unanticipated
216 results, such as mass-exchange minima along the channel length.

217 Figure 4b shows profiles of dimensionless velocities, pressures, and mass
218 fluxes along the dimensionless channel length with $\Gamma = 1.0$ and $\Omega = 0.5$.
219 Assuming, for example, that the channel dimensions and inlet velocity are
220 comparable to the results shown in Fig. 4a, the large increase in Ω is at-
221 tributed to increased permeability of the porous media (i.e., K). Under
222 these conditions, the shapes of the velocity profiles are qualitatively different
223 from those in Fig. 4a (i.e., there are no inflections in the profiles). Because
224 of the relatively low flow impedance through the porous media, the pressure
225 variations are much smaller than those in Fig. 4a, and qualitatively quite
226 different. Unlike the results shown in Fig. 4a, the pressure profile in the
227 feed channel *increases* toward the exit end. The mass-flux profile increases
228 monotonically, with a minimum near the inlet and maximum near the exit.
229 Although the magnitude of the dimensionless flux in Fig. 4b is comparable
230 to that shown in Fig. 4a, the behavior is qualitatively different.

231 Figure 4c shows profiles of dimensionless velocities, pressures, and mass
232 fluxes along the dimensionless channel length with $\Gamma = 1.0$ and $\Omega = 10^{-2}$.

233 Under these conditions, the flow impedance through the porous media is
 234 relatively large. This causes the pressures within the channels to be nearly
 235 spatially uniform. The result is nearly uniform mass flux between the chan-
 236 nels, which is usually a desirable result. The velocity profiles are nearly linear,
 237 caused by the nearly uniform flux between the channels. For a particular Ω ,
 238 the magnitude of the mass flux can be increased by increasing Γ . However,
 239 if Γ increases too greatly, the uniformity can be compromised (cf., Fig. 3).
 240 Another alternative for increasing mass fluxes, but retaining uniformity, is
 241 to reduce Ω and increase Γ . However, this comes at the cost of increasing
 242 overall pressure drop.

243 4. Porous-media flow

244 Flow through the porous media that connects the feed and exhaust chan-
 245 nels is assumed to be represented as Darcy flow. In this case, the superficial
 246 velocity $\mathbf{q}$ may be represented as

$$\mathbf{q} = -\frac{\mathcal{K}}{\mu} \nabla p, \quad (31)$$

247 where $\mathcal{K}$ is the porous-media permeability, μ is the fluid dynamic viscosity,
 248 and p is the pressure. The conservation equation to be solved is

$$\nabla \cdot \mathbf{q} = \nabla \cdot \left(-\frac{\mathcal{K}}{\mu} \nabla p \right) = 0. \quad (32)$$

249 Assuming constant values of $\mathcal{K}$ and μ , the Darcy problem reduces to

$$\nabla^2 p = 0. \quad (33)$$

250 Solving this elliptic partial differential equation requires boundary conditions.

251 Pressures are specified at the interface between the porous electrode and
 252 the floors of the flow channels. All other boundary conditions are specified
 253 as $\mathbf{n} \cdot \nabla p = 0$, meaning that no flow crosses these boundaries. The interfaces
 254 with the bipolar plate and the membrane are essentially solid boundaries.
 255 The right and left boundaries are symmetry planes.

256 As with the channel-flow problem, the porous-media problem can be gen-
 257 eralized in a dimensionless form. In this case, the two-dimensional Darcy
 258 problem uses the following dimensionless variables:

$$\hat{p} = \frac{p}{p_f - p_e}, \quad \hat{x} = \frac{x}{W_e}, \quad \hat{y} = \frac{y}{W_e}. \quad (34)$$

259 The pressure is normalized by the pressure difference between channels $\Delta p =$
 260 $p_f - p_e$ and the spatial coordinates are normalized by the electrode width W_e .
 261 Additionally, two geometric parameters are needed. The aspect ratio of the
 262 porous electrode can be represented as

$$\gamma = \frac{H_e}{W_e}. \quad (35)$$

263 The ratio of channel width and electrode width can be represented as

$$\delta = \frac{W_c}{W_e}. \quad (36)$$

264 With these definitions in mind, Fig. 5 illustrates the dimensionless electrode
 265 geometry.

266 In dimensionless variables, the conservation problem to be solved within
 267 the porous media is:

$$\nabla^2 \hat{p} = 0. \quad (37)$$

268 The dimensionless pressure at the feed channel is $\hat{p} = 1$ and at the exhaust
 269 channel is $\hat{p} = 0$. All other boundaries must satisfy $\mathbf{n} \cdot \nabla \hat{p} = 0$. It is important

270 to note that the solution to the dimensionless Darcy-flow problem depends
 271 *only* on the dimensionless domain dimensions γ and δ .

272 Once the pressure field is determined for a particular geometry and porous
 273 media, the solution can be used to establish the parameter K in Eq. 3. The
 274 process begins by evaluating the net flow rate Q (per unit depth, kg m^{-1}
 275 s^{-1}) between the feed and exhaust channels, as

$$Q = \rho \frac{\mathcal{K}}{\mu} \int_0^{H_e} \frac{\partial p}{\partial x} dy, \quad (38)$$

276 where ρ is the fluid's mass density. Because the domain is symmetric, the
 277 integral can be evaluated at the mid plane of the porous electrode (i.e., at
 278 $W_e/2$). The mass flux ($\text{kg m}^{-2} \text{s}^{-1}$), referenced to the full channel-floor
 279 width, follows as

$$m'' = \frac{2Q}{W_c}. \quad (39)$$

280 Note that the flux from each feed channel supplies two exhaust channels, one
 281 on the left and one on the right (cf., Fig. 6). Thus, the “2” in the numerator
 282 accounts for the flow between adjacent channels. The parameter K (Eq. 3)
 283 follows as

$$K = \frac{m''}{p_f - p_e}, \quad (40)$$

284 where $p_f - p_e$ is the local pressure difference between the feed and exhaust
 285 channels. Because the problem is linear and the properties are assumed to
 286 be constant, the value of K does not depend on the value of the pressure
 287 difference.

288 The parameter K can now be written in terms of the solution to the di-
 289 mensionless Darcy problem. The net mass flow rate at the electrode midplane

290 (Eq. 38) becomes

$$Q = \rho \left(\frac{\mathcal{K} \Delta p}{\mu} \right) \int_0^\gamma \frac{\partial \hat{p}}{\partial \hat{x}} d\hat{y}. \quad (41)$$

291 It then follows that

$$K = \frac{2\rho\mathcal{K}}{\mu W_c} \int_0^\gamma \frac{\partial \hat{p}}{\partial \hat{x}} d\hat{y} = \frac{2\rho\mathcal{K}}{\mu W_c} \hat{Q}. \quad (42)$$

292 To simplify the nomenclature somewhat, it is convenient to define

$$\hat{Q} \equiv \int_0^\gamma \frac{\partial \hat{p}}{\partial \hat{x}} d\hat{y}. \quad (43)$$

293 By solving the dimensionless Darcy problem for a wide range of γ and δ , a
 294 general expression for $\hat{Q}(\gamma, \delta)$ can be developed. The permeability $\mathcal{K}$ of a
 295 particular porous structure is needed to evaluate the parameter K . Perme-
 296 ability data is usually available from manufacturer’s specifications, but can
 297 certainly be measured independently.

298 There are numerous approaches for solving the Darcy problem (Eq. 37),
 299 which is a linear, elliptic, partial differential equation in two-dimensional
 300 cartesian coordinates. The present study begins with a finite-volume dis-
 301 cretization on a nonuniform (201×401) mesh network. The resulting al-
 302 gebraic equations are solved iteratively using a preconditioned conjugate-
 303 gradient method (incomplete LU preconditioner) as implemented in the MAT-
 304 LAB “bicgstab” function. Figure 7 shows representative solutions for fixed
 305 $\delta = 0.4$ and three values of γ .

306 The Darcy problem is solved for 100 values of both $0.05 \leq \gamma \leq 1.5$ and
 307 $0.05 \leq \delta \leq 1$, producing a matrix of 10,000 total simulations. Figure 8 shows
 308 contour maps of $\hat{Q}(\gamma, \delta)$. For any combination of γ and δ , $\hat{Q}(\gamma, \delta)$ can be
 309 evaluated from the contour maps (the Appendix provides analytical fits to

the contour maps). Then, the needed value of K follows directly from Eq. 42 for any particular fluid and porous structure.

5. Illustrative example

To illustrate the use of the dimensionless analysis, consider a particular problem motivated by a redox flow battery. Table 1 lists a set of physical parameters as well as some of the derived dimensionless groups. Based on these parameters and operating conditions, the first step is to evaluate $\hat{Q}$ using Fig. 8 or the fits in the Appendix. In this case, $\hat{Q} \approx 0.2$, from which, using Eq. 42, the mass-flux parameter $K \approx 6 \times 10^{-4}$. Once K is evaluated, the two dimensionless groups (Eqs. 16 and 17) describing the channel performance can be evaluated as $\Omega = 1.5 \times 10^{-2}$ and $\Gamma = 36.7$.

Using the values of Ω and Γ , Fig. 3 can be used to extract the dimensionless pressure drop through the channels and flow uniformity through the porous structure. Specifically, $\Delta\hat{p} \approx 10^{1.8}$ and $\Delta\hat{m} \approx 0.15$. For the parameters shown in Table 1, these circumstances lead to a net pressure drop of $\Delta p \approx 5300$ Pa, with approximately 15% variation in the local mass fluxes between channels.

Although $\Delta\hat{p}$ and $\Delta\hat{m}$ provide some very useful summary information, these results alone do not provide detailed information about profiles of velocity, pressure, and mass flux along the channels. Determining these details requires the full solution, which cannot be represented in a single graph. Nevertheless, the paper provides the information needed to program the solution. Assuming some working knowledge of computational mathematics and access to software such as MATLAB, writing the code is reasonably straightforward.

Table 1: Nominal parameters for an illustrative problem

Fluid and flow	
Density, ρ	1350 kg m ⁻³
Viscosity, μ	6×10^{-3} Pa s
Inlet velocity, U_{in}	0.25 m s ⁻¹
Channels	
Width, W_c	1.5 mm
Height, H_c	1.0 mm
Length, L_c	100 mm
Perimeter, P_c	5 mm
Area, A_c	1.5×10^{-6} m ²
Hydraulic diameter, D_h	1.2 mm
Aspect ratio, α	1.5
<i>Ref</i>	14.8
Porous electrode	
Width, W_e	5 mm
Height, H_e	1 mm
Aspect ratio, γ	0.2
Channel ratio, δ	0.3
Permeability, $\mathcal{K}$	10^{-11} m ²
Dimensionless groups	
Ω (Eq. 16)	1.5×10^{-2}
Γ (Eq. 17)	3.67

334 A particular channel solution requires only a few seconds in a typical personal
335 computer.

336 **6. Summary and conclusions**

337 This paper documents an analysis of the flow field within pairs of interdig-
338 itated channels that are connected by porous media beneath the channels.
339 Although the physical model itself can be even more general, a particular
340 objective here is to develop a model that can be expressed generally in di-
341 mensionless terms, with summary results being easily represented. Achiev-
342 ing this objective is facilitated by assuming incompressible and isothermal
343 flow with constant properties. The solutions are represented in terms of
344 three dimensionless groups, which involve fluid properties, channel geome-
345 try, porous-media configuration and properties, as well as operating condi-
346 tions. Net pressure drop and flow non-uniformity are presented in terms of a
347 single dimensionless contour map. However, the computational model itself
348 is needed to evaluate more details about velocity, pressure, and mass flux
349 profiles along the channel length.

350 **7. Acknowledgements**

351 This research is supported by the Office of Naval Research via grant
352 N00014-12-1-0201. We gratefully acknowledge insightful discussions with Dr.
353 Andrew Colclasure (ITN Energy Systems) on flow batteries and with Prof.
354 Nils Tilton (CSM) on aspects of porous-media transport.

References

- [1] Q. Xu, T.S. Zhao, and P.K. Leung. Numerical investigations of flow field designs for vanadium redox flow batteries. *Appl. Energy*, 105:47–56, 2013.
- [2] X. Ke, J.I.D. Alexander, J.M. Prah, and R.F. Savinell. Flow distribution and maximum current density studies in redox flow batteries with a single passage of the serpentine flow channel. *J. Power Sources*, 270:646–657, 2014.
- [3] C. Yin, Y. Gao, S. Guo, and H. Tang. A coupled three dimensional model of vanadium redox flow battery for flow field designs. *Energy*, 74:886–895, 2014.
- [4] R.M. Darling and M.L. Perry. The influence of electrode and channel configurations on flow battery performance. *J. Electrochem. Soc.*, 161:A1381–A1387, 2014.
- [5] T. Jyothi Latha and S. Jayanti. Hydrodynamic analysis of flow fields of redox flow battery applications. *J. Appl. Electrochem.*, 44:995–1006, 2014.
- [6] K.B. Shyam Prasad, S. Maharudrayya, and S. Jayanti. Flow maldistribution in interdigitated channels used in PEM fuel cells. *J. Power Sources*, 159:595–604, 2006.
- [7] X.-D. Wang, Y.-Y. Duan, W.-M. Yan, and X.-F. Peng. Effects of flow channel geometry on cell performance for PEM fuel cells with parallel and interdigitated flow fields. *Electrochimica Acta*, 53:5334–5343, 2008.

- [8] P. Hu, L. Peng, W. Zhang, and X. Lai. Optimization design of slotted-interdigitated channel for stamped thin metal bipolar plate in proton exchange membrane fuel cell. *J. Power Sources*, 187:407–414, 2009.
- [9] K.B. Shyam Prasad, P.V. Suresh, and S. Jayanti. A hydrodynamic network model for interdigitated flow fields. *Intl. J. Hydrogen Energy*, 34:8289–8301, 2009.
- [10] X.-D. Wang, X.-X. Zhang, W.-M. Yan, D.-J. Lee, and A. Su. Determination of the optimal active area for proton exchange membrane fuel cells with parallel, interdigitated or serpentine designs. *Intl. J. Hydrogen Energy*, 34:3823–3832, 2009.
- [11] G. Hu, J. Fan, S. Chen, Y. Liu, and K. Cen. Three-dimensional numerical analysis of proton exchange membrane fuel cells (PEMFCs) with conventional and interdigitated flow fields. *J. Power Sources*, 136:1–9, 2010.
- [12] J. Lobato, P. Cañizares, M. A. Rodrigo, F.J. Pinar, and D. Úbeda. Study of flow channel geometry using current distribution measurement in a high temperature polymer electrolyte membrane fuel cell. *J. Power Sources*, 196:4209–4217, 2011.
- [13] H.-W. Ku and H.-W. Wu. Influences of operational factors on proton exchange membrane fuel cell performance with modified interdigitated flow field design. *J. Power Sources*, 232:199–208, 2013.
- [14] A.D. Santamaria, N.J. Cooper, M.K. Becton, and J.W. Park. Effect of channel length on interdigitated flow-field PEMFC performance: A com-

- 401 putational and experimental study. *Intl. J. Hydrogen Energy*, 38:16253–
402 16263, 2013.
- 403 [15] A.D. Santamaria, J. Bachman, and J.W. Park. Design strategy for a
404 polymer electrolyte membrane fuel cell flow-field capable of switching
405 between parallel and interdigitated configurations. *Intl. J. Hydrogen*
406 *Energy*, 38:5807–5812, 2013.
- 407 [16] H. Guo, M.H. Wang, J.X. Liu, Z.H. Nie, F. Ye, and C.F. Ma. Tem-
408 perature distribution on anodic surface of membrane electrode assembly
409 in proton exchange membrane fuel cell with interdigitated flow bed. *J.*
410 *Power Sources*, 273:775–783, 2015.
- 411 [17] J.P. Feser, A.K. Prasad, and S.G. Advani. On the relative influence of
412 convection in serpentine flow fields of PEM fuel cells. *J. Power Sources*,
413 161:404–412, 2006.
- 414 [18] Q. Ye, T.S. Zhao, and C. Xu. The role of under-rib convection in mass
415 transport of methanol through the serpentine flow field and its neigh-
416 boring porous layer in a DMFC. *Electrochimica Acta*, 51:5420–5429,
417 2006.
- 418 [19] C. Xu and T.S. Zhao. A new flow field design for polymer electrolyte-
419 based fuel cells. *Electrochemistry Comm.*, 9:497–503, 2007.
- 420 [20] R.J. Kee, P. Korada, K. Walters, and M. Pavol. A generalized model
421 of flow distribution in channel networks of planar fuel cells. *J. Power*
422 *Sources*, 109:148–159, 2002.

- 423 [21] R.J. Kee, M.E. Coltrin, and P. Glarborg. *Chemically Reacting Flow*.
424 Wiley, Hoboken, NJ, 2003.

425 Appendix A. Fits to summary results

426 Figures 3 and 8 show summary results in graphical form. However, espe-
 427 cially in a computational setting, it can be practically useful to have analytic
 428 representations of these results. The following sections report fits to the
 429 surfaces shown as contours in Figs. 3 and 8.

430 Appendix A.1. Mass flow through porous structure

431 The dimensionless mass flow rate $\hat{Q}(\gamma, \delta)$ can be fit using

$$\hat{Q}(\gamma, \delta) = A_Q \left[1 - \exp \left(-\frac{\gamma}{B_Q} \right) \right] \quad (\text{A.1})$$

432 where $A_Q(\delta)$ and $B_Q(\delta)$ are the 10th-order polynomials as

$$A_Q(\delta) = \sum_{i=0}^{10} a_{Q,i} \delta^i, \quad B_Q(\delta) = \sum_{i=0}^{10} b_{Q,i} \delta^i. \quad (\text{A.2})$$

433 Table Appendix A.1 lists the polynomial coefficients $a_{Q,i}$ and $b_{Q,i}$.

Table A.1: Fitting parameters for $\hat{Q}(\gamma, \delta)$

i	$a_{Q,i}$	$b_{Q,i}$
0	1.101056×10^{-01}	1.152951×10^{-01}
1	$1.110693 \times 10^{+00}$	$1.670473 \times 10^{+00}$
2	$1.492183 \times 10^{+01}$	$-2.070772 \times 10^{+01}$
3	$-3.131708 \times 10^{+02}$	$1.695987 \times 10^{+02}$
4	$2.461064 \times 10^{+03}$	$-8.574868 \times 10^{+02}$
5	$-1.041577 \times 10^{+04}$	$2.742538 \times 10^{+03}$
6	$2.624356 \times 10^{+04}$	$-5.656151 \times 10^{+03}$
7	$-4.050821 \times 10^{+04}$	$7.494034 \times 10^{+03}$
8	$3.757392 \times 10^{+04}$	$-6.156397 \times 10^{+03}$
9	$-1.922792 \times 10^{+04}$	$2.852407 \times 10^{+03}$
10	$4.172088 \times 10^{+03}$	$-5.695567 \times 10^{+02}$

434 *Appendix A.2. Net pressure drop*

435 The net pressure drop (i.e., between inlet to the feed channel and exit of
436 the exhaust channel) can be fit using a 5th-order polynomial as

$$\tilde{p}(\tilde{\omega}, \tilde{\gamma}) = \sum_{i=0}^5 A_{p,i} \tilde{\omega}^i, \quad (\text{A.3})$$

437 where $\tilde{p} = \log_{10}(\Delta\hat{p})$, $\tilde{\omega} = \log_{10}(\Omega)$, and $\tilde{\gamma} = \log_{10}(\Gamma)$. The coefficients $A_{p,i}$
438 are fit using 10th-order polynomials as

$$A_{p,i}(\tilde{\gamma}) = \sum_{j=0}^{10} a_{p,ij} \tilde{\gamma}^j. \quad (\text{A.4})$$

439 Table A.2 lists the fitting coefficients.

440 *Appendix A.3. Mass-flux uniformity*

441 Taking the logarithm of $\Delta\hat{p}$, $\Delta\hat{m}$, Ω , and Γ , and denoting, $\tilde{p} = \log_{10}(\Delta\hat{p})$,
442 $\tilde{m} = \log_{10}(\Delta\hat{m})$, $\tilde{\omega} = \log_{10}(\Omega)$, and $\tilde{\gamma} = \log_{10}(\Gamma)$, the function of uniformity
443 $\tilde{m}(\tilde{\omega}, \tilde{\gamma})$ can be fitted using the following expression,

$$\tilde{m}(\tilde{\omega}, \tilde{\gamma}) = \left(\sum_{i=0}^5 A_{m,i} \tilde{\omega}^i \right) \exp(-B_m \tilde{\omega}) \quad (\text{A.5})$$

444 where $A_{m,i}$ and $B_{m,i}$ are functions of $\tilde{\gamma}$, and can be represented using the
445 10th-order polynomials as,

$$A_{m,i}(\tilde{\gamma}) = \sum_{j=0}^{10} a_{m,ij} \tilde{\gamma}^j, \quad B_m(\tilde{\gamma}) = \sum_{j=0}^{10} b_{m,j} \tilde{\gamma}^j. \quad (\text{A.6})$$

446 The fitted polynomial coefficients $a_{m,ij}$ and $b_{m,j}$ are listed in Table A.3 for
447 $\tilde{\gamma} \leq 1$, and Table A.4 when $\tilde{\gamma} \geq 1$.

Table A.2: Fitting parameters for $\tilde{p}(\omega, \tilde{\gamma})$

j	$a_{p,0j}$	$a_{p,1j}$	$a_{p,2j}$	$a_{p,3j}$	$a_{p,4j}$	$a_{p,5j}$
0	2.952916×10^{-1}	-4.768488×10^{-1}	3.035519×10^{-1}	5.370931×10^{-2}	-4.897395×10^{-3}	-1.607541×10^{-3}
1	2.975381×10^{-1}	1.397331×10^{-1}	-2.061557×10^{-1}	-1.786723×10^{-1}	-4.866813×10^{-2}	-4.521380×10^{-3}
2	1.637493×10^{-1}	3.728268×10^{-1}	3.864130×10^{-1}	1.760230×10^{-1}	3.705088×10^{-2}	3.167989×10^{-3}
3	3.199340×10^{-1}	-1.201263×10^{-1}	-6.613754×10^{-1}	-4.136090×10^{-1}	-9.878804×10^{-2}	-8.706759×10^{-3}
4	-3.673190×10^{-1}	-1.546238×10^{-1}	5.312483×10^{-1}	4.518205×10^{-1}	1.282925×10^{-1}	1.273242×10^{-2}
5	2.000377×10^{-1}	7.769896×10^{-2}	-2.665186×10^{-1}	-2.058297×10^{-1}	-5.256612×10^{-2}	-4.756107×10^{-3}
6	-6.987056×10^{-2}	1.055775×10^{-2}	9.983546×10^{-2}	3.646682×10^{-2}	-7.545186×10^{-4}	-9.615480×10^{-4}
7	1.673354×10^{-2}	-1.488955×10^{-2}	-2.807092×10^{-2}	1.358594×10^{-3}	6.155231×10^{-3}	1.086590×10^{-3}
8	-2.678815×10^{-3}	3.971189×10^{-3}	5.237407×10^{-3}	-1.409679×10^{-3}	-1.765385×10^{-3}	-2.911634×10^{-4}
9	2.542303×10^{-4}	-4.530965×10^{-4}	-5.503529×10^{-4}	1.842700×10^{-4}	2.076246×10^{-4}	3.415335×10^{-5}
10	-1.059536×10^{-5}	1.949852×10^{-5}	2.422543×10^{-5}	-7.797240×10^{-6}	-9.135636×10^{-6}	-1.525178×10^{-6}

Table A.3: Fitting parameters for $\tilde{m}(\tilde{\omega}, \tilde{\gamma})$ when $\tilde{\gamma} \leq 1$

j	$a_{m,0j}$	$a_{m,1j}$	$a_{m,2j}$	$a_{m,3j}$	$a_{m,4j}$	$a_{m,5j}$	$b_{m,j}$
0	-1.174523×10^{-1}	3.916152×10^{-1}	-5.226555×10^{-1}	2.069247×10^{-1}	3.537317×10^{-2}	1.018871×10^{-2}	-4.730262×10^{-1}
1	-1.061809×10^{-1}	3.770835×10^{-2}	2.434254×10^{-1}	-1.582434×10^{-1}	-1.313394×10^{-2}	-1.539116×10^{-2}	2.689791×10^{-1}
2	-4.913866×10^{-2}	1.155462×10^{-1}	5.434263×10^{-2}	-1.100394×10^{-1}	-2.468978×10^{-2}	2.949099×10^{-4}	1.097435×10^{-1}
3	2.093015×10^{-1}	-5.296718×10^{-2}	-4.963768×10^{-1}	3.213661×10^{-1}	1.065397×10^{-2}	3.390043×10^{-2}	-5.771348×10^{-1}
4	3.500668×10^{-1}	-7.147759×10^{-1}	3.748627×10^{-2}	2.951543×10^{-1}	1.157369×10^{-1}	-8.309008×10^{-4}	-6.776838×10^{-2}
5	-1.539076×10^{-1}	-3.207333×10^{-1}	$1.142300 \times 10^{+0}$	-6.432772×10^{-1}	1.951836×10^{-2}	-6.607554×10^{-2}	$1.367168 \times 10^{+0}$
6	-5.476059×10^{-1}	$1.197047 \times 10^{+0}$	-1.335939×10^{-1}	-4.606415×10^{-1}	-2.182530×10^{-1}	5.676734×10^{-3}	1.496701×10^{-2}
7	5.480271×10^{-2}	4.681452×10^{-1}	$-1.180167 \times 10^{+0}$	6.357828×10^{-1}	-3.857134×10^{-2}	6.475134×10^{-2}	$-1.422921 \times 10^{+0}$
8	4.143471×10^{-1}	$-1.002356 \times 10^{+0}$	2.002798×10^{-1}	3.379531×10^{-1}	1.946131×10^{-1}	-1.002785×10^{-2}	1.074867×10^{-1}
9	-5.051272×10^{-3}	-1.988496×10^{-1}	4.423315×10^{-1}	-2.313365×10^{-1}	1.884884×10^{-2}	-2.362500×10^{-2}	5.368818×10^{-1}
10	-1.236842×10^{-1}	3.300208×10^{-1}	-1.015483×10^{-1}	-8.791399×10^{-2}	-6.541124×10^{-2}	5.343843×10^{-3}	-8.170307×10^{-2}

Table A.4: Fitting parameters for $\tilde{m}(\tilde{\omega}, \tilde{\gamma})$ when $\tilde{\gamma} \geq 1$

j	$a_{m,0j}$	$a_{m,1j}$	$a_{m,2j}$	$a_{m,3j}$	$a_{m,4j}$	$a_{m,5j}$	$b_{m,j}$
0	-1.311010×10^{-1}	3.962813×10^{-1}	-4.775535×10^{-1}	1.862075×10^{-1}	2.931367×10^{-2}	8.072482×10^{-3}	-4.043438×10^{-1}
1	6.438208×10^{-2}	-6.403823×10^{-2}	-1.116558×10^{-1}	-2.372317×10^{-2}	4.600101×10^{-2}	-1.164392×10^{-3}	-3.194176×10^{-1}
2	-9.117564×10^{-1}	8.144598×10^{-1}	$1.038953 \times 10^{+0}$	-3.963680×10^{-1}	-2.483229×10^{-1}	-2.447999×10^{-2}	$1.979008 \times 10^{+0}$
3	$2.452722 \times 10^{+0}$	$-2.126668 \times 10^{+0}$	$-2.023366 \times 10^{+0}$	7.748498×10^{-1}	4.682953×10^{-1}	4.235916×10^{-2}	$-3.841431 \times 10^{+0}$
4	$-2.692825 \times 10^{+0}$	$1.869012 \times 10^{+0}$	$2.341197 \times 10^{+0}$	-7.181301×10^{-1}	-3.987703×10^{-1}	-2.564894×10^{-2}	$4.263094 \times 10^{+0}$
5	$1.615415 \times 10^{+0}$	-7.978889×10^{-1}	$-1.593843 \times 10^{+0}$	3.898706×10^{-1}	1.737858×10^{-1}	3.051504×10^{-3}	$-2.742465 \times 10^{+0}$
6	-5.883108×10^{-1}	1.726612×10^{-1}	6.583688×10^{-1}	-1.343775×10^{-1}	-3.841466×10^{-2}	3.501752×10^{-3}	$1.079398 \times 10^{+0}$
7	1.343107×10^{-1}	-1.347061×10^{-2}	-1.681157×10^{-1}	3.008053×10^{-2}	2.853006×10^{-3}	-1.876968×10^{-3}	-2.645503×10^{-1}
8	-1.886273×10^{-2}	-1.573203×10^{-3}	2.602982×10^{-2}	-4.261129×10^{-3}	4.513208×10^{-4}	4.164888×10^{-4}	3.945360×10^{-2}
9	1.494406×10^{-3}	3.840577×10^{-4}	-2.245164×10^{-3}	3.477740×10^{-4}	-1.038662×10^{-4}	-4.506979×10^{-5}	-3.281384×10^{-3}
10	-5.123284×10^{-5}	-2.154027×10^{-5}	8.291442×10^{-5}	-1.245851×10^{-5}	5.884669×10^{-6}	1.952644×10^{-6}	1.169541×10^{-4}

List of Figures

1. Illustration of a segment of redox flow battery with interdigitated channels in the bipolar plates. The anode and cathode structures are separated by ion-transport membranes.
2. Finite-volume stencil for modeling the two-channel interdigitated flow problem.
3. Dimensionless map of net pressure drop and porous-media flux uniformity.
4. Profiles of dimensionless velocities, pressures, and mass fluxes for three values of Γ and Ω . The black dots in the inset maps (cf., Fig. 3) show the position of the dimensionless channel and operating parameters.
5. Physical domain for modeling porous-media flow in dimensionless coordinates.
6. Illustration of the two-interdigitated-channel computational domain. The computational domain for the porous media is noted by the dashed line.
7. Example solutions of the dimensionless Darcy problem. The solutions illustrated here all use a channel width of $\delta = 0.4$, but with three different values of the porous-media aspect ratio γ .
8. Contour maps of $\hat{Q}(\gamma, \delta)$ as functions of dimensionless porous media aspect ratio γ and dimensionless channel width δ . The upper panel shows the full γ - δ space. The lower panel shows an expanded region as marked by the dashed rectangle in the upper panel.

Figure 1: Illustration of a segment of redox flow battery with interdigitated channels in the bipolar plates. The anode and cathode structures are separated by ion-transport membranes.

Figure 2: Finite volume stencil for the two-channel interdigitated flow problem.

Figure 3: Dimensionless map of net pressure drop and porous-media flux uniformity.

Figure 4: Profiles of dimensionless velocities, pressures, and mass fluxes for three values of Γ and Ω . The black dots in the inset maps (cf., Fig. 3) show the position of the dimensionless channel and operating parameters.

Figure 5: Physical domain for modeling porous-media flow in dimensionless coordinates.

Figure 6: Illustration of the two-interdigitated-channel computational domain. The computation domain for the porous media is noted by the dashed line.

Figure 7: Example solutions of the dimensionless Darcy problem. The solutions illustrated here all use a channel width of $\delta = 0.4$, but with three different values of the porous-media aspect ratio γ .

Figure 8: Contour maps of $\hat{Q}(\gamma, \delta)$ as functions of dimensionless porous media aspect ratio γ and dimensionless channel width δ . The upper panel shows the full γ - δ space. The lower panel shows an expanded region as marked by the dashed rectangle in the upper panel.